Information Economics

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Static Games of Complete Information

Main learning objectives:

. basic coordinates of static games of complete information

- Static games of complete information
  - strategic form games
  - dominant/dominated strategies
  - iterated elimination of strictly dominated strategies
  - Nash equilibrium (in pure strategies)
- Applications (Cournot duopoly, Bertrand duopoly)
- Mixed strategies
  - simplified Nash equilibrium tests
  - application (batter-pitcher duel)

- Strategic form games
  - N players $i = 1, ..., N$, strategies $s_i \in S_i$, payoffs u_i
  - payoff function $u_i : \times_{j=1}^{N} S_j \to \mathbf{R}$, $S \equiv \times_{j=1}^{N} S_j \equiv S_1 \times S_2 \times ... \times S_N$
  - definition: $G = \{S_1, S_2, ..., S_n; u_1, u_2, ..., u_n\}$
    - define G for the Prisoner's dilemma
    - define $\times_{j=1}^{N} S_j$ for the Prisoner's dilemma
    - explain u_i for the Prisoner's dilemma
- Solution concept 1:
  Iterated elimination of strictly dominated strategies (IESDS)

A few notational conventions

- individual strategy s_i

  set of individual strategies $S_i, \;\; s_i \in S_i$

- specific joint strategy $s = (s_1, s_2, ..., s_N)$

  joint strategy set $S \equiv \times_{j=1}^{N} S_j, \;\; s \in S$

- "others" joint strategy $s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_N)$

  "others" joint strategy set
  $S_{-i} = S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_N, \;\; s_{-i} \in S_{-i}$

  $(s_i, s_{-i}) = s$!

  - demonstrate all of these concepts for the Prisoner's dilemma

Iterated elimination of strictly dominated strategies

- Strictly dominant strategies

  A strategy $\hat{s}_i$, for player i is strictly dominant if $u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $(s_i, s_{-i}) \in S$.

  - identify strictly dominant strategies in the following game

		Player 2 L	Player 2 R
Player 1	U	3 , 0	0 , -4
	D	2 , 4	-1 , 8

- If $\hat{s}_i$ exists, i plays it.

Strictly dominated strategies

Player i's strategy $\hat{s}_i$ strictly dominates another of her strategies $\bar{s}_i$, if $u_i(\hat{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. In this case, $\bar{s}_i$ is strictly dominated in S.

If $\bar{s}_i$ exists, i does not play it.

- identify strictly dominated strategies in the following game

		Player 2		
		l	m	r
	u	3 , 0	0 , -5	0 , -4
Player 1	c	1 , -1	3 , 3	-2 , 4
	d	2 , 4	4 , 1	-1 , 8

- Strictly dominated strategies can be iteratively eliminated from game
- Game theory – outcome: strategies that survive IESDS

		Player 2		
		l	m	r
	u	3 , 0	0 , -5	0 , -4
Player 1	c	1 , -1	3 , 3	-2 , 4
	d	2 , 4	4 , 1	-1 , 8

  - calculate outcome according to IESDS

- Drawbacks
    - rationality
    - IESDS often not informative

		Player 2		
		l	m	r
	u	0 , 4	4 , 0	5 , 3
Player 1	c	4 , 0	0 , 4	5 , 3
	d	3 , 5	3 , 5	6 , 6

        - all strategies survive IESDS

players don't play SD strategies – what do they play?

- best responses

- Nash equilibrium: Given $G = (S_i, u_i)_{i=1}^N$, $\hat{s} \in S$ is a pure strategy Nash equilibrium of G if for each player i, $u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{i-1})$ for all $s_i \in S_i$.

  - NE is a joint strategy; observe the weak inequality

  - find NE in the Prisoner's dilemma

find pure strategy Nash equilibria

		Player 2		
		l	m	r
Player 1	u	0 , 4	4 , 0	5 , 3
	c	4 , 0	0 , 4	5 , 3
	d	3 , 5	3 , 5	6 , 6

- **Theorem.** (i) If $\hat{s}$ is a NE it survives IESDS.
  (ii) If only $\hat{s}$ survives IESDS, then $\hat{s}$ is the unique NE of the game.

