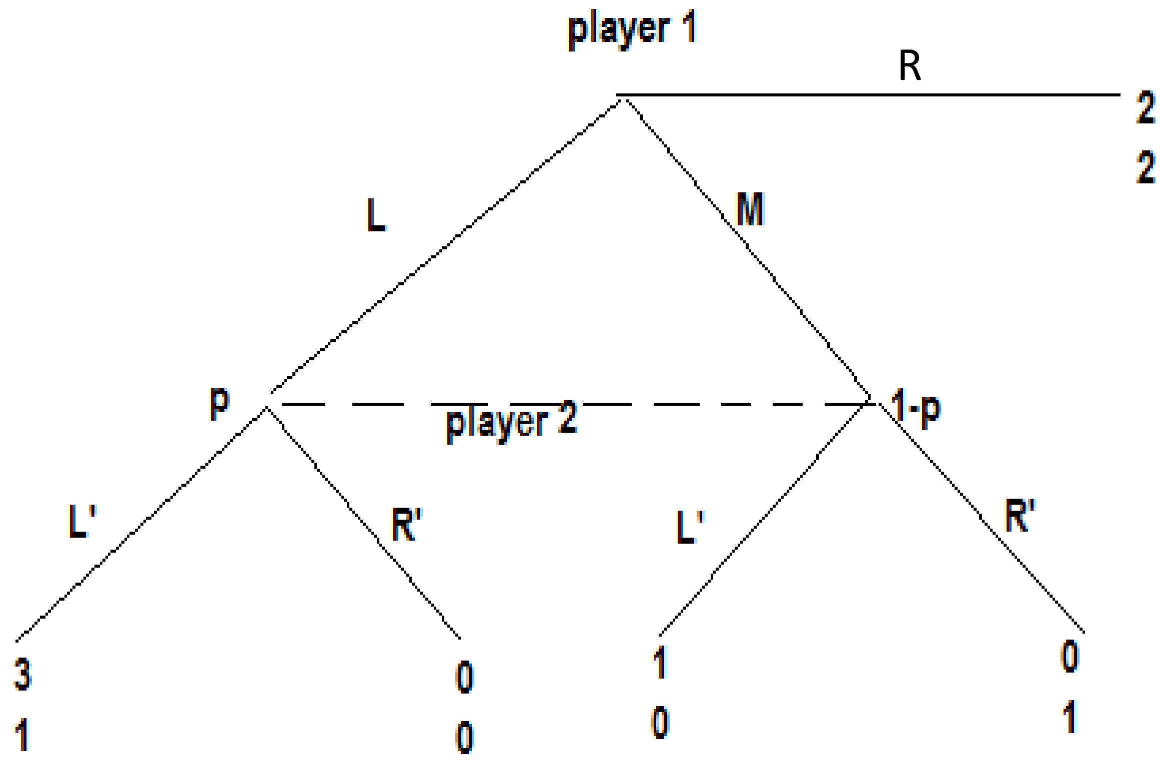
Refinements of PBE

Out of equilibrium beliefs and Refinements of PBE

- Requirement 1 and 2 of the PBE say that no player's strategy can be strictly dominated beginning at any information set.
- The problem is to prevent player *j* to believe that player *i* would play a dominated strategy, i.e. to impose restrictions on the beliefs off the equilibrium path.
- Consider for example the following game:



Strategy L’ and R’ can both be the better strategy for player 2 according to the values of p.

Expected payoff from L’: 1p+(1-p)0

Expected payoff from R’: 0p+(1-p)1

R’ is better than L’ if p<1/2. If player 2 plays R’ player 1 plays R.

Thus, there are two pure-strategy PBE in this game:

1. L, L', $p=1$ where belief are computed according to Bayes rule

2. R,R', $p \leq 1/2$ in this case beliefs are arbitrary because R is off equilibrium.

In the second equilibrium player 2 believes that player 1 plays M with probability $(1-p)>1/2$.

However, M for player 1 is strictly dominated by L.

