Dynamic Optimization
Introductory Remarks

optimization problems: defined over period of time

- date to be specified: $x(t), y(t)$

1 Two classes of variables: flows vs. stocks

- flow $x(t)$
  - measured as quantity **per unit of time**
    time dimension needed to give the variable a meaning

  - dimension of $x(t)$: *quantity/time*
    number of births ? → *per month*
    income of 20.000$? → *per month*
    rains at rate of 200 l/m^2 ? → *per day*
    investment at rate of 200 € ? → *per month*
    fisher harvests on average 2 tons of fish?
    → *per day*

- stock $y(t)$
  - measured as quantity at t
    not over a period of time

  - dimension of $y(t)$: *quantity*
    population
    wealth of a person
    depth of flood (water) at given location
    amount of money in pension fund
    stock of fish in the sea

- flows add to/detract from stocks: $\dot{y}(t) = g(x(t))$
  more generally: $\dot{y}(t) = g(t, y(t), x(t))$

- in mathematics
  stocks $\longrightarrow$ "state variables" or "states"
  flows $\longrightarrow$ "control variables"

- dynamic optimization

  - control variable:
    - decision variable for the agent
    - affects state variable
    - can jump from one t to another t

  - state variable:
    - describes state of dynamic system
    - evolves continuously (cannot jump)
    - is affected by some control variable

  $\longrightarrow$ is vehicle by which $x(t)$ has impact on future

  present decisions affect future events by making certain opportunities available only if
  $\dot{y}(t) = g(t, y(t), x(t))$

  if $x(t) \nrightarrow \dot{y}(t)$: **static** optimization problem

problem	oil extraction
control variable $x(t)$	rate of oil extraction per unit of time
state variable $y(t)$	amount of oil in reservoir: $\dot{y}(t) = -x(t)$

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problem	economic growth
control variable $x(t)$	rate of investment per unit of time
state variable $y(t)$	capital: $\dot{y}(t) = x(t) - \delta\, y(t)$

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problem	catching fish
control variable $x(t)$	harvest of fish per unit of time
state variable $y(t)$	fish stock in sea: $\dot{y}(t) = a\, y(t) - b\, y(t)^2 - x(t)$

- over a specified period, $t \in [t_0, t_1]$, choose $x(t) \in X(t)$, for all t, such as to optimize an objective function, and observe $\dot{y}(t) = g(t, x(t), y(t))$.

- objective function:

$$\int_{t_0}^{t_1} f(t, x(t), y(t))\, dt$$

2 Differential equations

$\dot{y}(t) = g(t, x(t), y(t))$

equation

unknown is not a variable but a function of t

includes derivative $\dot{y}(t)$

- examples

  $\dot{y}(t) = -x(t)$

  $\dot{y}(t) = x(t) - \delta\, y(t)$

  $\dot{y}(t) = a\, y(t) - b\, y(t)^2 - x(t)$

  $\dot{y}(t) = a\, y(t)$

  $\rightarrow$ first order ODE

  $\rightarrow$ linear ODE (except for 3^{rd})

- solution: $y(t)$ as a function of t and parameters
  infinitely many solutions exist

  first order ODE: sol includes 1 arbitrary constant
  definitized by one initial condition $y(t_0) = y_0$
  E.g.: $\dot{y}(t) = a\, y(t) \rightarrow y(t) = C\, e^{a\, t} \rightarrow y(t) = y_0\, e^{a\, t}$

3 Qualitative analysis

- equilibrium (stationary state): $\dot{x}(t) = 0$

- stability of equilibrium
  stability $\frac{\partial \dot{x}(t)}{\partial x(t)} < 0$
  instability $\frac{\partial \dot{x}(t)}{\partial x(t)} > 0$

E.g.: $\dot{x}(t) + a\,x(t) = b$

- equilibrium (stationary state): $\dot{x}(t) = 0 \rightarrow x(t) = x = b/a$

- stability of equilibrium

  stability $\frac{\partial \dot{x}(t)}{\partial x(t)} < 0$: $a > 0$

  instability $\frac{\partial \dot{x}(t)}{\partial x(t)} > 0$: $a < 0$

- phase diagram: $(x, \dot{x})$ space

  - different phase diagram with
    system of two differential equations

Example. $\dot{x}(t) + a\,x(t) = b$, $a = 0.1, b = 10$.

