Non-equilibrium lattice field theory

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1. Classical approximation
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3. Real-time fermions
4. Classical equilibrium and critical exponents
5. Cosmological applications of scalar fields
6. Gauge fields and heavy ion collisions
7. Turbulent scaling, Dual cascade
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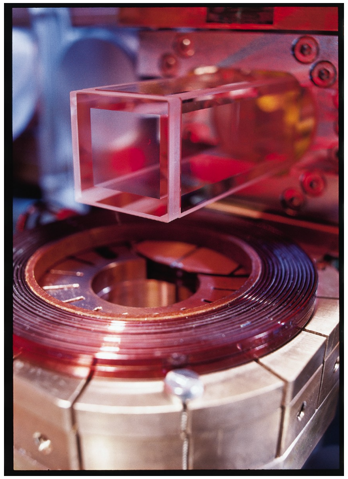
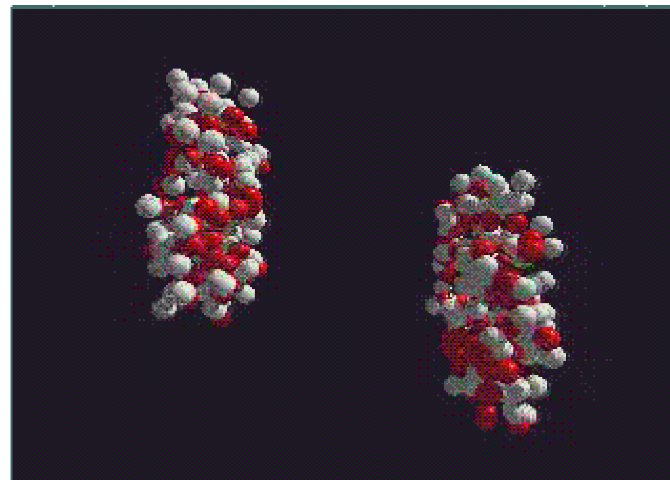
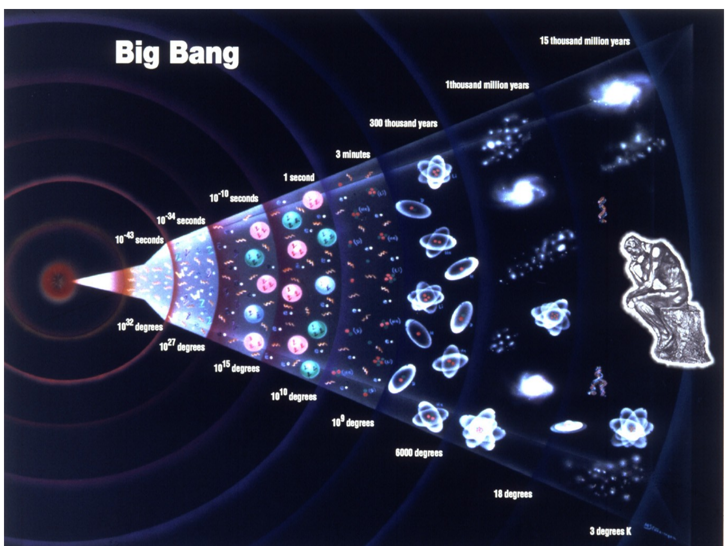
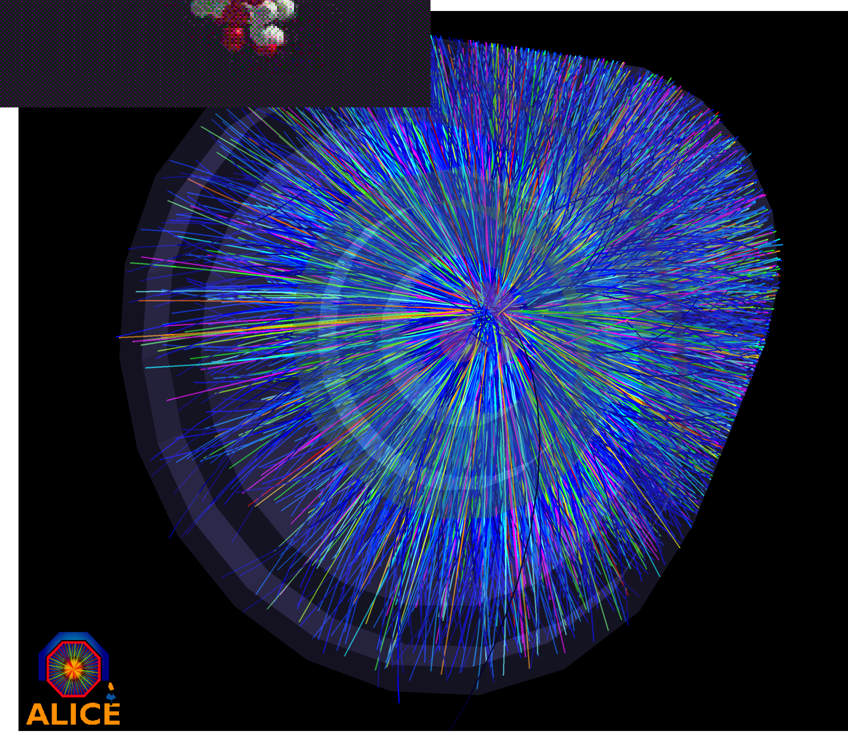
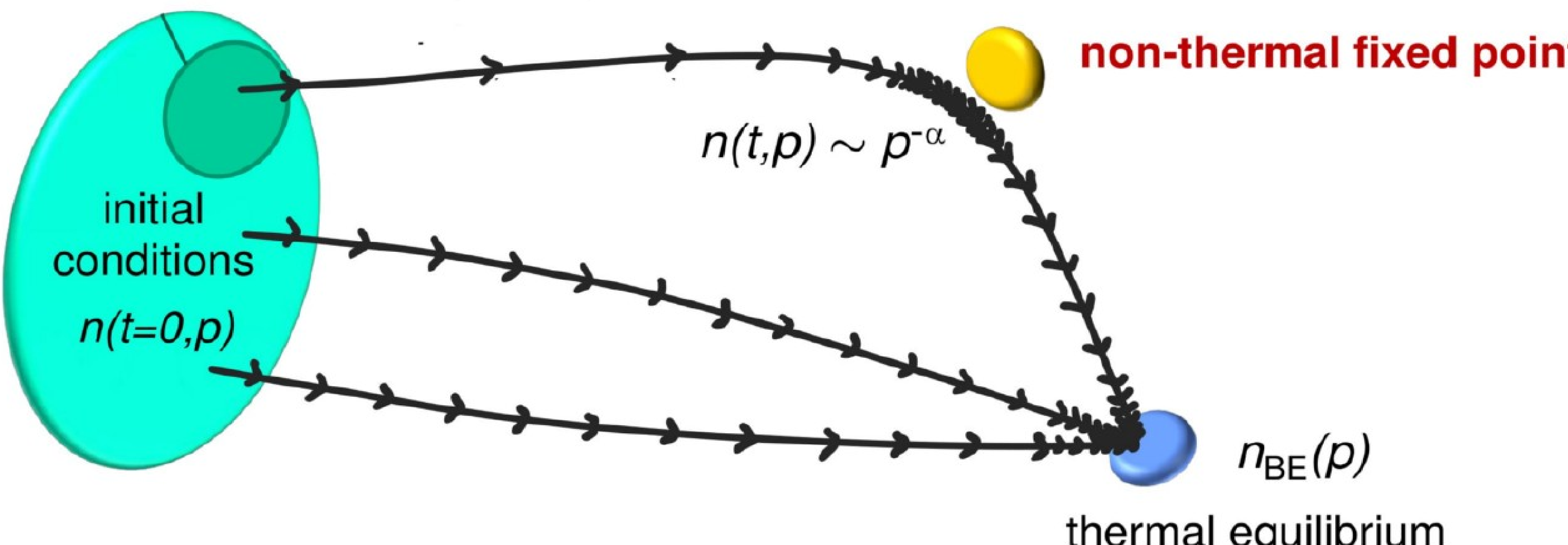





Big Bang
15 thousand million years
1thousand million years
300 thousand years
3 minutes
1 second
10^{-10} seconds
10^{-34} seconds
10^{-43} seconds
10^{32} degrees
10^{27} degrees
10^{15} degrees
10^{10} degrees
10^{9} degrees
6000 degrees
18 degrees
3 degrees K


ALICE



non-thermal fixed point
$n(t,p) \sim p^{-\alpha}$
initial
conditions
$n(t=0,p)$
$n_{BE}(p)$
thermal equilibrium

Non-equilibrium QFT

We want to describe Non-equilibrium QFT (HIC, reheating, cold atom experiments,...)
What can we do?

Need to develop and test different aprroximation schemes

- classical simulations
- 2PI studies
- HTL approximation
- stochastic quantisation
- Boltzmann equation
- transport coefficients in eq. + hydro

All methods have different range of applicability
Possibly need more than one of them to describe the physics

Classical approximation

QFT: $Z = \int D\phi \exp(i S[\phi])$

Integral over all possible paths $\phi(\boldsymbol{x}, t)$

Calculate on the lattice? sign problem
(Equilibrium is OK: imaginary time $e^{iS} \rightarrow e^{-\beta S}$)

Which path has the biggest contribution?
The one with stationary phase

$$\int D\phi \exp(i S[\phi]) \rightarrow \int D\phi\big|_{\delta S[\phi]/\delta\phi=0} \exp(i S[\phi])$$

Classical Equation of Motion: $\frac{\delta S[\phi]}{\delta \phi} = 0$

$$\langle f(\phi) \rangle = \frac{1}{Z} \int D\phi\, f(\phi)\, e^{iS} \rightarrow f(\phi)\big|_{\delta S[\phi]/\delta\phi}$$

Scalar fields

O(N) symmetric scalars:

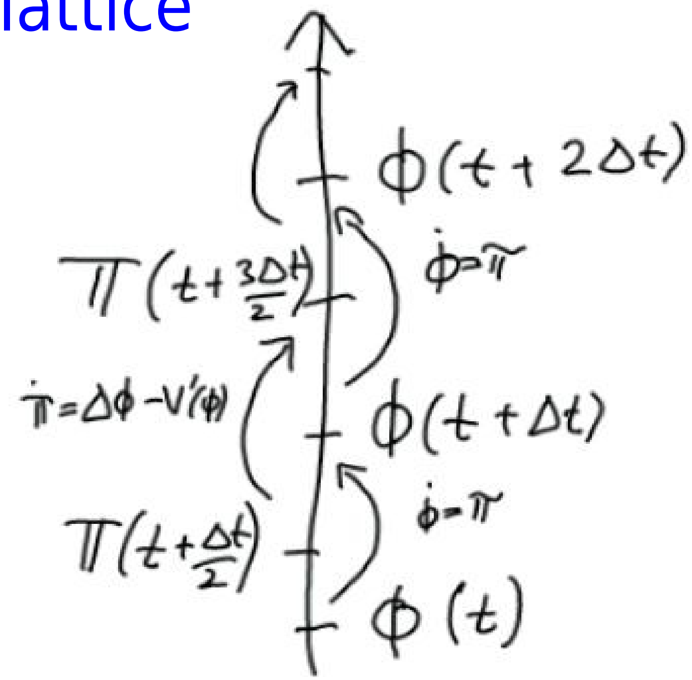
$$S[\phi_a]=\int d^d x\, dt\left[\frac{1}{2}(\partial_t\phi_a)^2-\frac{1}{2}(\nabla\phi_a)^2-V(\phi_a^2)\right]$$

$$V(\phi_a^2)=\frac{1}{2}m^2\phi_a^2+\frac{\lambda}{24}(\phi_a^2)^2$$

$$(\partial_t^2-\nabla_x^2)\phi_a(\boldsymbol{x},t)=-m^2\phi_a(\boldsymbol{x},t)-\frac{\lambda}{6}\phi_b^2(\boldsymbol{x},t)\phi_a(\boldsymbol{x},t)$$

O(4): Higgs of the weak interactions
sigma-pions effective modell
O(1): Inflaton field
O(N): Testground for field theoretical methods

Discretisation of the EOM on a space time lattice



Leap frog discretisation:

$$\dot{\phi}_a(\boldsymbol{x},t)=\pi_a(\boldsymbol{x},t)$$
$$\dot{\pi}_a(\boldsymbol{x},t)=\nabla^2\phi_a(\boldsymbol{x},t)-m^2\phi_a(\boldsymbol{x},t)-\frac{\lambda}{6}\phi_b^2(\boldsymbol{x},t)\phi_a(\boldsymbol{x},t)$$

Eliminate momentum:

$$\phi(t+\Delta t)=2\phi(t)-\phi(t-\Delta t)+\nabla^2\phi(t)-V'_a(\phi(t))$$

Cubic spacetime lattice $\quad x=n_\mu a_\mu \qquad n_i=0\ldots N_s-1,\quad n_t=0\ldots\infty$

$$\phi(\boldsymbol{x},t+\Delta t)=2\phi(\boldsymbol{x},t)-\phi(\boldsymbol{x},t-\Delta t)+\Delta_L\phi(\boldsymbol{x},t)-V'_a(\phi(\boldsymbol{x}.t))$$
$$\Delta_L\phi(\boldsymbol{x},t)=\sum_i(\phi(\boldsymbol{x}+a_i,t)-2\phi(\boldsymbol{x},t)+\phi(\boldsymbol{x}-a_i,t))$$

Improved Laplacian: $\quad A=-\frac{1}{12}\qquad B=\frac{4}{3}\qquad C=-\frac{5}{2}$

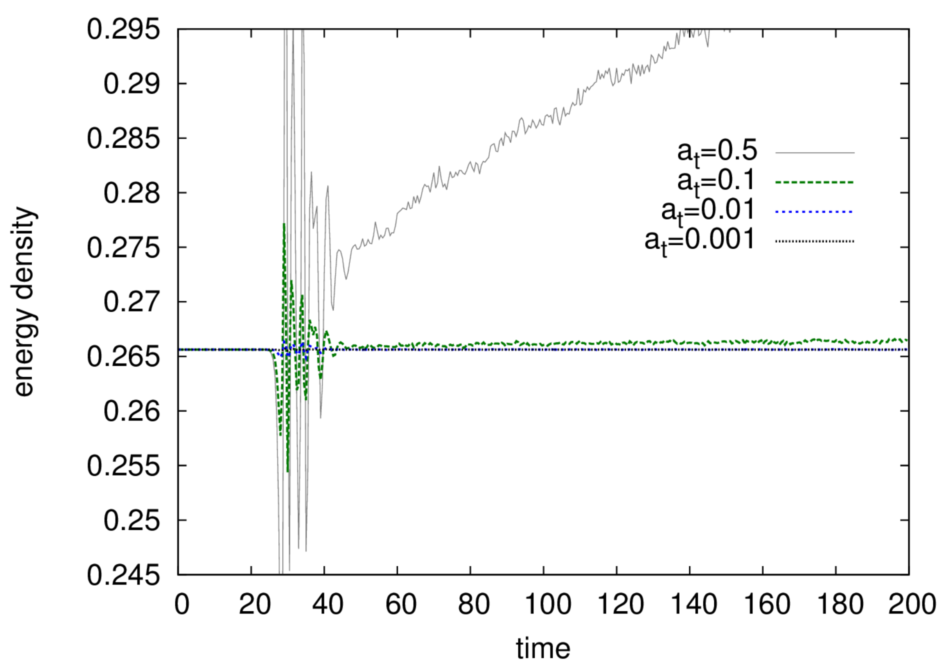
$$\Delta_L\phi(\boldsymbol{x},t)=\sum_i(A\phi(\boldsymbol{x}+2a_i,t)+B\phi(\boldsymbol{x}+a_i,t)+C\phi(\boldsymbol{x},t)+B\phi(\boldsymbol{x}-a_i,t)+A\phi(\boldsymbol{x}-2a_i$$

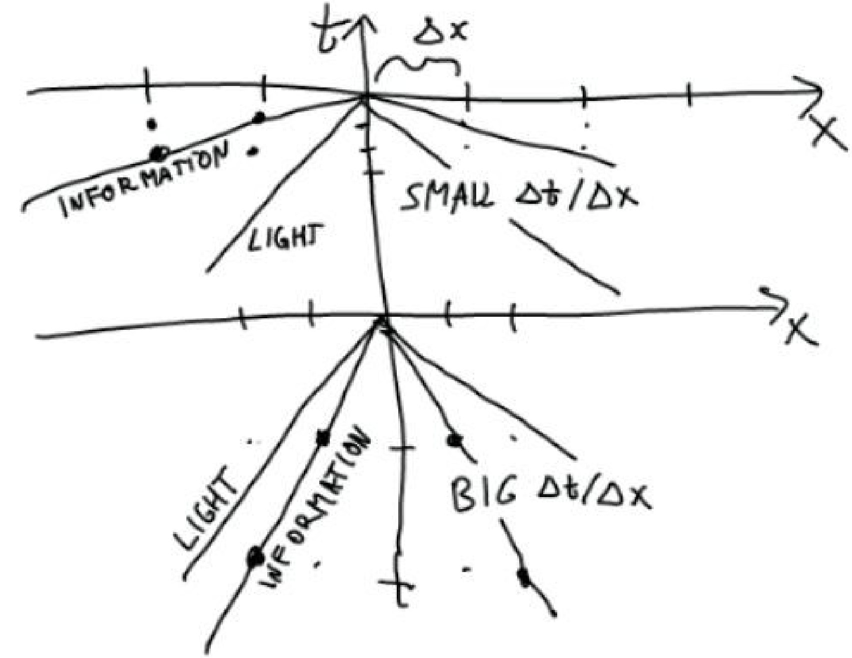
asymmetric lattice $a_i = a_s \quad a_0 = a_t$

Courant condition $\frac{a_t}{a_s} < 0.1$ Otherwise the solution is linearly instable

Speed of light should “fit into lattice”

Energy conservation fulfilled in limit $a_t \rightarrow 0$





Non-Relativistic Bose Gas

Dilute bose gas, s-channel interaction dominates (tuned with Feshbach resonance)

Gross-Pitaevskii equation

$$i\,\partial_t \Psi(x,t) = -\frac{1}{2\mathrm{m}}\Delta\Psi(x,t) + U\left|\Psi(x,t)\right|^2\Psi(x,t) \qquad \Psi\in\mathbb{C}$$

Describes time-evolution of a Bose-Einstein condensate

Particle number exactly conserved $\qquad n_{tot} = \int d^3x \left|\Psi(x,t)\right|^2$

$$L[\Psi,\dot{\Psi}] = \int d^d x \, -\frac{i}{2}\Psi^+\,\partial_t\Psi + \frac{i}{2}\Psi\,\partial_t\Psi^+ - \frac{1}{2\mathrm{m}}|\nabla\Psi|^2 - \frac{U}{2}\left(|\Psi|^2\right)^2$$

Canonical momenta

$$p_\Psi = \frac{\delta L}{\delta\,\partial_t\Psi} = \frac{i}{2}\Psi^+ \qquad\qquad p_{\Psi^+} = \frac{\delta L}{\delta\,\partial_t\Psi^+} = -\frac{i}{2}\Psi$$

$$H[\Psi,\Psi^+] = \int d^d x \, p_\Psi\dot{\Psi} + p_{\Psi^+}\dot{\Psi}^+ - L[\Psi,\dot{\Psi}] = \frac{1}{2\mathrm{m}}|\nabla\Psi|^2 + \frac{U}{2}\left(|\Psi|^2\right)^2$$

Split-step method

$$i\,\partial_t \Psi(x,t) = -\frac{1}{2\mathrm{m}}\Delta\Psi(x,t) + U\left|\Psi(x,t)\right|^2 \Psi(x,t) \qquad \Psi \in \mathbb{C}$$

Naive discretisation break charge conservation $\left|(1 - i R \Delta t)\right| < 1$

