A Note on Elasticities

1 Discussion of Problem 5.7

Before I write more on elasticities, I'll offer a solution to Problem 5.7. The solution will be helpful for understanding the discussion on elasticities below. So, we need to show:

$$s_x = \frac{d\,(\ln E)}{d\,(\ln p_x)} \,. \tag{1}$$

Remember that we defined the expenditure share for commodity x by $s_x = p_x x/E$, and E is total expenditure. In fact, we may expand (1) and re-write it a bit. Before we do this, we need to be aware of the following rule:

$$\frac{d\,p_x}{d\,(\ln p_x)} = p_x \,. \tag{2}$$

Why? You might think that p_x is not a function of $\ln p_x$. Well, but consider that

$$p_x = e^{\ln p_x} \,. \tag{3}$$

This can be easily verified by taking logarithms on both sides of (3). We are allowed to do this, as (3) is an equation. So let's do it:

$$\ln p_x = \ln\left(e^{\ln p_x}\right) = \ln p_x \, \ln e = \ln p_x \,, \tag{4}$$

so, in fact, (3) holds true. As a consequence,

$$\frac{d\,p_x}{d\,(\ln p_x)} = \frac{d\,e^{\ln p_x}}{d\,\ln p_x} = e^{\ln p_x} = p_x \,, \tag{5}$$

where the first equality follows from (3), the second equality follows from the rule of differentiation $d\,(e^y)/d\,y = e^y$, and the third equality, again, follows from (3).

We now manipulate expression (1) a little:

$$\frac{d\,(\ln E)}{d\,(\ln p_x)} = \frac{d\,(\ln E)}{d\,E}\,\frac{d\,E}{d\,p_x}\,\frac{d\,p_x}{d\,(\ln p_x)} \,. \tag{6}$$

The first derivative on the right-hand side equals $1/E$. The second derivative equals x by Shepard's lemma (please make sure you remember this lemma well). The third derivative equals p_x by (5). So, we can write this as

$$\frac{d\,(\ln E)}{d\,(\ln p_x)} = \frac{1}{E}\,x\,p_x = \frac{p_x\,x}{E} = s_x \,, \tag{7}$$

what was to be shown. Great, problem solved!

2 Elasticities in logarithmic form

It turns out that our typical elasticity formula

$$e_{x,y} = \frac{d\,x}{d\,y}\,\frac{y}{x}\,, \tag{8}$$

where x and y are arbitrary variables (e.g., x may be some demand, and y may be some price or income) , can be re-expressed in logarithmic form. To do so will make it much easier to calculate elasticities in many important cases. So,

$$e_{x,y} = \frac{d\,x}{d\,y}\,\frac{y}{x} = \frac{d\,\ln x}{d\,\ln y}\,. \tag{9}$$

First, we will have a look why this is so. Second, we will apply this to our constant elasticity of substitution utility functions.

Similar to what we did in Section 1, we re-write and expand the elasticity formula a bit:

$$\frac{d\,\ln x}{d\,\ln y} = \frac{d\,\ln x}{d\,x}\,\frac{d\,x}{d\,y}\,\frac{d\,y}{d\,\ln y} = \frac{1}{x}\,\frac{d\,x}{d\,y}\,y = \frac{d\,x}{d\,y}\frac{y}{x} = e_{x,y}\,. \tag{10}$$

We have established now that our typical elasticity formula and the logarithmic form are equivalent. Super! Let's make it work now.

3 Example: Constant elasticity of substitution

In class, we argued that the elasticity of substitution, σ, for utility functions of the form

$$u(x,y) = \alpha\frac{x^{\delta}}{\delta} + \beta\frac{y^{\delta}}{\delta}\,, \quad \delta < 1 \tag{11}$$

is constant and is equal to

$$\sigma = \frac{1}{1-\delta}\,. \tag{12}$$

Now, we can easily calculate this. However, first we need to make clear what the elasticity of substitution is. So, let's start with a definition first.

$$\sigma = \frac{d\,(y/x)/(y/x)}{d\,MRS_{x,y}/MRSx,y}\,. \tag{13}$$

Wow! Think about this for a minute or so. The denominator shows a relative change of the $MRS_{x,y}$. That is, it shows a percentage change of the slope of an indifference curve. The numerator refers to choices of (x, y). Specifically, it refers to different ratios of (y/x) (different points on an indifferent curve chosen). If you change the ratio of y/x – by how much is the slope of an indifference curve changing at the new point of choice? Well, that depends on its curvature, obviously. If an indifference curve is linear, you can change the choice y/x in any way you like, but the $MRS_{x,y}$ does not at all change. In contrast, if an indifference curve is very "curved", then a little change in y/x (different choice on the indifferent curve) is drastically changing the slope of the newly chosen point.

The elasticity of substitution relates both variables ((y/x) at the one hand and $MRS_{x,y}$ at the other hand). For a high elasticity of substitution, an indifference curve is close to linear, for a low elasticity of substitution, an indifference curve is highly curved. All of this, though, we discussed in class. Let us move forward.

Analytically, we can (and should!) write the elasticity in logarithmic form.

$$\sigma = \frac{d\,(y/x)/(y/x)}{d\,MRS_{x,y}/MRSx,y} = \frac{d\,\ln(y/x)}{d\,\ln MRS_{x,y}}\,. \tag{14}$$

With the logarithmic form at hand, we can easily compute the elasticity. For our CES utility function (11), we can easily calculate:

$$MRS_{x,y} = \frac{\alpha}{\beta}\,\frac{x^{\delta-1}}{y^{\delta-1}} = \frac{\alpha}{\beta}\,\left(\frac{y}{x}\right)^{1-\delta}\,.$$

Thus,

$$\frac{y}{x} = MRS_{x,y}^{1/(1-\delta)}\,\left(\frac{\beta}{\alpha}\right)^{1/(1-\delta)} \Rightarrow \ln(y/x) = 1/(1-\delta)\,\ln MRS_{x,y} + 1/(1-\delta)\,\ln(\beta/\alpha)\,,$$

and as a consequence,

$$\frac{d\,\ln(y/x)}{d\,\ln MRS_{x,y}} = \frac{1}{1-\delta} = \sigma\,, \tag{15}$$

which is a constant.