Dynamic Games with Incomplete Information

Notes based on Chapter 4 of Game Theory for Applied Economists,

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First Lecture:
Perfect Bayesian Equilibrium

(Why subgame perfect equilibrium is not «strong enough» in games with incomplete information)

The Role of Information

The existence of private information leads to attempt to communicate by informed parties and to attempts to learn by uninformed parties.

The simplest model of this has two players, one with private information and one without. There two stages in the game: first the informed party chooses a strategy (a signal) and then moves the uninformed party .

Classic example is the Spence’s signaling model (1973):
the informed party is the worker who has private information on the his/her ability and uniformed party is the potential employer.
The signal is the level of education and the response is a wage level.

To analyze dynamic games with incomplete information we introduce a new equilibrium concept: **perfect Bayesian equilibrium.**

The crucial feature is that beliefs are elevated to the level of importance of strategies.

That is the definition of equilibrium includes, in addition to the strategies a beliefs for each player whenever the player has to move but is uncertain about the history of prior play.

Previously we asked player to choose credible strategies (i.e subgame perfect) now we ask player to hold reasonable beliefs.

- Consider the following dynamic game. Player 1 moves first and chooses among L, M, R. If he chooses R the game ends. If he plays L or M player 2 is called upon to move. He knows that R was not chosen but he cannot distinguish between L and M. Player 2 chooses L' or M' and then the game ends.
- If we look at the normal-form representation we see that there are 2 Nash equilibria in this game: (L, L') and (R, R').

To verify whether they are also subgame perfect however we need to use the extensive-form representation.

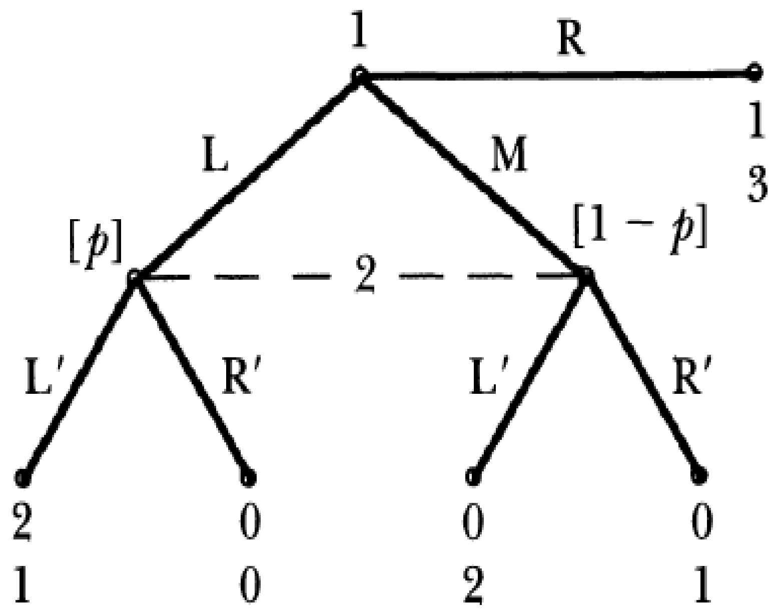
First we have to define the possible subgames in the game.

- A subgame begins from a decision node that is a singleton information set but is not the first decision node.

Thus there is no subgames in this game. => all Nash equilibria are also subgame perfect (the requirement is trivially satisfied).

Figure 4.1.3

Why Players' Beliefs are as Important as their Strategies



		Player 2 L′	Player 2 R′
Player 1	L	2, 1	0, 0
	M	0, 2	0, 1
	R	1, 3	1, 3

Despite the fact that both equilibria are subgame perfect, one relies on a non-credible threat.

Look at (R, R’) equilibrium. If player 2 gets to move he has no incentive to play R’. R’ is dominated by L’. However if player 1 believes that player 2 will play R’, then he prefers to choose R with a payoff of 1 because otherwise he gets 0.

If we want rule out equilibria that are based on non-credible threat we need to impose the following requirements.

Requirement 1: At each information set, the player with the move must have a belief about which node in the information set has been reached. For a non singleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton the player belief puts probability 1 on the single decision node.

Requirement 2: Given their beliefs, the players' strategies must be sequentially rational. That is, at each information set the action taken by the player with the move must be optimal given the players' belief at that information set and other players' subsequent strategies .

Subsequent strategy is a complete plan of action covering every contingency that may arise after the information set that has been reached.

