

Robust equilibria in binary cheap-talk games*

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Abstract

We study the robustness of equilibria in cheap-talk communication games with transparent motives and a binary state space. While most equilibria are not robust against relaxing the assumption of state-independent sender preferences, and many equilibrium sender payoffs cannot be attained in a robust equilibrium, there is always a robust equilibrium among all sender-optimal equilibria. This strengthens the empirical plausibility of such equilibria.

JEL codes: C72, D82, D83

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1 Introduction

Executive decision makers typically have to rely on others to provide them with the necessary information to make informed decisions. These others often have their own interests, not necessarily aligned with those of the decision maker, who has to be wary of this conflict of interest when interpreting what they are told.

The theory of strategic communication, starting with the seminal paper by Crawford and Sobel (1982), has been useful to understand problems in political science (e.g., the strategic communication of political leaders to their potential voters as in Smith (1998)), accounting (e.g., the strategic reporting decisions of firms as surveyed in Beyer et al. (2010)), finance (e.g., the strategic communication of a company's board to affect management as in Adams and Ferreira (2007)), and even macroeconomics (e.g., the strategic communication of a central bank as in Moscarini (2007)).

Researchers in all these areas are interested in understanding how much information can be expected to be transmitted and how informed the final decisions will be. In order to do so, analysts

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typically study strategic communication by means of sender-receiver games. This will require the analyst to have knowledge of the interests of the people involved in the interaction. Naturally, an analyst can never hope to have a perfect understanding of these interests, even if the parties involved themselves do. The analyst, appreciating this, would hope to provide predictions from their game-theoretic model that are at least somewhat robust to small model misspecification errors.

In the present paper, we take a step in this research agenda. We study robust equilibria of the transparent cheap-talk games with one informed sender and one uninformed receiver of Lipnowski and Ravid (2020) with a binary state space. The term transparent refers to the sender having state-independent preferences over the receiver's action choice; the receiver knows the sender's preferences. In such a setting a sender is only willing to transmit meaningful information, that is send different messages in different states, if they are indifferent between the actions that these different messages induce.

We study the robustness of equilibria to the sender having nearby but slightly state-dependent preferences. Equilibria based on indifference seem very fragile to even small changes and, indeed, many equilibria are not robust in this sense. Yet, we show that there is always a robust equilibrium among all sender-optimal equilibria. While many equilibrium sender payoffs are not obtainable through robust equilibria, the upper bound, which the main result in Lipnowski and Ravid (2020) identifies as the quasiconcave envelope of the sender's value function, is robustly obtainable. Our finding, thus, strengthens the empirical plausibility of these equilibria.

To be more precise, no equilibrium, other than those equivalent to a babbling equilibrium, is robust to changing the sender preferences to any nearby preferences. The sender-optimal equilibria, are, however, at least robust to changing the sender preferences to any preferences from a nearby open set.

The paper proceeds as follows. Section 2 provides the model of finite cheap-talk games, our notion of equilibrium and robustness, and the benchmark results of Lipnowski and Ravid (2020). Section 3 provides two key examples that illustrate the main result of this paper. Section 4 provides the main result of the paper along with a selection of results used in its proof that are of some independent interest. This section also provides a sketch of the proof. Section 5 concludes. All (nontrivial) proofs are relegated to the Appendix.

1.1 Related Literature

We here study cheap-talk games with one informed sender and one uninformed receiver, as in a large literature initiated by Crawford and Sobel (1982) and surveyed in Sobel (2013).¹

¹We do not study models of multiple experts trying to influence a single decision maker as, e.g., in Battaglini (2002), Lipman and Seppi (1995), Gilligan and Krehbiel (1989), Krishna (2001), Ambrus and Lu (2014), Ambrus and Takahashi (2008), Gentzkow and Kamenica (2017a), Gentzkow and Shapiro (2008), Board and Lu (2018), and Gentzkow and Kamenica (2017b). We also do not study models of one expert trying to influence multiple individuals as, e.g., in

In particular, we focus on transparent cheap-talk games, i.e. cheap-talk games with state-independent sender preferences, as in Chakraborty and Harbaugh (2010) and, more closely, Lipnowski and Ravid (2020). We utilize the belief-based approach to analysing these games as set out by, among others, Aumann et al. (1995), Aumann and Hart (2003), Kamenica and Gentzkow (2011), and Lipnowski and Ravid (2020).

In transparent cheap-talk games, communication seems fragile.² In particular, many of the equilibria in cheap-talk games require exact sender indifference between multiple messages. This seems somewhat of a knife-edge case, and, therefore, possibly nonrobust to small changes in the modeling assumptions.

Indeed, inspired by the Wilson doctrine (in economics), Diehl and Kuzmics (2021) consider equilibrium robustness with respect to the receiver being uncertain about the sender's preferences.³ They find that no influential equilibria are robust in this sense in the sender state-independent preference cheap-talk games considered by Chakraborty and Harbaugh (2010) (whose setting is a special case of Lipnowski and Ravid (2020)).

The robustness exercise we perform here is not with respect to the players' knowledge about the game, but with respect to the modeler's or analyst's understanding of the game. Specifically, we suppose that the analyst, while having a fairly good idea what the preferences of the sender and receiver are, does not know them perfectly. And we are interested in identifying equilibria that are robust to a small degree of misspecification error on behalf of the analyst. Such a robustness check for general games was already introduced by Wen-Tsün and Jia-He (1962) in their essential equilibrium concept, arguably the first equilibrium refinement in game theory. In order to apply such a robustness concept for cheap-talk games, we adopt it appropriately by taking into account that communication in nearby games needs to remain costless.

A different robustness question is studied in the same basic setting as ours, i.e., transparent cheap-talk games with binary state space, by Arieli et al. (2023). They keep the preferences of both sender and receiver fixed, but introduce a vanishing amount of private information for the receiver about the true state. Their main finding is that only in special cases is a sender-optimal equilibrium robust against such infinitesimal private information for receiver, namely if the receiver obtains no information, i.e., if the babbling equilibrium is sender-optimal, or if some state is fully revealed with positive probability.

Chakraborty et al. (2020) (combined with a voting setup), Alonso and Câmara (2016), Kosterina (2018), Wang (2013), Arieli and Babichenko (2019), Bardhi and Guo (2018) and Chan et al. (2019).

²Yet, many standard refinements have little or no power in cheap-talk games, see, e.g., Banks and Sobel (1987), Cho and Kreps (1987), Grossman and Perry (1986), Kohlberg and Mertens (1986), and McLennan (1985). An exception are the more recent, evolutionary inspired, refinements of Balkenborg et al. (2015) and Myerson and Weibull (2015). Specific refinements have been developed for cheap-talk games by, e.g., Farrell (1993) and Blume and Sobel (1995).

³Robert Wilson has stated that he expects most progress for our understanding of such problems by repeatedly weakening common knowledge assumptions. This motivated the recent literature on Robust Mechanism Design (see, e.g., Bergemann and Morris (2011)). For instance, Heifetz and Neeman (2006) show that the surplus extraction result of Crémer and McLean (1985) and Crémer and McLean (1988) does not extend to models that weaken the common knowledge assumption.

2 The model

We consider finite cheap-talk games. There are two players, a sender (she), and a receiver (he). There is a finite state space Θ containing at least two elements and a full-support prior distribution $\mu_0 \in \Delta\Theta$ that is common knowledge between the two players.⁴

The sender observes the true state, and, conditional on that state, sends a message from an unmodeled finite set of messages. The receiver observes the sender's message and then takes an action from the finite set A , that, again, contains at least two elements. The sender's and receiver's payoffs are, respectively, given by utility functions $u_S, u_R : A \times \Theta \rightarrow \mathbb{R}$. Messages do not enter the utility functions. A finite *cheap-talk game* is then given by the quintuple $\Gamma = (\Theta, \mu_0, A, u_S, u_R)$.

To analyze cheap-talk games, we adopt the belief-based approach, which concentrates on the *ex ante* distribution of the receiver's posterior beliefs that is induced by any combination of a sender strategy (for sending messages) with a receiver belief system.⁵ Thus, instead of modeling messages, we let the sender directly choose a (finite support) posterior belief distribution $p \in \Delta\Delta\Theta$. Consistency with Bayesian updating requires only that p is *Bayes plausible* (in the language of Kamenica and Gentzkow, 2011), which means that the ex ante expected posterior equals the prior belief:

$$\sum_{\mu \in \text{supp } p} \mu p(\mu) = \mu_0.$$

Sometimes, it is enough to consider the receiver's reaction to all possible posterior beliefs only in terms of the implied ex ante expected sender payoff, which is a single number.⁶ Here, however, we are interested in the sender's *interim* expected payoff, which means after learning the state but before releasing any information to the receiver. Therefore, we represent the receiver's behavior more explicitly. Specifically, we consider receiver strategies $\rho : \text{supp } p \rightarrow \Delta A$ that assign a (mixed) action to any posterior belief that is possible given p . For any $\mu \in \Delta\Theta$, let $A(\mu)$ be the set of the receiver's optimal actions given belief μ , that is $A(\mu) := \arg \max_{a \in A} \sum_{\theta \in \Theta} u_R(a, \theta) \mu(\theta)$.

Cheap talk is characterized by the inability of the sender to commit to any postulated strategy. In our setting this means that, after learning the state, the sender can "propose" to the receiver any of the ex ante possible posterior beliefs. In order to express the sender's incentives in any state θ conveniently, we enumerate the available actions in an arbitrary way and let, thus, $A = \{a_1, \dots, a_n\}$. This allows us to introduce (with a slight abuse of notation) the vector $u_S(\theta) := (u_S(a_1, \theta), \dots, u_S(a_n, \theta))^\top$ and to treat also any mixed action $r \in \Delta A$ as a vector $(r(a_1), \dots, r(a_n))^\top$. Hence, we can simply write $u_S(\theta) \cdot \rho(\mu)$ for the sender's interim expected payoff from any posterior

⁴For any given finite (or metric) space X , the set of all (Borel) probability measures over X is denoted by ΔX .

⁵The belief-based approach has proven to be very useful in various contexts. See, e.g., Lipnowski and Ravid (2020) for (transparent) cheap talk, Kamenica and Gentzkow (2011) for Bayesian persuasion, or Aumann and Maschler (1995) for repeated games.

⁶This is the case, e.g., for Kamenica and Gentzkow (2011), where the information structure is such that sender and receiver share the same posterior belief, and also for Lipnowski and Ravid (2020), where state independence implies that the sender's expected payoff does not change when learning the state.

belief μ and strategy ρ for the receiver.

Definition 1. For a cheap-talk game $\Gamma = \langle \Theta, \mu_0, A, u_S, u_R \rangle$, a pair (p, ρ) consisting of a posterior belief distribution $p \in \Delta\Delta\Theta$ and a receiver strategy $\rho: \text{supp } p \rightarrow \Delta A$ is an *equilibrium* if

- (i) p is Bayes plausible,
- (ii) $\text{supp}(\rho(\mu)) \subseteq A(\mu)$ for all $\mu \in \text{supp } p$, and
- (iii) $u_S(\theta) \cdot \rho(\mu) \geq u_S(\theta) \cdot \rho(\mu')$ for all $\mu, \mu' \in \text{supp } p$ and all $\theta \in \text{supp } \mu$.

Any equilibrium (p, ρ) induces the *outcome* (p, s^*) , where $s^*: \Theta \rightarrow \mathbb{R}$ is the sender's interim expected payoff function, which is given by $s^*(\theta) = \max_{\mu \in \text{supp } p} u_S(\theta) \cdot \rho(\mu)$.⁷ In Appendix A.1 we show that the outcomes that can result from any equilibrium as defined here are exactly the same as those induced by equilibria defined in the standard way including arbitrary (finite) sets of messages.

We say that an equilibrium (p', ρ') is *sender payoff-equivalent* to an equilibrium (p, ρ) if the sender's interim expected payoff functions induced by the two equilibria agree, i.e., if there is a function $s^*: \Theta \rightarrow \mathbb{R}$ such that the respective equilibrium outcomes are (p', s^*) and (p, s^*) .

A particular benchmark (e.g., for attainable sender payoffs) are the trivial *babbling* equilibria, which are those in which the receiver infers no information at all and sticks to the prior belief. Since the support of its belief distribution is the singleton $\{\mu_0\}$, any babbling equilibrium can be characterized by the single (mixed) action $\rho(\mu_0)$. In fact, the only requirement is that the taken action be from $A(\mu_0)$, so one does not even need to know u_S to construct any babbling equilibrium.

2.1 Robustness

We are interested in equilibria for cheap-talk games that are robust to small payoff perturbations. In particular, we are interested in the robustness of equilibria for the transparent cheap-talk games analyzed by Lipnowski and Ravid (2020) within the ambient space of general cheap-talk games.

