Signaling Games

Gibbons Ch.4
A signaling game is a dynamic game of incomplete information with 2 players: a Sender (S) and a receiver (R). The timing is:

1. Nature draws a type $t_i \in T=\{t_1, ..., t_I\}$ for the Sender according to the probability distribution $p(t_i)$, with $p(t_i) > 0$ and $p(t_1) + ... + p(t_I) = 1$.

2. Sender observes $t_i$ and chooses a message $m_j$ from a set $M=\{m_1, ..., m_J\}$.
3. Receiver observes \( m_j \) (but not the type) and then chooses an action \( a_k \in A = \{a_1, \ldots, a_k \} \)

4. Payoff are given by \( U_S(t_i, m_j, a_k) \) and \( U_R(t_i, m_j, a_k) \).

Note that the first move is by nature.

We can now translate the informal statements of requirement 1 through 3 for Perfect Bayesian Equilibrium into a formal definition of PBE in Signaling games.
**Signaling Requirement 1:** After observing any message $m_j$ the receiver must have a belief about which types could have sent $m_j$. Denote this belief by the probability distribution

$$\mu(t_j \mid m_j)$$

with $\mu(t_j \mid m_j) > 0$ and $\sum \mu(t_j \mid m_j) = 1$.

**Signaling Requirement 2R:** For each $m_j$, the Receiver action must maximize his/her expected utility given the belief:

$$\max_{a_k} \sum_{t_i}^T \mu(t_i \mid m_j) U_R(t_i, m, a_k,)$$
The same requirement applies to the Sender who has complete information so that the requirement simplifies to:

**Signaling Requirement 2S:** For each $t_j$, the Sender’s message $m^*(t_j)$ must maximize his/her expected utility given the Receiver strategy $a^*(m_j)$:

$$\max_{m \in M} U_s(t_i, m_j, a^*(m_j))$$
Finally, for messages on the equilibrium path we have to apply requirement 3 to the Receiver belief:

**Signaling Requirement 3**: For each \( m_j \) in \( M \), if there exists \( t_j \) in \( T \) such that \( m^*(t_j) = m_j \), then the Receiver’s belief at the information set corresponding to \( m_j \) must follow from Bayes’ rule and the Sender’s strategy:

\[
\mu(t_i | m_j) = \frac{p(t_i)}{\sum_{t_i \in T_i} p(t_i)}
\]
Definition: A pure-strategy Perfect Bayesian Equilibrium in a signaling game is a pair of strategies $m^*(t_j)$ and $a^*(m_j)$ and a belief $\mu(t_j | m_j)$ that satisfy Requirements (1), (2R), (2S), and (3).

If the sender’s strategy is pooling (all types send the same message) we call the equilibrium pooling, if different types send different messages we call the equilibrium separating.
Consider the following signaling game with 2 types, 2 messages. The game starts with a move by Nature that chooses the type of the Sender, who then selects a message for the receiver. In this game the message consists of the choice between Left and Right.
There are 4 possible pure-strategy perfect Bayesian equilibria in this game:
1. Pooling on L,
2. Pooling on R
3. Separating with $t_1$ playing L and $t_2$ playing R
4. Separating with $t_1$ playing R and $t_2$ playing L.

1. Pooling on L. If both types play L, there is no updating:
   $p = (1-p) = 1/2$.
   Receiver plays $u$ with expected payoff $3p + 4(1-p) > \text{Expected payoff from } d$ if L is observed ($=1-p$).
   $t_1$ gets 1, $t_2$ gets 2.

How would Receiver answer if R were observed? Is crucial to determine whether pooling on L is optimal.
If Receiver plays u, \( t_1 \) deviates and plays \( R \) because in this way gets 2 rather than 1. 
\( t_2 \) does not deviate \((2 > 1)\). To support pooling on \( L \) we need the Receiver to play \( d \). 
This is optimal only if:
\[
q_1 < 0q + 2(1-q) \Rightarrow q < 2/3.
\]
Thus the equilibrium is defined by:
\[
[(L, L), (u, d), p=1/2, q \leq 2/3]
\]

2. Pooling on \( R \). \( \Rightarrow \) \( q=(1-q)=1/2 \). In this case we know Receiver plays \( d \). 
If Receiver plays \( d \), \( t_1 \) does not want to play \( R \), (he would deviate and play \( L: \ 1 > 0 \)).

\[
\Rightarrow \quad \text{No pooling equilibrium on } R.
\]
3. Separating with $t_1$ playing $L \Rightarrow p = 1, q = 0$. Receiver plays $u$ if $L$ and $d$ if $R$. But if receiver plays $u$ when he observes $L$, $t_2$ finds more profitable playing $L$ which gives him a payoff of 2 instead of 1, $\Rightarrow t_2$ deviates. There is no equilibrium.

We have to check the last possible equilibrium:

4. Separating with $t_1$ playing $R \Rightarrow p = 0$ and $q = 1$. The Receiver best response is $(u,u)$. $t_1$ has no incentive to deviate, nor has it $t_2$.

Then $[(R, L), (u,u), p = 0, q = 1]$ is a separating perfect Bayesian equilibrium.

Summarizing, there are 2 equilibria: one pooling and one separating.