Example. Cournot duopoly (1838)

- Cournot game
- $P(Q) = a - Q,\ Q = q_1 + q_2,\ Q < a,\ C_i(q_i) = c\,q_i,\ 0 < c < a$

$S_i = [0, \infty)$, as for $Q \geq a,\ P(Q) = 0$, no firm produces $q_i \geq a$
$s_i = q_i \in S_i,\ i = 1, 2;\ s = (q_1, q_2) \in \times_{i=1}^{2} S_i;\ \times_{i=1}^{2} S_i = \mathbf{R}_+^2$

$u_i(q_i, q_j) = q_i\,[P(q_i + q_j) - c] = q_i\,[a - q_i - q_j - c]$

NE: $u_i(\hat{s}_i, \hat{s}_j) \geq u_i(s_i, \hat{s}_j)$

$\max_{q_i \in S_i}\ q_i\,[a - q_i - \hat{q}_j - c]$

$\Rightarrow$ $\hat{q}_1 = (a - \hat{q}_2 - c)/2$ and $\hat{q}_2 = (a - \hat{q}_1 - c)/2$

$\Leftrightarrow$ $\hat{q}_1 = \hat{q}_2 = (a - c)/3$

- prediction of game theory: $\hat{s} = ((a-c)/3, (a-c)/3)$
  - unique pure strategy NE
  - $P(Q) = a - 2(a-c)/3 = a/3 - 2c/3 > c$ (as $a > c$)
    $\Rightarrow$ oligopoly profits
  - complete info vs. incomplete info (cost structure)
    $\rightarrow$ Baysian version of Cournot game

Example. Bertrand duopoly (1883)

- Bertrand game
- $q_i(p_i, p_j) = a - p_i + b\, p_j$, $2 > b > 0$; constant marginal cost $c < a$
- $p_i \in S_i = [0, \infty)$, thus: $s = (p_1, p_2) \in S = \mathbf{R}_+^2$
- $u_i(p_i, p_j) = q_i(p_i, p_j)(p_i - c) = (a - p_i + b\, p_j)(p_i - c)$
- NE: $\max_{p_i \in S_i} (a - p_i + b\, \hat{p}_j)(p_i - c)$

$\Rightarrow$ $\hat{p}_1 = (a + b\, \hat{p}_2 + c)/2$ and $\hat{p}_2 = (a + b\, \hat{p}_1 + c)/2$

$\Leftrightarrow$ $\hat{p}_1 = \hat{p}_2 = (a + c)/(2 - b)$

- prediction of game theory: $\hat{s} = ((a+c)/(2-b), (a+c)/(2-b))$
  - unique pure strategy NE
  - Bertrand NE $\neq$ Cournot NE: $p_B > p_C$
  - $(a+c)/(2-b) > c$ (as $a > c$)
    $\Rightarrow$ oligopoly profits
  - complete info vs. incomplete info (cost structure)
    $\rightarrow$ Baysian version of Bertrand game

Mixed strategies

- Mixed strategies: Consider a finite $G = (S_i, u_i)_{i=1}^{N}$. A mixed strategy for i is a probability distribution, m_i, over S_i. That is, $m_i : S_i \to [0,1]$, $0 \leq m_i(s_i) \leq 1$ and $\sum_{s_i \in S_i} m_i(s_i) = 1$.

- Set of i's mixed strategies:
  $M_i \equiv \{m_i : S_i \to [0,1] \mid \sum_{s_i \in S_i} m_i(s_i) = 1\}$
  - simplex ($\to$ show simplex for 2 or 3 strategies)

    pure strategies $\subset$ mixed strategies

- Payoffs: expected $u-$ function
  - $s = (s_1, s_2, ..., s_N)$
  - probability of $s \in S$: $m_1(s_1)\, m_2(s_2) \ldots m_N(s_N)$

    if strategies are chosen independently, prob(s) = product of probabilities $m_i(s_i)$

    complication: not only i randomizes, but so do all others as well

- $u_i(m) = \sum_{s \in S} \underbrace{m_1(s_1)\, m_2(s_2) \ldots m_N(s_N)}_{\text{probability of } s}\, u_i(s)$

$$m = (m_1, m_2, \ldots, m_N)$$

$$m \in M \equiv M_1 \times M_2 \times \ldots \times M_N$$

$$m_{-i} = (m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_N)$$

$$M_{-i} \equiv M_1 \times M_2 \times \ldots M_{i-1} \times M_{i+1} \times \ldots \times M_N$$

$$m_{-i} \in M_{-i}$$

Example

		Player 2	
		l (pr=p)	r (pr=1-p)
Player 1	u (pr=q)	1 , 2	3 , 0
	d (pr=1-q)	2 , 4	1 , 1