We consider only sender payoff perturbations. From a given cheap-talk game $\Gamma = \langle \Theta, \mu_0, A, u_S, u_R \rangle$ we, thus, fix the state space Θ , the prior belief μ_0 , the action space A , and the receiver preferences u_R ; we vary only u_S . Let \mathcal{U} denote the space of all possible sender utility functions. Naturally, $u_S \in \mathcal{U}$. The space \mathcal{U} can be identified with the Euclidean space $\mathbb{R}^{|A| \cdot |\Theta|}$, which is naturally endowed with the Euclidean topology.

Similarly, we can identify any equilibrium outcome (p, s^*) with an element of the space $\Delta\Delta\Theta \times \mathbb{R}^{|\Theta|}$, which we equip with the product topology, using the topology of weak convergence on $\Delta\Delta\Theta$ and again the Euclidean topology on $\mathbb{R}^{|\Theta|}$.

⁷This follows from equilibrium condition (iii) and the fact that, by the Bayes plausibility of p and the full-support assumption for μ_0 , every state $\theta \in \Theta$ is indeed in the support of some posterior $\mu \in \text{supp } p$.

Definition 2. An equilibrium with outcome (p, s^*) is *fully robust* (to a small perturbation in sender preferences) if for any pair of neighborhoods V of p and W of s^* there is a neighborhood U of u_S such that for any $\tilde{u}_S \in U$ there is an equilibrium whose outcome is in $V \times W$.

Full robustness turns out to be too demanding; almost no equilibria are fully robust. Nevertheless, we still want to identify equilibria that are not “knife edge” with respect to the exact sender utility. We, thus, consider sets of perturbed utility functions with positive measure and not only approximating sequences.

Definition 3. An equilibrium with outcome (p, s^*) is *robust* if for any triple of neighborhoods U of u_S , V of p , and W of s^* there is a point $u'_S \in U$ with a neighborhood U' such that for any $\tilde{u}_S \in U'$ there is an equilibrium whose outcome is in $V \times W$.

In effect, arbitrarily closely to u_S , we want to have nonempty, open sets of perturbed sender utility functions \tilde{u}_S that support equilibrium outcomes arbitrarily closely to the given (p, s^*) ; see Figure 1 for a conceptual illustration.

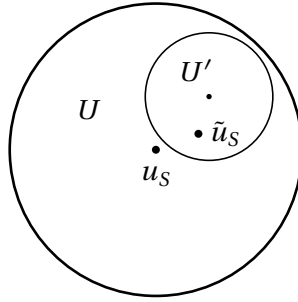


Figure 1: Admissible perturbations for a robust (but not fully robust) equilibrium.

For characterizing robust equilibria, the following two results are extremely helpful. Firstly, we need not consider any other equilibrium belief distributions than the given p when we perturb the sender’s utility function.

Lemma 1. *For any cheap-talk game, the set of (fully) robust equilibria does not change if, in the definition, requiring equilibrium outcomes to be in $V \times W$ is strengthened to them being in $\{p\} \times W$.*

Further, we can ignore all belief distributions that are supported by more posterior beliefs than there are states. This fact usually follows quite immediately from Carathéodory’s theorem. We prove that it still holds for our version of the belief-based approach, which considers the sender’s expected payoff at the interim and not only the ex ante stage, and, more importantly, also for robust equilibria.

Lemma 2. *Let Γ be a cheap-talk game and suppose (p, ρ) is an equilibrium with $|\text{supp } p| > |\Theta|$. Then there is another, sender payoff-equivalent, equilibrium (p', ρ') in which $\text{supp } p' \subseteq \text{supp } p$, $|\text{supp } p'| \leq |\Theta|$, and $\rho' = \rho|_{\text{supp } p'}$. Further, if (p, ρ) is (fully) robust, so is (p', ρ') .*

2.2 Transparent games

We call a cheap-talk game Γ *transparent* if, as under the main assumption in Lipnowski and Ravid (2020), the sender's preferences are independent of the state, i.e., if there is a function $v_S : A \rightarrow \mathbb{R}$ such that the sender's utility is $u_S(a, \theta) = v_S(a)$ for all $\theta \in \Theta$ and $a \in A$.

For transparent games, Lipnowski and Ravid (2020) introduce the *sender value correspondence*, which we here denote by

$$\mathcal{V} : \Delta\Theta \rightarrow 2^{\mathbb{R}},$$

such that

$$\mu \mapsto \text{co } v_S(A(\mu)),$$

and which contains, by the linearity of expected utility, all sender payoffs from mixed best replies of the receiver to the belief μ . In a transparent cheap-talk game, the sender, by the fact that her preferences are state-independent, must expect the same payoff in all states and for all messages sent in equilibrium. Otherwise, she would strictly prefer one message over another and would do better by avoiding sending the latter message.

Given this, Lipnowski and Ravid (2020) provide a characterization of equilibrium posterior belief distribution and sender (ex ante) expected payoff pairs (p, s) .

Lemma LR (Lipnowski and Ravid, 2020). *For a given transparent cheap-talk game with sender value correspondence \mathcal{V} , there is an equilibrium inducing the posterior belief distribution $p \in \Delta\Delta\Theta$ and the ex ante expected sender payoff $s \in \mathbb{R}$ if and only if*

- (i) *the posterior belief distribution p is Bayes plausible, and*
- (ii) $s \in \bigcap_{\mu \in \text{supp } p} \mathcal{V}(\mu)$.

The main result in Lipnowski and Ravid (2020) is the quasiconcavification theorem, which geometrically characterizes the sender's maximal equilibrium value. The quasiconcave envelope of the correspondence \mathcal{V} is the pointwise lowest quasiconcave and upper semicontinuous function that majorizes $\max \mathcal{V}$.

Quasiconcavification Theorem (Lipnowski and Ravid, 2020). *The sender's maximal equilibrium value (for any prior μ_0) is given by the quasiconcave envelope of $\max \mathcal{V}$, evaluated at μ_0 .*

3 Examples

Consider the leading example given by Lipnowski and Ravid (2020) with two states and three actions. It models the case of a political think tank (the sender) advising a lawmaker (the receiver) about which one of two possible reforms to implement. The action space is given by $A = \{0, 1, 2\}$, where 0 actually means keeping the status quo, and 1 and 2 stand for the two reforms. The sender's

preferences are given by the state independent utility function $u_S(a, \theta) = v_S(a) = a$, which means the think tank’s own agenda is perfectly known. The receiver’s preferences, however, depend on the state $\theta \in \Theta = \{\theta_1, \theta_2\}$; they are given in Figure 2. Figure 3 illustrates the induced sender value correspondence, which shows that the receiver only wants to choose one of the action 1,2 if the probability of the matching state is high enough.

	θ_1	θ_2
0	3	3
1	4	0
2	0	4

Figure 2: The receiver’s payoffs for Example 1.

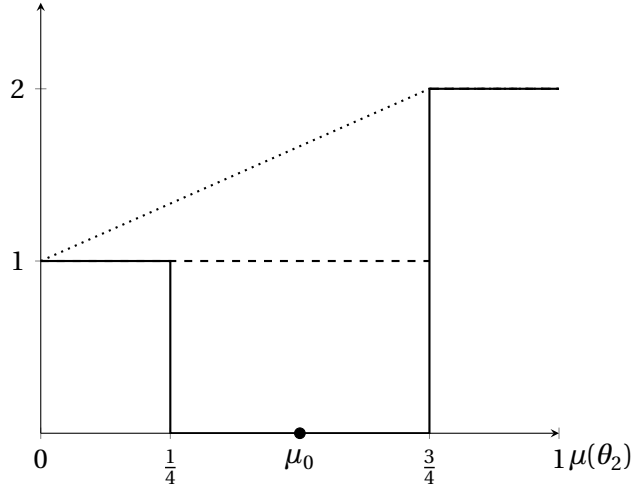


Figure 3: The sender value correspondence \mathcal{V} for Example 1, given by the thick black line. The dotted line is its concave envelope that characterizes the achievable sender payoffs under commitment (Kamenica and Gentzkow (2011)). The dashed line is the quasiconcave envelope that characterizes the achievable sender payoffs under cheap talk (without commitment, Lipnowski and Ravid (2020)).

Consider the uniform prior μ_0 with $\mu_0(\theta_2) = \frac{1}{2}$.⁸ In this transparent game, by the quasiconcavification theorem of Lipnowski and Ravid (2020) the sender’s maximal equilibrium payoff at μ_0 is 1. In fact every sender payoff $s \in [0, 1]$ is attainable in an equilibrium.

However, only the two extreme payoffs, $s = 1$ and $s = 0$, are also attainable in a robust equilibrium. All other equilibrium payoffs disappear when the sender’s utility function is perturbed in a “generic” way, even if the perturbation is so small that the ordinal preference remains the same (and, in particular, state independent).

⁸Analogous arguments apply to all priors with $\mu_0(\theta_2) \in (\frac{1}{4}, \frac{3}{4})$.

The lowest robust equilibrium payoff, $s = 0$, is the babbling equilibrium payoff. The robustness of any babbling equilibrium is in fact trivial. Since the communication is completely uninformative, we can fix the receiver's reply $\rho(\mu_0) \in A(\mu_0)$, so that the sender's equilibrium payoff depends continuously on the utility function.

In this example, it is also quite easy to see that all equilibria with a sender payoff $s \in (0, 1)$ fail to be robust. Lemma LR implies that any such equilibrium must induce a Bayes plausible posterior belief system with the two posteriors μ and μ' given by $\mu(\theta_2) = \frac{1}{4}$ and $\mu'(\theta_2) = \frac{3}{4}$. Both of these posteriors are interior, i.e., they attach strictly positive probability to both states. Any nearby equilibrium in a nearby game must then also attach a positive probability to posteriors close to (in fact one can show identical to) these two. Thus, also in the nearby game with the sender having slightly state-dependent preferences, the sender must induce both posteriors with positive probability in each state and hence be indifferent between inducing either of them.

Therefore, the receiver must choose action distributions for the two posteriors in a way that makes both sender types indifferent. The receiver can only choose among best response actions, which are 0 and 1 for posterior μ and 0 and 2 for posterior μ' . Denote by $x \in \mathbb{R}^3$ the difference between the two action distributions. Further, construct the 3×3 -matrix \tilde{U} as follows. The first row consists of the sender's payoffs from the three different actions in the first state, and the second row is analogous. Indifference means that the dot product of each of these rows with x is zero. The last row is a row of ones, because also $\sum_{i=0}^2 x_i$ must be zero by x being the difference between two probability vectors. Thus, necessarily $\tilde{U}x = 0$. However, except for degenerate cases (when \tilde{U} does not have full rank), the only solution of $\tilde{U}x = 0$ is $x = 0$. In the present example this means that the receiver chooses action 0 after both posteriors, which leads to a sender payoff close to 0 in either state. But robustness would require a payoff close to s , which we considered to be in $(0, 1)$. The essence of this argument is that there are not enough pairs of mixed replies for the receiver that make *both* sender types indifferent, other than choosing the *same* reply to both posteriors, but which would generate a different sender payoff.

The highest payoff, $s = 1$, in contrast, can be sustained in a robust equilibrium, using the two posteriors μ with $\mu(\theta_2) = 0$ and μ' with $\mu'(\theta_2) = \frac{3}{4}$. The crucial difference to the previous case is that the sender with type θ_2 induces only one posterior, μ' , and may also have a strict preference for it. If only one sender type needs to be indifferent, and the other needs to have only a preference for one particular posterior, these incentive constraints are weaker, and it turns out that they can indeed be satisfied for a large set of state-dependent sender preferences if the receiver has at least three actions that are optimal for some of the two posteriors. Showing this (and also for an arbitrary dimension of the action space), however, requires considerably more work, and in addition to dealing with the incentives, we need further steps to control the sender payoffs.

Our second example is a minimal extension of the first example and demonstrates that also some sender suboptimal values can be sustained by robust equilibria, at least in some cases, and that additional arguments are needed for the general result that the sender-optimal payoff can

always be sustained in a robust equilibrium.

The only difference to the previous example is that we now have four actions, with action set $A = \{0, 1, 2, 3\}$. The common prior is the same, $\mu_0 = \frac{1}{2}$, and so is the sender state-independent utility function, $u_S(a, \theta) = v_S(a) = a$. The receiver has payoffs given in Figure 4, and Figure 5 illustrates the sender value correspondence.

	θ_1	θ_2
0	3	3
1	5	-7
2	4	0
3	0	4

Figure 4: The receiver's payoffs for Example 2.