In the game represented above requirement 1 means that player 2 must have belief represented by probabilities p and 1-p attached to each node.

The expected payoff from playing R' is: $p0+(1-p)1 = 1-p$.

The expected payoff from playing L' is: $p1+(1-p)2 = 2-p$

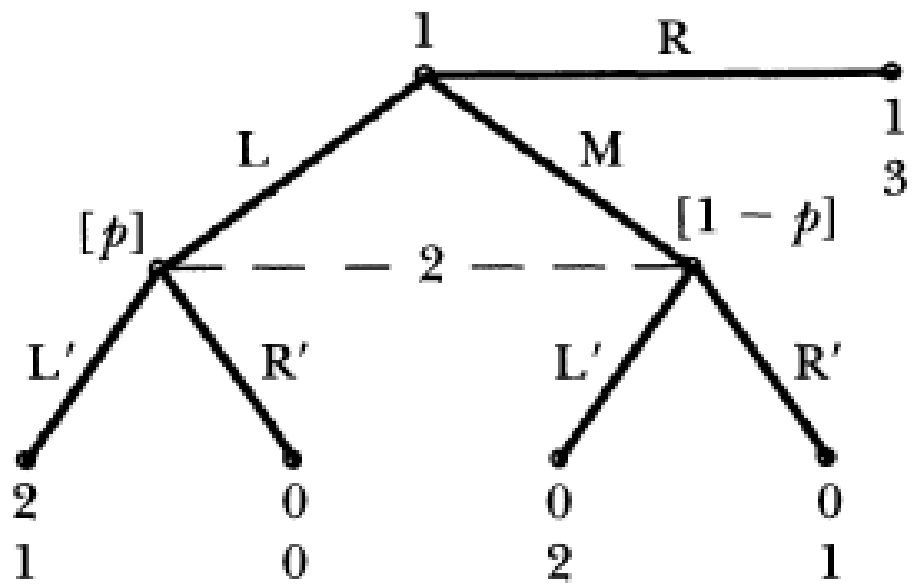
Thus L' gives a higher payoff for any value of p. Requirement 2 prevents player 2 from choosing R'.

So far we haven't said anything about belief being reasonable. To do so we need to distinguish between information nodes **on** the equilibrium path and **off** the equilibrium path.

Definition: An information set is on the equilibrium path if it will be reached with positive probability if the game is played according to the equilibrium strategies; and is off the equilibrium path if it is certain not to be reached if the game is played according to the equilibrium strategies.

Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

If we use requirement 3 in the game reported below we see that player 2 must hold belief p=1. Given player 1 strategy, player 2 gets to move when player 1 plays L, then p=1. With this belief player 2 is better off by playing L’ rather than R’.

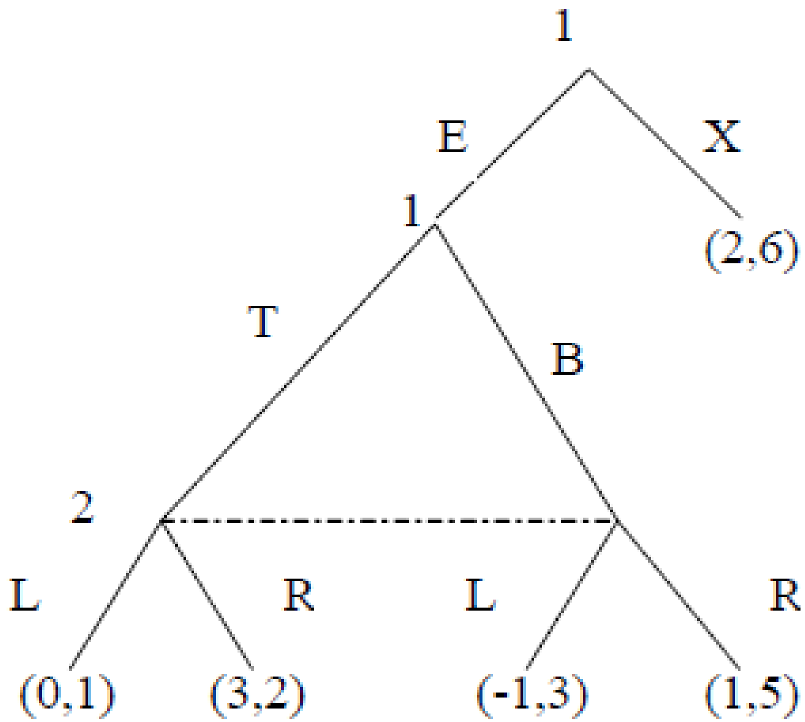


In simple economic applications Requirements 1-3 constitute the definition of perfect Bayesian equilibrium.
In more complex games however, more requirements need to be imposed to eliminate implausible equilibria. Most authors use requirement 4:

Requirement 4: At information set off the equilibrium path, beliefs are determined by Bayes rule and players equilibrium strategies where possible.