- $m_1 = (q, (1-q))$, $m_2 = (p, (1-p))$, $m = (m_1, m_2)$
- expected payoff of playing u: $p\,1 + (1-p)\,3$; d: $p\,2 + (1-p)\,1$
- player 1's expected payoff of m_1, given m_2:
  $u_1(m) = q\,p\,1 + q\,(1-p)\,3 + (1-q)\,p\,2 + (1-q)\,(1-p)\,1$
- calculate $u_2(m)$

- Interpretation of mixed strategies

  - sometimes in one's best interest to employ randomization mechanism

  $\rightarrow$ expected payoff of $m_i > s_i$,
  for probabilities chosen so to maximize expected payoff

- Mixed strategy Nash equilibrium

  - Given $G = (S_i, u_i)_{i=1}^N$, $\hat{m} \in M$ is a Nash equilibrium of G if for each player i, $u_i(\hat{m}) = u_i(\hat{m}_i, \hat{m}_{-i}) \geq u_i(m_i, \hat{m}_{-i})$ for all $m_i \in M_i$.

    - infinitely many strategies to be tested!

- Batter-pitcher duel (baseball)

		batter	
		F	C
pitcher	F	-1 , 1	1 , -1
	C	1, -1	-1 , 1

  - no SD strategies, no NE
  - how to find mixed strategy NE, $\hat{m}$:
    $u_i(\hat{m}) = u_i(\hat{m}_i, \hat{m}_{-i}) \geq u_i(m_i, \hat{m}_{-i}) \quad \forall m_i \in M_i, \forall i$
  - mixed strategy NE test

- **Theorem.** Mixed strategy Nash equilibrium test.

  For all i, the following holds:

  - $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$ for all s_i played with positive probability.
  - $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for all s_i played with zero probability.
  - batter-pitcher duel: $S_i = \{F, C\}$, both strategies played with positive probability:
    $u_i(F, \hat{m}_{-i}) = u_i(C, \hat{m}_{-i})$

  $\rightarrow$ calculate mixed strategy NE for batter pitcher duel

A few important results...

- **Proposition.** A pure strategy of i can be a best response to a mixed strategy by j, even if it is not a best response to any other pure strategy by j.

		Player 2	
		L	R
	T	3 , -	0 , -
Player 1	M	0 , -	3 , -
	B	2 , -	2 , -

  - Let $m_2 = (1/2, 1/2)$, and $s_1 = B$.

    Then, s_1 is a best response to m_2:
    $E[B] = 2 > E[M] = E[T] = 3/2$.

- **Proposition.** A pure strategy can be dominated by a mixed strategy, even if it is not dominated by any other pure strategy.

		Player 2: L	R
Player 1	T	6 , -	0 , -
	M	0 , -	6 , -
	B	2 , -	2 , -

  - B is not dominated by either T or M. Let $m_1 = (1/2, 1/2, 0)$, and $s_1 = B$.

    Then, for all m_2: $u_1(s_1, m_2) = 2 < 3 = u_1(m_1, m_2)$.

- **Theorem.** In any finite G, there exists a Nash equilibrium.
  - *Proof sketch.* (1) Define best-response correspondence and show that any fixed-point of correspondence is a NE;

    (2) By Kakutani's fixed-point theorem, best-response correspondence has a fixed point.

(1) For any given m, define player i's set of best responses $\phi_i(m) \subseteq M_i$

$\phi(m) \equiv \times_{i=1}^{N} \phi_i(m)$ (best response correspondence)

m^* is a fixed point of $\phi(m)$ if $m^* \in \phi(m)$

$m^* = \hat{m}$, obviously.

(2) As $m \in M$, the domain of $\phi(m)$ is nonempty, compact, convex.

$\phi(m)$ is upper hemicontinuous from M into M.

By Kakutani's fixed point theorem, there exists $m^* \in M$ such that $m^* \in \phi(m)$. ||