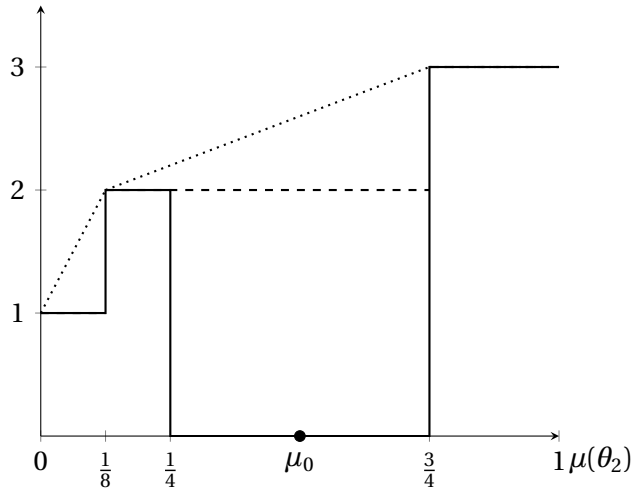


Figure 5: The sender value correspondence \mathcal{V} for Example 2, given by the thick black line. The dotted line is its concave envelope that characterizes the achievable sender payoffs under commitment (Kamenica and Gentzkow (2011)). The dashed line is the quasiconcave envelope that characterizes the achievable sender payoffs under cheap talk (without commitment, Lipnowski and Ravid (2020)).

Similarly as in the first example, the results of Lipnowski and Ravid (2020) imply that the set of sender equilibrium payoffs of the transparent game at prior μ_0 is the interval $[0, 2]$. Concerning all payoffs $s \in [0, 1]$, we have the same (non)robustness results as in the previous example, and for exactly the same reasons.

The crucial difference in the present example is that all payoffs $s \in [1, 2]$ can be sustained by an equilibrium using the two posteriors μ with $\mu(\theta_2) = \frac{1}{8}$ and μ' with $\mu'(\theta_2) = \frac{3}{4}$, which are such that the receiver now has four actions that are optimal for some of these two posteriors, because $A(\mu) = \{1, 2\}$ and $A(\mu') = \{0, 3\}$. It turns out that four actions give the receiver enough possibilities

to randomize such that the required indifference of both sender types can actually be achieved for a large set of state-dependent sender preferences.

Additionally, it turns out that there always is—as in both examples—at least one equilibrium that sustains the sender’s maximal equilibrium payoff with posteriors such that the receiver has enough optimal actions to randomize over.

4 Main Result

Theorem 1. *Let Γ be a transparent cheap-talk game with binary state space. Then the sender’s maximal equilibrium value is attainable in a robust equilibrium.*

The proof of Theorem 1 ultimately relies on a number of intermediate and auxiliary results. In this section, we first present those results that are of interest on their own. Together, they effectively provide a characterization of all sender payoffs that are attainable in a robust equilibrium (if the state space is binary). Towards the end of the section, we will then sketch a proof that the sender’s maximal equilibrium value satisfies this characterization. The full proofs are given in Appendix B, and all auxiliary results and their proofs are relegated to Appendix A.

We first do away with the trivial case of a *babbling* equilibrium, which means that with probability one the receiver sticks to the prior belief.

Proposition 1. *Any babbling equilibrium is fully robust.*

Proof. Let Γ be a cheap-talk game and (p, ρ) an equilibrium with $p(\mu_0) = 1$. Then the same equilibrium persists for *any* perturbed sender utility function \tilde{u}_S by the simple fact that there is no belief $\mu \neq \mu_0$ in the support of p . \square

By Lemma LR, the sender payoffs in a transparent cheap-talk game that can be sustained by a babbling equilibrium are all $s \in \mathcal{V}(\mu_0)$, so we now turn to payoffs $s > \max \mathcal{V}(\mu_0)$.

Assume, from now on, that the state space is indeed binary. Then, by Proposition 1 and Lemma 2, we only need to characterize the robust equilibria with exactly two posteriors in the support of the belief distribution. To simplify notation, let specifically $\Theta = \{\theta_1, \theta_2\}$; this allows us to identify any belief $\mu \in \Delta\Theta$ by the probability it assigns to θ_2 , so that we can treat μ as a number in $[0, 1]$.

Proposition 2. *Let Γ be a transparent cheap-talk game. Any equilibrium with an expected sender payoff $s > \max \mathcal{V}(\mu_0)$ and $|\text{supp } p| = 2$ is robust if and only if one of the following conditions holds.*

1. $\text{supp } p = \{0, 1\}$.
2. $\text{supp } p$ contains a posterior $\mu \in \{0, 1\}$ and $\left| \bigcup_{\mu \in \text{supp } p} A(\mu) \right| \geq 3$.
3. $\left| \bigcup_{\mu \in \text{supp } p} A(\mu) \right| \geq 4$.

Since Proposition 2 is the main characterization result, but its proof (given in Appendix B.2) involves a number of steps and technical results, we provide a sketch of the arguments here. Recall that, by Lemma 1, we can fix the belief distribution p .

The easier part is the proof of necessity. Condition 1. means that there is no further restriction if the equilibrium is fully informative. So suppose p involves a posterior belief that is supported by both states. Then the sender must be indifferent in some state. There are three ways to achieve this indifference. The first one is that the receiver uses the same (mixed) action for both posteriors. In this case, it is possible to show that there exists also a babbling equilibrium with the same expected sender payoff, which is in conflict with $s > \max \mathcal{V}(\mu_0)$. The second way is that the sender derives the same utility from different actions that the receiver uses. Such sender utility functions are, however, “nongeneric” and contained in a lower-dimensional space. The third possibility to make the sender indifferent is that the receiver has enough degrees of freedom to choose different mixed actions for the two posteriors (and that the corresponding ranges of sender payoffs overlap). This amounts to three optimal actions for the receiver when the sender needs to be indifferent in only one state (condition 2.), and to four optimal actions when indifference must hold in both states (condition 3.).

To see the latter in more detail, and to prepare for the sketch of sufficiency, let the two posteriors in the support of p be μ and μ' . Then any equilibrium $(p, \tilde{\rho})$ for a perturbed sender utility function \tilde{u}_S can be characterized in terms of the difference $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$. In particular, the set of all such x where $\tilde{\rho}$ satisfies equilibrium condition (ii) is

$$\mathcal{D} := \{r - r' \mid r, r' \in \Delta A, \text{supp } r \subseteq A(\mu), \text{ and } \text{supp } r' \subseteq A(\mu')\}.$$

Further, the sender is indifferent in state θ if and only if $\tilde{u}_S(\theta) \cdot x = 0$. We, thus, need a nonnull $x \in \mathcal{D}$ that solves the indifference equation(s). This requires \mathcal{D} to be big enough, and the dimension of the linear span of \mathcal{D} is precisely $|A(\mu) \cup A(\mu')| - 1$ (see Lemma 6 in Appendix A.3).

The proof for sufficiency consists of two steps and uses the same representation of equilibria in terms of the vectors x . But now we also need to take care of the sender’s (weak) preference if some state is revealed to the receiver, i.e., if some posterior belief is supported by only one state θ . Then equilibrium condition (iii) corresponds to an inequality of the form $\tilde{u}_S(\theta) \cdot x \geq 0$ or $\tilde{u}_S(\theta) \cdot x \leq 0$. Hence, letting $H_x = \{y \in \mathbb{R}^n \mid y \cdot x = 0\}$, $H_x^+ = \{y \in \mathbb{R}^n \mid y \cdot x \geq 0\}$, and $H_x^- = \{y \in \mathbb{R}^n \mid y \cdot x \leq 0\}$ for any $x \in \mathbb{R}^n$, the set of utility functions \tilde{u}_S for which there is an equilibrium $(p, \tilde{\rho})$ is $\bigcup_{x \in \mathcal{D}} H_x^+ \times H_x^-$ (if $\mu = 0$ and $\mu' = 1$), $\bigcup_{x \in \mathcal{D}} H_x \times H_x^-$ (if $\mu = 0$ and $\mu' < 1$), $\bigcup_{x \in \mathcal{D}} H_x^+ \times H_x$ (if $\mu > 0$ and $\mu' = 1$), or $\bigcup_{x \in \mathcal{D}} H_x \times H_x$ (if $\mu > 0$ and $\mu' < 1$). In the first step of the proof, we show that the interior of this set is nonempty if \mathcal{D} is big enough, that it actually intersects any neighborhood of the given, state-independent utility function u_S , and that the same is still true if we consider only those $x \in \mathcal{D}$ that are in an arbitrary neighborhood of $\rho(\mu) - \rho(\mu')$ (where ρ is from the given equilibrium for u_S). Most of the technical work is necessary for the case of one equality paired with an inequality and

the case of two equalities, so the specific arguments for the two cases are presented as separate propositions (in Appendix B.2).

In the second step, we then show that any $x \in \mathcal{D}$ sufficiently close to $\rho(\mu) - \rho(\mu')$ has in fact a representation $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$ with $\tilde{\rho}(\mu)$ arbitrarily close to $\rho(\mu)$ and $\tilde{\rho}(\mu')$ arbitrarily close to $\rho(\mu')$ (see Proposition 5 in Appendix B.2). This implies that we can make the interim expected sender payoff in the equilibrium $(p, \tilde{\rho})$ as close as desired to the one in (p, ρ) , which finally yields robustness.

Combining Proposition 1, Lemma 2, and Proposition 2, we now have a characterization of all sender payoffs that are attainable in a robust equilibrium. The proof of Theorem 1 consists of showing that the sender's maximal equilibrium value satisfies this characterization. It proceeds along the following lines.

Since any babbling equilibrium is robust, we only need to consider the case in which the sender's maximal equilibrium value s is not attainable in a babbling equilibrium, $s > \max \mathcal{V}(\mu_0)$. With the help of Lemma 2 we argue that, in this case, there is an equilibrium (p, ρ) attaining s with $\text{supp } p = \{\mu, \mu'\}$, where in particular $\mu = \min\{\tilde{\mu} \in \Delta\Theta \mid \tilde{\mu} < \mu_0 \text{ and } s \in \mathcal{V}(\tilde{\mu})\}$ and $\mu' = \max\{\tilde{\mu} \in \Delta\Theta \mid \tilde{\mu} > \mu_0 \text{ and } s \in \mathcal{V}(\tilde{\mu})\}$.

Then we show that this equilibrium satisfies one of the sufficient conditions from Proposition 2. The choice of μ guarantees that, if $\mu > 0$, necessarily $|A(\mu)| \geq 2$, because $s \in \mathcal{V}(\mu) \setminus \mathcal{V}(\tilde{\mu})$ and $A(\tilde{\mu}) \subseteq A(\mu)$ for any $\tilde{\mu} < \mu$ sufficiently close to μ . Analogously, $|A(\mu')| \geq 2$ if $\mu' < 1$. By these facts, Proposition 2 already yields robustness if $A(\mu) \cap A(\mu') = \emptyset$.

If $A(\mu) \cap A(\mu') \neq \emptyset$, we exploit the additional fact that then $A(\mu) \cap A(\mu') = A(\mu_0)$ (see Lemma 5 in Appendix A.2). Now $s > \max \mathcal{V}(\mu_0)$ implies that there must be some $a \in A(\mu) \setminus A(\mu_0)$ and some $a' \in A(\mu') \setminus A(\mu_0)$, which implies $|A(\mu) \cup A(\mu')| \geq 3$.

If $\mu > 0$ and $\mu' < 1$, we finally use the fact that s is the value of the quasiconcave envelope of $\max \mathcal{V}$, evaluated at μ_0 , which in fact implies $s \geq \sup \bigcup_{\tilde{\mu} \leq \mu_0} \mathcal{V}(\tilde{\mu})$ or $s \geq \sup \bigcup_{\tilde{\mu} \geq \mu_0} \mathcal{V}(\tilde{\mu})$. Using this (and Lemma 5 again), we show that $A(\mu)$ or $A(\mu')$ contains a subset $A(\tilde{\mu})$ (for some $\tilde{\mu}$ less than μ or greater than μ') that is disjoint from $A(\mu_0)$, $\{a\}$, and $\{a'\}$, which implies $|A(\mu) \cup A(\mu')| \geq 4$.

5 Conclusion

In transparent cheap-talk games, most equilibria are not robust to a small misspecification of the sender's true, and possibly slightly state-dependent, preferences, even if the ordinal preferences remain the same. We show, however, that, for a binary state space, the sender-optimal equilibrium payoff is always robustly attainable: The game always has a robust equilibrium that provides the sender-optimal payoff.

We show this result by providing a necessary and sufficient condition for an equilibrium to be robust. This condition amounts to a dimension counting exercise. Consider an equilibrium of a transparent game. If the number of actions that the receiver finds optimal for some of the

posterior beliefs induced in the equilibrium exceeds the number of indifference conditions for the sender by at least two, then and only then is the equilibrium robust.

There are two avenues we plan to follow in future research. One is to investigate if this argument extends to games with an arbitrary finite state space. The other is to explore what we can say about robust equilibria in general binary, and then also arbitrary finite, cheap-talk games.

