Information Economics

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Games of incomplete information

- Games of incomplete information
  - incomplete information
  - type space, beliefs
- Associated games that can be solved
- Cournot competition with incomplete information
- Incomplete information and mixed strategies (BoS)
- Further examples

Games of incomplete information

- Information: incomplete versus imperfect
  - *imperfect*: not all **information sets** are singletons; types of other agents are known
  - *incomplete*: **types** (payoff functions) of other agents are not known
  - examples: Cournot, insurance market, market for used cars, BoS
- Additional ingedients
  - type spaces
  - beliefs about others' types
  - *expected* payoff functions

Incomplete information

- type space
  - T_i set of possible types of player i
  - $T \equiv \times_{i=1}^{N} T_i$, $t \in T$ joint type vector (space)
- beliefs
  - $p_i(t_{-i}|t_i)$, for all i, for all t_i
  - $p_i(t_{-i}|t_i) \in [0,1]$, and $\sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) = 1$
  - prior beliefs identical $\rightarrow$ nature chooses type vector
    $\rightarrow$ posterior (conditional) beliefs formed via Bayes rule
- $u_i : S \times T \rightarrow \mathbb{R}$ expected payoff

Example. Prior vs. posterior beliefs

- Common joint distribution: $p(t) > 0$, $\sum_{t \in T} p(t) = 1$

Bayes rule: $p_i(t_{-i}|t_i) = \frac{p(t_i, t_{-i})}{\sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i})}$

		t_2			
		L	M	R	$\boldsymbol{p(t_1)}$
t_1	T	0.4	0.1	0.1	**0.6**
	D	0.3	0.1	0.0	**0.4**
	$\boldsymbol{p(t_2)}$	**0.7**	**0.2**	**0.1**	**1.0**

$T_i = \{T, D\}$, $T_j = \{L, M, R\}$

$T = \times_{i=1}^{2} T_i = \{(T, L), (T, M), (T, R), (D, L), (D, M), (D, R)\}$

$t \in T$, e.g., $t = (T, L)$, $p(t) = p(t_1, t_2) = 0.4$ (prior believe)

- Posterior beliefs
  - verify $\sum_{t \in T} p(t) = 1$
  - $p(t_i) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)$

    $p(T) = p(L, T) + p(M, T) + p(R, T) = 0.6$
  - $p_i(t_{-i}|t_i)$: e.g., $p_1(t_2|t_1) = p_1(M|T) =?$

    Bayes' rule: $p_1(M|T) = \frac{p(M,T)}{p(T)} = \frac{0.1}{0.6} = 0.17$
- Independence: $p(T, L) = p(T)p(L)$ but: $0.4 \neq 0.6 * 0.7 = 0.42$

  t_i not independent of $t_j \Leftrightarrow$ prior belief $\neq$ posterior belief
  - prior belief $p(T, M) = 0.1$
  - posterior belief $p_1(M|T) = 0.17$

Bayesian Game

$$G = (p, T_i, S_i, u_i)_{i=1}^{N}$$

- $u_i : S \times T \to \mathbf{R}$
- p prob. distribution over T

- Cournot duopoly w/ incomplete information

  - both players: constant marginal cost ; player 2: c_H, c_L.
    type space

	Player 2 c_L	c_H	$\boldsymbol{p(t_1)}$
Player 1	$1-\theta$	θ	**1.0**
$\boldsymbol{p(t_2)}$	$\boldsymbol{1-\theta}$	$\boldsymbol{\theta}$	**1.0**

$T_1 = \{1\},\ T_2 = \{c_L, c_H\},\ T = T_1 \times T_2 = \{(1, c_L), (1, c_H)\}$
independence: $(1-\theta) = p(1, c_L) = p(1)\, p(c_L) = 1\,(1-\theta)$.
$\to$ $p(c_L) = p_1(c_L|1)$ (prior = posterior belief)

Finding equilibria in Bayesian games

- Associate equivalent game G^*: each type is separate player
  - each type of player chooses her strategy
  - nature, using p, randomly chooses type vector
    those players actually play the game
  - joint pure strategy $s^* = (s_1(t_1), ..., s_N(t_N))_{t \in T}$ where $* \rightarrow$ types

    s^* is a joint pure strategy for every $t \in T$
    - Example. $N = 2$, player 1 has 1 type, player 2 has 2 types
      2 type vectors: $t, t' \in T$
      one joint strategy for t, $s(t)$; and one joint strategy for t', $s'(t')$

    $\rightarrow$ $s^* = (s(t), s(t'))$
  - expected payoffs (VNM!): note – payoff of type, not of player

    $v_{t_i}(s^*) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i)\, u_i(s^*, t_1, ..., t_N)$
- **Theorem.** A Bayesian NE of G is a NE of the associated G^*.

Example. Cournot duopoly w/ incomplete information

	Player 2 c_L	c_H	$\boldsymbol{p(t_1)}$
Player 1	$1-\theta$	θ	**1.0**
$\boldsymbol{p(t_2)}$	$\boldsymbol{1-\theta}$	$\boldsymbol{\theta}$	**1.0**

$$s(t) \equiv (q_1, q_{cL}),\ s'(t') \equiv (q_1, q_{cH})$$

expected payoffs (* = type dependent)

$$v_1(s^*) = (1-\theta)\, u_1(s, (1, c_L)) + \theta\ u_1(s', (1, c_H))$$

$$v_{2,cL}(s^*) = u_2(s, (1, c_L))$$
$$v_{2,cH}(s^*) = u_2(s', (1, c_H))$$

- Bayesian Nash equilibrium requires:

$$v_1(\hat{s}^*) \geq v_1(s_1, \hat{s}^*_{-1}),$$

$$v_{2_L}(\hat{s}^*) \geq v_{2_L}(\hat{s}^*_1, s_{2_L}),\ v_{2_H}(\hat{s}^*) \geq v_{2_H}(\hat{s}^*_1, s_{2_H})$$

every player chooses q_i so as to maximize expected payoff

- $P(Q) = a - Q,\ Q = q_1 + q_2,\ Q < a,\ C_1(q_1) = c\, q_1,\ 0 < c < a,$
  $C_2(q_2) = c_2\, q_2,\ c_2 \in \{c_L, c_H\}$

