A Note on Perfect Complements

1 Utility when Goods are Perfect Complements

At some point, we have been considering the case in which two goods, say (x_1, x_2), can only be consumed in a fixed proportion to each other. In such a case, we say that x_1 and x_2 are perfect complements. In this case, there is no way to substitute one good for the other one. In other words, the elasticity of substitution between x_1 and x_2 equals zero. How do we handle such a situation mathematically? Well, by, what we called a **min function** (Leontief function, limitational function). But what is a min function, and how do we maximize utility when the utility function is a min function (Leontief utility)? This note is addressing these very questions.

2 Digression: What is a Function?

Let us define the set of natural (counting) numbers by $\mathbb{N} = \{1, 2, 3, ...\}$. Furthermore, let the n-fold Cartesian product of $\mathbb{R}$, be given by $\mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$ (where the product is taken n times), and $n \in \mathbb{N}$

Generally, a (real-valued) function is a rule that assigns a *unique* real number to each element of its domain. Let us denote the domain by $X \subset \mathbb{R}^n$, with $n \in \mathbb{N}$. Each element of $\mathbb{R}^n$ is a vector of dimension n (an ordered n-tuple). So, the function may be a function of a single variable (in which case, $n = 1$) or of several variables (in which case $n > 1$).

We typically denote functions the following way. Let u denote our function of interest. Then we write:

$$u(x) : X \to \mathbb{R}\,. \tag{1}$$

This is understood the following way. The function u assigns to each element of its domain $x \in X$ a unique element from the set denoted at the right hand side of the right-arrow ($\mathbb{R}$ in our case — i.e., a real number).

In (1), it is understood that (i) $X \subset \mathbb{R}^n$; (ii) $x = (x_1, x_2, ..., x_n) \in X$. Let us work through three simple examples, next.

Example 1. Let $n = 2$ and $X = \mathbb{R}^2_+$. That is $x = (x_1, x_2) \geq 0$, and $u(x) = u(x_1, x_2) = ax_1 + bx_2$. Function $u(x)$ assigns a real value to each member of its domain $x = (x_1, x_2) \in X$. For example, consider $x = (1, 1)$. Then,$u(1, 1) = a + b$. Or, $u(0, 0) = 0$.

Example 2. Let $n = 2$, $X = \mathbb{R}^2_+$, and $u(x) = u(x) = x_1^{1/2} x_2^{1/2}$. Function $u(x)$ assigns a real value to each member of its domain $x = (x_1, x_2) \in X$. For example, consider $x = (1, 1)$. Then $u(1, 1) = 1$. Or $u(4, 4) = 4$.

Example 3. Let $n = 2$, $X = \mathbb{R}^2_+$, and $u(x) = \left(a\, x^\delta/\delta + b\, y^\delta/\delta\right)^{1/\delta}$, $\delta < 1$. As you recall, this is a CES utility function. Then $u(1, 1) = (a/\delta + b/\delta)^{1/\delta} = (1/\delta)^{1/\delta}\,(a + b)^{1/\delta}$.

Notice that utility is negative in case $\delta < 0$. This does not concern us in any way, as discussed in class.

We will come back to this CES utility function below. Specifically, we will argue that the min function is obtained as the limit of the CES utility function where the elasticity of substitution between x_1 and x_2 approaches zero.

3 The min Function

In order to keep things simple, we (1) interpret our function u as a utility function, and we (2) restrict ourselves to the case with two goods: $n = 2$; $X = \mathbb{R}^2_+$. That is, we focus on the case

$$u(x_1, x_2) : \mathbb{R}^2_+ \to \mathbb{R}\,. \tag{2}$$

To deal with perfect complements, we introduce the min function here:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}\,, \quad a, b \in \mathbb{R}_{++}\,.$$

So, for any given parameter values of a and b, this function assigns to each $(x_1, x_2) \in X$ the smaller value, either $a\, x_1$ or $b\, x_2$. Formally,

$$u(x_1, x_2) = \begin{cases} a\, x_1\,, & \text{if} \quad a\, x_1 < b\, x_2 \\ b\, x_2\,, & \text{if} \quad a\, x_1 \geq b\, x_2 \end{cases}\,. \tag{3}$$

Example 4. Let $n = 2$ and $X = \mathbb{R}^2_+$. Suppose $a < b$. Then $u(1, 1) = a$. However, if $a > b$, then $u(1, 1) = b$. And it is very simple to calculate $u(x_1, x_2)$ for any other values of (x_1, x_2). E.g., $u(2, 1) = 2a$ if $2a < b$, otherwise, $u(2, 1) = b$.

