A Market Economy: Positive Theory

Main objectives of today's session:

. study existence of competitive equilibrium
. (bounded consumption/production sets)

- preliminary remarks

- existence of equilibrium

1 Preliminary Remarks

- excess demand

$$z_l(p) = \sum_{i=1}^{I} x_{li}(p) - \sum_{j=1}^{J} y_{lj}(p) - \sum_{i=1}^{I} \omega_{li}$$

$$z(p) = \begin{pmatrix} z_1(p) \\ z_2(p) \\ \vdots \\ z_L(p) \end{pmatrix}$$

- HD0 of $x_i(p)$ and $y_j(p)$

  $\rightarrow$ HD0 of $z(p)$

  $\rightarrow$ normalization of p to unit simplex:

$$P = \{p \in \mathbb{R}^L_+ \,|\, p_l \geq 0\,, \sum_{l=1}^{L} p_l = 1\}$$

Definition 1 *A Walrasian equilibrium is a price vector $p^* \in P$ and an attainable allocation (x, y) such that:*

(i) all households maximize utility s.t. budget constraints,

(ii) all firms maximize profit s.t. technology constraints,

(iii) all markets clear: $z(p^*) \leq 0$, and if $z_l(p^*) < 0$ then $p^*_l = 0$.

- weak Walras law: $p \cdot z(p) \leq 0$

  and if $p \cdot z(p) < 0 \Rightarrow \exists\, l \;:\, z_l(p) > 0$

  $\rightarrow$ under which assumptions does weak Walras' law hold?

- continuity of $z(p)$

  $\rightarrow$ under which assumptions does continuity hold?

2 Existence of Equilibrium

$\rightarrow$ a story of price adjustment

Theorem 1 *Suppose weak Walras law and continuity of $z(p)$ hold. Then there exists a $p^* \in P$ so that p^* is a Walrasian equilibrium price vector.*

Proof steps:

1. Set up price adjustment functions $\wp(p) : P \rightarrow P$
   equivalently: $\wp_l(p) : P \rightarrow [0, 1], \;\; l = 1, ..., L$

2. Show denominator of $\wp_l(p)$ is strictly positive (...well defined)

3. Establish existence of a fixed point, $p^* \in P$, of $\wp(p)$

4. Show that this fixed point is a Walrasian equilibrium price vector

5. Argue that in equilibrium weak Walras law holds as an equality

6. Argue that in equilibrium, bound on X is not binding

- Step 1. price adjustment functions $\wp(p) : P \to P$

$$\wp_l(p) = \frac{\max[0,\, p_l + \gamma_l\, z_l(p)]}{\sum_{n=1}^{L} \max[0,\, p_n + \gamma_n\, z_n(p)]}\,, \quad l = 1, ..., L$$

  $\gamma_l > 0$ (sensitivity to excess demand)

- Step 2. $\wp(p)$ is well defined: denominator $\neq 0$
  - denominator $\geq 0 \Rightarrow$ claim: denominator > 0
  - **Proof.**
    Suppose not, then $\max[0,\, p_n + \gamma_n\, z_n(p)] = 0$ (*)
    for all $p_n > 0$, $z_n(p) < 0$!
    consequently, $p \cdot z(p) < 0$
    then, weak Walras law requires $z_l(p) > 0$ for some l but
    then, $\max[0,\, p_n + \gamma_n\, z_n(p)] > 0$, contradicting (*) QED

  - thus: $\max[0,\, p_n + \gamma_n\, z_n(p)] > 0$ ($\wp(p)$ is well defined)

- Notation alert.

$$\alpha \equiv \left[\sum_{n=1}^{L} \max[0,\, p_n + \gamma_n\, z_n(p)\right]^{-1} > 0$$

$$\wp_l(p) = \alpha \max[0,\, p_l + \gamma_l\, z_l(p)]\,, \quad l = 1, ..., L$$

- Step 3. Establish existence of a fixed point, $p^* \in P$, of $\wp(p)$
  - continuous (why?)
  - $\wp_l(p) \geq 0, \;\; \wp_l(p) \leq 1$
  - $\sum_{l=1}^{L} \wp_l(p) = 1$

  $\wp(p) : P \to P$, continuous

  by Brower's fixed point theorem:

  $\exists\, p^* \in P : \; \wp(p^*) = p^*$

  $\to$ claim: p^* is a Walrasian equilibrium price vector

- Step 4. A fixed point is a Walrasian equilibrium price vector

  Case 1: $p_l^* = 0$. Then $z_l(p^*) \leq 0$

  Case 2: $p_l^* > 0$

$$p_l^* = \alpha \, \max[0, \, p_l^* + \gamma_l \, z_l(p^*)] > 0$$
$$(1-\alpha)p_l^* = \alpha \, \gamma_l \, z_l(p^*)$$
$$(1-\alpha)p_l^* \, z_l(p^*) = \alpha \, \gamma_l \, z_l(p^*)^2 \geq 0 \quad (**)$$

- if $\alpha \geq 1$: $z_l(p^*) \leq 0$
  but for no l, $z_l(p^*) < 0$; otherwise $p^* \cdot z(p^*) < 0$
  then by weak Walras law: $\exists l : z_l(p^*) > 0$ contradicting (**)
  $\Rightarrow z_l(p^*) = 0$ for all markets with $p_l^* > 0$

from (**)

$$\sum_l (1-\alpha) p_l^* \, z_l(p^*) = \sum_l \alpha \, \gamma_l \, z_l(p^*)^2$$

- if $\alpha \leq 1$, by Weak Walras law:

$$\begin{aligned}
0 &\geq (1-\alpha) \sum_l [p_l^* \, z_l(p^*)] \\
&= (1-\alpha) \sum_{l \in Case1} p_l^* \, z_l(p^*) + (1-\alpha) \sum_{l \in Case2} p_l^* \, z_l(p^*) \\
&= 0 + (1-\alpha) \sum_{l \in Case2} p_l^* \, z_l(p^*) \leq 0 \\
&\Rightarrow \; \alpha \sum_{l \in Case2} \gamma_l \, z_l(p^*)^2 \leq 0 \\
&\Rightarrow \; z_l(p^*) = 0 \; \forall \, l \in \text{ Case 2}
\end{aligned}$$

- Step 5: In equilibrium, weak Walras law holds as an equality

$\nexists \, z_l(p^*) > 0 \Rightarrow \; p^* \cdot z(p^*) = 0$ (weak Walras law)

$\Rightarrow z_l(p^*) < 0 \; \Rightarrow p_l = 0$

- Step 6: In equilibrium, bound on X is not binding

  attainable aggregate consumption:
  $0 \leq x_l \leq y_l + \omega_l$, for all $l = 1, ..., L \quad \Leftrightarrow \; 0 \leq x \leq y + \omega$

    upper bound c on aggregate consumption set: $c > |x|$ for any attainable x

    $p^* \cdot z(p^*) = 0$

    by attainability: $|x_i(p)| < c$ for all $i = 1, ..., I$

  $\Rightarrow$ bound c not binding