Optimization / Maximization

- Ingredients

  x choice variable (control v., instrument, decision v.)

  X opportunity set: $x \in X$

  unconstrained problems: $X = \mathbb{R}$
  constrained problems: $X \subset \mathbb{R}$

  objective (related to x)

  objective function: $f(x)$

- choose x such as to max $f(x)$
  & observe $x \in X$

- solution: $x^* \in X$ $f(x^*) \geq f(x)$ for all $x \in X$
  x^* "optimizer" or "maximizer"

- $f'(x) = 0$ *necessary* condition for *interior* optimum

- complications
  - necessary vs. sufficient
  - inner vs. boundary solution
  - existence

- Weierstrass existence theorem

  $f(x)$ continuous
  X nonempty, closed, bounded

  - sufficient!

- local vs. global solutions

  - global solution: $x^* \in X$ $f(x^*) \geq f(x)$ for all $x \in X$
  - local solution: $x^* \in X$ $f(x^*) \geq f(x)$ for all $x \in N_\varepsilon(x)$

  - Every global solution = local solution.
  - sufficient condition(s) for x^* to be global sol.?

- Local-global theorem
  - $f(x)$ concave function
  - quasiconcavity is sufficient as well (either monotone or increasing-peak-decreasing)
  - uniqueness

    $f(x)$ *strictly* (quasi)concave

- Second order conditions (suppose $f'(x) = 0$)
  - x is a solution $\Rightarrow f''(x) \leq 0$ (necessary SOC)
  - $f''(x) < 0 \Rightarrow x$ is a solution (sufficient SOC)

Several Choice Variables

- Ingredients

  N choice variables: $x \equiv (x_1, x_2, ..., x_N)'$

  X opportunity set: $x \in X$

  unconstrained problems: $X = \mathbb{R}^{\mathbb{N}}$
  constrained problems: $X \subset \mathbb{R}^{\mathbb{N}}$

  objective (related to x)

  objective function: $f(x)$

- choose $x \in X$ such as to max $f(x)$

- 2 complications

  - necessary FOC: $\nexists\, f'(x)$

  - nec., suff. SOC: $\nexists\, f''(x)$

- FOC: derivative $\longrightarrow$ partial derivative
  - $f_i(x) \equiv \frac{\partial f(x)}{\partial x_i} = 0$ for $i = 1, ..., N$
  - N necessary FOC
- SOC: second order partial derivatives
  - necessary: H negative semidefinite
    $\Leftrightarrow f(x)$ concave
  - sufficient: H negative definite
    $\Rightarrow f(x)$ strictly concave