A Auxiliary results

A.1 Belief-based approach

Given any finite set of messages M , recall that an *equilibrium* for a cheap-talk game consists of a sender strategy $\sigma: \Theta \rightarrow \Delta M$, a receiver strategy $\rho: M \rightarrow \Delta A$, and a belief system $\beta: M \rightarrow \Delta\Theta$ such that

1. β is obtained from μ_0 , given σ , using Bayes's rule, which means that

$$\sigma(m|\theta)\mu_0(\theta) = \beta(\theta|m) \sum_{\theta' \in \Theta} \sigma(m|\theta')\mu_0(\theta') \quad \text{for all } m \in M \text{ and } \theta \in \Theta, \quad (1)$$

2. $\rho(m)$ is supported on $A(\beta(m))$ for all $m \in M$, and
3. $\sigma(\theta)$ is supported on $\arg\max_{m \in M} u_S(\theta) \cdot \rho(m)$ for all $\theta \in \Theta$.

However, instead of working with explicit message sets and sender strategies, we adopt the so-called *belief-based approach*. Since equilibrium payoffs for the sender may generally depend on the state θ in our case, we use the following characterization of equilibria.

Lemma 3. *Let $p \in \Delta\Delta\Theta$ be a finite support distribution over beliefs, and let $s^*: \Theta \rightarrow \mathbb{R}$ be a function that maps states to sender payoffs. Then there exists an equilibrium (σ, ρ, β) with a finite message set M such that*

$$p(\mu) = \sum_{\substack{\theta \in \Theta \\ m \in \beta^{-1}(\mu)}} \sigma(m|\theta)\mu_0(\theta) \quad (2)$$

for all $\mu \in \Delta\Theta$ and

$$s^*(\theta) = \max_{m \in M} u_S(\theta) \cdot \rho(m) \quad (3)$$

for all $\theta \in \Theta$ if and only if there exists a function $\rho': \text{supp } p \rightarrow \Delta A$ such that

- (i) $\sum_{\mu \in \text{supp } p} \mu p(\mu) = \mu_0$,
- (ii) $\text{supp}(\rho'(\mu)) \subseteq A(\mu)$ for all $\mu \in \text{supp } p$, and
- (iii) $s^*(\theta) = u_S(\theta) \cdot \rho'(\mu) \geq u_S(\theta) \cdot \rho'(\mu')$ for all $\mu, \mu' \in \text{supp } p$ and $\theta \in \text{supp } \mu$.

Proof. “ \Rightarrow ” Let $\text{supp } p = \{\mu_1, \dots, \mu_K\}$ and suppose (σ, ρ, β) is an equilibrium with finite message set M such that (2) holds for all $\mu \in \Delta\Theta$. Then, by (1), $\sum_{m \in \beta^{-1}(\mu_k)} \sigma(m|\theta)\mu_0(\theta) = \mu_k(\theta)p(\mu_k)$ for all $k = 1, \dots, K$ and $\theta \in \Theta$. Thus, condition (i) holds if $\sum_{k=1}^K \sum_{m \in \beta^{-1}(\mu_k)} \sigma(m|\theta) = 1$ for all θ , and the latter actually follows from the facts that $1 = \sum_{k=1}^K p(\mu_k) = \sum_{k=1}^K \sum_{\theta \in \Theta, m \in \beta^{-1}(\mu_k)} \sigma(m|\theta)\mu_0(\theta)$, $\sum_{\theta \in \Theta} \mu_0(\theta) = 1$, and $\sum_{k=1}^K \sum_{m \in \beta^{-1}(\mu_k)} \sigma(m|\theta) \leq 1$ for all θ .

Now suppose additionally (3) holds for all $\theta \in \Theta$. Consider any $\theta \in \Theta$ and $k = 1, \dots, K$ such that $\theta \in \text{supp } \mu_k$, i.e., $\mu_k(\theta) > 0$. Then, for any $m \in \beta^{-1}(\mu_k)$, (1) and $\mu_0(\theta) > 0$ together imply that $\sigma(m|\theta) > 0$ if and only if $\sum_{\theta' \in \Theta} \sigma(m|\theta')\mu_0(\theta') > 0$. Thus, by equilibrium condition 3., any $m_k \in \beta^{-1}(\mu_k)$ with $\sum_{\theta' \in \Theta} \sigma(m_k|\theta')\mu_0(\theta') > 0$ must satisfy $u_S(\theta) \cdot \rho(m_k) = s^*(\theta)$. Such m_k indeed exists because of $\sum_{\theta' \in \Theta, m \in \beta^{-1}(\mu_k)} \sigma(m|\theta')\mu_0(\theta') = p(\mu_k) > 0$. Fix one such m_k for each $k = 1, \dots, K$ and define the function $\rho': \{\mu_1, \dots, \mu_K\} \rightarrow \Delta A$ by $\rho'(\mu_k) = \rho(m_k)$. Then condition (ii) holds by construction and equilibrium condition 2., since $\beta(m_k) = \mu_k$. Moreover, by construction of m_k and ρ' , whenever $\theta \in \text{supp } \mu_k$, then $u_S(\theta) \cdot \rho'(\mu_k) = s^*(\theta) \geq u_S(\theta) \cdot \rho'(\mu_j)$ for all $j = 1, \dots, K$, so also condition (iii) holds.

“ \Leftarrow ” Suppose there is a function $\rho': \text{supp } p \rightarrow \Delta A$ such that conditions (i), (ii), and (iii) hold. Let $M := \text{supp } p = \{\mu_1, \dots, \mu_K\}$, define the belief system $\beta: M \rightarrow \Delta\Theta$ by $\beta(\mu_k) = \mu_k$, and fix the receiver strategy $\rho := \rho'$. Finally, construct the sender strategy $\sigma: \Theta \rightarrow \Delta M$ by $\sigma(\mu_k|\theta)\mu_0(\theta) = \mu_k(\theta)p(\mu_k)$, which is possible because the latter equation implies $\sigma(\mu_k|\theta) > 0$ and, by condition (i), $\sum_{k=1}^K \sigma(\mu_k|\theta) = 1$. Further, then $\sum_{\theta \in \Theta} \sigma(\mu_k|\theta)\mu_0(\theta) = p(\mu_k)$ by $\mu_k \in \Delta\Theta$, which implies that (1) holds. Equilibrium condition 2. holds by construction, since $\text{supp}(\rho(\mu_k)) \subseteq A(\mu_k) = A(\beta(\mu_k))$ for all $\mu_k \in M$. To verify also equilibrium condition 3. and that (3) holds for all $\theta \in \Theta$, suppose $\sigma(\mu_k|\theta) > 0$. Then by construction $\mu_k(\theta) > 0$, i.e., $\theta \in \text{supp } \mu_k$, so that condition (iii) indeed implies $s^*(\theta) = u_S(\theta) \cdot \rho(\mu_k) \geq u_S(\theta) \cdot \rho(\mu_j)$ for all $\mu_j \in M$. \square

A.2 Belief dependence of the receiver's optimal actions

Here we study how the belief-dependent sets of optimal actions for the receiver, $A(\mu)$, are related to each other for different beliefs $\mu \in \Delta\Theta$.

Lemma 4. *For every belief $\mu \in \Delta\Theta$ there exists a neighborhood U_μ such that $A(\tilde{\mu}) \subseteq A(\mu)$ for all $\tilde{\mu} \in U_\mu$.*

Proof. This follows from upper hemicontinuity (by Berge's Maximum Theorem) of the correspondence that associates to each $\mu \in \Delta\Theta$ the set $A(\mu)$ and finiteness of A . \square

Lemma 5. *Let $\mu, \mu' \in \Delta\Theta$ be two given beliefs. If $A(\mu)$ and $A(\mu')$ have any common element, then, for all $\tilde{\mu} \in \text{co}\{\mu, \mu'\}$, $A(\tilde{\mu}) = A(\mu) \cap A(\mu')$ or $\tilde{\mu} \in \{\mu, \mu'\}$.*

Proof. Suppose $a \in A(\mu) \cap A(\mu')$ and $\tilde{\mu} = \alpha\mu + (1-\alpha)\mu'$ for some $\alpha \in (0, 1)$. For notational simplicity, let $u_R(a') \cdot \mu$ denote $\sum_{\theta \in \Theta} u_R(a', \theta)\mu(\theta)$ for any $a' \in A$. Proceeding likewise also for other beliefs,

$(u_R(a) - u_R(a')) \cdot \tilde{\mu} = \alpha(u_R(a) - u_R(a')) \cdot \mu + (1 - \alpha)(u_R(a) - u_R(a')) \cdot \mu'$, which is nonnegative by $a \in A(\mu) \cap A(\mu')$. Thus, also $a \in A(\tilde{\mu})$, and it follows that $A(\mu) \cap A(\mu') \subseteq A(\tilde{\mu})$. Now suppose $a' \in A(\tilde{\mu})$. Then in fact $(u_R(a) - u_R(a')) \cdot \tilde{\mu} = 0$, so that, by $\alpha \in (0, 1)$, also both inequalities $(u_R(a) - u_R(a')) \cdot \mu \geq 0$ and $(u_R(a) - u_R(a')) \cdot \mu' \geq 0$ must hold with equality. Thus, $a' \in A(\mu) \cap A(\mu')$, and it follows that $A(\tilde{\mu}) \subseteq A(\mu) \cap A(\mu')$. \square

A.3 The set \mathcal{D}

A crucial aspect for an equilibrium with two posterior beliefs $\mu, \mu' \in \Delta\Theta$ is the “size” of the set $\mathcal{D} = \{r - r' \mid r, r' \in \Delta A, \text{supp } r \subseteq A(\mu), \text{ and } \text{supp } r' \subseteq A(\mu')\}$.

Lemma 6. *The linear subspace spanned by \mathcal{D} has dimension $|A(\mu) \cup A(\mu')| - 1$.*

Proof. We can identify ΔA and \mathcal{D} with subsets of $\mathbb{R}^{|A|}$. The restrictions that $r(a) - r'(a) = 0$ for $a \notin A(\mu) \cup A(\mu')$ and $\mathbf{1} \cdot (r - r') = 0$ show that the dimension of the span of \mathcal{D} is at most $|A(\mu) \cup A(\mu')| - 1$. It remains to find as many linearly independent vectors in \mathcal{D} . If this number is zero, there is nothing to show. Otherwise, $A(\mu)$ and $A(\mu')$ are nonempty sets whose union contains at least two elements. We can then partition their union into two nonempty disjoint sets E and E' , with $E \subseteq A(\mu)$ and $E' \subseteq A(\mu')$, respectively. For any $a \in A$, let $\delta_a \in \Delta A$ be the Dirac measure that assigns probability one to $\{a\}$. Fix some $a \in E$ and $a' \in E'$. Then \mathcal{D} contains $|E'|$ vectors of the form $\delta_a - \delta_{a''}$ with $a'' \in E'$, and $|E| - 1$ vectors of the form $\delta_{a''} - \delta_{a'}$ with $a'' \in E \setminus \{a\}$, and all these vectors are linearly independent. \square

A.4 Hyperplanes and half spaces

Recall that, for any $x \in \mathbb{R}^n$, H_x , H_x^- , and H_x^+ respectively denote the sets $\{y \in \mathbb{R}^n \mid y \cdot x = 0\}$, $\{y \in \mathbb{R}^n \mid y \cdot x \leq 0\}$, and $\{y \in \mathbb{R}^n \mid y \cdot x \geq 0\}$. If $x \neq 0$, thus, H_x is the hyperplane with normal vector x , and H_x^- and H_x^+ are the two associated (closed) half spaces (whereas $H_0 = H_0^- = H_0^+ = \mathbb{R}^n$).

Lemma 7. *Let $v_1, v_2 \in \mathbb{R}^n$ and $D = \text{co}\{v_1, v_2\}$. Then*

- (i) $\bigcap_{x \in D} H_x^- = H_{v_1}^- \cap H_{v_2}^-$,
- (ii) $\bigcup_{x \in D} H_x = (H_{v_1}^- \cap H_{v_2}^+) \cup (H_{v_1}^+ \cap H_{v_2}^-)$, and
- (iii) $H_{v_1}^- \cap H_{v_2}^-$ is convex, and it has nonempty interior unless $v_2 = \lambda v_1 \neq 0$ for some $\lambda < 0$.

Proof. (i) Clearly, the set on the right is a subset of the set on the left. For the other direction, note that $y \cdot v_1 \leq 0$ and $y \cdot v_2 \leq 0$ implies $y \cdot (\alpha v_1 + (1 - \alpha)v_2) = \alpha y \cdot v_1 + (1 - \alpha)y \cdot v_2 \leq 0 + 0$ for $\alpha \in (0, 1)$.