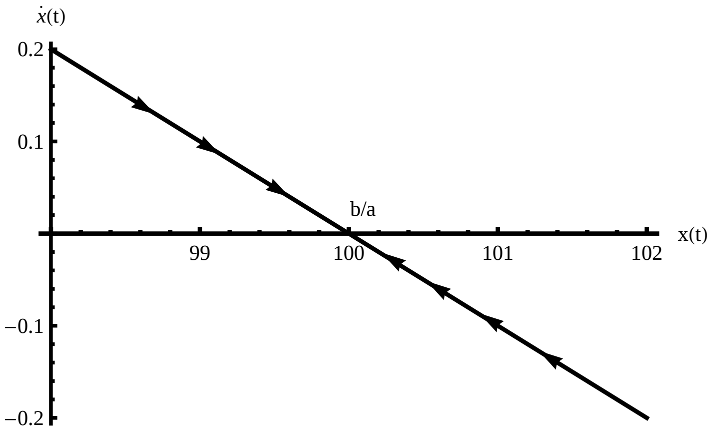

Figure 1: Phase diagram (single ODE)

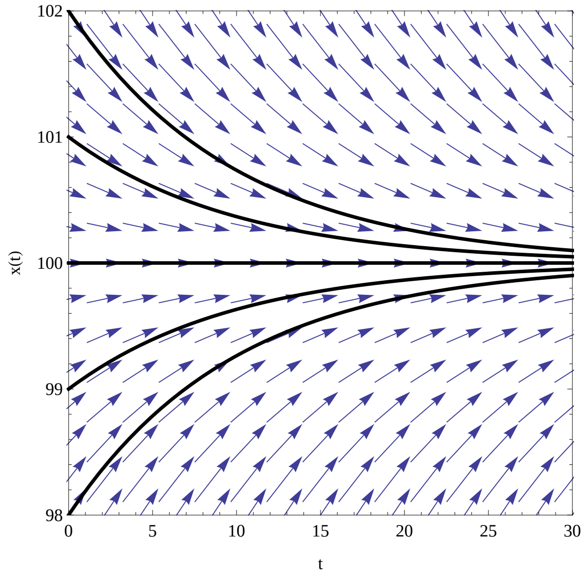

Figure 2: Trajectories and Vector Field

4 Digression

Integration

$$\int f(x)\,dx = F(x) + C, \quad \text{where } F'(x) = f(x)\,, \tag{1}$$

$$\int x^a\,dx = \frac{1}{1+a}\,x^{1+a} + C\,, \tag{2}$$

$$\int \frac{1}{x}\,dx = \ln\,x + C\,, \tag{3}$$

$$\int e^{a\,x}\,dx = \frac{1}{a}\,e^{a\,x} + C\,. \tag{4}$$

Growth rates

constant growth rate, γ, then: $x(t_1) = x(t_0)\,e^{\gamma(t_1 - t_0)}$.

growth rate, $\gamma(t)$, then: $x(t_1) = x(t_0)\,e^{\int_{t_0}^{t_1}\gamma(s)\,ds}$.

First-order, linear ODE

$$\dot{x}(t) + a\,x(t) = b\,, \tag{5}$$
$$\dot{x}(t) + a\,x(t) = b(t)\,, \tag{6}$$
$$\dot{x}(t) + a(t)\,x(t) = b(t)\,. \tag{7}$$

For $x(t_0) = x_0$, given:

$$x(t) = \frac{b}{a} + e^{-a(t-t_0)}\,\left[x_0 - \frac{b}{a}\right]\,, \tag{8}$$
$$x(t) = x_0\,e^{-a(t-t_0)} + \int_{t_0}^{t} b(\tau)\,e^{-a(t-\tau)}\,d\tau\,, \tag{9}$$
$$x(t) = x_0\,e^{-\int_{t_0}^{t} a(s)\,ds} + \int_{t_0}^{t} b(\tau)\,e^{-\int_{\tau}^{t} a(s)\,ds}\,d\tau\,. \tag{10}$$