Pareto Efficiency and Competitive Equilibrium

Main objectives of today's session:

. normative questions
. market equilibrium and efficiency

- Pareto efficiency

- 1st FUN Theorem

- 2nd FUN Theorem

1 Pareto Efficient Allocations

- allocation (x, y)

  $\rightarrow$ attainable allocation: $0 \leq \sum_i x_{li} \leq \sum_j y_{lj} + \sum_i \omega_{li}$
  $l = 1, ..., L$

  $\rightarrow$ wasteful vs. efficient allocations

  – inefficient production

  – inefficient consumption: $MRS_i(x_i) \neq MRS_{i'}(x_{i'})$

Definition 1 *Consider x, x'. x' is Pareto superior to x (or Pareto dominates x) if $x'_i \succsim_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.*

Definition 2 *x is Pareto efficient if $\nexists$ attainable x' so that x' is Pareto superior to x.*

$\rightarrow$ Pareto efficiency vs. equity (distribution)

Definition 3 *A Walrasian equilibrium is a price vector $p \in P$ and an allocation (x, y):*

(i) $y_j \in Y_j$ and $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in Y_j$, $j = 1, ..., J$

(ii) $x_i \in X_i$, $p \cdot x_i \leq M_i(p) = p \cdot \omega_i + \sum_j \alpha_{ij}\, p\, y_j$
and $x_i \succsim_i x'_i$ for all $p \cdot x'_i \leq M_i(p)$, $i = 1, ..., I$

(iii.a) $\sum_i x_{li} - \sum_j y_{lj} - \sum_i \omega_{li} \leq 0$, $l = 1, ..., L$
(iii.b) if $\sum_i x_{l'i} - \sum_j y_{l'j} - \sum_i \omega_{l'i} < 0$ for some l', then $p_{l'} = 0$.

2 1st FUN Theorem

Theorem 1 *Suppose C.IV, C.V holds. Let (p, x, y) be a Walrasian equilibrium. Then x is Pareto efficient.*

- $x_i' \succ_i x_i(p) \Rightarrow p \cdot x_i' > p \cdot x_i(p)$

  $p \cdot y_j' > p \cdot y_j(p) \Rightarrow y_j' \notin Y_j$

  $\sum_i x_i \leq \sum_j y_j + \omega$

  $p \cdot x_i(p) = M_i(p) = p \cdot \omega_i + \sum_j \alpha_{ij}(p \cdot y_j(p))$

- summing over all i:

  $\sum_i p \cdot x_i(p) = \sum_i M_i(p) = p \cdot \omega + p \cdot \sum_j y_j(p)$

- suppose there were a Pareto superior allocation (x', y'):

  $\sum_i p \cdot x_i' > \sum_i p \cdot x_i(p) = \sum_i M_i(p) = p \cdot \omega + p \cdot \sum_j y_j(p)$
  $\Rightarrow \sum_i p \cdot x_i' > p \cdot \omega + p \cdot \sum_j y_j(p)$
  suppose x' attainable, then $\exists y'$:
  $\omega + \sum_j y_j' \geq \sum_i x_i' \Rightarrow$
  $p \cdot \omega + p \cdot \sum_j y_j' \geq \sum_i p \cdot x_i' > p \cdot \omega + p \cdot \sum_j y_j \Rightarrow$
  $p \cdot \sum_j y_j' > p \cdot \sum_j y_j(p) \Rightarrow \exists j : p \cdot y_j' > p \cdot y_j(p)$
  $\Rightarrow y_j' \notin Y \Rightarrow x'$ not attainable.