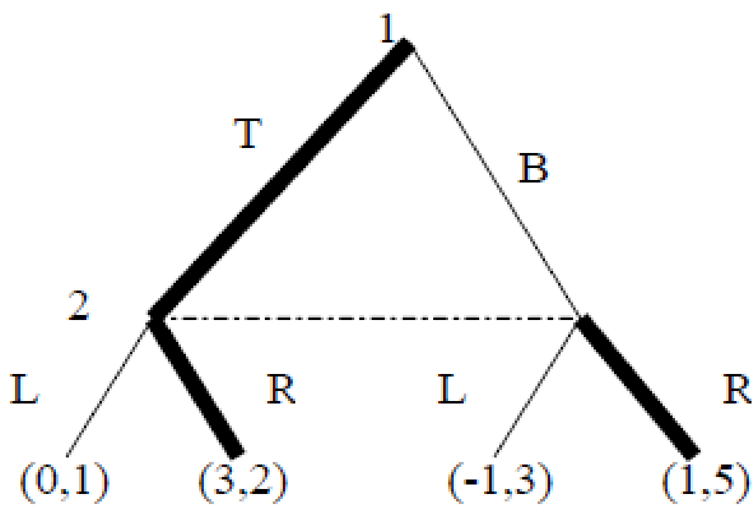
Definition: A **perfect Bayesian equilibrium** consists of strategies and beliefs satisfying Requirements 1 through 4.

Example 2: Difference between subgame perfect equilibria and perfect Bayesian. Consider the game represented below. It has a proper subgame starting after player 1 plays E.

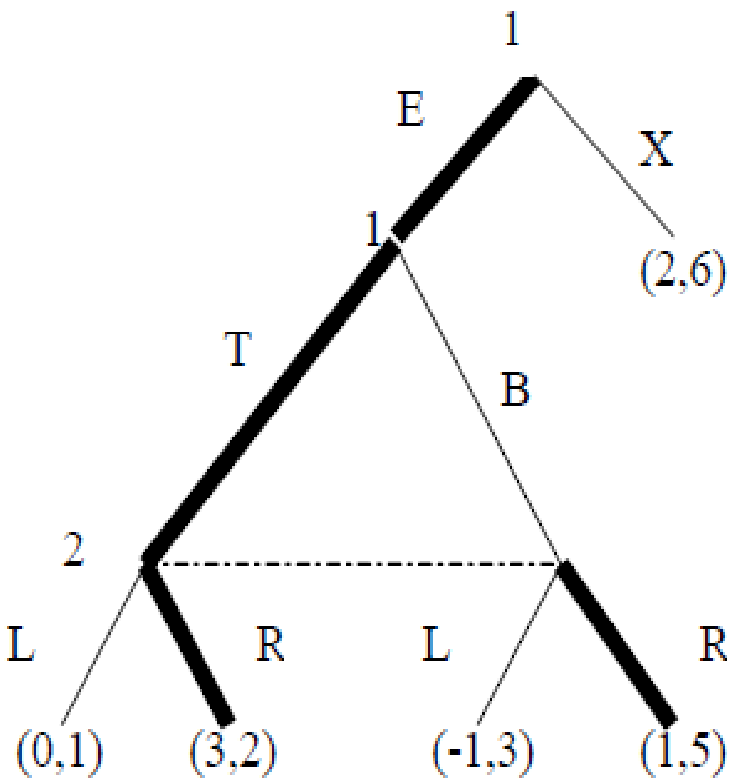


Since there is a subgame we can look for a subgame perfect equilibrium. Consider the subgame reported below. Each player has a dominant (or dominated) strategy: strategy B is weakly dominated for Player 1 and strategy L is dominated for player 2.