Split-step method:

$$\Psi(t+\Delta t) = \exp(-i K \Delta t)\exp(-i V \Delta t)\Psi(t)$$

$$V = U\,|\Psi|(x,t)^2$$

$$K = -\frac{1}{2\mathrm{m}}\Delta = \frac{k^2}{2\mathrm{m}}$$

$$\Psi'(x) = \exp(-i V \Delta t)\Psi(x,t)$$

calculate $\Psi'(k,t)$ with FFT

$$\Psi''(k) = \exp(-i K \Delta t)\Psi'(k)$$

calculate $\Psi''(x,t)$ with invFFT

$$\Psi(x, t+\Delta t) = \Psi''(x)$$

$$\Psi(t+\Delta t) = \exp(-i K \Delta t)\Psi(t)$$

↓ Expand in linear order

$$\frac{\Psi(t+\Delta t) - \Psi(t)}{\Delta t} = -i K \Psi(t)$$

Always multiplying with phase factor

Discretisation correct to some order in timestep but conserves particle number exactly

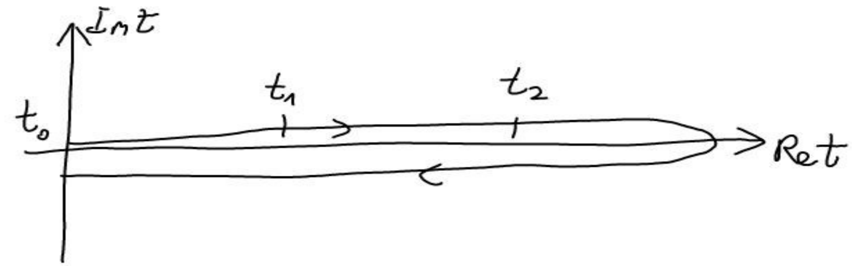
Functional represenatation of nonequilibrium physics

$\langle \hat{\phi}(t_1)\hat{\phi}(t_2)\rangle = ?$ Initial density matrix ρ_i

Schrödinger picture $\hat{\phi}(t) \rightarrow U(0,t)\hat{\phi}\,U(t,0)$ with $U(t,0)=\exp(-i\hat{H}t)$

$$\langle \hat{\phi}(t_1)\hat{\phi}(t_2)\rangle = \mathrm{Tr}\left(\rho_i U(t_0,t_1)\hat{\phi}\,U(t_2,t_1)\hat{\phi}\,U(t_1,t_0)\right)$$

Schwinger-Keldysh contour



Path integral representation:

$$Z=\int D\phi^+ D\phi^- \langle \phi^+(t_0)|\rho_i|\phi^-(t_0)\rangle \exp\left(iS[\phi^+]-iS[\phi^-]\right)$$

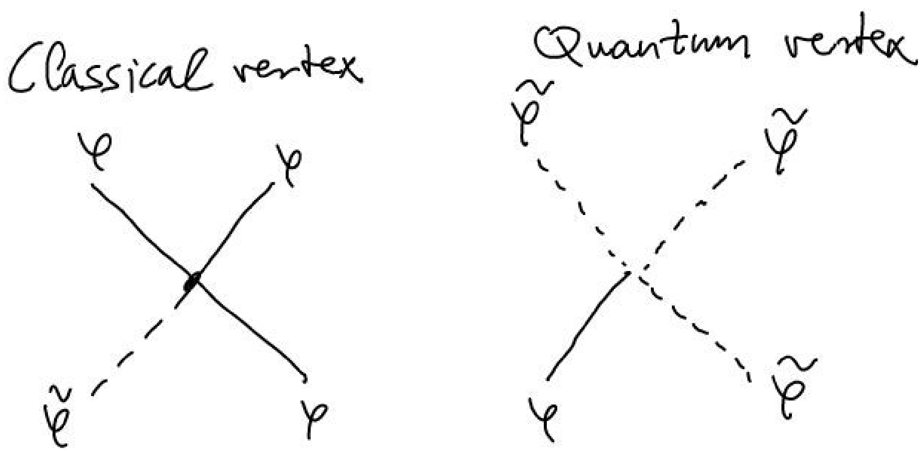
Now define $\phi = \dfrac{\phi^+ + \phi^-}{2}$, $\tilde{\phi} = \phi^+ - \phi^-$

$$S[\phi,\tilde{\phi}] = S[\phi^+] - S[\phi^-]$$

For O(N) scalars:

$$S_0[\phi,\tilde{\phi}]=\int_{x,t_0}[\partial^{\mu}\tilde{\phi}_a(x)\partial_{\mu}\phi_a(x)-m^2\tilde{\phi}_a(x)\phi_a(x)]$$

$$S_{\mathrm{int}}[\phi,\tilde{\phi}]=-\frac{\lambda}{6\mathrm{N}}\int_{x,t_0}\tilde{\phi}_a(x)\phi_a(x)\phi_b(x)\phi_b(x)+$$
$$-\frac{\lambda}{24\mathrm{N}}\int_{x,t_0}\tilde{\phi}_a(x)\tilde{\phi}_a(x)\tilde{\phi}_b(x)\phi_b(x)$$



$$\langle\phi_a(x)\phi_b(y)\rangle=F_{ab}(x,y)$$
$$\langle\phi_a(x)\tilde{\phi}_b(y)\rangle=-i\,G_{R,ab}(x,y)$$
$$\langle\tilde{\phi}_a(x)\phi_b(y)\rangle=-i\,G_{A,ab}(x,y)$$
$$\langle\tilde{\phi}_a(x)\tilde{\phi}_b(y)\rangle=0$$

Classical statistical field theory

Use some ensemble of initial conditions, and classical EOM $\left.\frac{\delta S}{\delta \phi}\right|_{\phi_{cl}} = 0$

$$\langle f(\phi)\rangle = \frac{1}{Z}\int D\phi\, D\pi\, P(\phi_0, \pi_0)\, f(\phi)\,\delta[\phi(x,t) - \phi_{cl}(x,t)]$$

$$\text{e.g. } f(\phi) = \phi(t_1)\ldots\phi(t_n)$$

Functional delta function

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \longrightarrow \delta[f(\phi(x))] = \sum_i \frac{\delta[\phi - \phi_i(x)]}{|\det f'_\phi[\phi]|}$$

Fourier representation of the delta function $\delta(x) = \int \frac{dk}{2\pi} \exp(i k x)$

Define classical action with to variables:

$$S_{cl}[\phi, \tilde{\phi}] = \mathrm{EOM}[\phi]\tilde{\phi} \text{ such that } EOM[\phi] = \delta S_{cl}[\phi, \tilde{\phi}]/\delta\tilde{\phi}$$

$$\delta[EOM[\phi]]=\delta[\delta S_{cl}[\phi,\tilde{\phi}]/\delta\tilde{\phi}]=\frac{\delta[\phi-\phi_{cl}]}{\left|\det\frac{\delta^2 S_{cl}[\phi,\tilde{\phi}]}{\delta\phi\,\delta\tilde{\phi}}\right|}=\frac{\delta[\phi-\phi_{cl}]}{J(\phi)}$$

Use this in the averages:

$$\langle f(\phi)\rangle=\frac{1}{Z}\int D\phi\, D\pi\, P(\phi_{0,}\pi_0)\, f(\phi)\,\delta[\phi(x,t)-\phi_{cl}(x,t)]$$

$$=\frac{1}{Z}\int D\phi\, D\pi_0\, P(\phi_{0,}\pi_0)\, f(\phi)\, J(\phi)\,\delta[EOM[\phi]]$$

$$=\frac{1}{Z}\int D\phi\, D\tilde{\phi}\, D\pi_0\, P(\phi_{0,}\pi_0)\, f(\phi)\, J(\phi)\exp(i(\delta S_{cl}/\delta\tilde{\phi})\tilde{\phi})$$

$$\int D\pi_0\, P(\phi_0,\pi_0)=\rho_D(\phi+\frac{1}{2}\tilde{\phi},\phi-\frac{1}{2}\tilde{\phi})$$

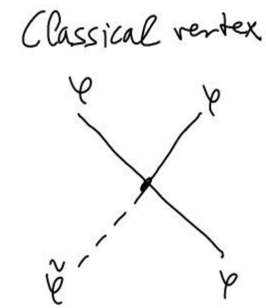
normalization

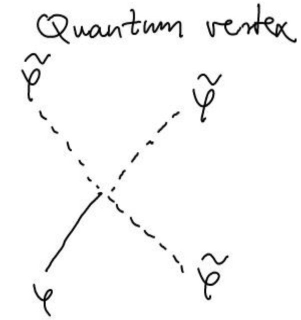
$$S_{cl}[\phi,\tilde{\phi}]$$

$$\langle f(\phi)\rangle_{cl}=\frac{1}{Z}\int D\phi\, D\tilde{\phi}\,\rho_D\, f(\phi)\exp(iS_{cl}[\phi,\tilde{\phi}])$$

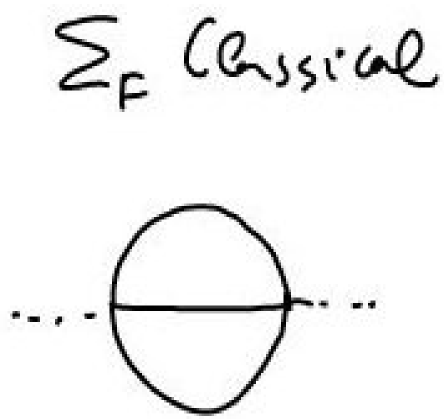
$$S_{cl}[\phi,\tilde{\phi}]=\mathrm{EOM}[\phi]\tilde{\phi}=\int_x \tilde{\phi}_a\left((\partial_t^2-\Delta)\phi_a-m^2\phi_a-\frac{\lambda}{6\mathrm{N}}\phi_b\phi_b\phi_a\right)$$

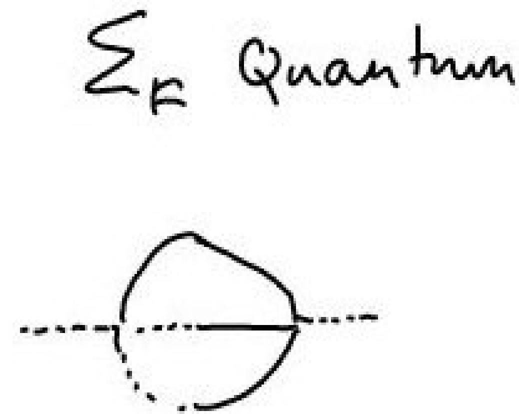
$$S[\phi,\tilde{\phi}]=S_{cl}[\phi,\tilde{\phi}]+\text{quantum vertex}$$





Feynman rules:



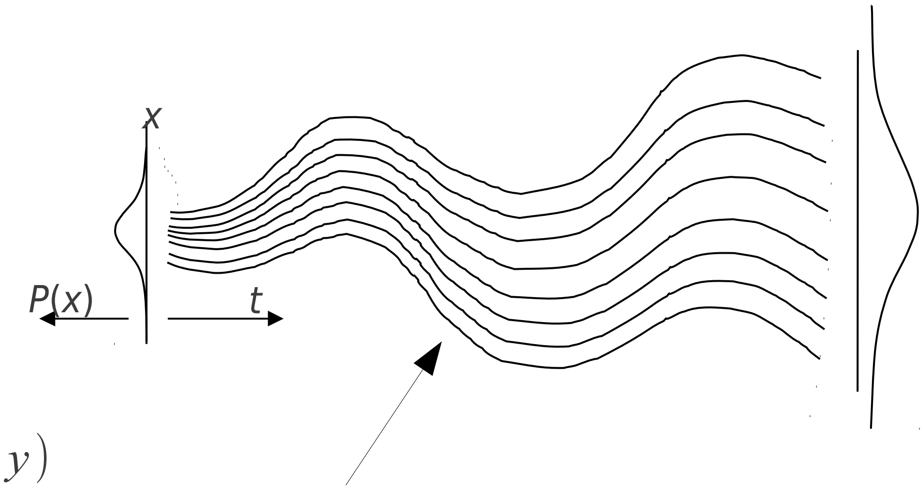


Classical field theory $\phi = \langle \hat{\phi} \rangle$ Schwinger-Dyson eq. + Mean field

$$\text{given } \phi(x, t=0)\, \pi(x, t=0) \quad \text{solve } \ddot{\phi} - \nabla^2 \phi + \frac{1}{2} m^2 \phi + \frac{\lambda}{6} \phi^3 = 0$$

Classical statistical field theory

Classical fluctuations dominate over quantum for $n > 1$



Using classical EOM

$$F(x, y) = \langle \{\varphi(x), \varphi(y)\} \rangle \gg \langle [\varphi(x), \varphi(y)] \rangle = \rho(x, y)$$

$$\langle O(\varphi, \pi) \rangle = \int D\varphi(0)\, D\pi(0)\, P(\varphi(0), \pi(0))\, O(\varphi(t), \pi(t))$$

Classical statistical field theory describes loop effects

Matching classical statistical simulations to quantum systems

Quantum mechanics $:\Psi(x)$

Classical mechanics: $P(x,p)$

Wigner function matches quantum states and distributions of classical phase-space

$$\langle s|f(\hat{x},\hat{p})|s\rangle=\int dx\,dp\,W_s(x,p)\,f(x,p)$$

Generalizing for density matrix

$$\mathrm{Tr}(\hat{\rho}\,f(\hat{x},\hat{p}))=\int dx\,dp\,W_\rho(x,p)\,f(x,p)$$

$W(x,p)\in\mathbb{R}$, but it might be negative $\quad W_\rho(x,p)=\frac{1}{\pi}\int\langle x+y|\hat{\rho}|x-y\rangle e^{-2ipy}\,dy$

For interesting states (Gaussian states, vacuum state,...)
Wigner function is positive

$$\text{density matrix}\rightarrow P(\phi(t=0),\pi(t=0))\rightarrow\text{Classical EOM}$$

$$\langle O(\varphi,\pi)\rangle=\int D\varphi(0)\,D\pi(0)\,P(\varphi(0),\pi(0))\,O(\varphi(t),\pi(t))$$

What about fermions?

$n \leq 1$ always in quantum regime

Fermions are always quadratic (if not, do a Hubbard-Stratonovich)

$$S = S[U,\phi] + \bar{\Psi} D(U,\phi) \Psi$$

Mode function expansion [Aarts, Smit (1998)]

$$\Psi(x,t) = \int \frac{dk}{(2\pi)^d} \left(\hat{b}_\alpha(k) \Phi^u_\alpha(x,k,t) + \hat{d}^+_\alpha(k) \Phi^v_\alpha(x,k,t) \right)$$

$$D(U,\phi)\Psi(x,t) = 0 \;\rightarrow\; D(U,\phi)\Phi_\alpha(x,k,t) = 0$$

$$\Phi^{uv}_\alpha(x,k,t) \;\rightarrow\; \Phi_i(x,t)$$

$4N^3$ modefunctions $\rightarrow$ $O(N^6)$ cost

Observables (and backreaction to bosons) calculated as

$$\bar{\Psi} O \Psi = \sum_i F_i \bar{\Phi}_i(x,t) O \Phi_i(x,t)$$

$$F_k = (1 - n^u_{ini,k}) \text{ or } n^v_{ini,k}$$

$$\Psi O' \bar{\Psi} = \sum_i F'_i \bar{\Phi}_i(x,t) O' \Phi_i(x,t)$$

$$F'_k = (1 - n^v_{ini,k}) \text{ or } n^u_{ini,k}$$

Bosons treated classically

Physics should be such that $n_\Phi, n_U \gg 1$

Hubbard Stratonovich transformation

$$\exp(-a\varphi^2+b\varphi)=\exp(-a(\varphi-\frac{b}{2a})^2+\frac{b^2}{4a})$$

Use it in a path integral:

$$\int D\varphi(x)\exp(\int d^d x(-a\varphi(x)^2+b(x)\varphi(x)))=$$

$$\int D\varphi(x)\exp\left(\int d^d x(-a\left(\varphi-\frac{b(x)}{2a}\right)^2+\frac{b(x)^2}{4a})\right)=\exp\left(\frac{b(x)^2}{4a}\right)$$

Practically:

$$\lambda(\bar{\Psi}(x)\Psi(x))^2 \rightarrow -\frac{1}{4\lambda}\varphi(x)^2+\bar{\Psi}(x)\Psi(x)\varphi(x)$$

φ is a constraint field, its EOM has no time derivatives.

Also possible if the interaction is not a square:

$$\frac{1}{U}\chi_1(x)\chi_2(x)+A(x)\chi_1(x)+B(x)\chi_2(x) \rightarrow -U\,A(x)B(x)$$

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Bosons treated classically

Physics should be such that $n_\Phi, n_U \gg 1$

Low cost fermions

[Borsanyi, Hindmarsh (2009)]

$$D(U,\phi)\Phi_i(x,t)=0 \qquad \sum_i F_i \bar{\Phi}_i(x,t) O \Phi_i(x,t)$$

EOM is linear $\qquad j=1\dots N_n$

With random numbers such that $\langle \xi_{j,i}\xi_{k,l}\rangle = F_i \delta_{jk}\delta_{il}$

initialize $\quad N_j(x,t=0)=\sum_i \xi_{j,i}\Phi_i(x,t=0)$

evolve $\quad D(U,\phi) N_j(x,t)=0$

$$\langle \sum_j \bar{N}_j(x,t) O N_j(x,t)\rangle = \sum_i F_i \bar{\Phi}_i(x,t) O \Phi_i(x,t)$$

On average, we get back the mode function method

In practice the convergence is fast (in at least 2 dimensions)

$N_n \ll N^3$ → low cost fermions

Observables

classical

$$\langle \phi(x) \rangle$$

$$F_{cl}(x,y) = \langle \phi(x)\phi(y) \rangle$$

$$\rho_{cl}(x,y) = \langle \{\phi(x),\phi(y)\}_{\mathrm{PB}} \rangle$$

quantum

$$\langle \hat{\phi}(x) \rangle$$

$$F(x,y) = \frac{1}{2}\langle \{\hat{\phi}(x),\hat{\phi}(y)\} \rangle$$

$$\rho(x,y) = i\langle [\hat{\phi}(x),\hat{\phi}(y)] \rangle$$

Poisson bracket = “classical commutator”

$$\{A(x),B(y)\}_{\mathrm{PB}} = \int_z \left(\frac{\partial A(x)}{\partial \phi(z)} \frac{\partial B(y)}{\partial \pi(z)} - \frac{\partial B(y)}{\partial \phi(z)} \frac{\partial A(x)}{\partial \pi(z)} \right)$$

$$\rho_{cl}(x,y)\big|_{x_0=y_0} = 0 \qquad \partial_{x_0}\rho_{cl}(x,y)\big|_{x_0=y_0} = \delta(x-y)$$

Homogeneous ensemble:

$$F(x,y) = F(x-y)$$

$$F(\omega,\boldsymbol{k}) = \int d\boldsymbol{x}\, dt\, F(\boldsymbol{x},t)\, e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}$$

$$F(t,k) = \langle |\phi(k,t)|^2 \rangle$$

Particle number, dispersion relation

Solve free field theory:

$$F(t,t',p)=(n_p+1/2)\frac{1}{\omega_p}\cos(\omega_p(t-t'))$$
$$\rho(t,t',p)=\frac{1}{\omega_p}\sin(\omega_p(t-t'))$$

$$\partial_t\partial_t{}'F(t,t',p)=(n_p+1/2)\,\omega_p\cos(\omega_p(t-t'))$$

One possible (and useful) definition:

$$n_p(t)+1/2=\sqrt{F(t,t',p)\,\partial_t\partial_{t'}F(t,t',p)\big|_{t=t'}}$$

In classical simulations:

$$n_p(t)=\sqrt{\langle|\phi(t,p)|^2\rangle\langle|\dot{\phi}(t,p)|^2\rangle}$$

If dispersion is known:

$$n_p(t)=\omega_p\langle|\phi(t,p)|^2\rangle$$

Dispersion relation:

$$\omega_p^2=\frac{\partial_t\partial_t{}'F(t,t',p)\big|_{t=t'}}{F(t,t,p)}=\frac{\langle|\dot{\phi}^2|\rangle}{\langle|\phi^2|\rangle}$$

Initial conditions

Given $n_{ini,k}, \langle \phi \rangle = \bar{\phi}$, calculate $\phi(x, t=0), \pi(x, t=0) = \dot{\phi}(x, t=0)$