Now, let us consider our min function graphically. In the graph, we distinguish between three regions. First, $x_2 = (a/b)\, x_1$. Along this line, $u(x_1, x_2) = a x_1 = b x_2$. Second, below this line (shaded area in Figure 1), $x_2 < (a/b)\, x_1$. That is, for all $x_2 < (a/b)\, x_1$, $u(x_1, x_2) = b x_2$. Third, above this line $x_2 > (a/b) x_1$, in which case $u(x_1, x_2) = a\, x_1$. The following graph illustrates our min utility function.

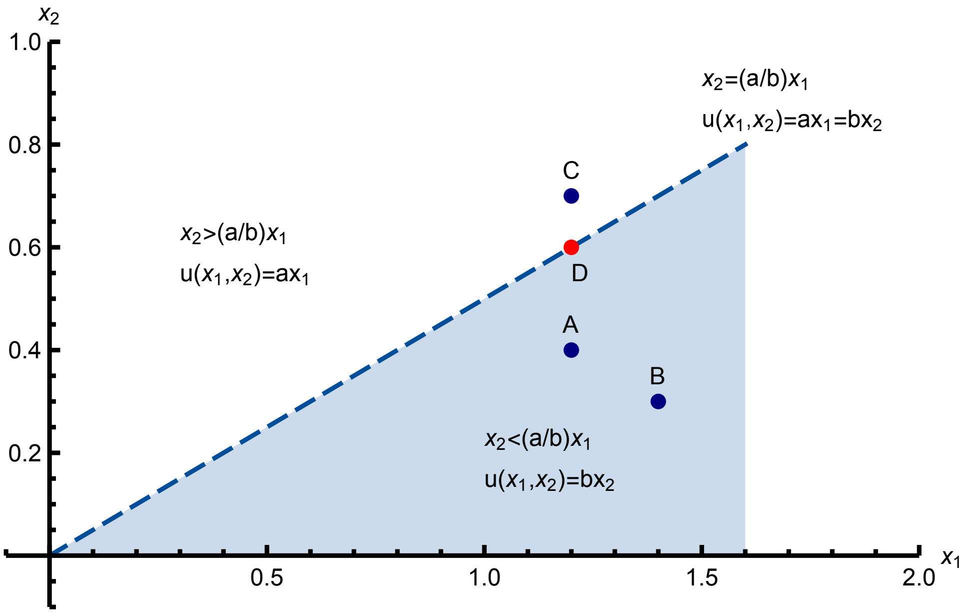

Figure 1: Perfect complements . Note: $a = 1, b = 2$.

In Figure 1, consumption bundle A is strictly preferred over consumption bundle B, as $u(A) > u(B)$. However, how does $u(C)$ compare to $u(A)$? One way to think about this is the following. Consider consumption bundles A, D and C. All three of them contain the same amount of x_1, but D contains a higher amount of x_2. That is, $u(D) > u(A)$. Moreover, as consumption bundles C and D contain the same amount of x_1, we know (from the utility function in the unshaded area of the figure) that $u(D) = u(C)$. Consequently, $u(C) > u(A)$, i.e., C is strictly preferred to A.

Notice that $u(C) = u(D) > u(A)$. That is, points C and D lie on the same indifference curve, while point A lies on a different indifference curve (obviously one that is closer to the origin). How do indifference curves look like, then? This is the question we discuss in the next section.

4 Marginal Utility and Indifference Curves

Given our utility function (3), marginal utilities are easily derived.

$$u_1(x_1,x_2) \equiv \frac{\partial u(x_1,x_2)}{\partial x_1} = \begin{cases} a\,, & \text{if} \quad a\,x_1 < b\,x_2 \\ 0\,, & \text{if} \quad a\,x_1 \geq b\,x_2 \end{cases}\,, \tag{4}$$

$$u_2(x_1,x_2) \equiv \frac{\partial u(x_1,x_2)}{\partial x_2} = \begin{cases} 0\,, & \text{if} \quad a\,x_1 < b\,x_2 \\ b\,, & \text{if} \quad a\,x_1 \geq b\,x_2 \end{cases}\,. \tag{5}$$

Consequently, the marginal rate of substitution of x_1 for x_2 is given by $u_1(x_1,x_2)/u_2(x_1,x_2) = 0$ for $ax_1 \geq bx_2$. So, the slope of the indifference curve is zero for $ax_1 \geq bx_2$ and is represented as a horizontal line in (x_1,x_2) space. Likewise, the marginal rate of substitution of x_2 for x_1 is given by $u_2(x_1,x_2)/u_1(x_1,x_2) = 0$ for $ax_1 < bx_2$. That is, it is represented as a vertical line in (x_1,x_2) space. Figure 2 illustrates Indifference curves for different values of utility $\bar{u}$.