(ii) Being element of the set on the right means having a nonnegative dot product with one of the values v_1, v_2 and a nonpositive one with the other. Between them, at an element x of D , the value must be zero, which means being in some H_x . On the other hand, if $y \in H_x$ for some $x \in D$,

we clearly cannot have both $y \cdot v_1 < 0$ and $y \cdot v_2 < 0$, or both $y \cdot v_1 > 0$ and $y \cdot v_2 > 0$. But this means y must be in the set on the right.

(iii) Convexity is straightforward. Concerning the nonempty interior, consider first the case that v_1 or v_2 is the null vector, say w.l.o.g. v_2 . Then $H_{v_2}^- = \mathbb{R}^n$, so the intersection equals $H_{v_1}^-$. We always have $-v_1 \in \text{int } H_{v_1}^-$, by $H_{v_1}^- = \mathbb{R}^n$ if also $v_1 = 0$, and otherwise by $(-v_1) \cdot v_1 = -v_1^2 < 0$ and continuity of the dot product. Hence, suppose neither v_1 nor v_2 is the null vector. If they are linearly dependent, then $v_2 = \lambda v_1$ for some $\lambda \neq 0$. So suppose now this is the case with $\lambda > 0$. Then $H_{v_2}^- = H_{v_1}^-$, which has nonempty interior as already shown. Finally, consider the case that v_1 and v_2 are linearly independent. Then we can write $v_2 = \lambda v_1 + w$ with $v_1 \cdot w = 0$ and $w \neq 0$. Since also $v_1 \neq 0$, this implies both $-(v_1 + w) \cdot v_1 = -v_1^2 < 0$ and $-(v_1 + w) \cdot v_2 = -\lambda v_1^2 - w^2 < 0$. Thus, again by continuity of the dot product, $-(v_1 + w)$ is in the interior of $H_{v_1}^- \cap H_{v_2}^-$. \square

For later reference, the following lemma collects some standard facts about orthogonal complements; see Lax (1997, Chapter 2).

Lemma 8. *Let D be a linear subspace of \mathbb{R}^n and U the orthogonal complement of D , that is the space of all $y \in \mathbb{R}^n$ such that $x \cdot y = 0$ for all $x \in D$. Then D is also the orthogonal complement of U and $\dim D + \dim U = n$. Moreover, U is the orthogonal complement of any set of vectors that span D . Similarly, D is the orthogonal complement of any set of vectors that span U .*

B Proofs for main results

B.1 Lemmas 1 and 2

Proof of Lemma 1. The result follows from the following claim.

Claim. *Let Γ be a cheap-talk game and p a given Bayes plausible posterior belief distribution with finite support. Then there exists a neighborhood W of p such that, if $(\tilde{p}, \tilde{\rho})$ is an equilibrium with $\tilde{p} \in W$, there is a sender payoff-equivalent equilibrium with posterior belief distribution p . Moreover, the neighborhood W does not depend on the sender's utility function u_S .*

For each $\mu \in \text{supp } p$, there exists a neighborhood U_μ such that $A(\mu') \subseteq A(\mu)$ for all $\mu' \in U_\mu$ (see Lemma 4). Further, since Θ is finite and the support correspondence is lower hemicontinuous by Theorem 17.14 in Aliprantis and Border (2006), there exists for each $\mu \in \text{supp } p$ a neighborhood V_μ such that $\text{supp } \mu' \supseteq \text{supp } \mu$ for each $\mu' \in V_\mu$.⁹ Again, since the support correspondence is lower hemicontinuous, there exists a neighborhood W of p such that for every $p' \in W$, and every $\mu \in \text{supp } p$, one has $\text{supp } p' \cap U_\mu \cap V_\mu \neq \emptyset$. So far, nothing depended on u_S .

Now suppose $(\tilde{p}, \tilde{\rho})$ is an equilibrium with $\tilde{p} \in W$. Since $\text{supp } \tilde{p} \cap U_\mu \cap V_\mu \neq \emptyset$ for all $\mu \in \text{supp } p$, there is a function $f_{\tilde{p}} : \text{supp } p \rightarrow \text{supp } \tilde{p}$ such that $f_{\tilde{p}}(\mu) \in U_\mu \cap V_\mu$ for all $\mu \in \text{supp } p$. Let $\rho = \tilde{\rho} \circ f_{\tilde{p}}$.

⁹In detail: For each $\theta \in \text{supp } \mu$, there must exist a neighborhood V_μ^θ of μ such that $\text{supp } \mu' \cap \{\theta\} \neq \emptyset$ for $\mu' \in V_\mu^\theta$. Let V_μ be the intersection of these V_μ^θ .

We claim that (p, ρ) is an equilibrium that is sender payoff-equivalent to $(\tilde{p}, \tilde{\rho})$. By assumption, p is Bayes plausible. Moreover, since $A(\mu') \subseteq A(\mu)$ for $\mu' \in U_\mu$ and $f_{\tilde{p}}(\mu) \in U_\mu$, we have $\text{supp}(\tilde{\rho} \circ f_{\tilde{p}}(\mu)) \subseteq A(\mu)$ for all $\mu \in \text{supp } p$. Now take any $\mu^* \in \text{supp } p$ and $\theta \in \text{supp } \mu^*$. Since $(\tilde{p}, \tilde{\rho})$ is an equilibrium, it must be optimal to choose any $\mu' \in \text{supp } \tilde{p}$ that has θ in its support at θ . This gives us the second equality in the following:

$$\begin{aligned} u_S(\theta) \cdot \rho(\mu^*) &= u_S(\theta) \cdot \tilde{\rho}(f_{\tilde{p}}(\mu^*)) = \max_{\mu' \in \text{supp } \tilde{p}} u_S(\theta) \cdot \tilde{\rho}(\mu') \\ &\geq \max_{\mu \in \text{supp } p} u_S(\theta) \cdot \tilde{\rho}(f_{\tilde{p}}(\mu)) \\ &= \max_{\mu \in \text{supp } p} u_S(\theta) \cdot \rho(\mu) \\ &\geq u_S(\theta) \cdot \rho(\mu^*). \end{aligned}$$

Therefore, all inequalities are actually equalities. The equality

$$u_S(\theta) \cdot \rho(\mu^*) = \max_{\mu \in \text{supp } p} u_S(\theta) \cdot \rho(\mu)$$

means that the last condition in the definition of an equilibrium is satisfied for (p, ρ) and the equality

$$u_S(\theta) \cdot \rho(\mu^*) = \max_{\mu' \in \text{supp } \tilde{p}} u_S(\theta) \cdot \tilde{\rho}(\mu')$$

together with $(\tilde{p}, \tilde{\rho})$ being an equilibrium guarantees that (p, ρ) is sender payoff-equivalent. \square

Proof of Lemma 2. By equilibrium condition (i), p must be Bayes plausible, i.e., μ_0 must be a convex combination of all beliefs $\mu \in \text{supp } p$. By Carathéodory's theorem, there is, thus, a subset B of $\text{supp } p$ with $|B| \leq |\Theta|$ such that μ_0 is a convex combination of all beliefs $\mu \in B$ (since one can identify beliefs $\mu \in \Delta\Theta$ with vectors in $\mathbb{R}^{|\Theta|-1}$ by a linear injective mapping). This convex combination defines a Bayes plausible belief distribution p' with $\text{supp } p' \subseteq B \subseteq \text{supp } p$. Now let $\rho' = \rho|_{\text{supp } p'}$. Then $\rho' = \rho \circ f$, where f is the identity function on $\text{supp } p'$. This function f trivially satisfies the properties that were used in the proof of Lemma 1, so the arguments given there (applied to p' in place of p and (p, ρ) in place of $(\tilde{p}, \tilde{\rho})$) show that (p', ρ') is an equilibrium that is sender payoff-equivalent to (p, ρ) .

Next, suppose (p, ρ) is robust and let, for any neighborhoods U, V , and W , U' be the corresponding set of perturbed sender utility functions \tilde{u}_S . Then the same set U' and the same construction that was used for ρ' can also be applied for showing that (p', ρ') is robust. Indeed, consider any $\tilde{u}_S \in U'$ and the corresponding equilibrium $(\tilde{p}, \tilde{\rho})$. By Lemma 1, we may assume $\tilde{p} = p$. Now let $\tilde{\rho}' = \tilde{\rho}|_{\text{supp } p'}$. Then, by the arguments given for (p', ρ') , also $(p', \tilde{\rho}')$ is an equilibrium, and $(p', \tilde{\rho}')$ and $(p, \tilde{\rho})$ are sender payoff-equivalent. Since also (p', ρ') and (p, ρ) are sender payoff-equivalent, it follows that (p', ρ') inherits robustness from (p, ρ) . The argument for a fully robust equilibrium is completely analogous. \square

B.2 Proof of Proposition 2

Preliminaries. Recall that, for any $x \in \mathbb{R}^n$, H_x , H_x^- , and H_x^+ respectively denote the sets $\{y \in \mathbb{R}^n \mid y \cdot x = 0\}$, $\{y \in \mathbb{R}^n \mid y \cdot x \leq 0\}$, and $\{y \in \mathbb{R}^n \mid y \cdot x \geq 0\}$.

To start the proof, suppose (p, ρ) is an equilibrium with $\text{supp } p = \{\mu, \mu'\}$ and where $\mu < \mu'$. By Lemma 1, we only need to consider equilibria $(\tilde{p}, \tilde{\rho})$ for any perturbed utility function \tilde{u}_S such that $\tilde{p} = p$. Thus, we only consider receiver strategies $\tilde{\rho}$ with domain $\{\mu, \mu'\}$, and it will be convenient to map any such $\tilde{\rho}$ to the difference $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$.

The set of these differences such that $(p, \tilde{\rho})$ satisfies equilibrium condition (ii) is

$$\mathcal{D} := \{r - r' \mid r, r' \in \Delta A, \text{supp } r \subseteq A(\mu), \text{ and } \text{supp } r' \subseteq A(\mu')\}.$$

Further, given $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$, and since $\mu < \mu'$ implies that $\theta_1 \in \text{supp } \mu$ and $\theta_2 \in \text{supp } \mu'$, equilibrium condition (iii) is satisfied if and only if

$$\tilde{u}_S(\theta_1) \cdot x \geq 0 \quad \text{and} \quad \tilde{u}_S(\theta_2) \cdot x \leq 0$$

for $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$, where the first inequality must hold with equality if $\mu' < 1$ (because then also $\theta_1 \in \text{supp } \mu'$), and the second must be an equality if $\mu > 0$ (because then also $\theta_2 \in \text{supp } \mu$).

Thus, there exists an equilibrium $(p, \tilde{\rho})$ for \tilde{u}_S if and only if there is some $x \in \mathcal{D}$ that satisfies the two given (in)equalities. In particular, since (p, ρ) is an equilibrium for $\tilde{u}_S = u_S$, $x = \rho(\mu) - \rho(\mu')$ satisfies $x \in \mathcal{D}$ and in fact $u_S(\theta_1) \cdot x = u_S(\theta_2) \cdot x = 0$, because transparency of Γ means $u_S(\theta_1) = u_S(\theta_2)$.

Proof of necessity. Suppose now that (p, ρ) is robust and generates an expected sender payoff $s > \max \mathcal{V}(\mu_0)$. Note that $\mu < \mu'$ implies in fact $\mu < \mu_0 < \mu'$. By way of contradiction, suppose none of the conditions 1.–3. holds. Specifically, assume $\mu' < 1$, as the case in which $\mu' = 1$ and $\mu > 0$ is symmetric to $\mu = 0$ and $\mu' < 1$.

Fix any neighborhoods U , V , and W from the definition of a robust equilibrium and let U' be the associated set of perturbed sender utility functions. Take any $\tilde{u}_S \in U'$ and let $(\tilde{p}, \tilde{\rho})$ be the corresponding equilibrium, where we may assume $\tilde{p} = p$ by Lemma 1 and hence consider $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu')$. As argued at the beginning of the proof, necessarily $x \in \mathcal{D}$, and, since $\mu' < 1$, $\tilde{u}_S(\theta_1) \cdot x = 0$ and $\tilde{u}_S(\theta_2) \cdot x \leq 0$, where the inequality cannot be strict if $\mu > 0$. We are now going to argue that this implies $x = 0$ (if \tilde{u}_S is a generic element of U').