We must statisfy $\langle |\phi_k|^2 \rangle = \left(n_k + \frac{1}{2} \right) \frac{1}{\omega_k}$ $\quad \langle |\pi_k|^2 \rangle = \left(n_k + \frac{1}{2} \right) \omega_k$

$$E_k = E_{kin} + E_{pot} = \left\langle \frac{1}{2} \dot{\phi}_k^2 + \frac{1}{2} \omega_k^2 \phi_k^2 \right\rangle = (n_k + 1/2) \omega_k$$

Dephasing (equilibration between kinetic and potential energy) is very fast

→ Only the sum matters

Initial conditions

Lazy choice:

$$\phi_k(t=0)=\delta_{k,0}\bar{\phi}+\sqrt{\frac{2(n_{\mathrm{ini},k}+1/2)}{\omega_k}}\,e^{i\varphi_k}, \qquad \pi_k(t=0)=0 \quad \text{with random phase } \varphi_k$$

“Democratic” choice (Wigner function: also absolute value is a Gaussian random)

$$\phi_k(t=0)=\delta_{k,0}\bar{\phi}+\sqrt{\frac{(n_{\mathrm{ini},k}+1/2)}{\omega_k}}\,e^{i\varphi_k}, \qquad \pi_k(t=0)=\sqrt{(n_{ini,k}+1/2)\omega_k}\,e^{i\varphi'_k}$$

Rayleigh Jeans problem:

$$E_\Lambda=\int_0^\Lambda dk\,k^2\,\frac{1}{2}\omega_k \;\rightarrow\; \infty$$

Quantum half should not dominate physics

High occupation in IR modes
Choose parameters such that

$$E_{1/2}\ll E_{physics}$$

Instability ⟶ turbulence, universality ⟶ initial conditions don't really matter

Rescaling with the coupling

Dimensionful EOM:
$$\ddot{\phi} - \nabla^2 \phi + m^2 \phi + \frac{\lambda}{6} \phi^3 = 0$$

Introduce dimensionless quantities:
$$\phi_L = \frac{\phi}{a_s} \qquad x_L = x\, a_s \qquad t_L = t\, a_s$$

further rescaling:
$$\phi' = \sqrt{\frac{\lambda}{6}}\, \phi_L$$

Rescaled, dimensionless EOM:
$$\ddot{\phi}' - \nabla^2 \phi' + m^2 a_s^2 \phi' + \phi'^3 = 0$$

Coupling drops out

To simplify notation, drop prime, but rescale observables:

$$\frac{\lambda}{6} n_p(t) = \sqrt{\langle |\phi(t,p)|^2 \rangle \langle |\dot{\phi}(t,p)|^2 \rangle}$$

In initial conditions:

$$\phi_k(t=0) = \sqrt{\frac{(\lambda n_{\mathrm{ini},k}/3 + \lambda/6)}{\omega_k}}\, e^{i \varphi_k}$$

→ Quantum half proportional to coupling

Classical equilibrium vs Quantum equilibrium

Classical equilibrium: equipartition

All modes have energy $E_k = T = K_k + V_k$

$$K_k = \left\langle \frac{|\dot{\phi}_k|^2}{2} \right\rangle = T/2 \qquad V_k = \left\langle \frac{(k^2+m^2)|\phi_k|^2}{2} \right\rangle = \frac{\omega_k^2}{2}\langle|\phi_k|^2\rangle = T/2$$

$$n_k = \sqrt{\langle|\dot{\phi}_k{}^2|\rangle\langle|\phi_k{}^2|\rangle} = \frac{T}{\omega_k}$$ Classical limit of $$n_{BE} = \frac{1}{e^{\beta\omega_k}-1} \quad \text{at } T \gg \omega_k$$

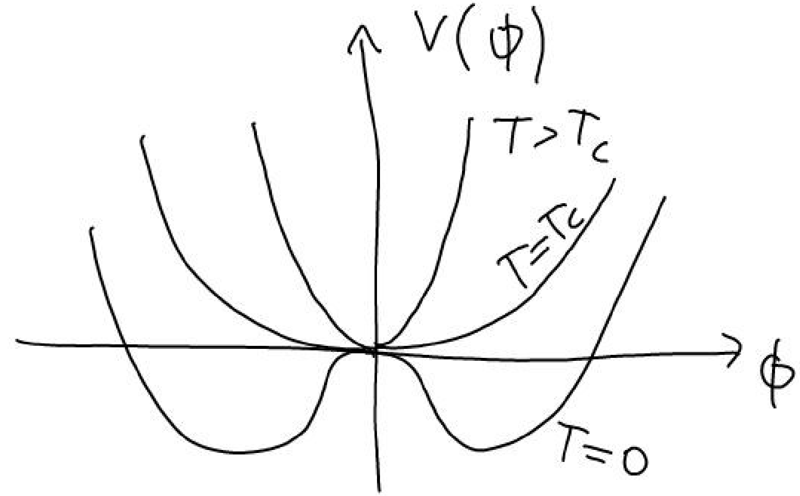
Rayleigh-Jeans problem

Energy content up to some cutoff $E_\Lambda = \int_0^\Lambda dk\, k^2\, T\, \omega_k \;\rightarrow\; \infty$

Classical field theory does not thermalize
(In the absence of a cutoff)

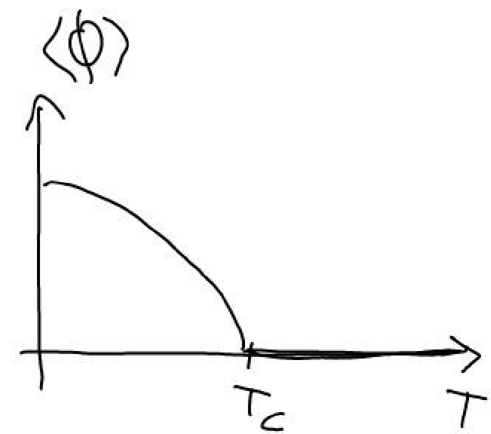
→ Classical simulations don't reach
the correct quantum equilibrium state

Second order phase transition



Bare mass squared negative

Temperature contribution $\sim T^2$



2nd order phase transition ⟶ critical scaling

$$\rho(s^z \omega, s\, p, s^{1/\nu} T_r) = s^{-(2-\eta)} \rho(\omega, p, T_r)$$

↓

$$\rho(\omega, 0, 0) \sim \omega^{\frac{-2-\eta}{z}}$$

Fourier transform $\rho(s^{-z} t, s\, p, s^{1/\nu} T_r) = s^{-(2-\eta-z)} \rho(t, p, T_r)$

$$\rho(t, 0, T_r) \sim t^{\frac{2-\eta}{z}-1} g\left(\frac{t}{\xi_t}\right)$$

Temporal correlation length $\xi_t \sim T_r^{-nu\, z}$

$$g\left(\frac{t}{\xi_t}\right) = \rho\left(1, 0, (t/\xi_t)^{1/(\nu z)}\right)$$

Classification of dynamical universality classes

Halperin and Hohenberg

Model A: No conserved charge
Model B: conserved order parameter
Model C: auxiliary conserved density
...

Universality given by slowest processes

→ Transport processes of conserved charges dominate

Scalar field theory   Energy density
Momentum density

Relativistic fluid: pressure   transverse part of the momentum density

linear dispersion   Quadratic dispersion

Model C: one conserved density $z = 2 + \alpha / \nu$

$\alpha = 0$ Susceptibility exponent vanishes in 2d

Classical KMS

Quantum KMS from periodicity of the Greens function

$$F(\omega, p) = -i\, n_{\mathrm{BE}}(\omega)\, \rho(\omega, p)$$

$$n_{BE} = \frac{1}{e^{\beta\omega_k} - 1}$$

In the far IR $\omega \ll T$

$$F(\omega, p) = -i \frac{T}{\omega} \rho(\omega, p)$$

$$i\rho_{cl}(\omega, p, T_r) = \rho(\omega, p, T_r)$$

Classical KMS relation

$$\rho_{cl}(t, p, T_r) = -\frac{1}{T} \partial_t F_{cl}(t, p, T_r)$$

$$F_{cl}(t-t', x-x', T_r) = \langle \phi(t, x)\, \phi(t', x') \rangle_{T_r} - \langle \phi(t, x) \rangle_{T_r} \langle \phi(t', x') \rangle_{T_r}$$

$$\rho_{cl}(t, p, T_r) = -\frac{1}{T} \int d^d x\, e^{-ip(x-y)} \langle \pi(t, x)\, \phi(0, y) \rangle_{T_r}$$

Near a 2nd order phase transition, physics is dominated in the IR
no Rayleigh-Jeans problem

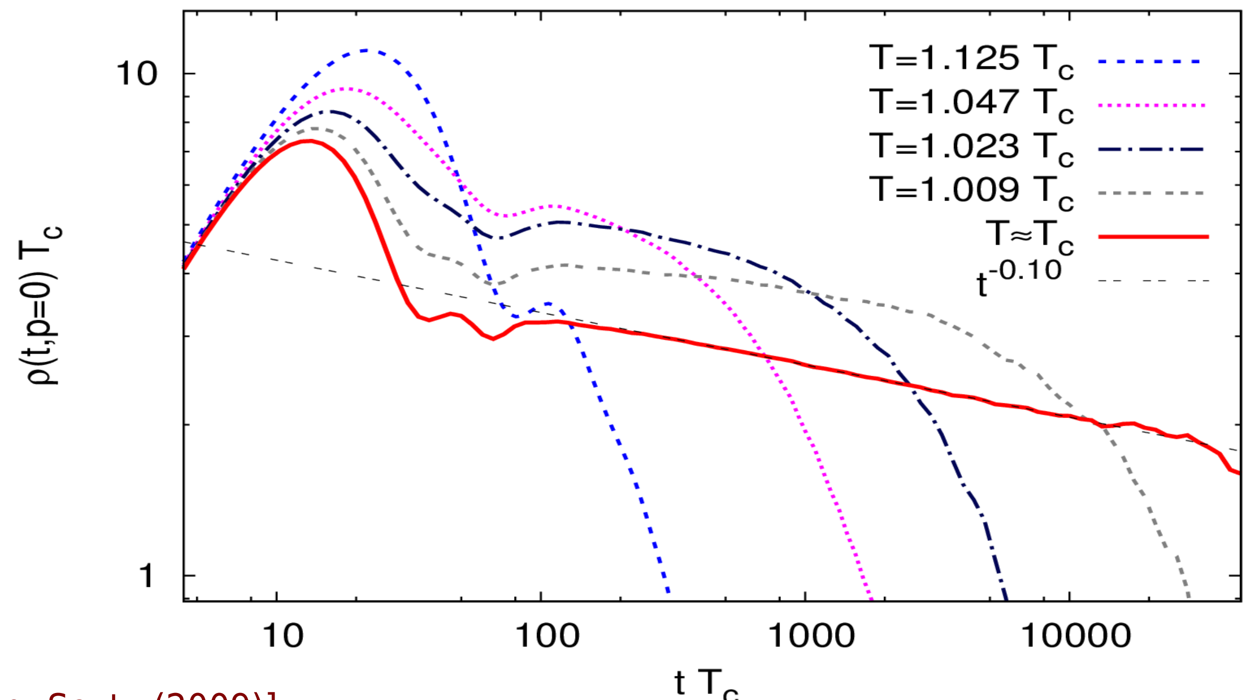
2D scalars with O(1) symmetry

$$H=\int d^d x\left(\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{24}\phi^4\right)$$

$$(\partial_t^2-\nabla_x^2)\phi(\boldsymbol{x},t)=-m^2\phi(\boldsymbol{x},t)-\frac{\lambda}{6}\phi(\boldsymbol{x},t)^3$$

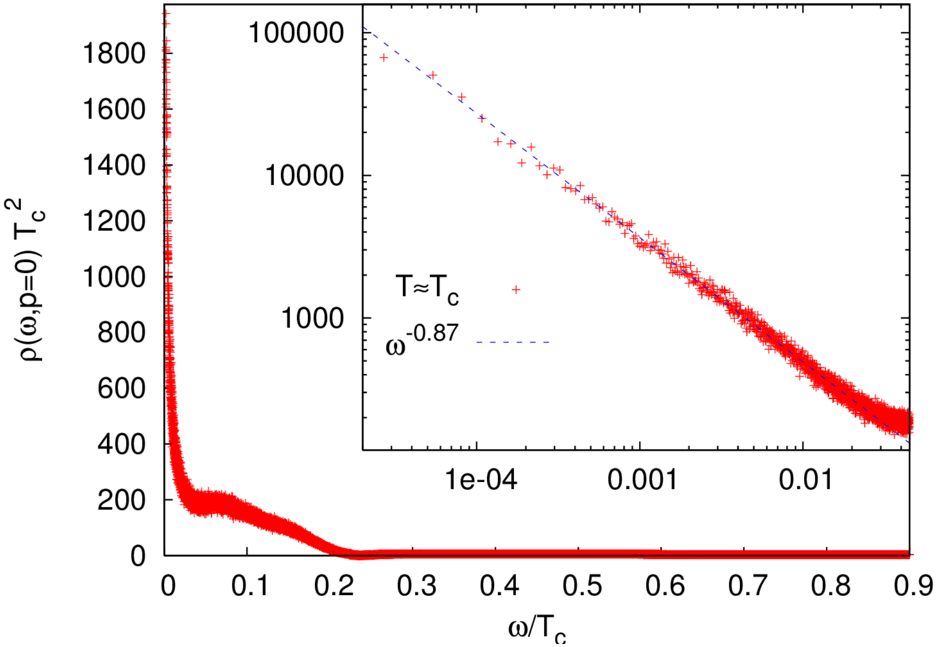
Initial conditions: Gaussian distribution for momenta
Field part is thermalized with the Metropolis algorithm.

Choose $m^2<0,\lambda,$ tune T to criticality



[Berges, Schlichting, Sexty (2009)]

Spectral function in Fourier space

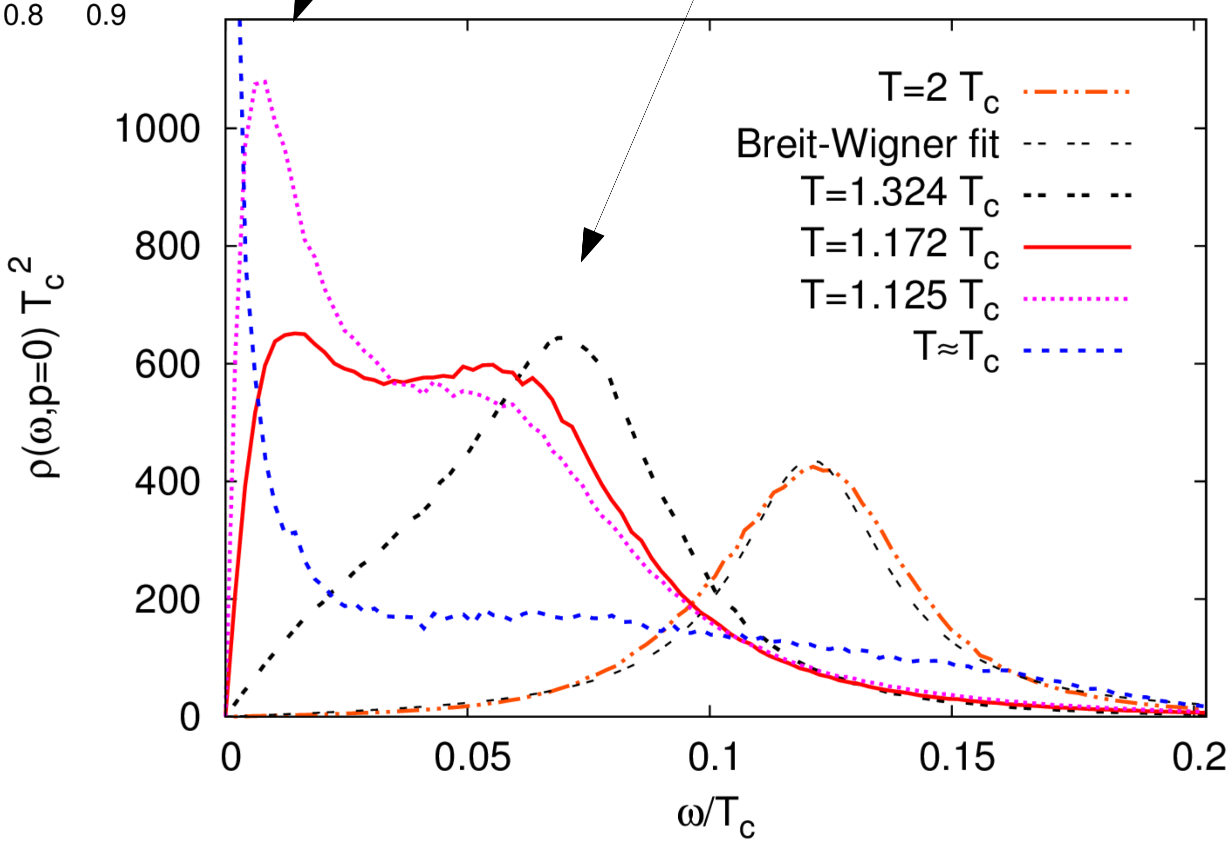


$$\frac{2-\eta}{z} \approx 0.87$$

$$z = 2.0 \pm 0.1$$

Critical peak

Thermal peak



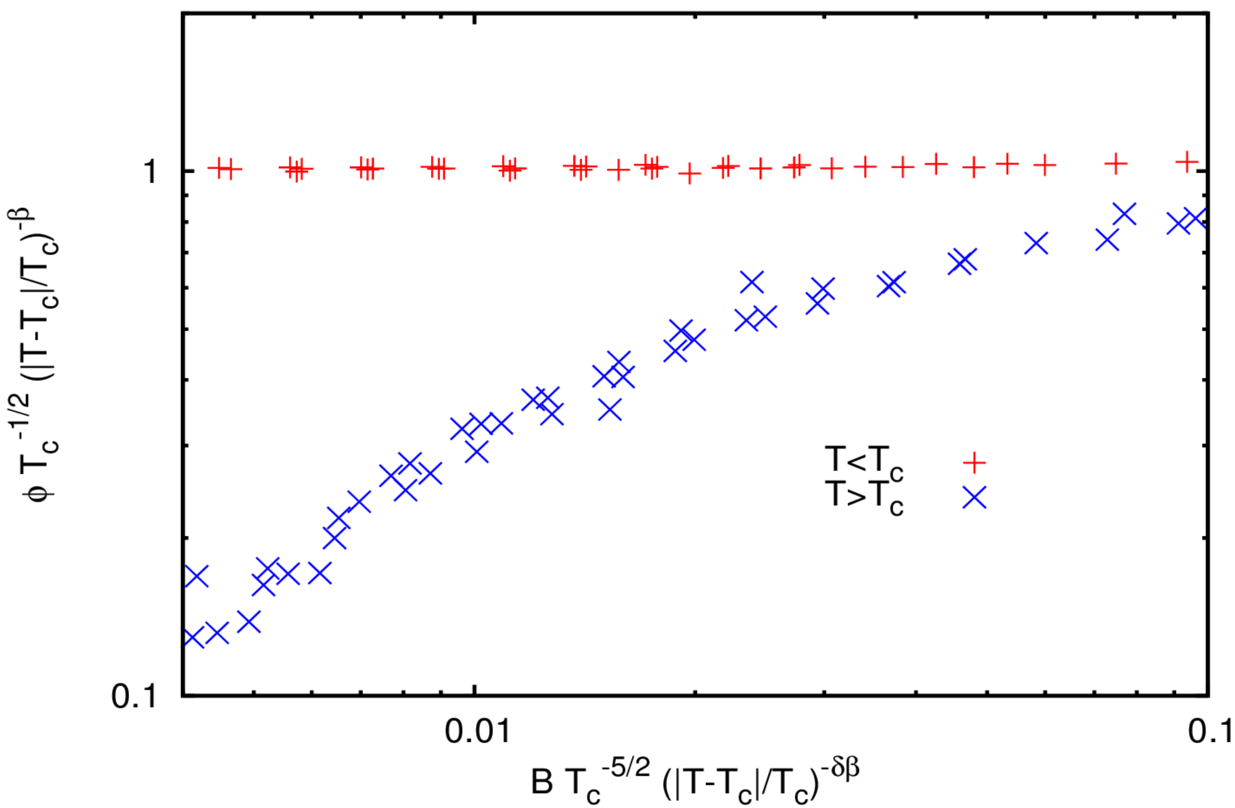
Checking for scaling

Introduce magentic field $$H_{new} = H + \int d^d x\, B\,\phi(x) = H + V\, B\, \Phi$$

$$\Phi(sT_r, s^{\beta\delta} B) = s^{\beta}\, \Phi(T_r, B)$$

set $s\, T_r = \mathrm{sgn}(T_r)$ $$\Phi(T_r, B) = \left|T_r^{\beta}\right| \Phi_{12}\left(B\left|T_r^{-\beta\delta}\right|\right)$$