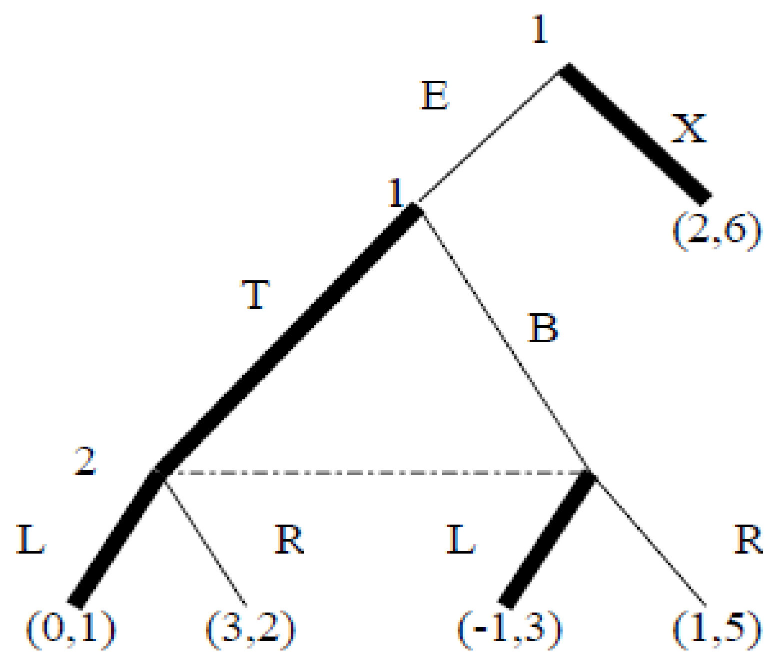
Player 1 plays T and Player 2 plays R yielding a payoff of 3,2.



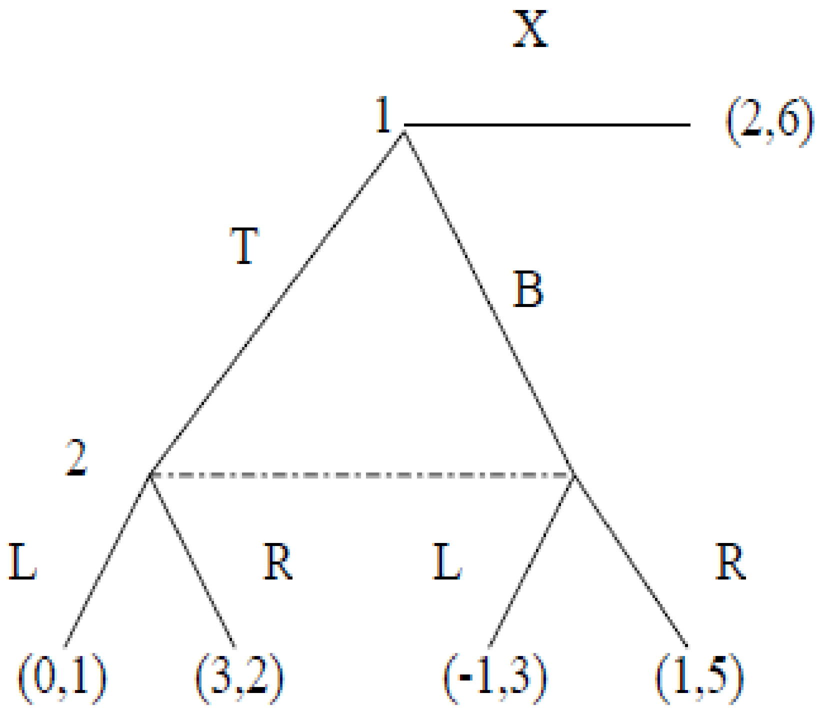
Given this, we can solve for the (subgame perfect) equilibrium in the whole game: (E, T; R).