The payoff from M is always smaller than the payoff from L no matter what player 2 does, strategy L is better for player 1.

We now introduce the requirement that player 2 should not believe that player 1 might have played a strategy that is strictly dominated beginning at any information set.

To do this we need to define what is a strictly dominated strategy beginning at a given information set.

Definition: Consider an information set at which player i has the move.
The strategy s_i' is **strictly dominated beginning at this information set** if there exists another strategy s_i such that, for every belief that *i* could hold at the given information set, and for each possible combination of the other players' subsequent strategies, i's expected payoff from taking the action specified by s_i at the information set and playing the subsequent strategy specified by s_i is strictly greater than the expected payoff from taking the action and playing the subsequent strategy specified by s_i'.

Requirement 5: If possible, each player's beliefs off the equilibrium path should place zero probability on nodes that are reached only if another palyer playes a strategy that is strictly dominated at some information set.

We have a similar requirement for signaling games.

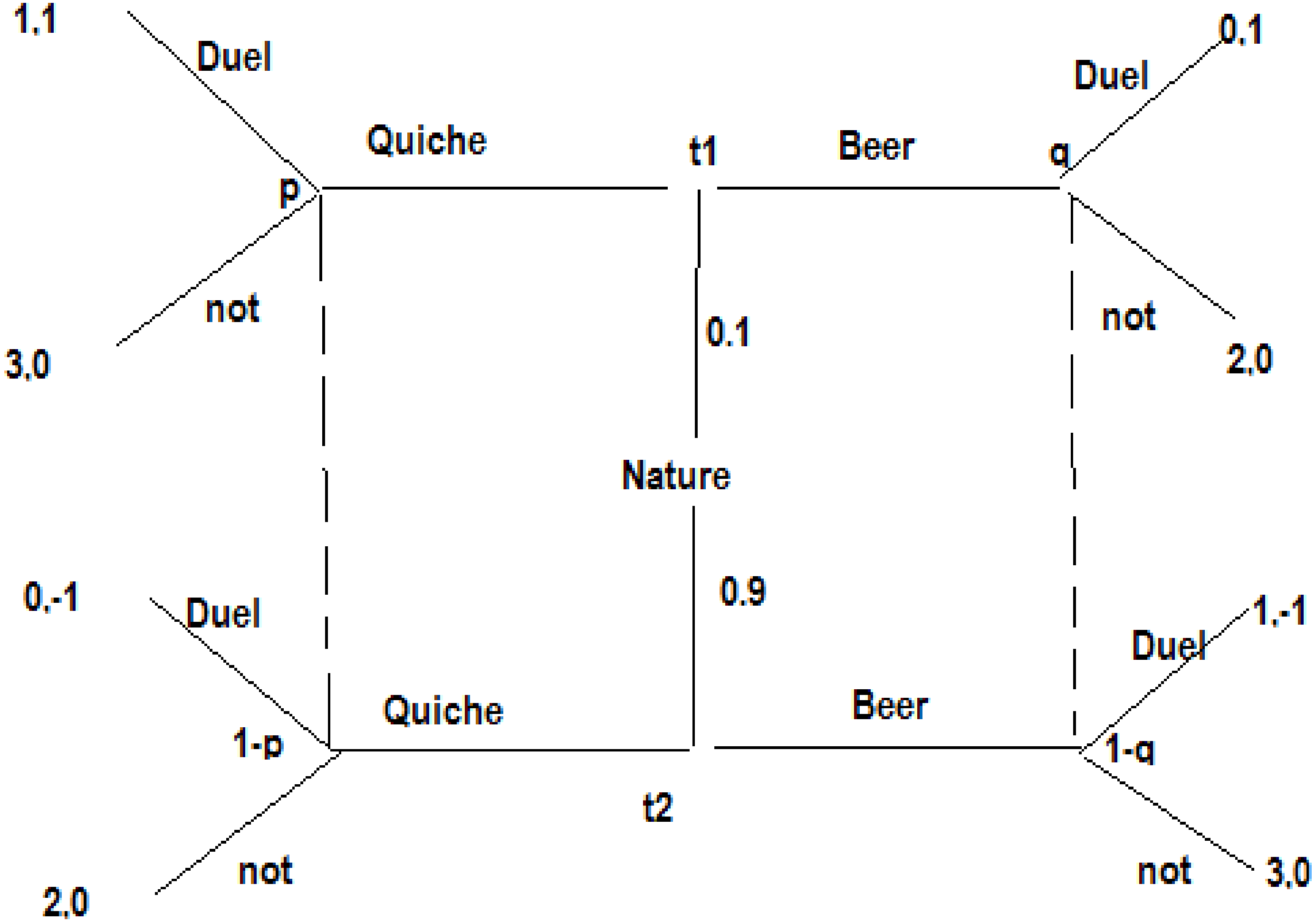
Definition: In a signaling game the message m_j from M is dominated for type t_i if there exists another message m_j' from M such that t_i's lowest possible payoff from $m_{j'}$ is greater than t_i' s highest possible payoff from m_j :

