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# Application. AK Model and Extensions

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# (Fully) Endogenous Growth

- Households
- Firms
- Endogenous growth (BGP)
- Examples

# Households

- households consume and save; consumption raises utility - savings do not; savings raise wealth (capital stock) thereby future consumption possibilities

  representative household

$$U_0 = \int_0^\infty (\ln c_t)\, e^{-\rho t} dt = \int_0^\infty (\ln c_t)\, e^{-\rho t} dt$$

  household budget (flow) constraint

$$\dot{k}_t = r_t k_t - c_t\;, \quad k_0 \text{ given}$$

  transversality

$$\lim_{t\to\infty} \mu_t\, k_t\, e^{-\rho\, t} = 0$$

# Technology

- simplest case: $y_t = Ak_t$
- factor prices

$$r_t = \frac{\partial y_t}{\partial k_t} - \delta = A - \delta > 0$$

- household budget constraint

$$\dot{k}_t = (A - \delta)k_t - c_t\,, \quad k_0 \text{ given}$$

# Ingredients of optimal control problem

- intertemporal utility function

$$U_0 = \int_0^\infty (\ln c_t)\, e^{-\rho t} dt$$

household budget constraint

$$\dot{k}_t = (A - \delta)k_t - c_t \,, \quad k_0 \text{ given}$$

transversality

$$\lim_{t \to \infty} \mu_t \, k_t \, e^{-\rho\, t} = 0$$

- 1. Set up Hamiltonian function

$$H(c_t, k_t, \mu_t) = \ln c_t + \mu_t[(A - \delta)k_t - c_t]$$

- 2. Maximize the Hamiltonian w.r.t. control variable $c_t$

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0 \quad \Rightarrow \quad \mu_t = c_t^{-1} \quad \Rightarrow \quad \frac{\dot{\mu}_t}{\mu_t} = -\frac{\dot{c}_t}{c_t}$$

- 3. Differentiate $H$ w.r.t. state variable $k_t$ and set = to $\rho\mu_t - \dot{\mu}_t$

$$\frac{\partial H}{\partial k_t} = \mu_t(A - \delta) = \rho\mu_t - \dot{\mu}_t \quad \Rightarrow \quad \frac{\dot{\mu}_t}{\mu_t} = -(A - \delta - \rho)$$

- Use 2. and 3. to calculate the consumption growth rate per-capita consumption growth rate

$$\frac{\dot{c}_t}{c_t} = (A - \delta - \rho)$$

per-capita capital growth rate: rewrite dynamic constraint

$$\frac{\dot{k}_t}{k_t} = (A - \delta) - \frac{c_t}{k_t}$$

- BGP requires $\frac{\dot{k}_t}{k_t} = \frac{\dot{c}_t}{c_t}$. Why? $\Rightarrow \frac{\dot{k}_t}{k_t} = (A - \delta - \rho)$

per-capita income growth: as $y_t = Ak_t$:

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} = (A - \delta - \rho)$$

Parameter restrictions

- positivity of growth: $\rho < A - \delta$
- boundedness of utility integral: $\rho > 0$

Endogenous growth: rises in $A$ and declines in $\rho$

$\rightarrow$ via saving rate

$$s = \frac{y - c}{y} = \frac{y/k - c/k}{y/k} = \frac{A - \rho}{A}$$

- AK-model explains the saving rate *and* the growth rate of per capita income and -consumption.