By Lemma 6, \mathcal{D} is contained in a subspace with dimension $m = |A(\mu) \cup A(\mu')| - 1 < n$. Therefore, by Lemma 8, $\mathcal{D} \subseteq \bigcap_{i=1}^{n-m} H_{v_i}$ for some linearly independent vectors v_1, \dots, v_{n-m} , which we fix. Since condition 3. does not hold, we have $m < 3$, so there are three cases to consider. If $m = 0$, then $\mathcal{D} = \{0\}$, which trivially implies $x = 0$. If $m = 1$, then necessarily $x \in H_{\tilde{u}_S(\theta_1)} \cap \bigcap_{i=1}^{n-1} H_{v_i}$, where, generically for all $\tilde{u}_S \in \tilde{U}$, the vectors $\tilde{u}_S(\theta_1), v_1, \dots, v_{n-1}$ are linearly independent, which implies $x = 0$. If $m = 2$, i.e., $|A(\mu) \cup A(\mu')| = 3$, then, since condition 2. does not hold, we must

have $\mu > 0$, so necessarily $x \in H_{\tilde{u}_S(\theta_1)} \cap H_{\tilde{u}_S(\theta_2)} \cap \bigcap_{i=1}^{n-2} H_{v_i}$, where, again generically, the vectors $\tilde{u}_S(\theta_1), \tilde{u}_S(\theta_2), v_1, \dots, v_{n-2}$ are linearly independent, which once more requires $x = 0$.

However, $x = 0$ means $\tilde{\rho}(\mu) = \tilde{\rho}(\mu')$, and then the sender's interim expected equilibrium payoff in any state is $\tilde{s}^*(\theta) = \tilde{u}_S(\theta) \cdot \tilde{\rho}(\mu)$. Further, $x = 0 \in \mathcal{D}$ only if $\text{supp}(\tilde{\rho}(\mu)) = \text{supp}(\tilde{\rho}(\mu')) \subseteq A(\mu) \cap A(\mu')$. Then, by Lemma 5 and $\mu < \mu_0 < \mu'$, also $\text{supp}(\tilde{\rho}(\mu)) \subseteq A(\mu_0)$. The latter means that there is also a babbling equilibrium in which the receiver uses the mixed action $\tilde{\rho}(\mu)$ (for the ‘‘posterior’’ belief μ_0). Using the state-independent utility function u_S , Lemma LR implies $u_S(\theta) \cdot \tilde{\rho}(\mu) \in \mathcal{V}(\mu_0)$ for any θ . Thus, since $\tilde{s}^*(\theta) = \tilde{u}_S(\theta) \cdot \tilde{\rho}(\mu)$, $\tilde{s}^*(\theta)$ must be in the vicinity of $\mathcal{V}(\mu_0)$ if we take U to be small enough and U' a subset of U (which is w.l.o.g.). This, however, contradicts that $\tilde{s}^*(\theta)$ approaches $s > \max \mathcal{V}(\mu_0)$ if we take W to be small enough.

Proof of sufficiency. For the first step, we are going to use the facts established at the beginning of the proof to show that, under any of the conditions 1.–3., and for any neighborhoods U of u_S and X of $\rho(\mu) - \rho(\mu')$, there is some $u'_S \in U$ with a neighborhood $U' \subseteq U$ such that, for any $\tilde{u}_S \in U'$, there is an equilibrium $(p, \tilde{\rho})$ such that $x = \tilde{\rho}(\mu) - \tilde{\rho}(\mu') \in X$. The candidate $x = \rho(\mu) - \rho(\mu')$, which is in $\mathcal{D} \cap X$, trivially works for all \tilde{u}_S if $x = 0$. Hence, suppose $\rho(\mu) \neq \rho(\mu')$.

If condition 1. holds, i.e., $\mu = 0$ and $\mu' = 1$, then $x = \rho(\mu) - \rho(\mu') \neq 0$ still works for a perturbed utility function \tilde{u}_S if and only if $\tilde{u}_S(\theta_1) \cdot x \geq 0$ and $\tilde{u}_S(\theta_2) \cdot x \leq 0$. This condition can be written as $\tilde{u}_S \in H_x^+ \times H_x^-$. Being half spaces, both H_x^+ and H_x^- are convex and have nonempty interior, and then the same holds for their product. Moreover, since $u_S(\theta_1) \cdot x = u_S(\theta_2) \cdot x = 0$, in particular $u_S \in H_x^+ \times H_x^-$. Therefore, by Lemma 5.28 (1.) in Aliprantis and Border (2006), the set $U \cap H_x^+ \times H_x^-$ has nonempty interior. Thus, we can take this interior as U' if condition 1. holds.

Now suppose condition 2. or 3. holds, but condition 1. not. Specifically, assume again $\mu' < 1$, as the case in which $\mu' = 1$ and $\mu > 0$ is symmetric to $\mu = 0$ and $\mu' < 1$. Then the equilibrium (in)equalities for x are $\tilde{u}_S(\theta_1) \cdot x = 0$ and $\tilde{u}_S(\theta_2) \cdot x \leq 0$, where the inequality cannot be strict if $\mu > 0$. Now we cannot use $x = \rho(\mu) - \rho(\mu')$ for many perturbed utility functions \tilde{u}_S anymore, and the more involved arguments that establish the desired set U' are relegated to Proposition 3 (in case $\mu = 0$) and Proposition 4 (in case $\mu > 0$). To see that the prerequisites for these propositions are satisfied, recall that $x = \rho(\mu) - \rho(\mu')$ satisfies $x \in \mathcal{D}$ and $u_S(\theta_1) \cdot x = u_S(\theta_2) \cdot x = 0$. Moreover, Lemma 6 implies that \mathcal{D} contains $m = |A(\mu) \cup A(\mu')| - 1$ linearly independent vectors, which, since condition 2. or 3. holds, is at least two, and at least three if $\mu > 0$. Further, if we pick any number of linearly independent vectors from \mathcal{D} , we may, by the Steinitz exchange lemma, assume that $x = \rho(\mu) - \rho(\mu')$ is one of them, because then $x \neq 0$ by hypothesis. Therefore, as \mathcal{D} is convex, we can indeed apply Proposition 3 or Proposition 4 to obtain U' .

For the second step of proving sufficiency, note that, if U' is the subset of U established in the first step, then we can ensure that all $\tilde{u}_S \in U'$ are arbitrarily close to u_S by choosing U small enough. Additionally, by choosing X small enough and using Proposition 5, we can guarantee that $\tilde{\rho}(\mu)$ and $\tilde{\rho}(\mu')$ are, respectively, arbitrarily close to $\rho(\mu)$ and $\rho(\mu')$ (uniformly for all $\tilde{u}_S \in U'$). This way, we can let the sender's equilibrium payoffs $\tilde{s}^*(\theta) = \max(\tilde{u}_S(\theta) \cdot \tilde{\rho}(\mu), \tilde{u}_S(\theta) \cdot \tilde{\rho}(\mu'))$ be

as close to $s^*(\theta) = \max(u_S(\theta) \cdot \tilde{\rho}(\mu), u_S(\theta) \cdot \tilde{\rho}(\mu'))$ as we want for every θ (again uniformly for all $\tilde{u}_S \in U'$), which yields the robustness of (p, ρ) .

The proof of Proposition 2 is now complete up to the three more technical Propositions 3, 4, and 5, which are central for sufficiency.

Proposition 3. *Let $u_0 \in \mathbb{R}^n$, $D = \text{co}\{v_1, v_2\}$ for two linearly independent vectors $v_1, v_2 \in \mathbb{R}^n$, and $x_0 \in D$ such that $u_0 \cdot x_0 = 0$. Then, for any given $\epsilon > 0$, there exists a nonempty, open set $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $U \subseteq B_\epsilon(u_0, u_0)$ and for all $(u_1, u_2) \in U$ there is some $x \in D \cap B_\epsilon(x_0)$ such that $u_1 \cdot x = 0 \geq u_2 \cdot x$ (i.e., $U \subseteq \bigcup_{x \in D \cap B_\epsilon(x_0)} H_x \times H_x^-$).*

Proof. Since $x_0 \in D$, there is some $\lambda_0 \in \mathbb{R}_+^2 \setminus \{0\}$ such that, in matrix notation, $(v_1, v_2)\lambda_0 = x_0$. In particular, thus, $x_0 \neq 0$ by linear independence of v_1, v_2 . First suppose $\lambda_0 \in \text{int}\mathbb{R}_+^2 = \mathbb{R}_{++}^2$.

The proof strategy is to identify a convex subset D' of $D \cap B_\epsilon(x_0)$ and two nonempty open sets $U_1, U_2 \subseteq B_{\epsilon/2}(u_0)$ such that for every $u_1 \in U_1$ we have $u_1 \cdot x = 0$ for some $x \in D'$, and for every $u_2 \in U_2$ we have $u_2 \cdot x \leq 0$ for all $x \in D'$. Then, for every pair $(u_1, u_2) \in U_1 \times U_2$, by $u_1 \in U_1$ there is some $x \in D'$ such that $u_1 \cdot x = 0$, and in particular for this x also $u_2 \cdot x \leq 0$ by $u_2 \in U_2$, so together indeed $u_1 \cdot x = 0 \geq u_2 \cdot x$. (In the set notation, we exploit the fact that $\bigcup_{x \in D'} H_x \times \bigcap_{x \in D'} H_x^- \subseteq \bigcup_{x \in D'} H_x \times H_x^-$ for any subset D' of D .) Hence, we can choose $U = U_1 \times U_2$, because it is a subset of $B_\epsilon(u_0, u_0)$ due to $U_1, U_2 \subseteq B_{\epsilon/2}(u_0)$.

We are now going to construct a set D' such that $u_0 \cdot x \leq 0$ for all $x \in D'$, and then we will verify the actually needed properties. Therefore, note that $\lambda_0 \in \mathbb{R}_{++}^2$ and $u_0 \cdot x_0 = (u_0 \cdot v_1, u_0 \cdot v_2)\lambda_0 = 0$ together imply that either $u_0 \cdot v_1 < 0 < u_0 \cdot v_2$, or $u_0 \cdot v_2 < 0 < u_0 \cdot v_1$, or $u_0 \cdot v_1 = u_0 \cdot v_2 = 0$. In the last case, in fact $u_0 \cdot x \leq 0$ for all $x \in D$, so let $D' = \text{co}\{x_0, x'\}$ for an arbitrary other vector $x' \neq x_0$ from $D \cap B_\epsilon(x_0)$. In the first case, $u_0 \cdot x \leq 0$ for all $x \in \text{co}\{v_1, x_0\}$, so let $D' = \text{co}\{x_0, x'\}$ for an arbitrary $x' \neq x_0$ from $\text{co}\{v_1, x_0\} \cap B_\epsilon(x_0)$, noting that such an x' exists due to $x_0 \neq v_1$. In the second case let analogously $D' = \text{co}\{x_0, x'\}$ for an arbitrary $x' \neq x_0$ from $\text{co}\{v_2, x_0\} \cap B_\epsilon(x_0)$. In any case we now have a set $D' \subseteq D \cap B_\epsilon(x_0)$ such that $u_0 \cdot x \leq 0$ for all $x \in D'$, which we can write as $u_0 \in \bigcap_{x \in D'} H_x^-$. Further, in any case $D' = \text{co}\{x_0, x'\}$ for some $x' \in D$ that is distinct from x_0 , so that these two vectors inherit linear independence from v_1 and v_2 .

Consider the property $u_0 \in \bigcap_{x \in D'} H_x^-$. By Lemma 7, the latter intersection is convex and has nonempty interior. Thus, also the open set $B_{\epsilon/2}(u_0) \cap \text{int}\bigcap_{x \in D'} H_x^-$ is nonempty by Lemma 5.28 (1.) in Aliprantis and Border (2006), so we can use it as our set U_2 , because for every $u_2 \in U_2$ then $u_2 \cdot x \leq 0$ holds by construction for all $x \in D'$ (i.e., $U_2 \subseteq \bigcap_{x \in D'} H_x^-$).

By $x_0 \in D'$ and $u_0 \cdot x_0 = 0$, we further have $u_0 \in \bigcup_{x \in D'} H_x$. By Lemma 7, and since $D' = \text{co}\{x_0, x'\}$ for two linearly independent vectors x_0 and x' , $\bigcup_{x \in D'} H_x = (H_{x_0}^- \cap H_{x'}^+) \cup (H_{x_0}^+ \cap H_{x'}^-)$. Each of the two latter intersections, again by Lemma 7, is convex and has nonempty interior. Choose one of them so that it contains u_0 . Then the interior of this convex set has a nonempty intersection with $B_{\epsilon/2}(u_0)$ by Lemma 5.28 (1.) in Aliprantis and Border (2006). Therefore, we can use the latter intersection as our set U_1 , because for every $u_1 \in U_1$ then $u_1 \cdot x = 0$ by construction

for some $x \in D'$ (i.e., $U_1 \subseteq \bigcup_{x \in D'} H_x$). Letting $U = U_1 \times U_2$ as argued in the beginning completes the proof for $\lambda_0 \in \mathbb{R}_{++}^2$.