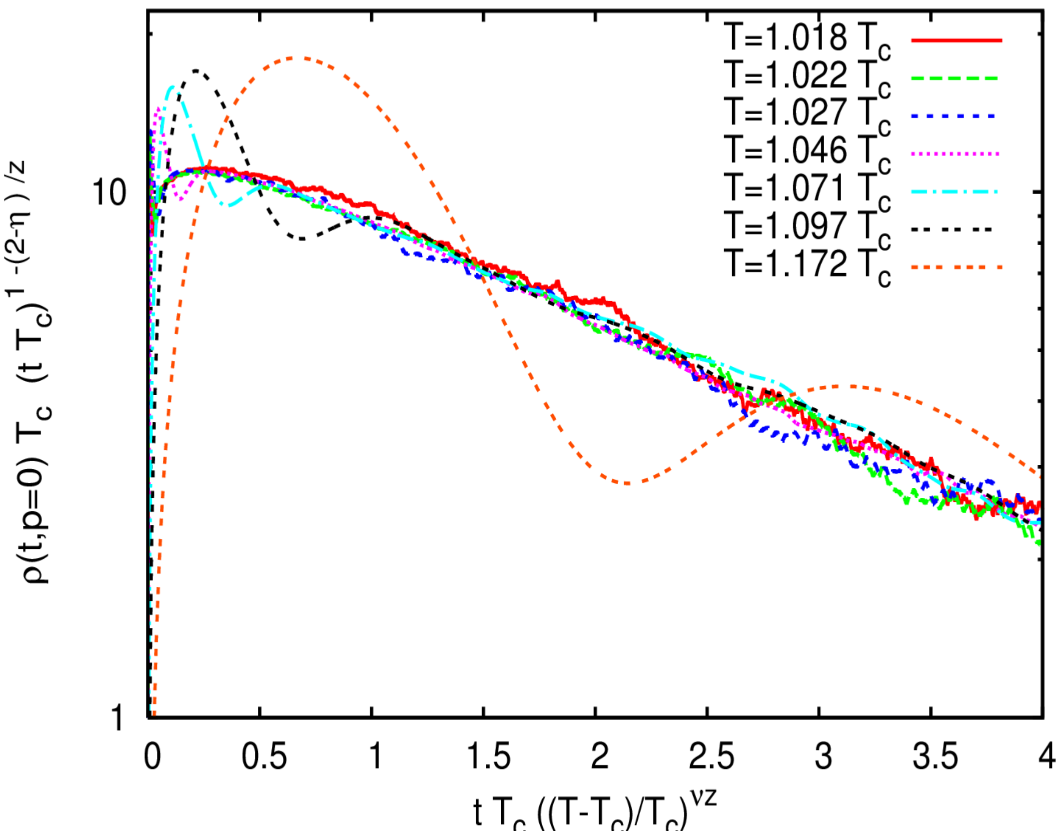
Onsager solution: $\beta = 1/8, \delta = 15, \nu = 1$



Dynamical scaling

$$\rho(t,0,T_r) \sim t^{\frac{2-\eta}{z}-1} g\left(\frac{t}{\xi_t}\right)$$

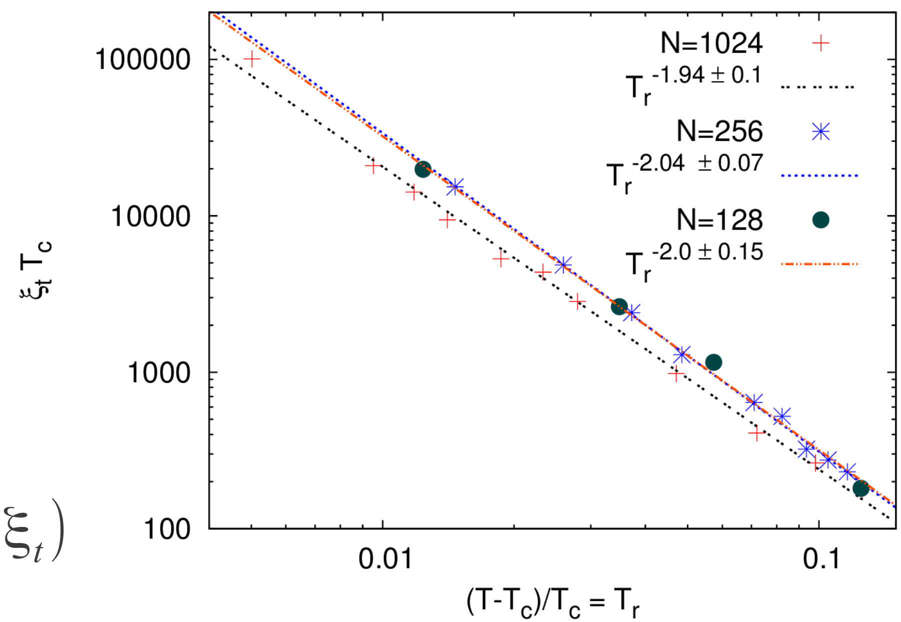
Scaling function is approached close to critical temperature



Fit z for best data collapse:

$$z = 2.05 \pm 0.15$$

$$\text{fit } \rho(t,0,T_r) \sim t^{2-\frac{\eta}{z}-1} \exp(-t/\xi_t)$$



Non-equilibrium lattice field theory

2nd Lecture

Dénes Sexty

Uni Heidelberg

Schladming Winterschool 2013

1. Classical approximation
2. QFT and classical statistical field theory
3. Real time fermions
4. Classical equilibrium and critical exponents
5. Cosmological applications
6. Gauge fields
7. Turbulent scaling, Dual cascade
   Non-equilibrium condensation

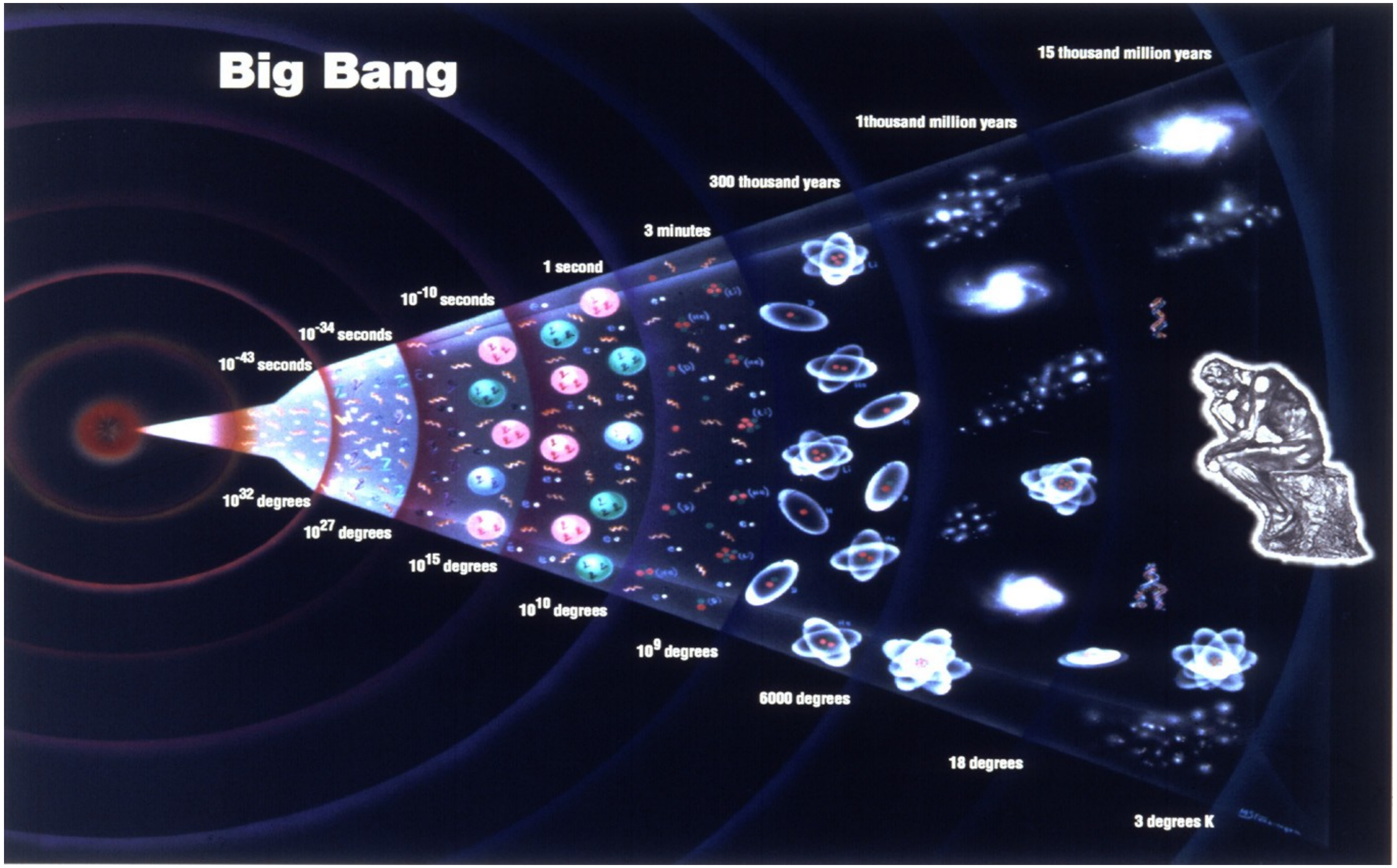
Cosmological applications

Inflation
slow roll

reheating
Far from equlibrium

Which mechanism?
Temperature?
Equilibration time?

Standard cosmology



Non-equilibrium scalar fields on expanding background

Cosmology: isotropic expansion

Expanding comoving metric $$ds^2 = dt^2 - a^2 dx^2$$

$$H = \frac{\dot{a}}{a}$$

$$\ddot{\phi} - 3H\dot{\phi} - \Delta\phi + m^2\phi + \frac{\lambda}{6}\phi^3 = 0$$

Friedmann equation $$H^2 = \frac{8\pi}{3\mathrm{m}_{pl}^2}\rho$$

Conformal time $$d\eta = dt/a(t),\, dx = dx_{phys}/a(t),\, \varphi = a(t)\phi$$

EOM simplifies $$\varphi'' - \Delta\varphi - \frac{a''}{a}\varphi + m^2 a^2\varphi + \frac{\lambda}{6}\varphi^3 = 0$$

Parametric resonance

[Kofman, Linde, Starobinsky (1994)]

End of inflation   Slow roll ends   Condensate starts oscillating

$$\Phi(x,t)=\bar{\Phi}(t)+\varphi(x,t)$$

$$\ddot{\bar{\Phi}}+\ddot{\varphi}-\nabla^2\varphi+\frac{1}{2}m^2(\bar{\Phi}+\varphi)+\frac{\lambda}{6}(\bar{\Phi}+\varphi)^3=0$$

EOM of zero mode: $$\ddot{\bar{\Phi}}+\frac{1}{2}m^2(\bar{\Phi})+\frac{\lambda}{6}(\bar{\Phi}^3)=0$$

EOM of fluctuations: $$\ddot{\varphi}_k+(m^2+k^2)\varphi_k+\frac{\lambda}{2}(\bar{\Phi}^2\varphi_k+\text{interactions})=0$$

$$m^2_{eff,k}=m^2+k^2+\frac{\lambda}{2}\bar{\Phi}^2$$

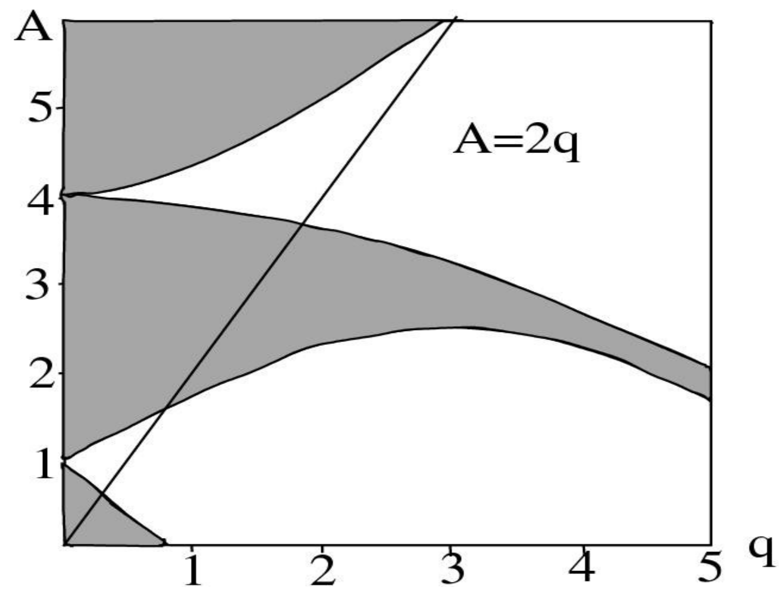
$$\Phi(t)\,\text{oscillates:}\;\;\Phi(t)=\Phi_0\sin(\omega_0 t)$$

Matthieu equation

$$\frac{d^2y}{dx^2}+[A-2\mathrm{q}\cos(2\mathrm{x})]y=0$$

Instability bands of Matthieu equation:

$$A = m^2 + k^2, \quad 2q = \frac{\lambda}{2}\Phi^2$$

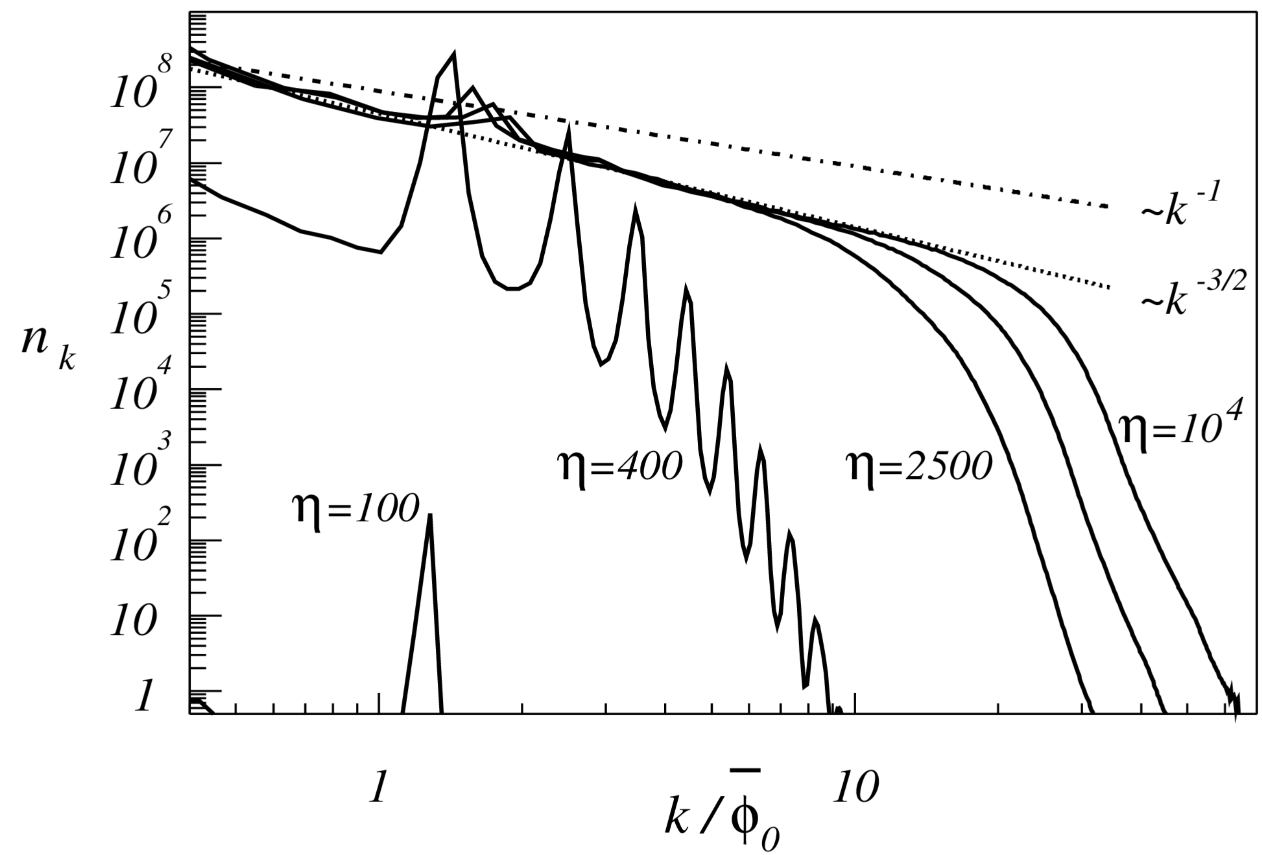


$2q > A$ Wide band parametric resonance

Resonant modes amplifyed

Scattering fills up
the whole IR

Distribution slowly evolves
towards equilibrium

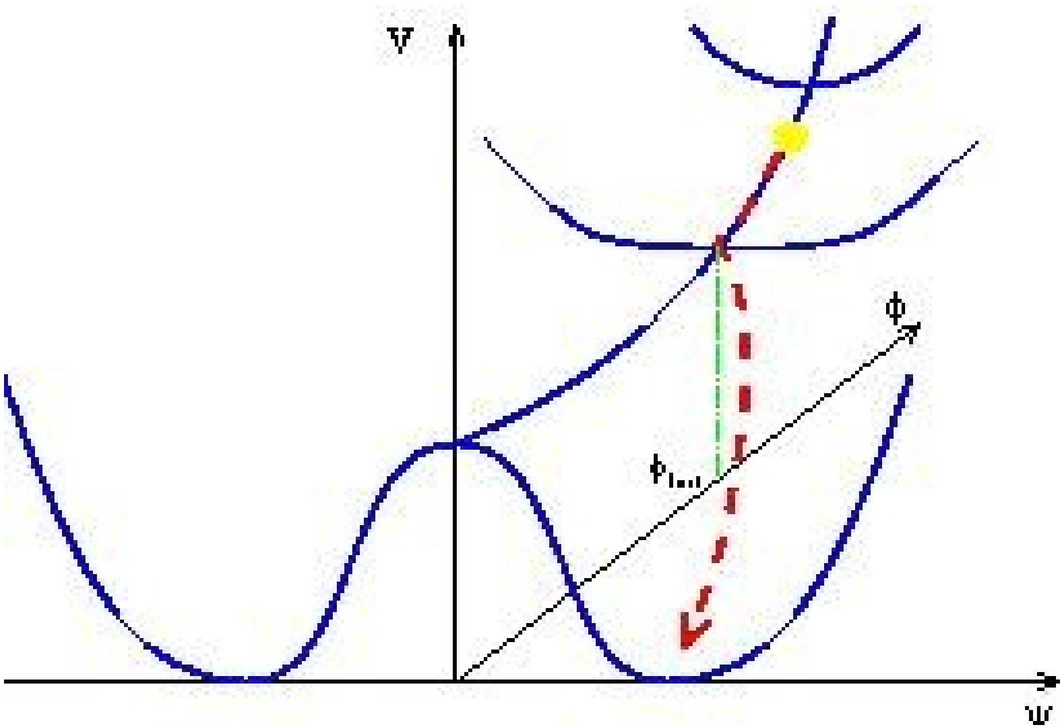


[fig: Micha, Tkachev (2004)]

Hybrid inflaton

$$S=\int_x \left(\frac{1}{2}(\partial_\mu \Psi \, \partial^\mu \Psi)+\frac{1}{2}(\partial_\mu \Phi \, \partial^\mu \Phi)-m^2\Psi^2-\frac{\lambda}{24}\Psi^4-\frac{1}{2}g\,\Phi^2\Psi^2+\frac{1}{2}m_\Phi \Phi^2\right)$$