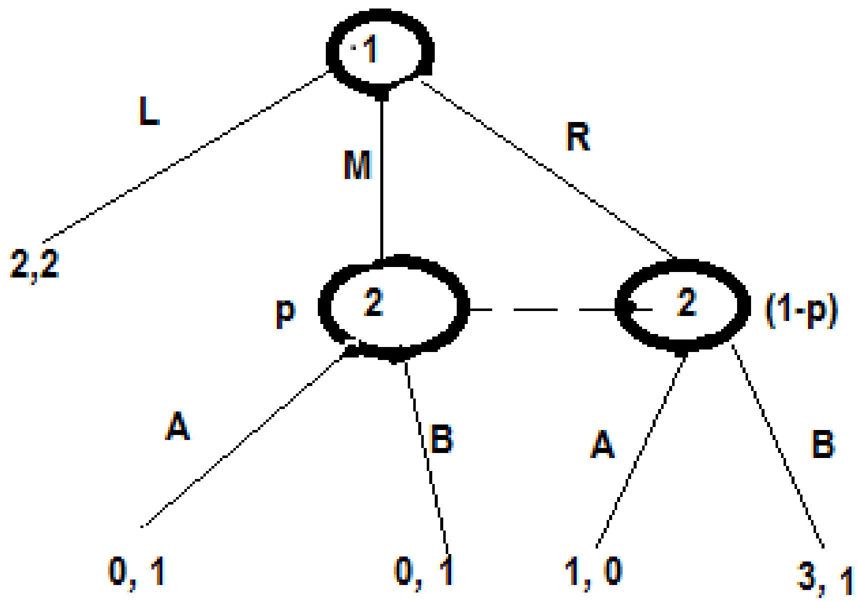
However there is also a Nash equilibrium that is not subgame perfect. If player 2 threaten to play L, then player 1 is better off playing X rather than E. This allows player 2 to get a payoff of 6 (rather than 2 as before).



Subgame perfect equilibrium can be very sensitive to the way we model the game. Consider now this game that is a slightly modified version of the previous one: player 1 makes his choice at once. There is no subgame in this game => all Nash equilibria are also subgame perfect.



Example 3: Another example in which subgame perfect equilibrium has no bite and we need Perfect Bayesian.



There are 2 subgame perfect equilibria in this game: (L, A) and (R,B) but (L,A) makes no sense. The problem is again that there is no proper subgame here.

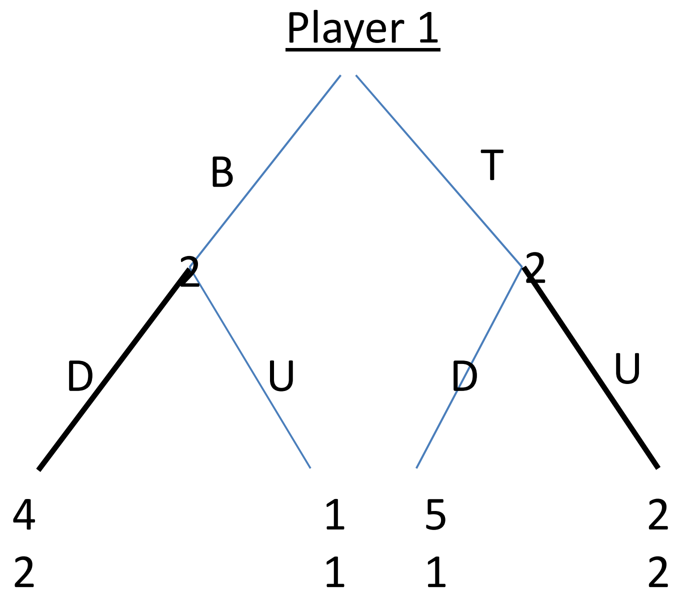
The problem with the equilibrium (L,A) is that player 2 will never choose A (the strategy A is dominated by B). However, if player 2 threaten player 1 to play A, then L is optimal for player 1.

Given that player 2 chooses B player 1 is better off playing R. Then, the belief of player 2 must be 0 on the left node in the information set and 1 on the right node.

The Perfect Bayesian Equilibrium of the game is (R,B p=0).

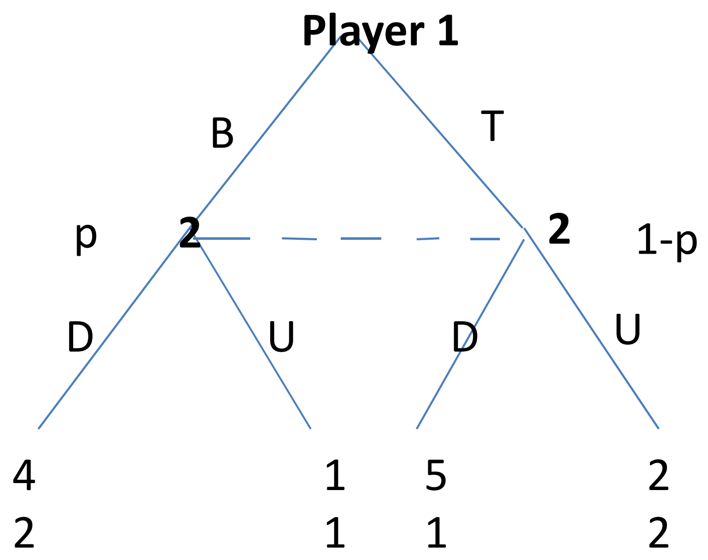
(L, A) cannot be part of a PBE because there is no belief that can make player 2 play A.