$\rightarrow$ formalization of invisible hand

3 Second FUN Theorem

- Every Pareto efficient allocation of "convex economy" is a WE for suitably chosen prices – subject to initial redistribution of endowments and ownership shares.

  any PE (x, y) can be achieved through market mechanism

- Steps

1. duality UMP – EMP
2. hyperplanes & Separating hyperplane theorem
3. proof strategy
4. sets & vectors employed in proof
5. theorem: existence of a p supporting PE allocation (x, y)
6. corollary: 2nd fundamental theorem of welfare economics
7. critical assumption: convexity; the role of government; and critical comments

- Step 1. UMP & EMP: dual problems

  UMP
  $\max_{x_i} u_i(x_i)$ s.t. $p \cdot x_i \leq M_i(p)$

  optimizer: $x_i(p)$ Walrasian demand
  value function: $v_i(p)$ indirect utility

  EMP
  $\min_{x_i} p \cdot x_i$ s.t. $u_i(x_i) \geq u_0$

  optimizer: $h_i(p, u_0)$ Hicksian demand
  value function: $e_i(p, u_0)$ expenditure function

  Proposition 1 *Suppose $u(.)$ is cont., representing monotone $\succsim$, $p \gg 0$, and fix $u_0 = v_i(p)$. Then:*

  (i) Suppose $M_i(p) > 0$, and $x_i^* \equiv x_i(p)$ solves UMP. Then $h_i(p, v_i(p)) = x_i^*$.

  (ii) Suppose $x_i^* \equiv h_i(p, v_i(p))$ solves EMP. Then $x_i(p) = x_i^*$, with wealth $M_i(p) = p \cdot h(p, v_i(p))$.

  2nd FUN employs a given PE (x, y)
  idea: if $x = (x_1, x_2, ..., x_I)$ expenditure min. for all i
  then x is utility max. for all i

$\rightarrow$ demonstrate the proposition graphically

$\rightarrow$ given a budget "line" show the price vector graphically

- Step 2. Hyperplanes & Separating hyperplane theorem

  Consider $p \in \mathbb{R}^L$, $p \neq 0$

  hyperplane $H(p, k) \equiv \{x \in \mathbb{R}^L \,|\, p \cdot x = k\}$

  $\rightarrow$ budget "line"
  (p = prices, x = consumption bundle, k = wealth)

  $\rightarrow$ isoprofit "line" (k = profit)

Theorem 2 (Separating hyperplane theorem (SHT)) *Consider $A, B \subset \mathbb{R}^L$, nonempty, convex, disjoint: $A \cap B = \varnothing$. Then, there exists $p \in \mathbb{R}^L \backslash \{0\}$: $p \cdot a \geq p \cdot b$, for all $a \in A$, $b \in B$.*

$\rightarrow$ illustrate theorem by figure

- Step 3. Proof strategy

  (i) characterize PE (x, y) as on boundaries of two convex, disjoint sets
  – attainable consumptions
  – preferable consumptions

  (ii) employ SHT to establish existence of hyperplane b/w them

  (iii) normal to hyperplane = p supporting (x, y)

  (iv) redistribute endowments so that p is WE price vector

- Step 4. Sets & vectors employed

  $x = (x_1, x_2, ..., x_I)$

  $A_i(x_i) \equiv \{x \in X_i \,|\, x \succsim_i x_i\}$

  $A(x) = \sum_i A_i(x_i)$

  $\mathcal{A}(x)$ set of all allocations Pareto superior to x

  $B \equiv (Y + \{\omega\}) \cap \mathbb{R}^L_+$

→ in which space lives x? $x \in \mathbb{R}^?_?$?

→ what are the properties of $A_i(x_i)$, $A(x)$, $\mathcal{A}(x)$?

→ show graphically the differences between: Y, $Y + \{\omega\}$, and set B?

→ what are the properties of set $B = (Y + \{\omega\}) \cap \mathbb{R}^L_+$?

→ give a graphical representation of those sets

- set $\mathcal{A}(x)$ captures preferable consumptions
- set B captures attainable consumptions

- Step 5. Existence of p supporting PE allocation (x, y)

Theorem 3 *Assume Y_j is convex for all $j = 1, ..., J$, and C.I–C.VI. Let (x, y) be an attainable, Pareto efficient allocation. Then there exists $p \in P$ such that:*

(i) $x_i(p)$ minimizes $p \cdot x$ in $A_i(x_i)$, $i = 1, ..., I$,

(ii) $y_j(p)$ maximizes $p \cdot y$ in Y_j, $j = 1, ..., J$.