  $S_i = [0, \infty)$, as for $Q \geq a,\ P(Q) = 0$, no firm produces $q_i \geq a$

  firm 2, type c_H: $\max_{q_2 \in S_2}\ q_2\,[a - q_2 - \hat{q}_1 - c_H]$

  firm 2, type c_L: $\max_{q_2 \in S_2}\ q_2\,[a - q_2 - \hat{q}_1 - c_L]$

  firm 1

  $\max_{q_1 \in S_1}\ \theta\,[a - q_1 - \hat{q}_2(c_H) - c]\,q_1 + (1-\theta)\,[a - q_1 - \hat{q}_2(c_L) - c]\,q_1$

$\rightarrow$ $\hat{q}_2(j) = (a - \hat{q}_1 - c_j)/2,\ j \in \{H, L\}$

$\rightarrow$ $\hat{q}_1 = [\theta\,[a - \hat{q}_2(c_H) - c] + (1-\theta)\,[a - \hat{q}_2(c_L) - c]]\,/2$

- Result
  - $\hat{q}_2(c_H) = \frac{a-2c_H+c}{3} + \frac{1-\theta}{6}(c_H - c_L)$
  - $\hat{q}_2(c_L) = \frac{a-2c_L+c}{3} - \frac{\theta}{6}(c_H - c_L)$
  - $\hat{q}_1 = \frac{a-2c+\theta c_H+(1-\theta)c_L}{3}$
- Interpretation
  - incomplete information affects Bayesian *Nash equilibrium*
    - $\hat{q}_2(c_H) > \hat{q}_2 = \frac{a-2c_2+c_1}{3}$
    - $\hat{q}_2(c_L) < \frac{a-2c_2+c_1}{3}$
    - $\hat{q}_1 = \frac{a-2c+\theta c_H+(1-\theta)c_L}{3}$
  - information rent

- Information rent
  - suppose 2 is of type c_H
    - $\hat{q}_2(c_H) > \hat{q}_2 \quad$ b/c $\quad \hat{q}_1(\theta < 1) < \hat{q}_1(\theta = 1)$
  - suppose 2 is of type c_L
    - $\hat{q}_2(c_L) < \hat{q}_2 \quad$ b/c $\quad \hat{q}_1(\theta > 0) > \hat{q}_1(\theta = 0)$

Incomplete information and mixed strategies (BoS)

- Incomplete information: payoff (rather than type)

		Pat: Opera	Pat: Fight
Chris	Opera	$2+t_c$, 1	0 , 0
	Fight	0 , 0	1 , $2+t_p$

  private information: t_c, t_p

  - $t_c \sim$ Uniform$[0,x]$, $t_p \sim$ Uniform$[0,x]$
  - type spaces: $T_c = [0,x]$, $T_p = [0,x]$

  beliefs about x

  - density for $x \in [0,x] = 1/(x-0) = 1/x$

- Chris plays Opera if $t_c > c$, $c =$ critical value

  Pat plays Fight if $t_p > p$, $p =$ critical value

Chris: $Pr(t_c > c) = 1 - Pr(tc \leq c) = 1 - F(c) = 1 - (c-0)/(x-0) = 1 - c/x$

Pat: $Pr(t_p > p) = 1 - Pr(tp \leq p) = 1 - F(p) = 1 - (p-0)/(x-0) = 1 - p/x$

		Pat	
		Opera (p/x)	Fight $(1-p/x)$
Chris	Opera $(1-c/x)$	$2+t_c$, 1	0 , 0
	Fight (c/x)	0 , 0	1 , $2+t_p$

- Chris
  - expected payoff of Opera: $(2+t_c)p/x + 0(1-p/x) = (2+t_c)p/x$
  - expected payoff of Fight: $0\,p/x + 1\,(1-p/x) = 1-p/x$
  - play Opera if $(2+t_c)p/x \geq 1-p/x \Leftrightarrow t_c \geq x/p-3 \equiv c$ (*)
- Pat
  - expected payoff of Opera: $1(1-c/x) + 0\,c/x = 1-c/x$
  - expected payoff of Fight: $0\,(1-c/x) + (2+t_p)c/x = (2+t_p)c/x$
  - play Fight if $(2+t_p)c/x \geq 1-c/x \Leftrightarrow t_p \geq x/c-3 \equiv p$ (**)
- (*)+(**) $\Rightarrow c = p$ and...

- Pure strategy Bayesian NE
  - Chris: play Opera if $t_c \geq c = [-3 + \sqrt{9 + 4x}]/2$

    Pat: play Fight if $t_p \geq p = [-3 + \sqrt{9 + 4x}]/2$
- Probabilities

$$\frac{x - p}{x} = \frac{x - c}{x} = 1 - \frac{-3 + \sqrt{9 + 4x}}{2x}$$

  - $\lim_{x \to 0} \frac{x-p}{x} = \lim_{x \to 0} \frac{x-c}{x} = \frac{2}{3}$
  - In limit, incomplete information BoS approaches behavior in mixed strategy NE in BoS with complete information.
  - Interpretation of mixed strategies (Harsanyi, 1973): A mixed-strategy NE in a game with complete information can be interpreted as a pure strategy Bayesian NE in a closely related game with a bit of incomplete information.