Now suppose $\lambda_0 \notin \mathbb{R}_{++}^2$. Consider any $\delta > 0$ and let $\hat{\lambda} = \lambda_0 + \delta \mathbf{1}$, which is in \mathbb{R}_{++}^2 (in particular $\hat{\lambda} \neq 0$). Then let $\hat{x} = \frac{1}{\mathbf{1} \cdot \hat{\lambda}}(\nu_1, \nu_2)\hat{\lambda}$, which is in D . We can induce \hat{x} to be arbitrarily close to x_0 by starting with sufficiently small δ , because $(\nu_1, \nu_2)\lambda_0 = x_0$ and $\mathbf{1} \cdot \lambda_0 = 1$. Next, let \hat{u} be the orthogonal projection of u_0 onto the hyperplane $H_{\hat{x}}$ (so that $\hat{u} \cdot \hat{x} = 0$), noting that $\hat{x} \in D$ implies $\hat{x} \neq 0$. To make $\|\hat{u} - u_0\|$ arbitrarily small, we only need to make $\|\hat{x} - x_0\|$ small enough—which we can do through δ (cf. footnote 10). Let δ in fact be sufficiently small so that $\hat{u} \in B_{\epsilon/2}(u_0)$ and $\hat{x} \in B_\epsilon(x_0)$. Then there exists also a sufficiently small $\epsilon' > 0$ so that $B_{\epsilon'}(\hat{u}) \subseteq B_{\epsilon/2}(u_0)$ and $B_{\epsilon'}(\hat{x}) \subseteq B_\epsilon(x_0)$. Fix such an ϵ' . Since $\hat{\lambda} \in \mathbb{R}_{++}^2$, we can apply the already proven results for $\lambda_0 \in \mathbb{R}_{++}^2$ to \hat{u} , \hat{x} , and ϵ' in place of u , x_0 , and ϵ , respectively. Thus, there exists a nonempty, open set $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $U \subseteq B_{\epsilon'}(\hat{u}, \hat{u})$ and $U \subseteq \bigcup_{x \in D \cap B_{\epsilon'}(\hat{x})} H_x \times H_x^-$. We can actually use the same set U for u , x_0 , and ϵ , because $B_{\epsilon'}(\hat{x}) \subseteq B_\epsilon(x_0)$, and also $B_{\epsilon'}(\hat{u}, \hat{u}) \subseteq B_\epsilon(u_0, u_0)$ by $B_{\epsilon'}(\hat{u}) \subseteq B_{\epsilon/2}(u_0)$. \square

Proposition 4. *Let $u_0 \in \mathbb{R}^n$, $D = \text{co}\{\nu_1, \nu_2, \nu_3\}$ for three linearly independent vectors $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^n$, and $x_0 \in D$ such that $u_0 \cdot x_0 = 0$. Then, for any given $\epsilon > 0$, there exists a nonempty, open set $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $U \subseteq B_\epsilon(u_0, u_0)$ and for all $(u_1, u_2) \in U$ there is some $x \in D \cap B_\epsilon(x_0)$ such that $u_1 \cdot x = u_2 \cdot x = 0$ (i.e., $U \subseteq \bigcup_{x \in D \cap B_\epsilon(x_0)} H_x \times H_x$).*

Proof. Note that the condition $u_1 \cdot x = u_2 \cdot x = 0$ holds for some $x \in D$ if and only if it holds for some $x = \sum_{i=1}^3 \lambda_i \nu_i$ such that $\lambda = (\lambda_1, \lambda_2, \lambda_3)^\top \in \mathbb{R}_+^3 \setminus \{0\}$. Hence, if we first set aside the additional requirement that $x \in B_\epsilon(x_0)$, we are looking for pairs $(u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^n$ such that the homogeneous system of linear equations

$$\begin{pmatrix} u_1 \cdot \nu_1 & u_1 \cdot \nu_2 & u_1 \cdot \nu_3 \\ u_2 \cdot \nu_1 & u_2 \cdot \nu_2 & u_2 \cdot \nu_3 \end{pmatrix} \lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}_+^3, \quad (4)$$

has a solution in $\mathbb{R}_+^3 \setminus \{0\}$. The latter is the case if $u_1 = u_2 = u_0$, since $u_0 \cdot x_0 = 0$ and $x_0 \in D$; then a suitable solution is the unique $\lambda_0 \in \mathbb{R}_+^3$ such that, in matrix notation, $(\nu_1, \nu_2, \nu_3)\lambda_0 = x_0$. Because we ultimately want to use the implicit function theorem to obtain the open set U , we are going to construct another, but near, starting pair (\hat{u}_1, \hat{u}_2) , which is such that the two rows of the matrix in (4) are linearly independent and there is a solution $\hat{\lambda}$ in the *interior* of the positive orthant. We indeed have $\hat{\lambda} \in \text{int} \mathbb{R}_+^3 = \mathbb{R}_{++}^3$ if $\hat{\lambda} = \lambda_0 + \delta \mathbf{1}$ for some $\delta > 0$. Fix an arbitrary such $\hat{\lambda}$ (so in particular $\hat{\lambda} \neq 0$), let $y_0 = (u_0 \cdot \nu_1, u_0 \cdot \nu_2, u_0 \cdot \nu_3)^\top$, which lies in $H_{\lambda_0} \subseteq \mathbb{R}^3$, and let \hat{y} be the projection of y_0 onto the hyperplane $H_{\hat{\lambda}}$. To have $\|\hat{y} - y_0\|$ arbitrarily small, we only need $\|\hat{\lambda} - \lambda_0\|$ to be small enough—which we can achieve by starting with sufficiently small δ .¹⁰ Next, consider any two linearly independent vectors $b_1, b_2 \in H_{\hat{\lambda}}$ and any $\delta' > 0$, and let $y_i = \hat{y} + \delta' b_i$ for both $i = 1, 2$. Then $y_1, y_2 \in H_{\hat{\lambda}}$ by construction, and y_1, y_2 are also linearly independent whenever δ' is small

¹⁰Specifically, $\hat{y} = y_0 + \mu \hat{\lambda}$ for $\mu = -y_0 \cdot \hat{\lambda} / \hat{\lambda}^2$, so $\|\hat{y} - y_0\|^2 = (y_0 \cdot \hat{\lambda})^2 / \hat{\lambda}^2$. The latter vanishes as $\hat{\lambda} \rightarrow \lambda_0$, because the numerator tends to $(y_0 \cdot \lambda_0)^2 = 0$ and the denominator to λ_0^2 , which is strictly positive since $\lambda_0 \neq 0$.

enough.¹¹ By linear independence of v_1, v_2, v_3 , the linear function f that maps every $u \in \mathbb{R}^n$ to $(u \cdot v_1, u \cdot v_2, u \cdot v_3)^\top = (v_1, v_2, v_3)^\top u \in \mathbb{R}^3$ is surjective. Hence, there are two vectors $\hat{u}_1, \hat{u}_2 \in \mathbb{R}^n$ such that $f(\hat{u}_i) = y_i$ for both $i = 1, 2$, so that the starting pair (\hat{u}_1, \hat{u}_2) has the desired properties by construction. Moreover, we may assume it to be arbitrarily close to (u_0, u_0) by choosing sufficiently small δ and δ' . Indeed, f is an open mapping by Theorem 5.18 in Aliprantis and Border (2006), and hence the inverse correspondence f^{-1} is lower hemicontinuous by Theorem 17.7 (ibid.), so if y_i is close enough to y_0 (which we can achieve via the triangle inequality by making δ and δ' small), there is some $\hat{u}_i \in f^{-1}(y_i)$ as close as desired to $u_0 \in f^{-1}(y_0)$.

In summary, for any such pair (\hat{u}_1, \hat{u}_2) , the matrix in (4) equals $(y_1, y_2)^\top$ for some linearly independent vectors $y_1, y_2 \in \mathbb{R}^3$, and there is a corresponding solution $\hat{\lambda} \in \text{int } \mathbb{R}_+^3$, so that actually $y_1, y_2 \in H_{\hat{\lambda}}$. Since $\mathbf{1} \cdot \hat{\lambda} > 0$, the vector $\mathbf{1} \in \mathbb{R}^3$ is outside $H_{\hat{\lambda}}$, and thus the square matrix $(y_1, y_2, \mathbf{1})^\top$ has full rank. Moreover, $\hat{\lambda}$ satisfies $(y_1, y_2, \mathbf{1})^\top \hat{\lambda} = (0, 0, \mathbf{1} \cdot \hat{\lambda})^\top$ by construction.

Fixing \hat{u}_1, \hat{u}_2 , and $\hat{\lambda}$, we are now in the position to apply the implicit function theorem. Let F be the (continuously differentiable) mapping that assigns the value

$$F(u_1, u_2, \lambda) = \begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & u_1 \cdot v_3 \\ u_2 \cdot v_1 & u_2 \cdot v_2 & u_2 \cdot v_3 \\ 1 & 1 & 1 \end{pmatrix} \lambda - \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \cdot \hat{\lambda} \end{pmatrix} \in \mathbb{R}^3$$

to every triple $(u_1, u_2, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^3$. Then $F(\hat{u}_1, \hat{u}_2, \hat{\lambda}) = 0$, and the Jacobian of F with respect to λ and evaluated at $(u_1, u_2, \lambda) = (\hat{u}_1, \hat{u}_2, \hat{\lambda})$ is $(y_1, y_2, \mathbf{1})^\top$, which has full rank. Hence, there exists a neighborhood $\hat{U} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of (\hat{u}_1, \hat{u}_2) and a continuously differentiable function $g: \hat{U} \rightarrow \mathbb{R}^3$ such that $g(\hat{u}_1, \hat{u}_2) = \hat{\lambda}$ and $F(u_1, u_2, g(u_1, u_2)) = 0$ for all $(u_1, u_2) \in \hat{U}$. Since $\hat{\lambda} \in \text{int } \mathbb{R}_+^3$ and g is continuous, there is another neighborhood $U \subseteq \hat{U}$ of (\hat{u}_1, \hat{u}_2) such that $g(U) \subseteq \text{int } \mathbb{R}_+^3$. Therefore, for every pair $(u_1, u_2) \in U$, there is some $\lambda = g(u_1, u_2) \in \text{int } \mathbb{R}_+^3$ such that $F(u_1, u_2, \lambda) = 0$, so that in particular (4) holds. Forcing \hat{u}_1 and \hat{u}_2 to be sufficiently close to u_0 (through small δ and δ'), and keeping the radius of U small enough (but still positive), we can further ensure that $U \subseteq B_\epsilon(u_0, u_0)$.

The last requirement to fulfill is that $x = (v_1, v_2, v_3)\lambda = (v_1, v_2, v_3)g(u_1, u_2)$ stays in $B_\epsilon(x_0)$ for all $(u_1, u_2) \in U$. By $x_0 = (v_1, v_2, v_3)\lambda_0$, it is enough to keep λ close to λ_0 , or λ close to $\hat{\lambda}$ and $\hat{\lambda}$ close to λ_0 . We can indeed make $\|\hat{\lambda} - \lambda_0\| = \|\delta \mathbf{1}\|$ arbitrarily small through δ , whereas $\|\lambda - \hat{\lambda}\| = \|g(u_1, u_2) - g(\hat{u}_1, \hat{u}_2)\|$ can be kept arbitrarily small for all $(u_1, u_2) \in U$ by a sufficiently small radius of U , because g is continuous. \square

¹¹If y_1 and y_2 are linearly *dependent*, then one of them, say w.l.o.g. y_2 , is a scalar multiple of the other, i.e., $y_2 = \mu y_1$ for some $\mu \in \mathbb{R}$. This requires that $(1 - \mu)\hat{y} = \delta'(\mu b_1 - b_2)$, which is a nontrivial linear combination of b_1, b_2 (due to $\delta' > 0$) and thus not null, implying that also $1 - \mu \neq 0$. Hence, \hat{y} must be a linear combination of b_1, b_2 , with some coefficients $\mu_1, \mu_2 \in \mathbb{R}$ that are uniquely determined by linear independence, and which by the previous equation must satisfy $\mu_1 = \delta' \mu / (1 - \mu)$ and $\mu_2 = -\delta' / (1 - \mu)$. The two latter equations yield $\mu(\delta' + \mu_1) = \mu_1$ and $\mu \mu_2 = \delta' + \mu_2$, so $(\delta' + \mu_1)(\delta' + \mu_2) = \mu_1 \mu_2$, which holds for $\delta' = 0$ and hence for at most one $\delta' > 0$.

To state Proposition 5, let $\mu, \mu' \in \Delta\Theta$ now be two arbitrary beliefs and consider the set

$$\mathcal{R} := \{(r, r') \in (\Delta A)^2 \mid \text{supp } r \subseteq A(\mu) \text{ and } \text{supp } r' \subseteq A(\mu')\}$$

and the (onto) mapping from \mathcal{R} to \mathcal{D} that maps (r, r') to $x = r - r'$. We need an inverse that is continuous in an arbitrary given point $x_0 = r_0 - r'_0$.