A and B are convex sets. Let $x(p) = \sum_i x_i(p)$, $y(p) = \sum_j y_j(p)$. $\mathcal{A}$ is a convex set with closure A. $\mathcal{A}$ and B are convex, disjoint sets. $x(p) \in A$ and $x(p) \in B$. But $x(p) \notin \mathcal{A}$, and $x(p) \notin$ interior of B. By the SHT,

– Separating hyperplane theorem: $\exists p$:

$p \cdot x' \geq p \cdot (y' + \omega)$ for all $x' \in \mathcal{A}(x)$ and $(y' + \omega) \in B$

– continuity of $\succsim$ and dot product:

$p \cdot x' \geq p \cdot (y' + \omega)$ for all $x' \in A(x)$ and $(y' + \omega) \in B$ (*)

– $x(p) \leq y(p) + \omega$, $p \geq 0 \Rightarrow p \cdot x(p) \leq p \cdot (y(p) + \omega)$ (**)

from (*) and (**):
$x(p)$ minimizes x' on A, $(y(p) + \omega)$ maximizes $(y' + \omega)$ on B

- additive structure of (x, y)

  not only aggregate but all individual firms' profits are maximal by $y_j(p)$

  for given utility values $u_i(x_i)$, $i = 1, ..., I$:
  all individual hh minimize expenditure by $x_i(p)$

households (parallel argument holds for firms)

$$
\begin{aligned}
p \cdot x(p) = \min_{x \in A} p \cdot x &= \min_{x_i \in A_i(x_i(p))} p \cdot \sum_i x_i \\
&= \sum_i \min_{x_i \in A_i(x_i(p))} p \cdot x_i
\end{aligned}
$$

→ p not only characterizes expenditure minimum in aggregate, but also for every i

→ in parallel, p not only characterizes a profit maximum in aggregate, but also for every j

→ but hh may or may not be able to effort x at p

- Step 6. 2nd FUN Theorem
- reallocation (budget neutral lump sum tax system)

  $\hat{\omega}_i$ such that $\sum_i \hat{\omega}_i = \omega$

  $\hat{\alpha}_{ij}$ such that $\sum_i \hat{\alpha}_{ij} = 1$

Corollary 1 *Assume P.I–P.IV and C.I–C.VI. Let (x, y) be an attainable, Pareto efficient allocation. Then there exists $p \in P$, and a reallocation such that:*

(i) y_j maximizes $p \cdot y$ in Y_j, $j = 1, ..., J$,

(ii) $p \cdot x_i = p \cdot \hat{\omega}_i + \sum_j \hat{\alpha}_{ij}(p \cdot y_j) = \hat{M}_i(p)$, $i = 1, ..., I$,

(iii) if $p \cdot x_i > \min_{x \in X_i} p \cdot x$:
$x_i \succsim_i x$ for all $x \in X_i \, : \, p \cdot x \leq \hat{M}_i(p)$.

- Step 7. Comments

(i) Critical assumption convexity: show a graph demonstrating that the theorem may fail under nonconvexisties

(ii) According to the 2nd FUN theorem of welfare economics, what is an important role for the government?

(iii) Criticize the 2nd FUN theorem of welfare economics.

The usual U-shaped cost curve model of undergraduate intermediate economics includes a small nonconvexity (diminishing marginal cost at low output levels). This is a violation of our usual convexity assumptions on production (P.I or PV). Consider the general equilibrium of an economy displaying U-shaped cost curves. It is possible that a general equilibrium exists despite the small violation of convexity. After all, P.I and P.V are sufficient, not necessary, conditions. If a general equilibrium does exist despite the small nonconvexity, will the allocation be Pareto efficient? Does the First Fundamental Theorem of Welfare Economics apply? Explain.