Example 4: Consider now the following game of perfect information (MasColell 9.C.7) There are 2 proper subgames . We can solve the game by backward induction.



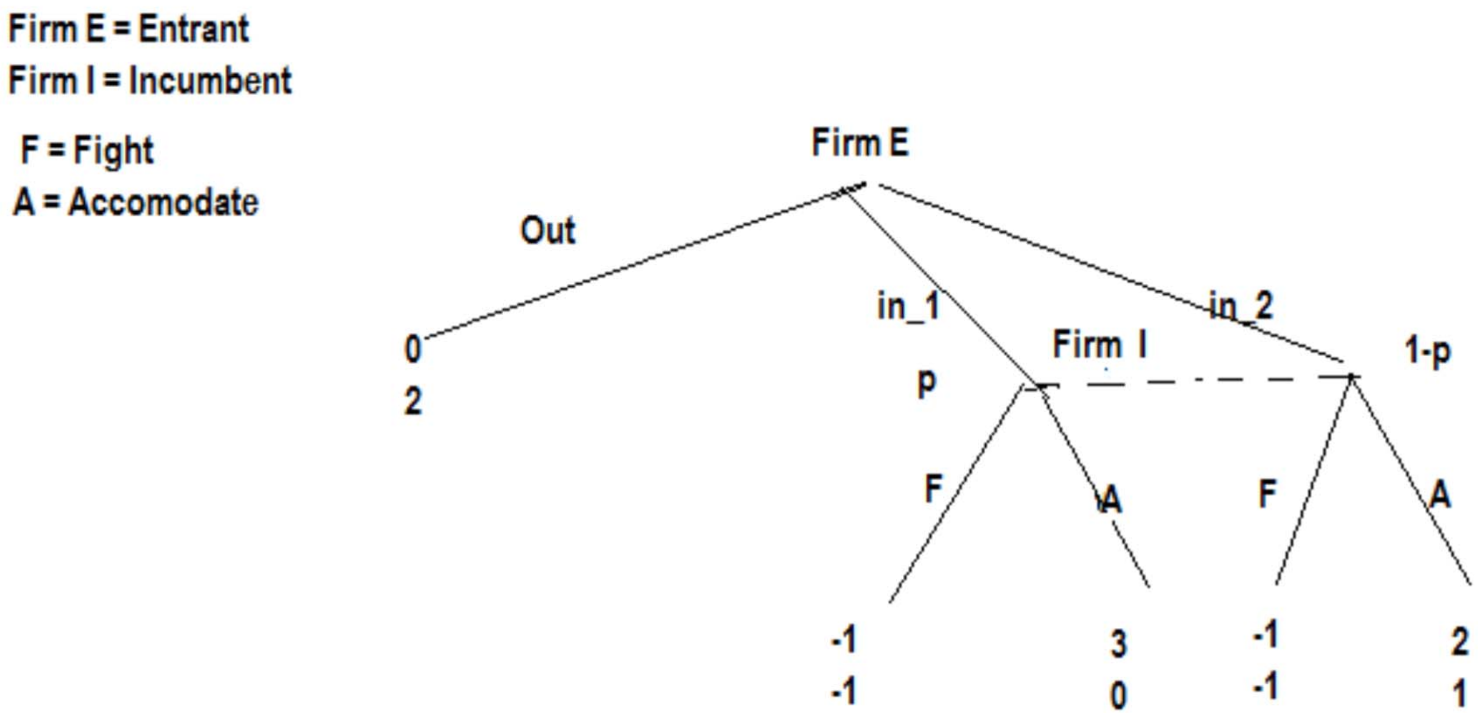
The unique subgame perfect N.E. is (B, DU).

Consider the game under the assumption that player 2 cannot observe player's 1 move. The solution in this case is different.



The solution of the game is easy since player 1 has a dominant strategy: T dominates B. Then the unique PBE is (T, U and p=0)

Consider the following Example from MasColell (Example 9 C1 page 283)



There is only one PBE in this game: the Entrant chooses in_1 and the Incumbent accomodates entry. Then the belief is p=1 . It is easy to see that Fight is never optimal for Firm I (payoff always negative).
However, there are 2 Nash equilibria. The other one is Out, Fight. There is no systemt of belief that support this equilibrium.