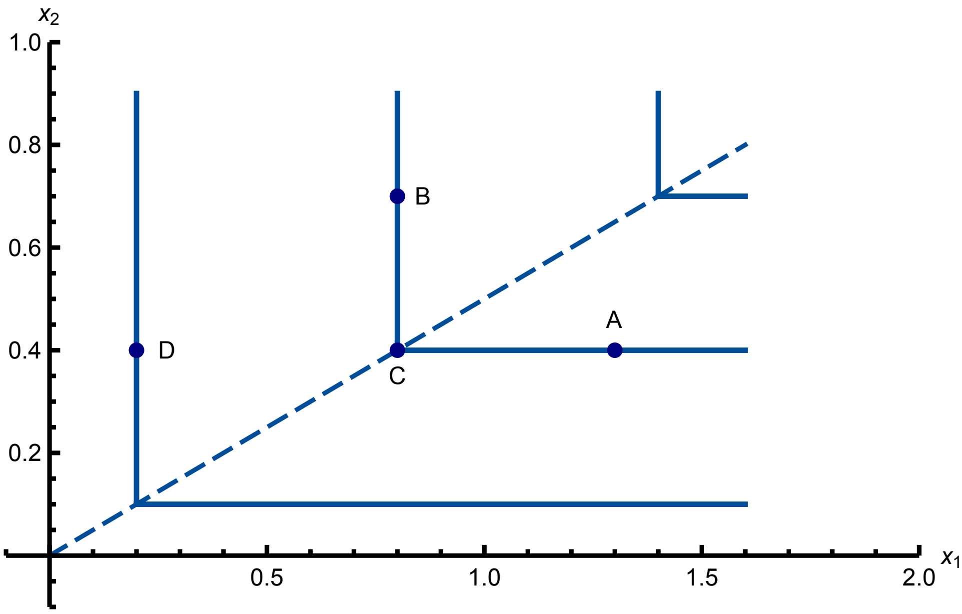

Figure 2: Indifference curves. Note: $a = 1, b = 2$.

Figure 2, displays three indifference curves for our min function. Clearly, $u(A) = u(B) = u(C) > u(D)$. The $MRS_{x_1,x_2}(A) = 0$, and the $MRS_{x_2,x_1}(B) = 0 = MRS_{x_2,x_1}(D)$.

There is one question remaining. What about the $MRS_{x_1,x_2}(C)$? The answer is rather simple: the $MRS_{x_1,x_2}(C)$ is not defined, as division by zero is against the rules. In other words, the slope of the indifference curve at C (or, more generally, along the ray from the origin along which $ax_1 = bx_2$) is not defined. This has significant implications for utility maximization, as discussed in the proceeding section of this note.

5 The min Function and Utility Maximization

Let (p_1, p_2) denote the prices of (x_1, x_2), and I some given income. Obviously, with a min utility function, we cannot apply the necessary first order conditions for an interior solution

$$\frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{p_1}{p_2}, \tag{6}$$

as the left hand side (marginal rate of substitution of x_1 for x_2) is not defined for all (x_1, x_2). Also, our Kuhn-Tucker conditions are not of help in this case. In order to make some progress, consider the following Figure 3.