Proposition 5. *Let $x_0 = r_0 - r'_0 \in \mathcal{D}$. For any neighborhood R of (r_0, r'_0) , there is a neighborhood X of x_0 such that every $x \in X \cap \mathcal{D}$ has a representation $x = r - r'$ with $(r, r') \in R \cap \mathcal{R}$.*

The proof of Proposition 5 will use the following lemma, which characterizes the whole preimage $\mathcal{R}_x := \{(r, r') \in \mathcal{R} \mid r - r' = x\}$ of $x \in \mathcal{D}$. For any vector $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, let x^+ denote $(\max(x_1, 0), \dots, \max(x_n, 0))^\top \in \mathbb{R}_+^n$, and let x^- denote $(-x)^+$ (so that $x = x^+ - x^-$).

Lemma 9. *Let $x \in \mathcal{D}$. Then $(r, r') \in \mathcal{R}_x$ if and only if $(r, r') = (x^+ + d, x^- + d)$ for some $d = (d_1, \dots, d_n)^\top \in \mathbb{R}_+^n$ such that $\mathbf{1} \cdot d = 1 - \mathbf{1} \cdot x^+$ and $d_i = 0$ whenever $a_i \notin A(\mu) \cap A(\mu')$.*

Proof. “ \Rightarrow ”: Suppose $(r, r') \in \mathcal{R}_x$ and let $d = r - x^+$. Then $r = x^+ + d$ and $r' = r - x = r - (x^+ - x^-) = d + x^-$. Further, since $r \geq 0$ and $r = x + r'$, where also $r' \geq 0$, we have $r \geq x^+$, which implies $d \in \mathbb{R}_+^n$, and $\mathbf{1} \cdot d = 1 - \mathbf{1} \cdot x^+$ by $\mathbf{1} \cdot r = 1$. Now let d_i be the i -th coordinate of d . If $a_i \notin A(\mu)$, then $r(a_i) = 0$, which implies $d_i = 0$, because $r = x^+ + d$ and both x^+ and d are nonnegative. Likewise, if $a_i \notin A(\mu')$, then $r'(a_i) = 0$, and hence $d_i = 0$ by $r' = x^- + d$ and nonnegativity of x^- and d . Thus, d is as claimed.

“ \Leftarrow ”: Suppose $r = x^+ + d$ and $r' = x^- + d$ for some d as stated in the lemma. Then both r and r' are in \mathbb{R}_+^n , and $r - r' = x$. Further, $\mathbf{1} \cdot r = \mathbf{1} \cdot x^+ + \mathbf{1} \cdot d = 1$ and, since $x \in \mathcal{D}$ implies $\mathbf{1} \cdot x = 0$ and thus $\mathbf{1} \cdot x^- = \mathbf{1} \cdot x^+$, also $\mathbf{1} \cdot r' = \mathbf{1} \cdot x^- + \mathbf{1} \cdot d = 1$. Hence, $(r, r') \in \Delta A$. Now let x_i be the i -th coordinate of x and suppose $a_i \notin A(\mu)$. Then $x \in \mathcal{D}$ implies $x_i \leq 0$, and $d_i = 0$ by hypothesis, so $r(a_i) = 0$. Likewise, if $a_i \notin A(\mu')$, then $r'(a_i) = 0$ by $x_i \geq 0$ and $d_i = 0$. It follows that $(r, r') \in \mathcal{R}_x$. \square

We remark that the decomposition $x = r - r'$ is in fact unique if x has *any* decomposition with $\text{supp } r \cap \text{supp } r' = \emptyset$, because then $\mathbf{1} \cdot x^+ = 1$, or if $A(\mu) \cap A(\mu')$ has at most one element.

Proof of Proposition 5. Consider any $x \in \mathcal{D}$. By Lemma 9, there is a decomposition $x = r - r'$ with $(r, r') \in \mathcal{R}$ if and only if $(r, r') = (x^+ + d, x^- + d)$ for some d as in Lemma 9. Suppose this is the case. Analogously, since $x_0 = r_0 - r'_0 \in \mathcal{D}$, $(r_0, r'_0) = (x_0^+ + d_0, x_0^- + d_0)$ for $d_0 = r_0 - x_0^+ = r'_0 - x_0^-$. Thus, $r - r_0 = x^+ + d - x_0^+ - d_0$ and $r' - r'_0 = x^- + d - x_0^- - d_0$, which implies $\|r - r_0\| \leq \|x^+ - x_0^+\| + \|d - d_0\|$ and $\|r' - r'_0\| \leq \|x^- - x_0^-\| + \|d - d_0\|$. Therefore, it is enough to show that for every (small enough) $\delta > 0$ and every $x \in \mathcal{D}$ with $\|x - x_0\| < \delta$ there is some d as in Lemma 9 such that $\|d - d_0\|$ vanishes as $\delta \rightarrow 0$.

First suppose $A(\mu) \cap A(\mu') = \emptyset$. For any $x \in \mathcal{D}$, there is by definition some $(r, r') \in \mathcal{R}_x$, and then, by Lemma 9, $(r, r') = (x^+ + d, x^- + d)$, where now necessarily $d = 0$. Analogously, $d_0 = 0$, so in particular $\|d - d_0\| = 0 < \delta$.

Now suppose there is some $a_i \in A(\mu) \cap A(\mu')$. If $d_0 \neq 0$, assume w.l.o.g. (by Lemma 9) that a_i is such that the corresponding i -th coordinate of d_0 is positive. Let $d = d_0 + \mathbf{1} \cdot (x_0^+ - x^+) \delta_{a_i}$ (where $\delta_{a_i} \in \Delta A$ is the Dirac measure that assigns probability one to $\{a_i\}$), which agrees with d_0 except for the i -th coordinate. Thus, if $d_0 \neq 0$, the choice of a_i implies $d \geq 0$ for all δ small enough. If $d_0 = 0$, Lemma 9 implies $\mathbf{1} \cdot x_0^+ = 1 - \mathbf{1} \cdot d_0 = 1$ and, since also x has a corresponding representation with *some* $d \geq 0$, $\mathbf{1} \cdot x^+ \leq 1$, so that also the constructed d satisfies $d \geq d_0 = 0$. Further, by construction $\mathbf{1} \cdot d = \mathbf{1} \cdot d_0 + \mathbf{1} \cdot (x_0^+ - x^+)$, so as required $\mathbf{1} \cdot d = 1 - \mathbf{1} \cdot x^+$ by $\mathbf{1} \cdot d_0 = 1 - \mathbf{1} \cdot x_0^+$. Now consider any $a_j \notin A(\mu) \cap A(\mu')$. Then $a_j \neq a_i$, so the j -th coordinate of d agrees with the j -th coordinate of d_0 , and the latter is zero by Lemma 9. Therefore, d satisfies all properties in Lemma 9 (if δ is small enough). Finally, $\|d - d_0\| = |\mathbf{1} \cdot (x_0^+ - x^+)|$, which indeed vanishes as $\delta \rightarrow 0$. \square

B.3 Proof of Theorem 1

Recall that, by the quasiconcavification theorem of Lipnowski and Ravid (2020), there exists an equilibrium that achieves the maximal equilibrium payoff for the sender, and this payoff is $s = c(\mu_0)$, where c denotes the quasiconcave envelope of \mathcal{V} . Fix this value of s .

If $s \in \mathcal{V}(\mu_0)$, then, by Lemma LR, there exists a babbling equilibrium (where $\text{supp } p = \{\mu_0\}$) with sender payoff s . Any babbling equilibrium is robust by Proposition 1.

Hence, suppose $s \notin \mathcal{V}(\mu_0)$. Since $s = c(\mu_0)$, this means in fact $s > \max \mathcal{V}(\mu_0)$. Moreover, by Lemma LR, any equilibrium that achieves the sender payoff s has $|\text{supp } p| \geq 2$. Since such an equilibrium exists, there is, by Lemma 2, in particular one with $|\text{supp } p| = |\Theta| = 2$, i.e., where $\text{supp } p = \{\mu, \mu'\}$ and $\mu < \mu'$. Further, by Lemma LR, there is an equilibrium with a belief distribution of the latter form and payoff s if and only if the two posteriors satisfy in fact $\mu < \mu_0 < \mu'$ and $s \in \mathcal{V}(\mu) \cap \mathcal{V}(\mu')$. Hence, we may assume $\mu = \min\{\tilde{\mu} \in \Delta\Theta \mid \tilde{\mu} < \mu_0 \text{ and } s \in \mathcal{V}(\tilde{\mu})\}$ and $\mu' = \max\{\tilde{\mu} \in \Delta\Theta \mid \tilde{\mu} > \mu_0 \text{ and } s \in \mathcal{V}(\tilde{\mu})\}$, where the minimum and the maximum indeed exist, because the respective sets are nonempty and \mathcal{V} is upper hemicontinuous (cf. Lemma 4). This equilibrium is robust if one of the conditions 1.–3. in Proposition 2 holds.

Since condition 1. is satisfied if $\mu = 0$ and $\mu' = 1$, suppose $\mu > 0$ or $\mu' < 1$. We are going to show that $s = c(\mu_0)$ implies that condition 2. or 3. is satisfied, i.e., $|A(\mu) \cup A(\mu')| \geq 3$ and either $\mu = 0$ or $\mu' = 1$, or actually $|A(\mu) \cup A(\mu')| \geq 4$. Specifically, assume $\mu' < 1$; the case in which $\mu' = 1$ and $\mu > 0$ is symmetric to $\mu = 0$ and $\mu' < 1$.

Using Lemma 4, consider any neighborhood U_μ of μ that is small enough such that $A(\tilde{\mu}) \subseteq A(\mu)$ for all $\tilde{\mu} \in U_\mu$, where the inclusion must now be strict whenever $\tilde{\mu} < \mu$, because then $s \in \mathcal{V}(\mu) \setminus \mathcal{V}(\tilde{\mu})$ by construction of μ . Thus, $|A(\mu)| \geq 2$ if $\mu > 0$, and analogously $|A(\mu')| \geq 2$ given $\mu' < 1$. If $A(\mu) \cap A(\mu') = \emptyset$, it already follows that $A(\mu) \cup A(\mu')$ has at least three elements and at least four if $\mu > 0$.

Therefore, suppose $A(\mu) \cap A(\mu')$ is nonempty, so in fact $A(\mu) \cap A(\mu') = A(\mu_0)$ by Lemma 5. Since $s \in \mathcal{V}(\mu)$ but $s > \max \mathcal{V}(\mu_0)$, there must be some $a \in A(\mu) \setminus A(\mu_0)$ such that $v_S(a) \geq s$. Analogously, there is another $a' \in A(\mu') \setminus A(\mu_0)$ such that $v_S(a') \geq s$. It follows that $A(\mu) \cup A(\mu')$ has at least three

elements.

Thus, suppose $\mu > 0$. Since $s = c(\mu_0)$, we have $s \geq \sup \bigcup_{\tilde{\mu} \leq \mu_0} \mathcal{V}(\tilde{\mu})$ or $s \geq \sup \bigcup_{\tilde{\mu} \geq \mu_0} \mathcal{V}(\tilde{\mu})$.¹² Suppose the former holds. In this case, for all $\tilde{\mu} < \mu$, the fact that s is not in $\mathcal{V}(\tilde{\mu})$ again means $s > \max \mathcal{V}(\tilde{\mu})$, so the a from the previous argument, which satisfies $a \in A(\mu) \setminus A(\mu_0)$ and $v_S(a) \geq s$, cannot be in $A(\tilde{\mu})$, either. Further, for all $\tilde{\mu} < \mu$, necessarily $A(\tilde{\mu}) \cap A(\mu_0) = \emptyset$, because otherwise Lemma 5, $\tilde{\mu} < \mu < \mu_0 < \mu'$, and $A(\mu_0) \subseteq A(\mu')$ would imply $A(\mu) = A(\mu_0)$, which is a contradiction to $a \in A(\mu) \setminus A(\mu_0)$. Therefore, choosing any $\tilde{\mu} < \mu$ in a sufficiently small neighborhood U_μ of μ such that, by Lemma 4, $A(\tilde{\mu}) \subseteq A(\mu)$, we obtain four disjoint subsets $A(\tilde{\mu})$, $\{a\}$, $A(\mu_0)$, and $\{a'\}$ of $A(\mu) \cup A(\mu')$ (where a' is still the element of $A(\mu') \setminus A(\mu_0)$ with $v_S(a') \geq s$). This implies $|A(\mu) \cup A(\mu')| \geq 4$, and the case in which $s = \max \bigcup_{\tilde{\mu} \geq \mu_0} \mathcal{V}(\tilde{\mu})$ is completely analogous.

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¹²In fact, since A is finite, $c(\mu_0) = \min(\max \bigcup_{\tilde{\mu} \leq \mu_0} \mathcal{V}(\tilde{\mu}), \max \bigcup_{\tilde{\mu} \geq \mu_0} \mathcal{V}(\tilde{\mu}))$.

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