[Linde (1994)]



Φ Inflaton

Ψ Higgs field

$m^2<0$

Simplification: fast quench

$$S=\int_x \left(\frac{1}{2}(\partial_\mu \Psi \, \partial^\mu \Psi)-m^2\Psi^2-\frac{\lambda}{24}\Psi^4\right)$$

At $t=0$ start from $\bar{\Psi}=0$

Tachyonic instability

EOM in fourier space neglecting interactions:

$$\ddot{\Psi}_k(t) = -(k^2 + m^2)\Psi_k(t)$$

$$\text{Instability for } k^2 < |m|^2: \ \Psi_k(t) \sim \exp\left(\left(\sqrt{|m|^2 - k^2}\right) t\right)$$

Saturation from coupling backreaction

$$\Psi(x,t) = \bar{\Psi}(t) + \psi(x,t) \qquad \frac{\lambda}{24}(\bar{\Psi} + \psi)^4 \rightarrow \left(\frac{\lambda}{4}\right)\bar{\Psi}^2 \psi^2$$

Mass term

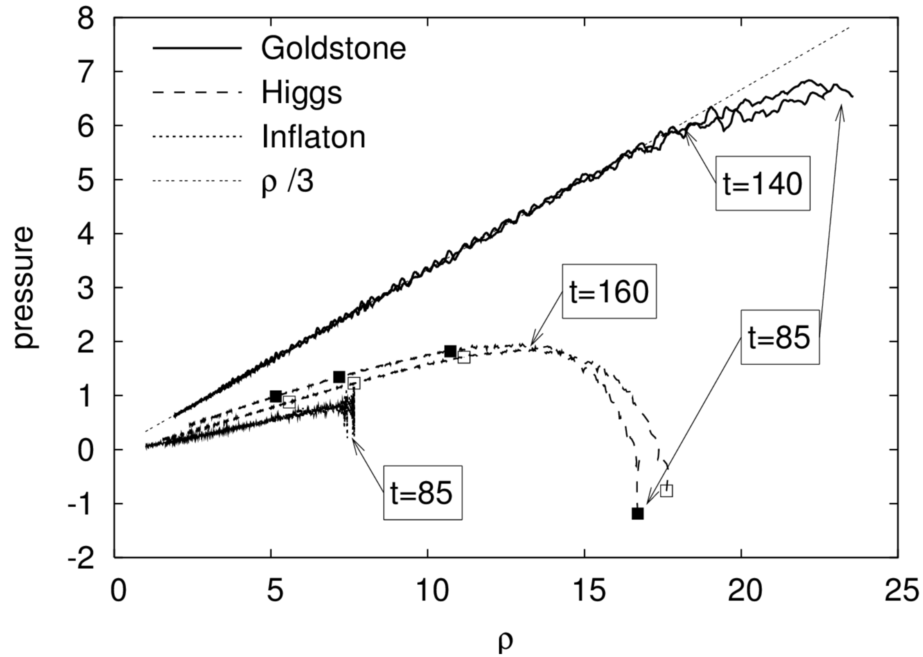
Saturation when $\lambda \bar{\Psi}^2 \sim |m|^2$

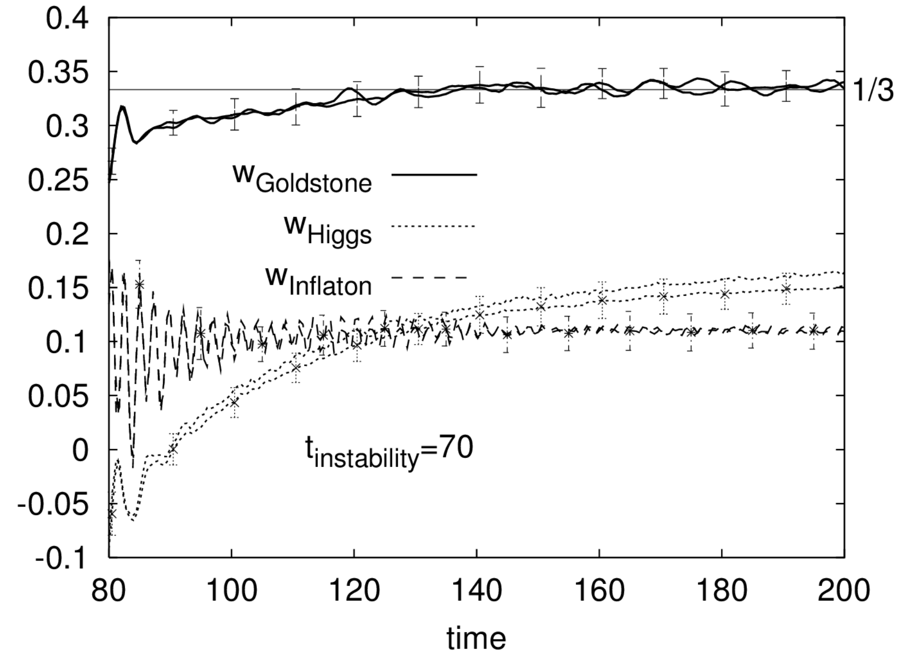
Particle number $n_k = \omega_k |\Psi_k|^2 \approx \omega_0 |\Psi_0|^2 \sim \omega_0 \frac{|m|^2}{\lambda}$

At the end of tachyonic instability

Overoccupation $n_k \sim \frac{1}{\lambda}$ for $k < |m|$

Tachyonic instability with O(2) Higgs field





Equation of state

$$p = w\,\epsilon$$

Equation of state is measured as the universe expands isotropically

Spectra far from equilibrium

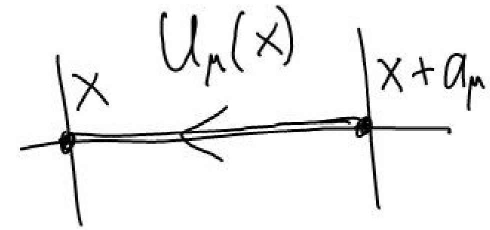
Well defined equation of state

Prethermalisation

[Borsányi, Patkós, Sexty (2003)]

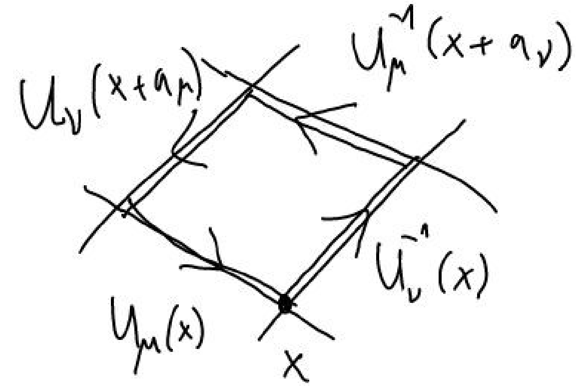
Gauge fields

Link variable: $U_\mu(x) = \exp(i\, a_\mu A_\mu(x))$



$$U \in SU(N) \qquad U = \exp(i\, \lambda_a \varphi_a)$$

Plaquette variable:



$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+a_\mu)\, U_\mu^{-1}(x+a_\nu)\, U_\nu^{-1}(x)$$

$$U_{\mu\nu} = \exp(-i\, F_{\mu\nu} a_\mu a_\nu) \qquad F_{\mu\nu} = g\, F_{\mu\nu}^a \frac{\lambda^a}{2}$$

$$U_{\mu\nu} + U_{\mu\nu}^{-1} = 2 - g^2 F_{\mu\nu}^a F_{\mu\nu}^b (a_\mu a_\nu)^2 \frac{\lambda_a \lambda_b}{4}$$

$$\frac{4\mathrm{N}}{g^2}\left(1 - \frac{\mathrm{Tr}(U_{\mu\nu} + U_{\mu\nu}^{-1})}{2\mathrm{N}}\right) = F_{\mu\nu}^a F_{\mu\nu}^a (a_\mu a_\nu)^2$$

asymetric lattice

$$a_i = a_s \qquad a_0 = a_t$$

Courant condition $\dfrac{a_t}{a_s} < 0.1$

Electric field squared   Magnetic field squared

$$S[U_\mu(x)] = \frac{2N}{g^2}\frac{a_s}{a_t}\sum_{\text{t Plaqs}}\left(1-\frac{\mathrm{Tr}(U_{\mu\nu}+U_{\mu\nu}^{-1})}{2N}\right) - \frac{2N}{g^2}\frac{a_t}{a_s}\sum_{\text{s Plaqs}}\left(1-\frac{\mathrm{Tr}(U_{\mu\nu}+U_{\mu\nu}^{-1})}{2N}\right)$$

Minkowski metric

EOM: $\frac{\delta S}{\delta U}=0$ group structure?

Left derivative: $D_a f(U) = \left.\frac{\partial f(e^{i\lambda_a \epsilon}U)}{\partial \epsilon}\right|_{\epsilon=0}$

$$D_a \mathrm{Tr}(UV) = i\,\mathrm{Tr}(\lambda_a U V)$$
$$D_a \mathrm{Tr}(U^{-1}V) = -i\,\mathrm{Tr}(V U^{-1}\lambda_a)$$

$D_{a,x,i} S[U_\mu(x)] = 0$ Equation of motion $\partial_t E = \dots, \partial_t B = \dots$

$D_{a,x,0} S[U_\mu(x)] = 0$ Gauss constraint $\mathrm{Div}\,E = \rho$

Gauge symmetry $U_\mu(x) \rightarrow g(x) U_\mu(x) g^{-1}(x+a_\mu)$ with $g(x) \in SU(N)$

For calculation of the dynamics, choose temporal gauge $U_0(x) = \mathbf{1}$

Any gauge can be fixed afterwards.

Coulomb gauge

Find local minimum of

$$M[g] = \sum_{x,i=1..3} 1 - \mathrm{ReTr}\, U_i^{(g)}(x) = \sum_{x,i=1..3} 1 - \mathrm{ReTr}\left(g(x) U_i(x) g^{-1}(x+a_i)\right)$$

Use $g(x) = h_a(x) = i\lambda_a \alpha$

$$\frac{dM}{d\alpha} = i \sum_j \left(Tr\left(U_j(x)\lambda_a\right) - Tr\left(U_j(x-a_j)\lambda_a\right)\right) = 0$$

Using $U_\mu(x) = \exp\left(i a_\mu A_\mu(x)\right)$ $\sum_j A_j(x) - A_j(x-a_j) = \mathrm{div}\, A = 0$

Search for the minimum using a relaxation algorithm

Overrelaxation, stochastic overrelaxation, Fourier acceleration ...

Equations of Motion

$$D_{a,x,i} S[U_\mu(x)] = 0$$

Define: $V_\mu(x) = U_{\mu 0} = U_\mu(x) U_\mu^{-1}(x + a_0)$

$$E_\mu^a(x) = -\frac{i}{2} \mathrm{Tr}(V_\mu(x) \sigma_a)$$

$$E_n^a(x) - E_n^a(x - a_0) = -\frac{i}{2} \frac{a_t^2}{a_s^2} \sum_{m \neq n} \Big(\mathrm{Tr}\, U_n(x) U_m(x + a_n) U_n^{-1}(x + a_m) U_m^{-1}(x) \sigma_a + \mathrm{Tr}\, U_n(x) U_m^{-1}(x + a_n - a_m) U_n^{-1}(x - a_m) U_m(x - a_m) \sigma_a \Big)$$

$$U_\mu(x + a_0) = V_\mu^{-1}(x) U_\mu(x)$$

To build the new link, we need inversion of $Tr(U \sigma^a)$

$$Tr(U \sigma^a) = b^a \;\rightarrow\; U = ?$$

For SU(2) this is easy:

$U = a + i\, b_i \sigma_i$ with $a^2 + b_i b_i = 1$ $\qquad Tr(U \sigma_a) = 2\mathrm{i}\, b_a$

For SU(3) there's no analytic formula
numerical solution with Newton iteration

Gauss Constraint

$$D_{a,x,0} S[U_\mu(x)] = 0$$

$$\sum_i Tr(V_i(x)\sigma_a) - Tr(\bar{V}_i(x-a_i)\sigma_a) = 0$$

$$\bar{V}_i(x) = U_i^{-1}(x+a_0) U_i(x)$$

Initial conditions: two timeslices

$$U_\mu(t_0) \quad U_\mu(t_0+a_0) \rightarrow E_\mu^a(x)$$

Must respect Gauss constraint

Simplest solution: $U_\mu(t_0+a_0) = U_\mu(t_0) \rightarrow E_\mu^a = 0$

dephasing

Otherwise use Gauss quenching

Minimize Gauss violation

fixed $U_\mu(t_0)$ change $E_\mu(t=0)$

Steepest descent

Gauss constraint fulfilled initially

Gauss constraint fulfilled at all times

Hamiltonian vs. Lagrangian formulation

Lagrange formulation:

Space time lattice with temporal and spatial links

EOM is calculated from the action: $$\frac{DS}{DU_\mu(x)}=0$$

Electric fields live in group space (represented by temporal plaquettes)

Needed: inversion of $Tr(U\lambda^a)$ $\qquad Tr(U\lambda^a)=b^a \;\rightarrow\; U=?$

Hamiltonian formulation

Space discretisation first: $$H(E^a_\mu, U_\mu)=\frac{1}{2}(E^a_\mu)^2+\beta\sum_{spatial\ plaq.} Tr(U)$$

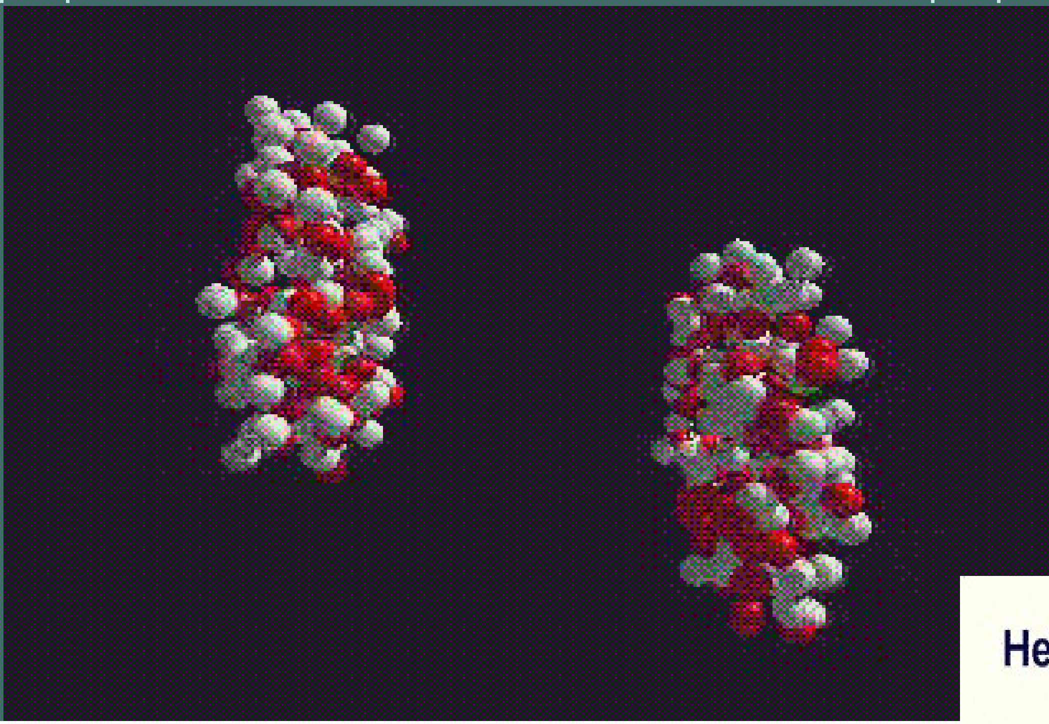
Then Hamiltonian EOM is discretised in time

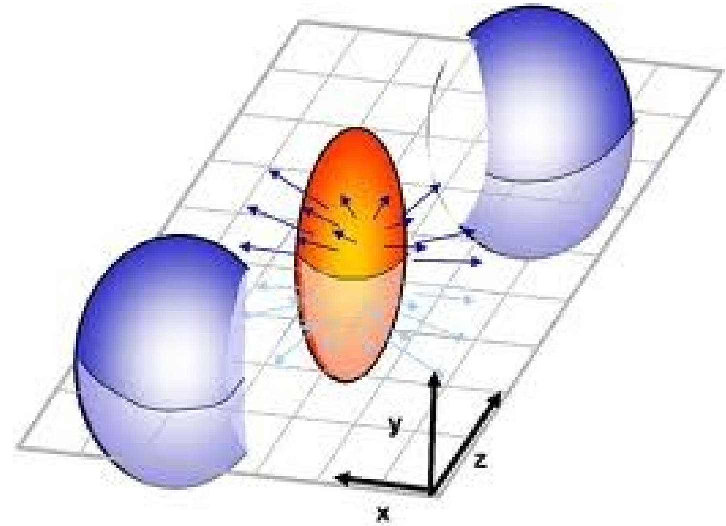
$$E^a_\mu(t+\Delta t/2)=E^a_\mu(t-\Delta t/2)-D_a H(E^a_\mu, U_\mu)$$