$$\underset{a_k \in A}{Min} \; U_s(t_i, m_{j'}, a_k) > \underset{a_k \in A}{Max} \; U_s(t_i, m_j, a_k)$$

Signaling requirement 5. If the information set following m_j is off the equilibrium path and m_j is dominated for type t_i then (if possible) the Receiver's belief $\mu(t_i|m_j)$ should place zero probability on type t_i (this is possible provided that m_j is not dominated for all types in T).

There are Perfect Bayesian equilibria that satisfy requirement 5 and nevertheless are unreasonable. Consider the following signaling game.

The sender can be of two types ($t1$= wimpy or $t2$=surly) with pr. 0.1 and 0.9 respectively. The message is the type of breakfast chosen: quiche or beer.
The receiver chooses beween duel and not duel.
The wimpy would prefer to have quiche, the surly would prefer beer.
Both would prefer not to duel, and the receiver would prefer to duel with the wimpy but not with the surly.
The payoff from having the preferred breakfast is 1 for both types of sender, and the additional payoff from avoiding the duel is 2.
The payoff from a duel for the receiver depends on the sender's type: -1 if surly, 1 if wimpy.
Payoffs from not duel are zero.


1,1
Duel
Quiche
t1
Beer
0,1
Duel
q
p
not
3,0
0.1
not
2,0
Nature
0,-1
Duel
0.9
1,-1
Duel
Quiche
Beer
1-p
1-q
t2
not
2,0
not
3,0

Consider the possible pooling equilibria.

1. Pooling on Beer.

Receiver chooses not duel because :

$$0.1(1) + 0.9(-1) < 0.1\ (0\ )+0.9\ (0)\ .$$

Then payoffs are:

Wimpy: 2   Surly: 3.

Let see if there are messages that are dominated according to Requirement 5. Surly has the highest possible payoff by playing Beer. Wimpy gets 2 by sending the message beer and the lowest payoff by sending quiche is 0.

Then [(Beer, Beer) (not, duel), $p \geq 1/2$, $q=0.1$] is an equilibrium.

2. Consider now pooling on Quiche.

[(Quiche, Quiche), (not, duel), $p=0.9$, $q \geq 1/2$]
is a PBE in our signaling game and satisfies Signaling requirement 5.

Beer is not dominated for either type. However, to prevent the wimpy to deviate and choose Beer we need the receiver to duel, which is optimal when $q \geq 1/2$.

This implies that if the receiver observes a deviation (a choice of Beer) it believes that the wimpy type is at least as likely as the surly type .

But:

i) The wimpy cannot improve on the equilibrium payoff
ii) The surly type can improve its payoff by choosing beer if the off-equilibrium belief is $q<1/2$

Several refinements of perfect bayesian equilibria have been proposed to solve this type of problem. This has been a very active research area at the end of last century.
One of the most commonly used is the Intuitive criterion proposed by Cho and Kreps (1987).