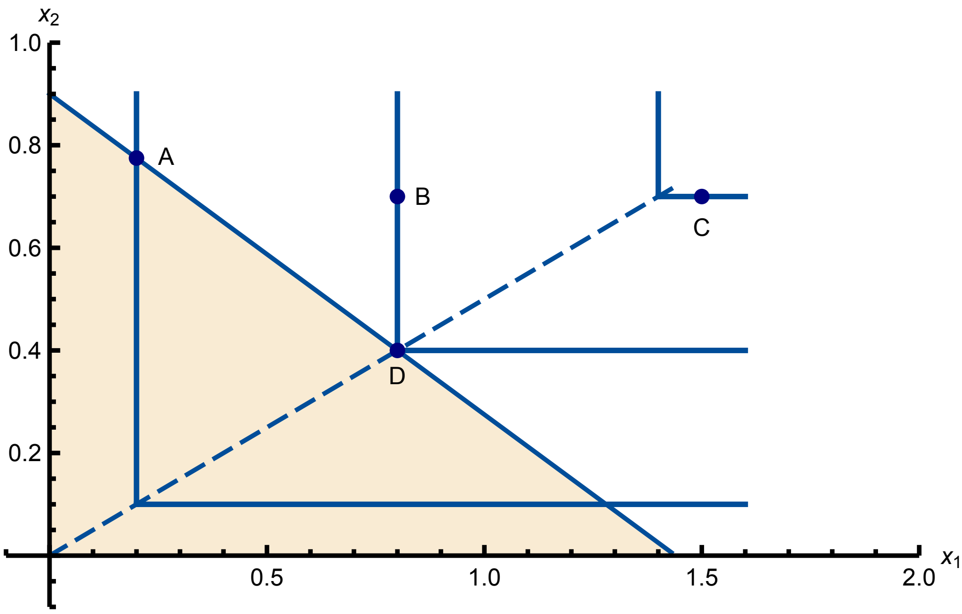

Figure 3: Optimal choice. Note: $a = 1, b = 2$.

In Figure 3, the budget set (all consumption bundles (x_1, x_2) that cost less than the income I) is depicted as the shaded area (in yellow). This set corresponds to the opportunity set, as we discussed in class. All consumption bundles in this area can be afforded. Consumption bundles outside this set are either not affordable or not available (e.g., negative quantities of either x_1 or x_2). What is the best — that is, utility-maximizing – choice?

The answer is not a big challenge. Among the indicated consumption bundles in Figure 3, C would be the best one, as $u(C) > u(B) = u(D) > u(A)$, according to our indifference curves. Consumption bundle C, however, cannot be afforded, as it is not an element of the budget set. Well, consumption bundle A can be afforded (as it belongs to the budget set). However, there are other consumption bundles that can be afforded and give a higher utility. Clearly, consumption bundle D is such a candidate.

Considering consumption bundle B and D, we see that both lie on the same indifference curve — though only consumption bundle D can be afforded (while B cannot be afforded). As a more general insight then, we see that the utility-maximizing consumption bundle must be on the kink of an indifference curve, like D. On the kink, $ax_1 = bx_2$. Starting from D and increasing x_1 would not raise utility, as then $ax_1 > bx_2$. We would consume "too much" of x_1 and could raise utility by reducing x_1 a little and increasing consumption of x_2 instead. Particularly, as we see in Figure 3, the optimal choice is located on that kink of an indifference curve that is also located on the budget line.

Equipped with this insight, we are ready to solve our optimization problem. We have two equations with two unknowns. First, we know that the optimal choice occurs at the kink of an indifference curve, so that

$$a\,x_1 = b\,x_2\,. \tag{7}$$

Second, we also know that the optimal choice occurs at some point of the budget line, so that

$$p_1\,x_1 + p_2\,x_2 = I\,. \tag{8}$$

Now we have two equations in two unknowns. Easily, it follows that our Marshallian demand functions are given by

$$x_1^* = \frac{I}{p_1 + p_2(a/b)}\,, \quad x_2^* = \frac{I}{p_1(b/a) + p_2}\,. \tag{9}$$

6 Indirect Utility, Expenditure Function and Compensated Demand

From here, things become very simple.

$$V(p_1, p_2, I) = \frac{a\,b\,I}{bp_1 + ap_2}\,, \tag{10}$$

$$E(p_1, p_2, \bar{u}) = \frac{bp_1 + ap_2}{a\,b}\,\bar{u}\,, \tag{11}$$

$$x_1^c(p_1, p_2, \bar{u}) = \frac{\partial E(p_1, p_2, \bar{u})}{\partial p_1} = \frac{\bar{u}}{a}\,, \quad x_2^c(p_1, p_2, \bar{u}) = \frac{\partial E(p_1, p_2, \bar{u})}{\partial p_2} = \frac{\bar{u}}{b}\,, \tag{12}$$

where the last line follows from Shephard's lemma. Furthermore, the substitution effects are zero: $\partial x_1^c/(\partial p_i) = \partial x_2^c/(\partial p_i) = 0$, $i = 1, 2$.