$$U_\mu(t+\Delta t)=\exp(i\Delta t\lambda_a E^a_\mu)U_\mu(t)$$

Electric fields live in Lie algebra space

Needed: matrix exponentialization



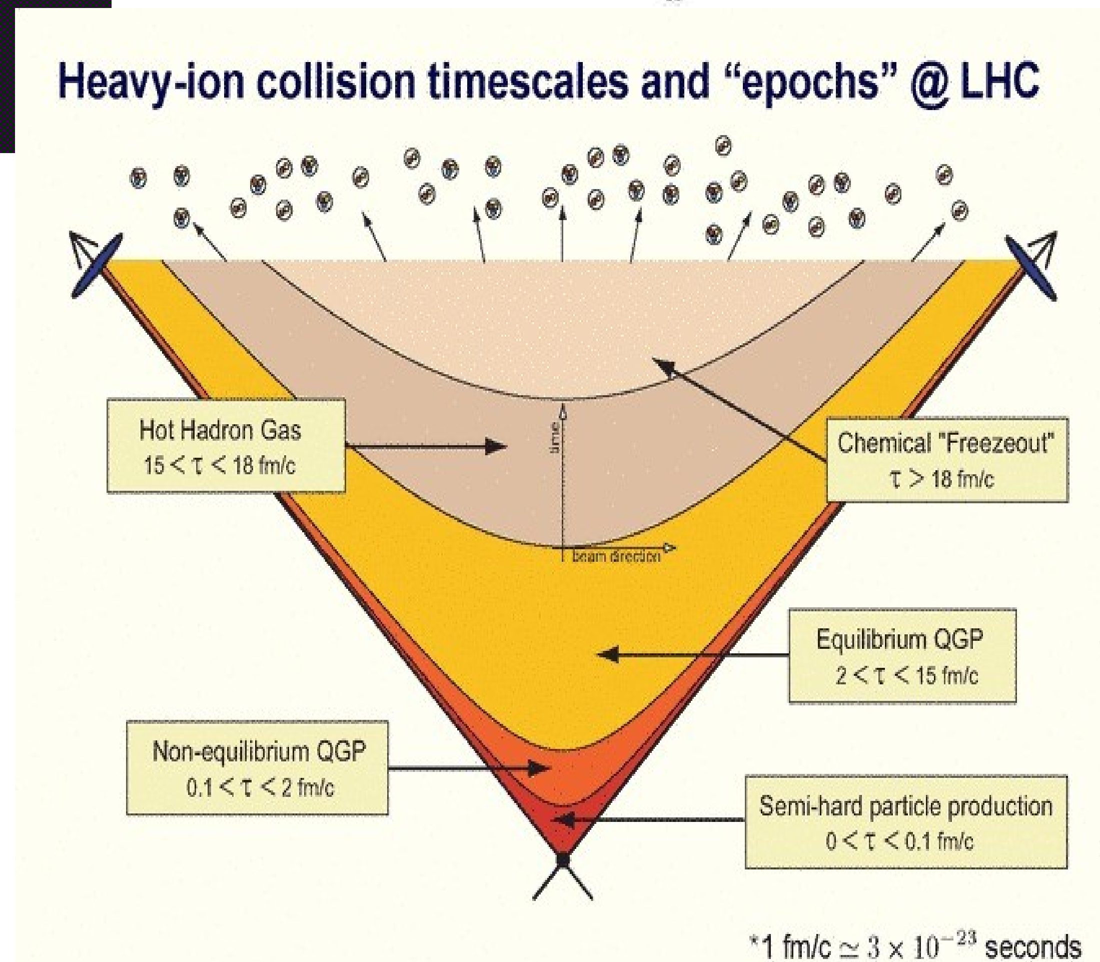


Initial occupation $\sim 1/\alpha_s$

Fermions are neglected

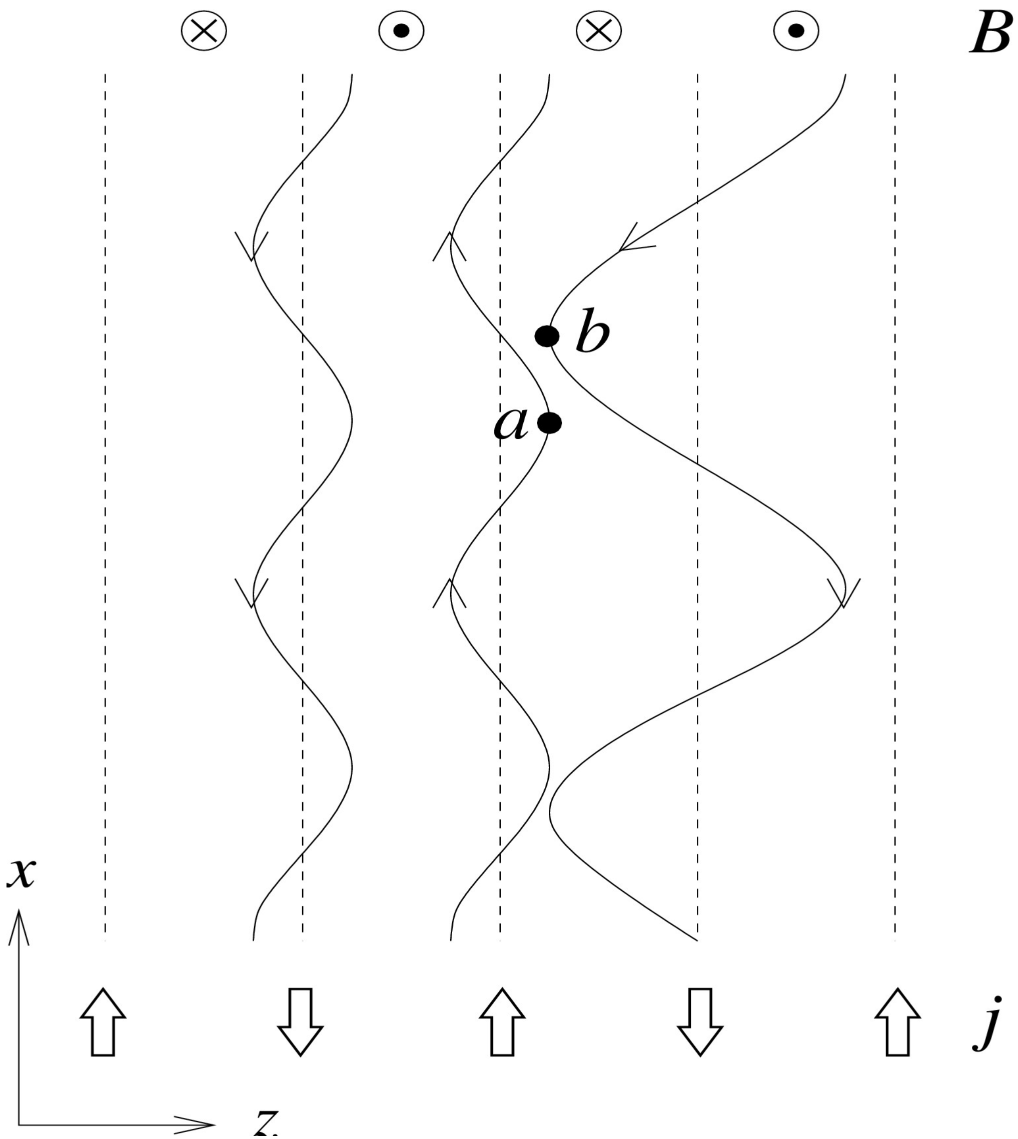
Large initial anisotropy

Fixed box vs longitudinal expansion



Plasma Instability (aka. Weibel Instability)

(Weibel, PRL 2(1959) 83, Mrowczynski, PRC 49 (1994) 2191)



anisotropic particle distribution

seed of magnetic waves

↓

The current is bent to amplify magnetic field (filamentation)

↓

Exponential growth of certain modes

Phenomena also exsits in pure gauge theories (without charged fermions)

Initial Conditions

anisotropic distribution: lots of particles in the transverse plane
few particles in the longitudinal planes

Gaussian initial configuration with:

$$\langle |A_i^a(\vec{k}, t=0)|^2 \rangle = C \exp\left(-\frac{k_x^2 + k_y^2}{2\Delta^2} - \frac{k_z^2}{2\Delta_z^2} \right) \qquad \Delta \gg \Delta_z$$

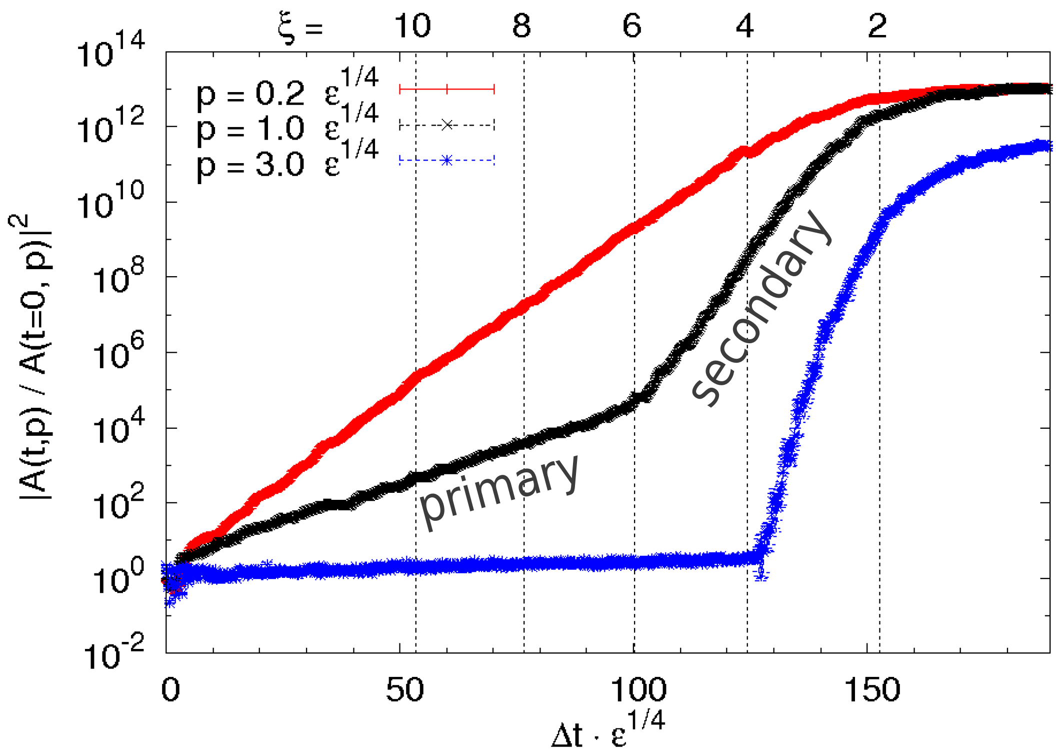
C is given by fixing energy density

zero initial momenta → Gauss constraint is trivially fulfilled

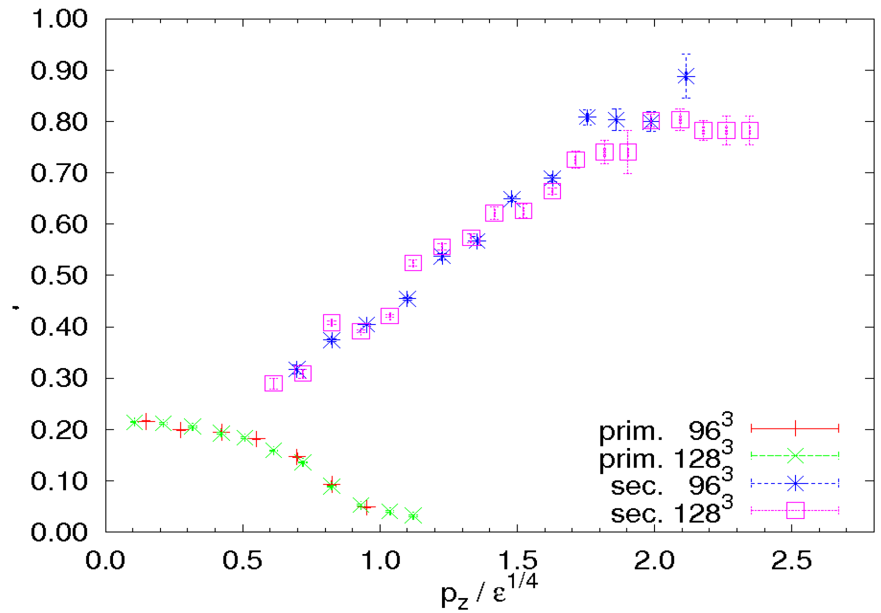
To avoid numerical problems, a small plateau is added
mimicing the quantum n=1/2

$$\langle |A_i^a(\vec{k}, t=0)|^2 \rangle = MAX\left(C \exp\left(-\frac{k_x^2 + k_y^2}{2\Delta^2} - \frac{k_z^2}{2\Delta_z^2} \right), \frac{Amp}{|\vec{k}|} \right) \qquad Amp \approx 10^{-12}$$

Results



Growth rates



Timescales from secondary rates:

$1/\gamma_{sec} \approx 0.3\,\mathrm{fm/c}$ optimistically ($\epsilon = 30\,\mathrm{Gev/fm^3}$)

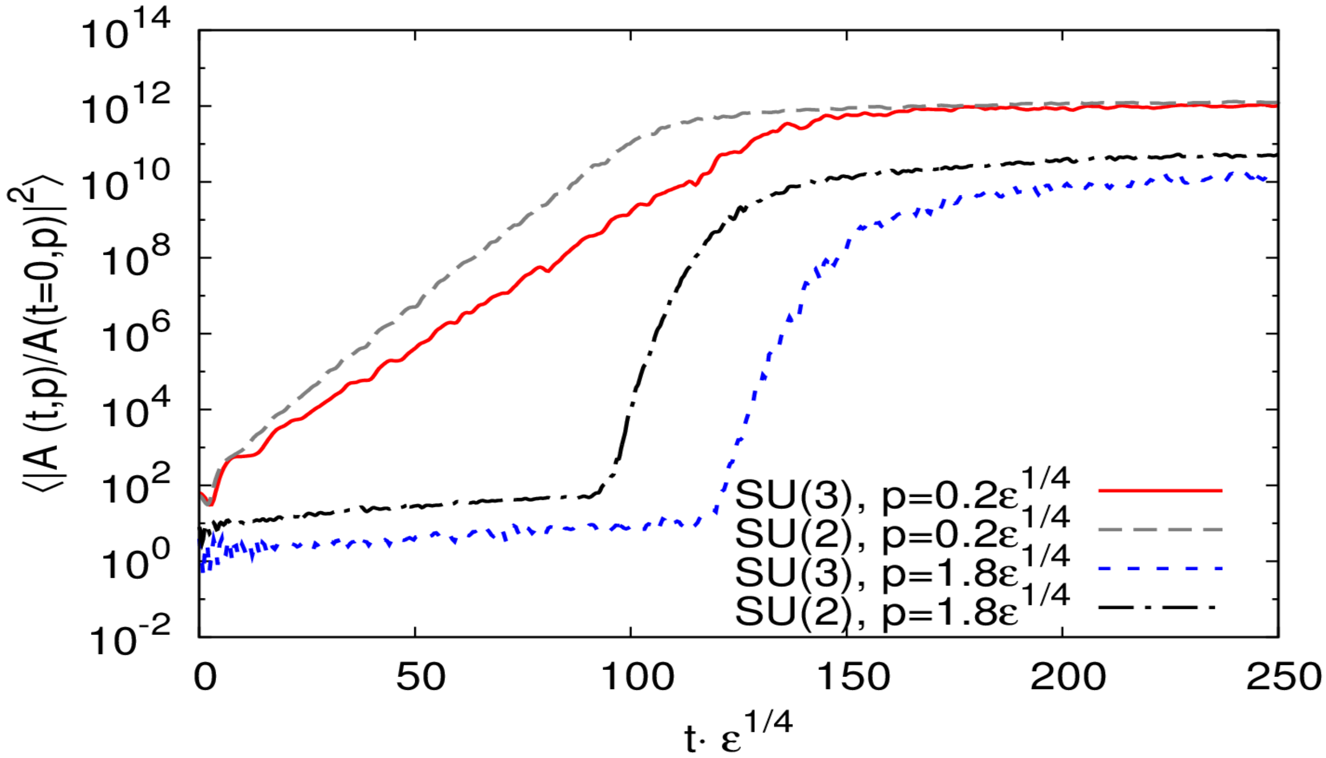
$1/\gamma_{sec} \approx 0.8\,\mathrm{fm/c}$ pessimistically ($\epsilon = 1\,\mathrm{Gev/fm^3}$)

Comparison of SU(2) and SU(3) results

Solving the EOM: $Tr(U\lambda^a) \rightarrow U = ?$

SU(2) is simple: $b_a = -\frac{1}{2} Tr(U\sigma^a) \qquad U = \sqrt{1 - b_a b_a} + i\, b_a \sigma_a$

SU(3) : Newton method is used for numerical solution

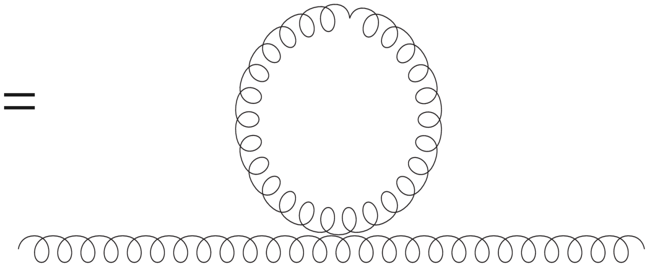


Berges, Gelfand, Scheffler, Sexty PLB (2009)