Definition: Given a perfect Bayesian equilibrium in a signaling game, the message m_j from M is equilibrium-dominated for type t_i from T if t_i's equilibrium payoff denoted $U^*(t_i)$, is greater than t_i's highest possible payoff from m_j:

$$U^*(t_i) > \max_{a_k \in A} U_s(t_i, m_j, a_k)$$

Signaling requirement 6: (Intuitive Criterion) If the information set following m_j is off the equilibrium path and m_j, is equilibrium dominated for type t_i then (if possible) the receiver belief belief $\mu(t_i|m_j)$ should place zero probability on type t_i. (this is possible provided that m_j is not dominated for all types in T).

An example of the intuitive criterion

- Consider the following game where the two players are the monetary authority and a labour union. Nature plays first and chooses the type of the monetary authority, strong with probability 0,6 and weak with probability 0,4.
- The monetary authority chooses the announcement to make to the labor union, high or low inflation. This announcement will be used by the labor union in the wage bargaining.
- The first payoff from the left is monetary authority's payoff.

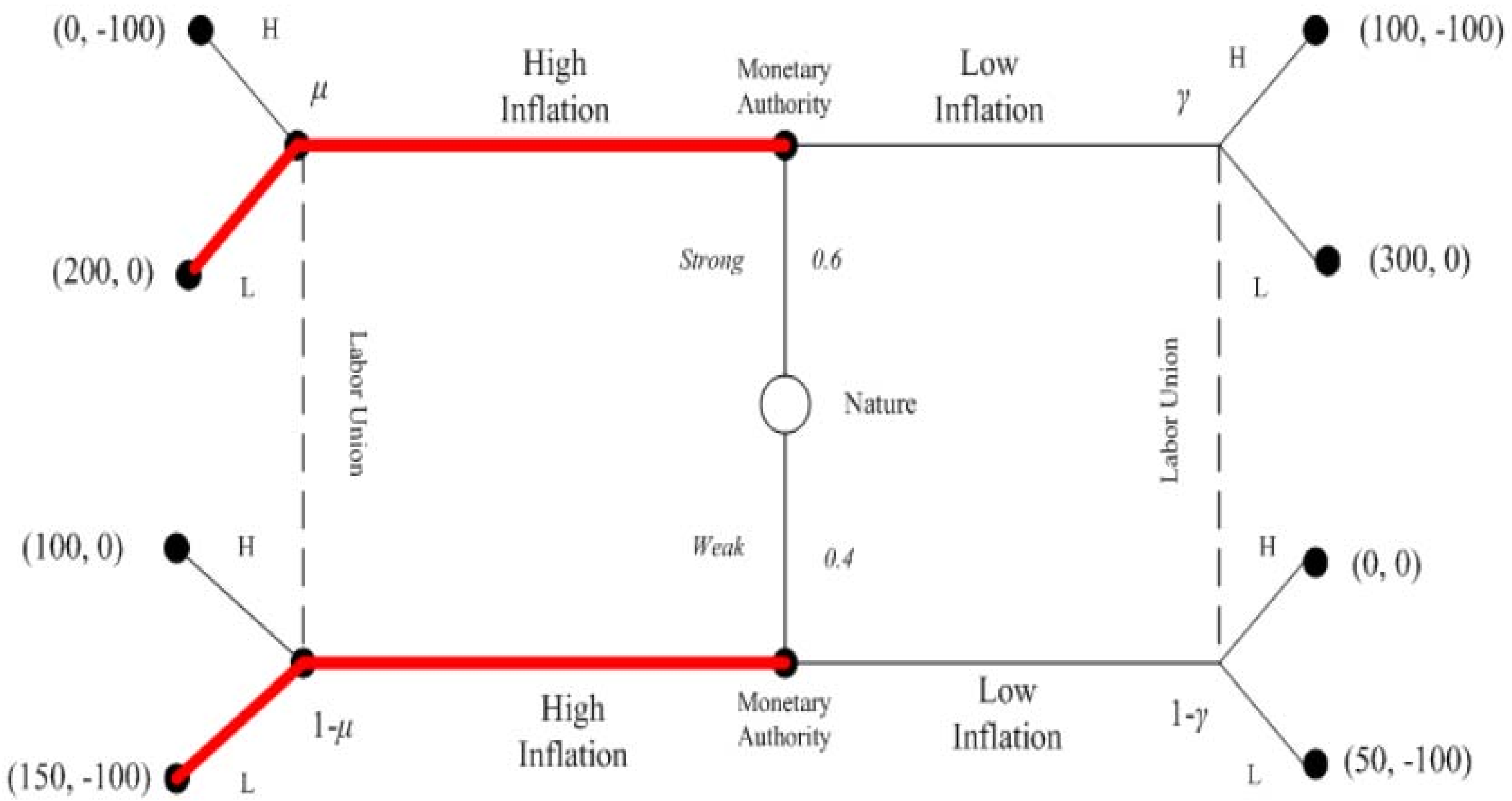


Figure 1. Monetary authority announcements game.

Consider the <u>pooling</u> equilibrium in which both types announce <u>high</u> inflation (the one highlighted in red).

The response by the labor union is to play L (Low wage at the contract stage).

The strong authority has a payoff 200, the weak has a payoff 150 and labor union has a payoff 0 if authority is strong and -100 if authority is weak.

A message of low inflation is a off-equilibrium message.

Can we rule out this message for both types? (Can we be sure that no type wants to deviate?)

WEAK TYPE: The answer is yes for the weak authority. The equilibrium payoff i s higher than what the weak authoriy can get by announcing low inflation. Low inflation is equlibrium dominated by high inflation for the weak type .

STRONG TYPE: the strong authority could profitably deviate because the highest payoff it could obtain by announcing Low inflation is 300. (The equilibrium payoff of the strong authority is 200).

Requirement 6 says that we should set (1- γ)=0, i.e. if an announcement of Low inflation is observed the union should believe that it comes from the strong type .
