# Classical Programming

## 1 Lagrange Function

- Setup (1 constraint)

  $N$ choice variables: $x = (x_1, ..., x_N)'$

  objective function: $f(x)$

  $g(x) - b = 0$ (equality constraint)

  $b$ constraint constant

  opportunity set: $X = \{x \in \mathbb{R}^N \mid g(x) - b = 0\}$

- choose $(x_1, ..., x_N) \in X$ such as to max $f(x)$

- set up Lagrange function:

$$\mathcal{L}(\lambda, x) = f(x_1, ..., x_N) + \lambda \left[b - g(x_1, ..., x_N)\right]$$

  – notice:

  $\mathcal{L}(\lambda, x) = f(x)$

- necessary first order conditions

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \ i = 1, ..., N; \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

$$\rightarrow x_1^*, ..., x_N^*, \lambda^*$$

- sufficient second order conditions

  negative definiteness of bordered Hessian matrix

## 2 Lagrange Multiplier

- marginal benefit to cost ratio

  - principle of optimality:

    $\lambda$ same for all choice variables:

$$\lambda = \frac{\partial f(.)/\partial x_1}{\partial g(.)/\partial x_1} = ... = \frac{\partial f(.)/\partial x_N}{\partial g(.)/\partial x_N}$$

- marginal addition to $f^*$ as $b$ is released by a marginal unit

  measured in units of the objective

## 3 $M \geq 1$ Equality Constraints

$$\begin{aligned} g^1(x_1, ..., x_N) - b_1 &= 0 \\ g^2(x_1, ..., x_N) - b_2 &= 0 \\ \vdots \qquad\qquad & \vdots \\ g^M(x_1, ..., x_N) - b_M &= 0 \end{aligned}$$

- $M < N$ !

$$\begin{aligned} \mathcal{L} = & f(x_1, ..., x_N) + \lambda_1 \left[ b_1 - g^1(x_1, ..., x_N) \right] + ... \\ & ... + \lambda_M \left[ b_M - g^M(x_1, ..., x_N) \right] \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \ i = 1, ..., N; \qquad \frac{\partial \mathcal{L}}{\partial \lambda_j} = 0, \ j = 1, ..., M;$$

## 4 Constraint Qualification

- $M = 1$: $\nabla g(x) \neq 0$

- $M > 1$: Jacobian of constraint functions must be of (row-)rank $M$ (= rank condition)

## 5 SOC

- $B$ = bordered Hessian of $\mathcal{L}(\lambda_1, ..., \lambda_M, x_1, ..., x_N)$

- necessary SOC: $B$ negative semidefinite

- sufficient SOC: $B$ negative definite

- $\operatorname{sgn} |B_{2M+1}| = \operatorname{sgn}(-1)^{M+1}$, i.e., starting with $(2M + 1)$-th principal minor

Example: $M = 1, N = 2$;
sufficient SOC: $|B| > 0$.