Explanation of slower growth

Initial growth of the longitudinal modes caused by tadpole

$m_T =$

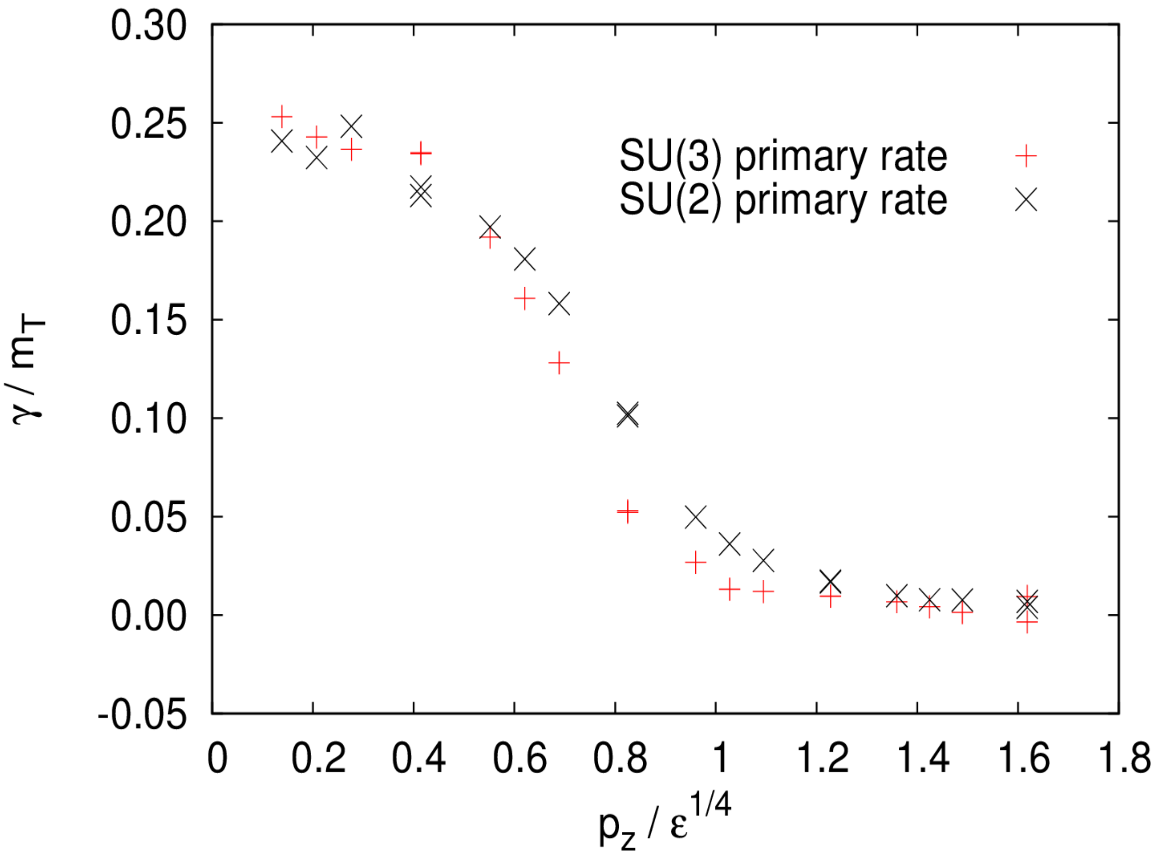


Other diagrams suppressed by the anisotropy ratio

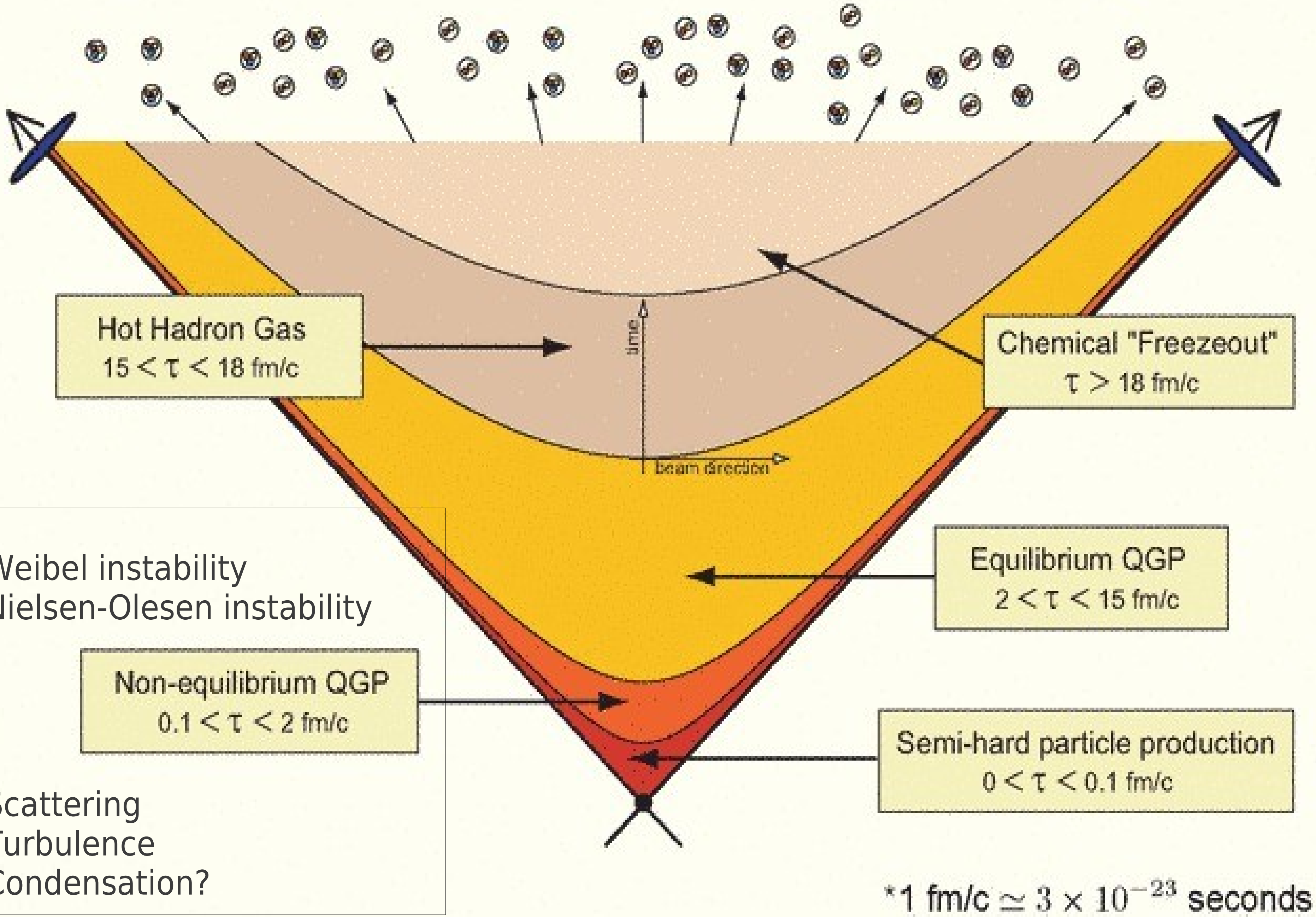
Energy density is kept fixed

$$m_T^2 \propto \frac{N}{N^2-1}$$

$$m_{T,SU(3)} = \frac{3}{4} m_{T,SU(2)}$$



Heavy-ion collision timescales and “epochs” @ LHC

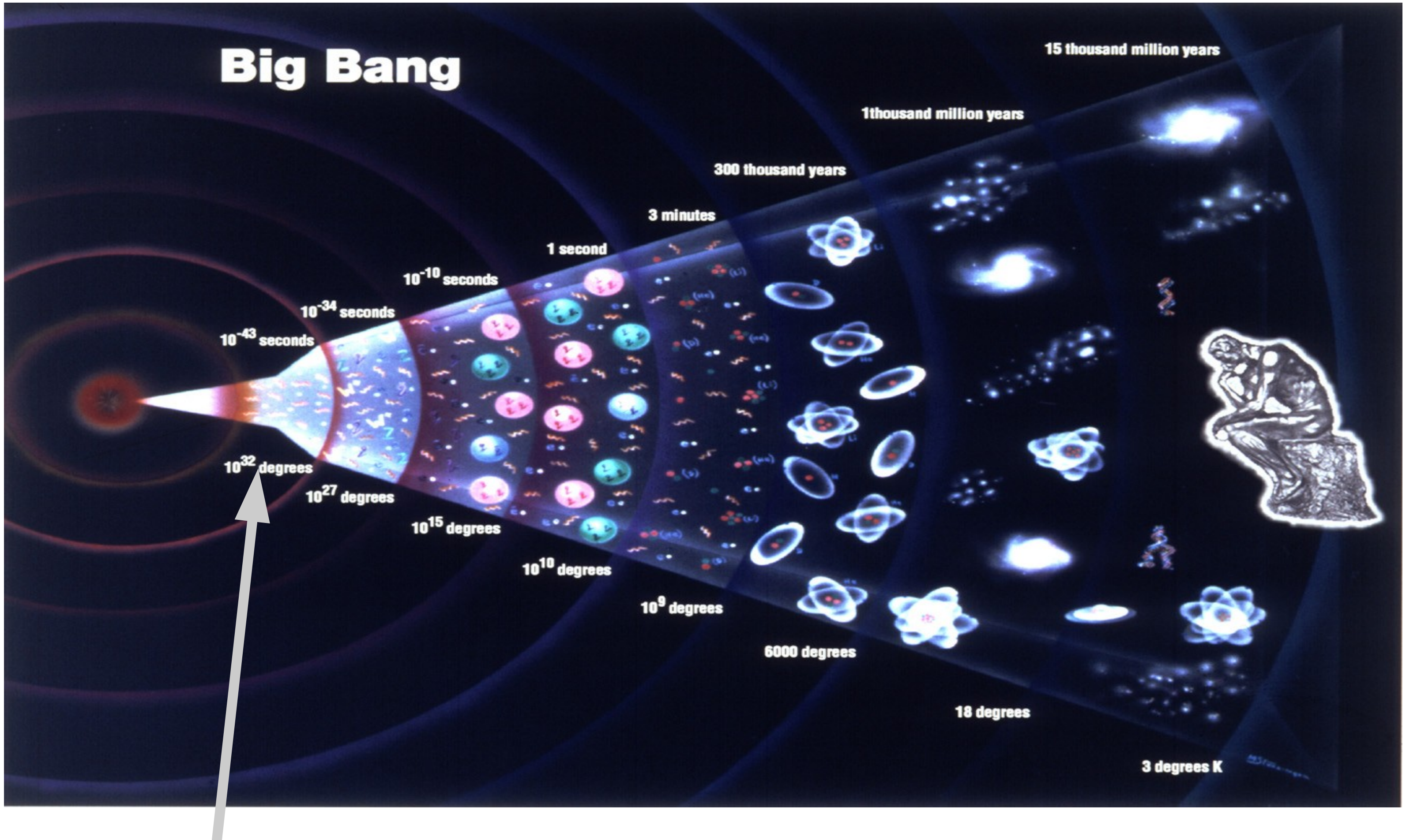


Weibel instability
Nielsen-Olesen instability

Scattering
Turbulence
Condensation?

[Fig.:Strickland]

Cosmological reheating



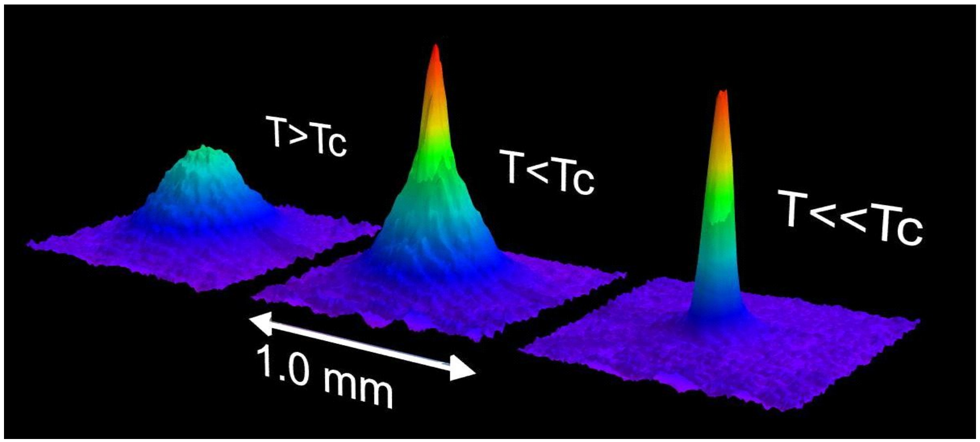
Inflation → reheating → standard cosmology

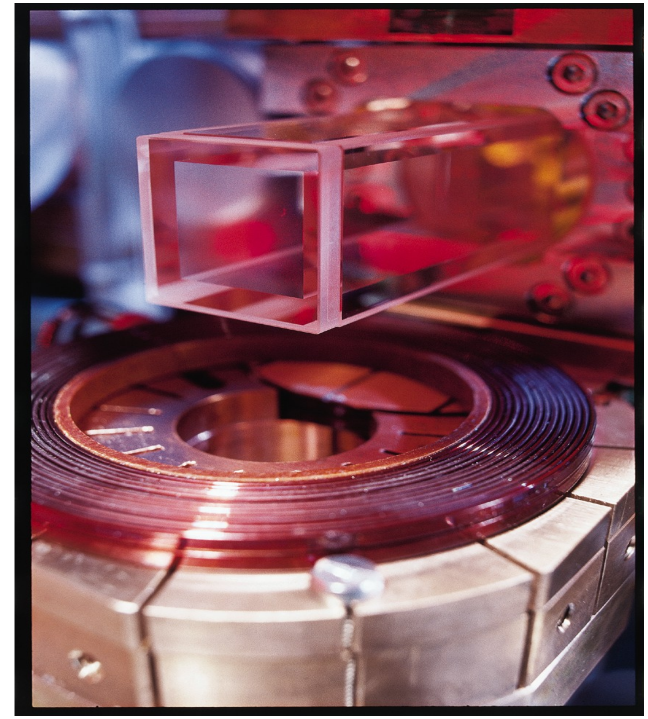
Conversion of vacuum-energy to particles

Parametric resonance
Tachyonic instability → Overoccupation → Turbulence → Thermal eq.

Ultracold Atoms

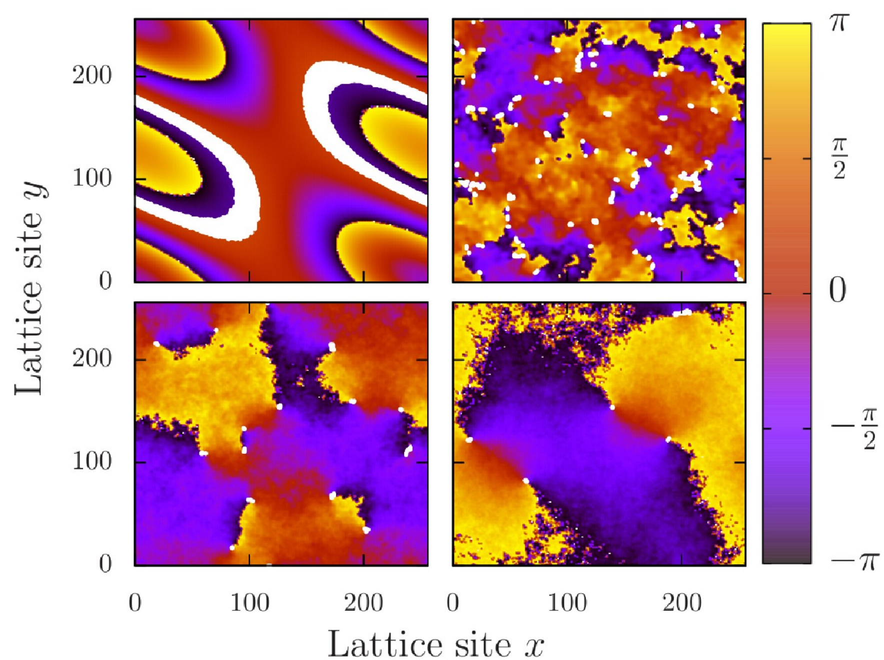
Bose-Einstein Condensation
In equilibrium





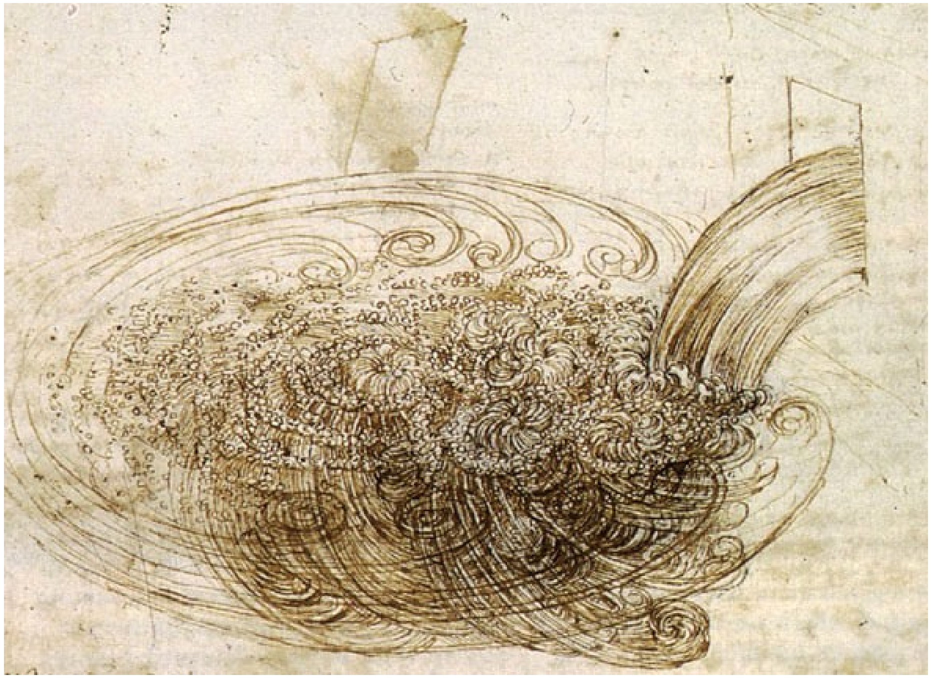
Excite at low momenta

Topological defects
Quantum turbulence



[Nowak,Sexty,Gasenzer (2010)]

Turbulence



Kinetic energy cascade

Large scales → small scales

Richardson (1920)

"Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

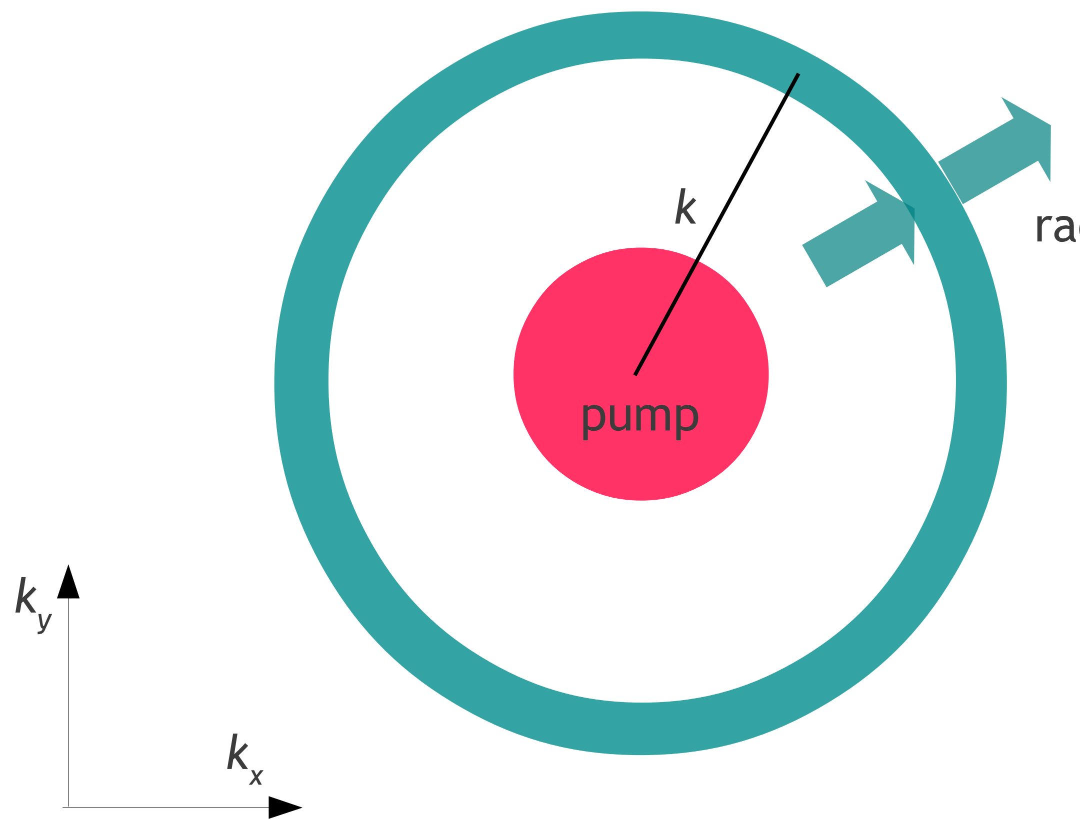
Kolmogorov (1941) Turbulent stationarity

Scaling law for radial energy density: $E(k) \sim P^{2/3} \rho^{1/3} k^{-5/3}$

Kolmogorov turbulence

Energy pumped into the system in IR

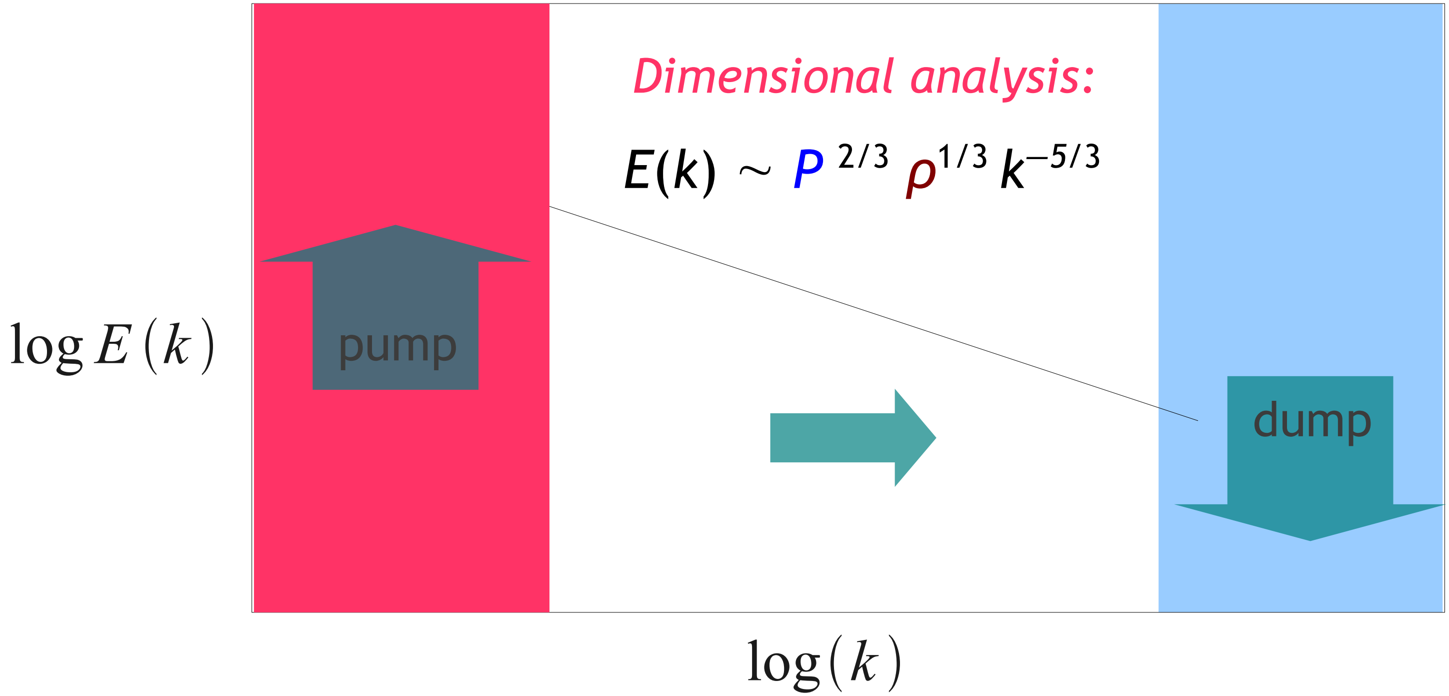
Scale invariant flow 2 types(at least): constant energy flow
constant particle number flow



$$\partial_t \epsilon(k) = -\nabla_k \boldsymbol{p}(k)$$

Energy density : ϵ
Energy flux : $\boldsymbol{p}$

Kolmogorov turbulence



"Local" evolution in k space

Stationary state: k independent energy flow

Nonequilibrium, but stationary dynamics

Driven turbulence

No source → free turbulence

only parameters present:

Radial energy density	E	[kg s^{-2}]
Radial energy flux	P	[kg m^{-1} s^{-3}]
Density	ρ	[kg m^{-3}]

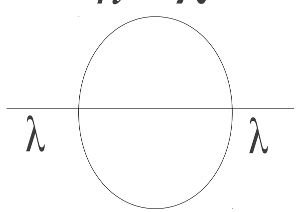
$E(k) = \text{const} \rightarrow \text{Power laws}$