In this case the union responds by playing L.

Given this belief and the union response to the announcement the strong monetary authority prefers to deviate from the pooling equilibrium.

Therefore the pooling equilibrium (high, high) violates the Intuitive Criterion.

To avoid deviation by the strong type we need a value of γ smaller than ½.

If γ >1/2 the labor union in the right branches of the game would play L and the strong type would deviate, i.e. it would announce Low inflation.

Union prefers H is preferred to L if

$$-100\,\gamma + (1-\gamma)0 > 0\,\gamma - 100(1-\gamma)$$

$$\gamma < 1/2$$

But this belief violates requirement 6.

Pooling on Low inflation

It is easy to verify that there is no pooling equilibrium on Low. The weak type never wants to play Low because its payoff is always lower than the payoff obtained by playing high (the strategy is strictly dominated).

There are 2 possible separating equilibria:

1. Strong plays High, weak plays Low
2. Strong plays Low, weak plays High

We can disregard the first one since we have seen that weak never plays low.

Eq.2: Separating with Strong playing low and weak playing high.

Since the equilibrium is separating, it follows that γ=1 and μ=0.

The labor union plays L if the message is low and H if the message is high.

The payoffs are (300,0) for the strong authority and the union respectively, and (100,0) for the weak authority and the union).

No type has incentive to deviate and this is indeed an equilibrium, [(low, high) (H,L) (γ=1, μ=0)].

Let consider more carefully the candidate pooling equilibrium given by:

[(high, high) (L,H) (μ=0,6)($\gamma < 1/2$)].

and check whether it satisfies the Intuitive criterion.

However, the deviation to Low announcement is equilibrium dominated for the weak type.

Then, (1-γ)=0, => γ=1.

To prevent the strong type from deviating we need

$\gamma < \frac{1}{2}$. i.e. an announcement of low inflation is considered more likely to come from the weak type than from the strong one.

$\gamma = 1$ (as required by Intuitive criterion) destroyes the equilibrium , i.e. this pooling equilibrium does not survive the Intuitive Criterion.