7 The min Function as a Special Case of the CES Function

In our last section, we derive the min Function as the limit of the CES Function when the elasticity of substitution approaches zero. As a starting point, consider a CES utility function. In a simple version, we may consider

$$u(x_1, x_2) = [(a\,x_1)^{-\rho} + (b\,x_2)^{-\rho}]^{-1/\rho}\,, \quad a, b > 0,\ \rho > -1\,, \tag{13}$$

with $(x_1, x_2) \in \mathbb{R}_+^2$. In class, we denoted the exponent by $\delta = -\rho$. For this section, it is slightly easier to use the ρ (as it will turn out to be a positive exponent, rather than a negative one).

With this notation at hand, we can define the elasticity of substitution of x_1 for x_2 by

$$\sigma = \frac{1}{1+\rho}\,. \tag{14}$$

For perfect complements, the elasticity of substitution equals zero. That is, we aim to show that in the limit, as ρ approaches plus infinity, CES function (13) becomes

$$u(x_1, x_2) = \min\{a\,x_1,\ b\,x_2\}\,. \tag{15}$$

Assume, without loss of generality, that $ax_1 \geq bx_2$.[1] That is,

$$\min\{ax_1, bx_2\} = bx_2\,. \tag{16}$$

We consider the limit as ρ approaches plus infinity. That is, we do not care about non-positive of ρ and, without loss of generality, assume that $\rho > 0$.

As we assume $ax_1 \geq bx_2$, and noting that $\rho > 0$,

$$(ax_1)^{-\rho} \leq (bx_2)^{-\rho}\,. \tag{17}$$

Moreover, let us write the utility function as:

$$u^{-1} = [(a\,x_1)^{-\rho} + (b\,x_2)^{-\rho}]^{1/\rho}\,. \tag{18}$$

[1] You may assume $ax_1 \leq bx_2$. The analysis presented is the same, just then, $\min\{ax_1, bx_2\} = ax_1$.

Now, we construct two inequalities. First, we replace (ax_1) with the weakly smaller (bx_2). Observing (17),

$$\begin{aligned} u^{-1} &= [(a\,x_1)^{-\rho} + (b\,x_2)^{-\rho}]^{1/\rho} \\ &\le [(b\,x_2)^{-\rho} + (b\,x_2)^{-\rho}]^{1/\rho} = [2(bx_2)^{-\rho}]^{1/\rho} = 2^{1/\rho}(bx_2)^{-1}\,. \end{aligned}$$

So, we know that

$$u^{-1} \le 2^{1/\rho}(bx_2)^{-1}\,. \tag{19}$$

Second, obviously

$$u^{-1} = [(a\,x_1)^{-\rho} + (b\,x_2)^{-\rho}]^{1/\rho} \ge [(b\,x_2)^{-\rho}]^{1/\rho} = (bx_2)^{-1}\,,$$

so we know that

$$u^{-1} \ge (bx_2)^{-1}\,. \tag{20}$$

Putting inequalities (19) and (20) together, we know that for all $\rho > 0$:

$$2^{1/\rho}(bx_2)^{-1} \ge u^{-1} \ge (bx_2)^{-1}\,. \tag{21}$$

As a final step, consider the limit as ρ goes to plus infinity:

$$\lim_{\rho\to\infty} 2^{1/\rho}(bx_2)^{-1} = (bx_2)^{-1}\,, \tag{22}$$

as $2^{1/\rho}$ approaches unity in the limit. Now, we "sandwiched" u^{-1} in the limit:

$$\lim_{\rho\to\infty} u^{-1} = (bx_2)^{-1}\,. \tag{23}$$

It follows that $\lim_{\rho\to\infty} u = b\,x_2$:

$$\lim_{\rho\to\infty} u(x_1, x_2) = \lim_{\rho\to\infty} [(a\,x_1)^{-\rho} + (b\,x_2)^{-\rho}]^{-1/\rho} = b\,x_2 = \min\{ax_1, bx_2\}\,, \tag{24}$$

as was to be shown. As mentioned in the footnote, a parallel argument can be given for the case of $ax_1 \le bx_2$.

References

[1] Arrow, K.J., H.B. Chenery, B.S. Minhas, R.M. Solow (1961), Capital-labor substitution and economic efficiency, *The Review of Economics and Statistics* **43**, 225-250.