“Natural” nonequilibrium initial conditions

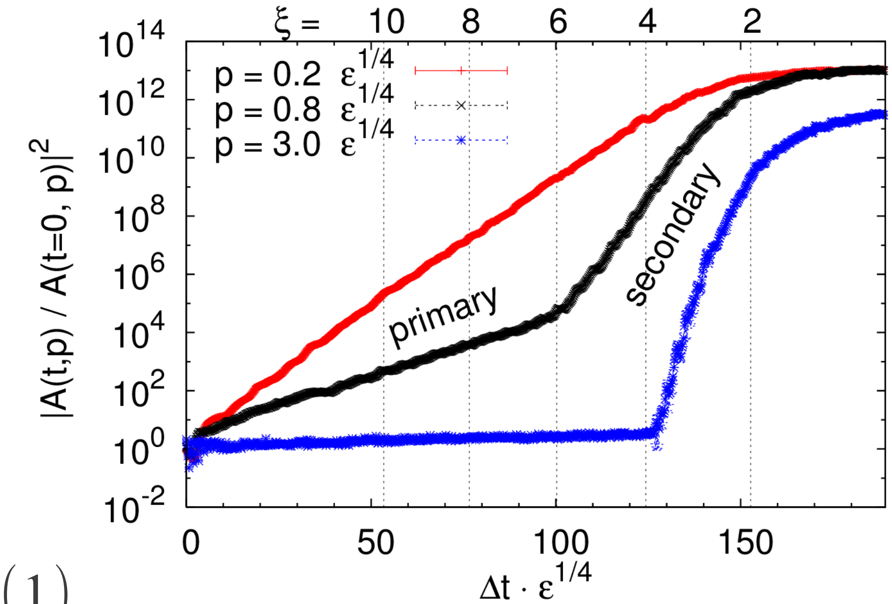
Instability for some momentum band

Amplitude grows

$n \sim \lambda^{-2/3}$

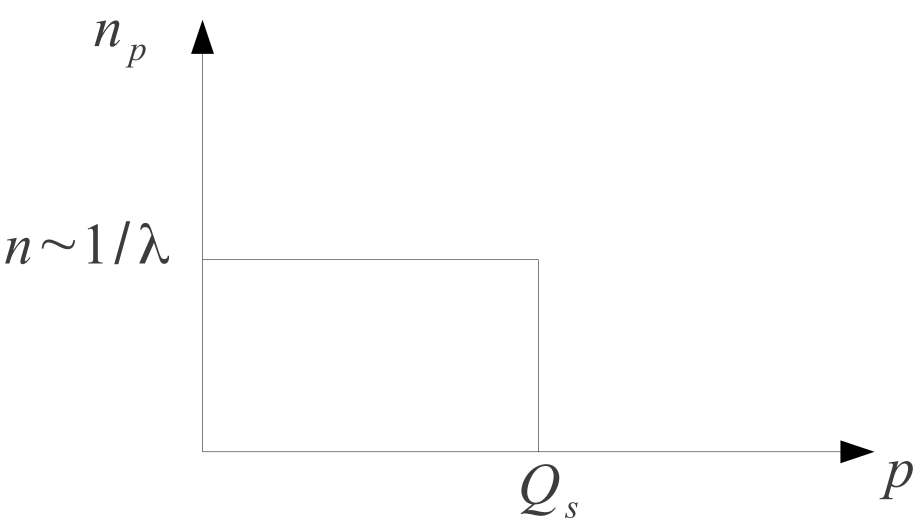


Secondary modes



At $n \sim 1/\lambda$ all diagrams become $\sim O(1)$

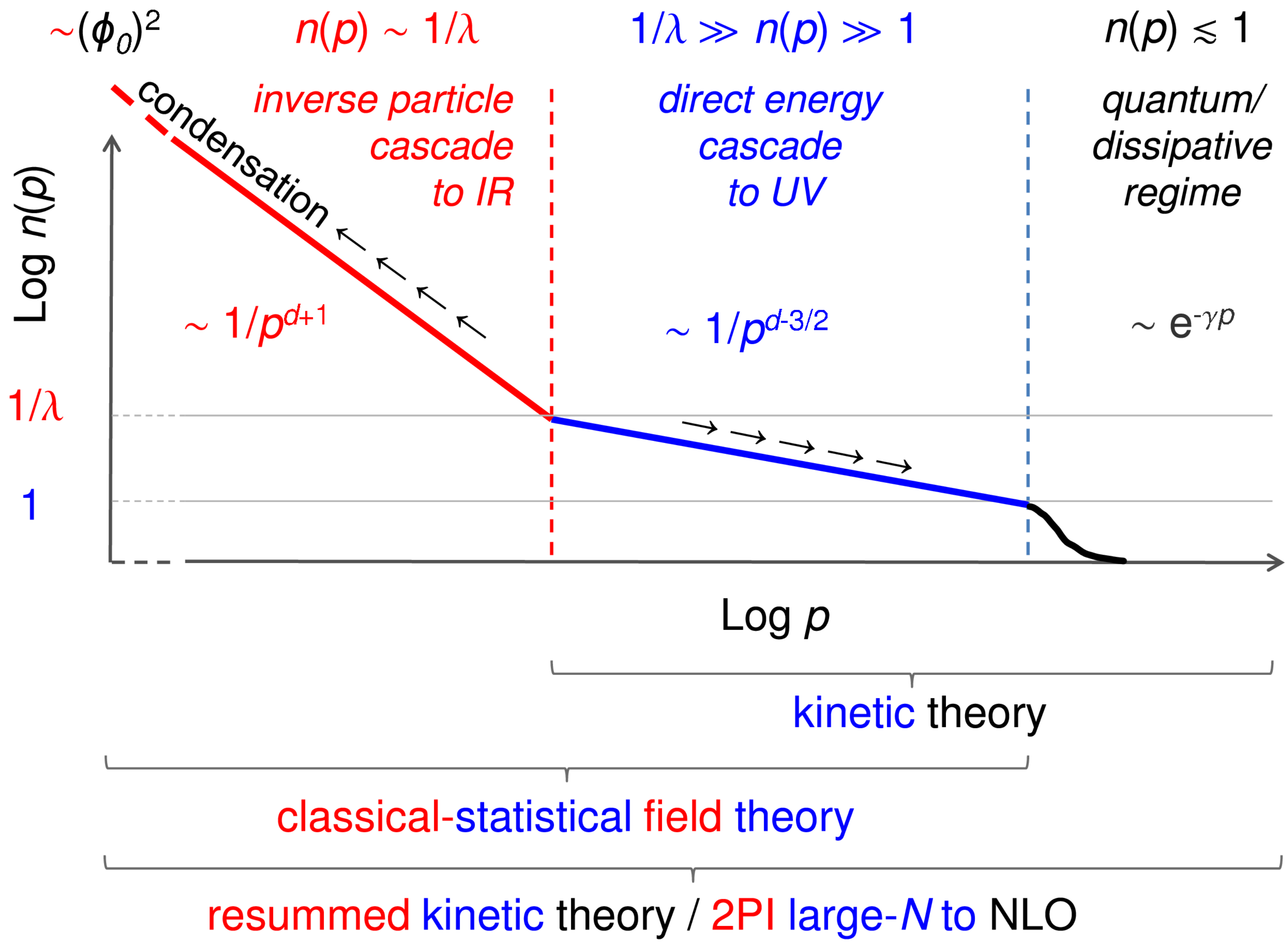
Non-linear physics; instability saturates



Overoccupation

Tachyonic reheating
Parametric reheating
Weibel instability
CGC

Turbulence and condensation in scalar field theories



Cascade in wave turbulence

Boltzmann equation:

$$\frac{\partial n_p}{\partial t} = C(t, p) = gain - loss$$

By integration one finds the flux in momentum space

Power law ansatz + constant flux ⟶ exponents

Local interactions in Fourier space:

$$\partial_t(\omega n_{\boldsymbol{p}}) + \nabla_p j_{\boldsymbol{p}}^{ener} = 0$$

Energy cascade

$$\partial_t n_{\boldsymbol{p}} + \nabla_{\boldsymbol{p}} j_{\boldsymbol{p}}^{part} = 0$$

Particle number cascade

Weak wave turbulence

$$\partial_t n_{\boldsymbol{p}} + \nabla_{\boldsymbol{p}} j_{\boldsymbol{p}}^{part} = 0$$

$$\frac{\partial n_p}{\partial t} = C(t,p) = gain - loss$$

Collision integral

$$C = \int d\Omega^{2\Leftrightarrow 2}(p,l,q,r)[(1+n_p)(1+n_l)n_q n_r - n_p n_l (1+n_q)(1+n_r)]$$

Classical regime: $n \gg 1$

$$C = \int d\Omega^{2\Leftrightarrow 2}(p,l,q,r)[(n_p+n_l)n_q n_r - n_p n_l (n_q+n_r)]$$

Integrated on a sphere = flux

$$\int_0^k d^d p\, \partial_t(\omega_p n_{\boldsymbol{p}}) \propto A(k) = \int^k dp\, p^{d-1}\omega_{\boldsymbol{p}} \partial_t n_{\boldsymbol{p}}(t)$$

Using the scaling ansatz $n_{\boldsymbol{p}} \propto p^{-\kappa_W}$ $\omega_{\boldsymbol{p}} \propto p$

Plug in collision integral: $A(k) \propto k^{d+1-3\kappa+2\mathrm{d}-5}$

$$C=\int d\Omega^{2\Leftrightarrow 2}(p,l,q,r)[(n_p+n_l)n_q n_r-n_p n_l(n_q+n_r)]$$

$$d\Omega^{2\Leftrightarrow 2}(p,l,q,r)=\lambda^2\frac{(N+2)}{18N}\int_{lqr}(2\pi)^{d+1}\delta^{(d)}(p+l-q-r)$$

$$\delta(\omega_p+\omega_l-\omega_q-\omega_r)\frac{1}{\omega_p\omega_l\omega_q\omega_r}$$

$$d\Omega^{2\Leftrightarrow 2}(sp,sl,sq,sr)=s^{(3d-4)-(d+1)}d\Omega^{2\Leftrightarrow 2}(p,l,q,r)$$

Using the scaling ansatz $n_p \propto p^{-\kappa_W}$ $\omega_p \propto p$

Plug in collision integral: $A(k)=\int^k dp\, p^{d-1}\omega_p C(t,p)\sim k^{d+1-3\kappa+2d-5}=k^0$

Well known Zakharov results $\kappa_W^{ener}=d-\frac{4}{3}$ $\kappa_W^{part}=d-\frac{5}{3}$

Similarly for interaction with 3-vertex: $\kappa_W^{ener}=d-\frac{3}{2}$ $\kappa_W^{part}=d-1$

Turbulence = Universality

Weak wave turbulence

Power law distributions $n(k) \sim k^{-\kappa}$

Self similar evolution: $n(k,t) \sim t^{-\gamma p} n_0(k t^{-p})$

Fluctuation decay $\langle \varphi^2 \rangle_{conn} \sim t^{-\nu}$

[Micha, Tkachev (2004)]

$\dot{n}(k) = I[n(k)]$

Interaction through m-vertex

k-independent stationary flow

Particle cascade
Energy cascade

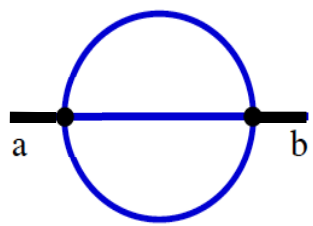
$\mu = d(m-2) - 1 - m$ $\kappa = \frac{d+\mu}{m-1}$ $\gamma = d$ $p = \frac{1}{2m+1}$ $\nu = \frac{2}{2m-1}$

Universality far from equilibrium

Scaling beyond kinetic theory

Obtaining the scaling solution analytically using 2PI

$$\partial_t n(p) = \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p)$$

$\Sigma_{ab}(x,y) =$ 

$p = (p_0, \mathbf{p})$:

Scaling ansatz for the spectral function (commutator) $\rho(x,y) = \langle [\varphi(x), \varphi(y)] \rangle$

$$\rho(s^z \omega, s\, p) = s^{-2+\eta} \rho(\omega, p)$$

Statistical function (anticommutator) $F(x,y) = \langle \{\varphi(x), \varphi(y)\} \rangle$

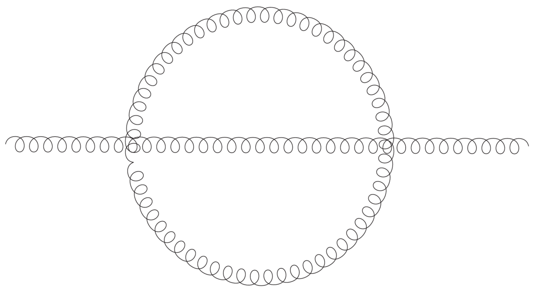
$$F(s^z \omega, s\, p) = s^{-2-\kappa} F(\omega, p)$$

Stationarity condition

$$J(p) = \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p) \rho_{ba}(p) = 0$$ → Scaling laws

Classicality condition $F(p) \gg \rho(p)$

Scaling analysis with sunset diagram



$$\Sigma(p)=\int_{qkl} G(q)G(k)G(l)\delta^{(4)}(p+q+k+l)$$

Classical part of the stationarity condition:

$$0=\int_{\mathbf{p}qkl} V(p,q,k,l)^2\delta^{(4)}(p+q+k+l)\big[F(p)F(q)F(k)\rho(l)+F(p)F(q)\rho(k)F(l)+F(p)\rho(q)F(k)F(l)+\rho(p)F(q)F(k)F(l)\big]$$

Zakharov transformation:

swapping momenta

$$l'=\xi p;\quad p'=\xi l;\quad k'=\xi k;\quad l'=\xi l$$

$$F(p)F(q)F(k)\rho(l)\Rightarrow\rho(p)F(q)F(k)F(l)$$

$$0=\int_{pqkl} V(p,q,k,l)^2\delta^{(4)}(p+q+k+l)\rho(p)F(q)F(k)F(l)\left[1+\left|\frac{p_0}{q_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{q_0}\right)+\left|\frac{p_0}{k_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{k_0}\right)+\left|\frac{p_0}{l_0}\right|^{\Delta}\operatorname{sgn}\left(\frac{p_0}{l_0}\right)\right]$$

Solutions:

$\Delta=-1$

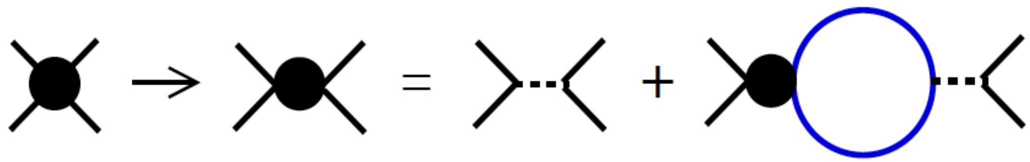
$\Delta=0$ On shell limit 2->2 dominates

$$\kappa=\frac{5}{3} \text{ and } \kappa=\frac{4}{3}$$

$$\Sigma = \frac{\partial \Phi(G)}{\partial G} \qquad J(p) = \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p) = 0$$

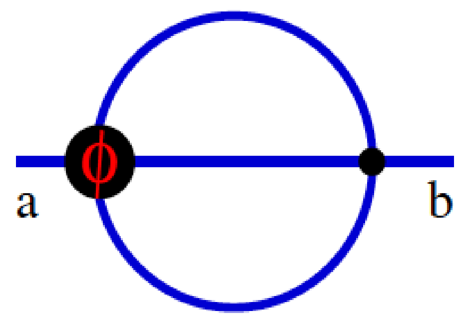
Loop expansion of $\Phi(G)$ ⟶ Boltzmann scaling $\kappa=4/3, \kappa=5/3$ (in 3D)

2PI to NLO in $1/N$: Vertex:



[J. Berges, NPA **699** (02) 847; G. Aarts et al., PRD **66** (02) 45008]

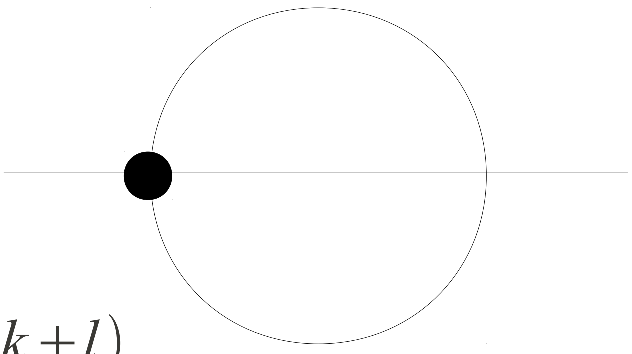
$$\Sigma_{ab}(x,y) =$$



Resummation of bubble chain ⟶ IR scaling $\kappa=4, \kappa=5$ (in 3D)

IR resummation – Strong turbulence

1/N resummation: effective vertex



$$\Sigma(p)=\int_{kql}\lambda_{eff}(p+q)G(q)G(k)G(l)\delta^{(4)}(p+q+k+l)$$

$$\lambda_{eff}(p)=\frac{\lambda}{(1+\Pi^R(p))(1+\Pi^A(p))}$$

With one loop bubble:

$$\Pi(p)=\int_q G(p)G(p-q)$$

In the IR: $\Pi(p)\gg 1$

The vertex scales:

$$\lambda_{eff}(s\,p)=s^{2r}\lambda_{eff}(p)\text{ with } r=3+\kappa-d$$

In the UV: $\lambda_{eff}=\lambda$

$$\lambda(sp)=\lambda(p)$$

Strong turbulence in the IR:

$\kappa=4$ or 5 (in d=3)

From 2PI to kinetic equations

Using Wigner coordinates

$$F_p(X)=\int d^4s\,\exp(-ip_\mu s^\mu)\,F(X+s/2,X-s/2)$$

Gradient expansion, spatially homogeneous ensemble:

$$\partial_t \rho_p(X)=0$$

$$2p_0\partial_t F_p(X)=\Sigma_p^\rho(X)F_p(X)-\Sigma_p^F(X)\rho_p(X)$$

Define:

$$F_p(X)=(n_p(X)+1/2)\rho_p(X)$$

$$n_{eff}(t,\boldsymbol{p})=\int_0^\infty \frac{dp_0}{2\pi}2p_0\rho_p(X)n_p(X)$$

On-shell limit, only 2->2 contributes

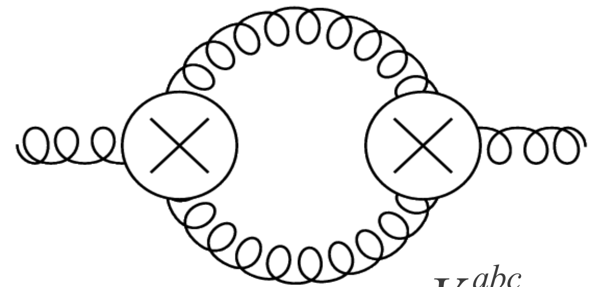
$$\partial_t n_{eff}(t,\boldsymbol{p})=\int d\Omega_{2\to 2}\left[(1+n_p)(1+n_l)n_q n_r-n_p n_l(1+n_q)(1+n_r)\right]\lambda_{eff}(p+l)$$

Effective kinetic description also valid at $\lambda n \gtrsim 1$

[Berges, Sexty (2011)]

Weak wave turbulence in gauge theories

Lowest order contribution
to self energy:



$$V^{abc}_{\mu\nu\gamma} \sim g$$

$$V^{abc}_{\mu\nu\gamma} = V^{abc}_{0,\mu\nu\gamma} + V^{abc}_{A,\mu\nu\gamma}$$

$$V^{abc}_{0,\mu\nu\gamma}(p,q,k) = g f^{abc}\left(g_{\mu\nu}(p-q)_\gamma + g_{\nu\gamma}(q-k)_\mu + g_{\gamma\mu}(k-p)_\nu\right)$$

$$V^{abc}_{A,\mu\nu\gamma}(x,y,z) = \left(C_{ac,bd}\, g_{\mu\nu} A^d_\gamma(x) + C_{ab,dc}\, g_{\nu\gamma} A^d_\mu(x) + C_{ab,cd}\, g_{\mu\nu} A^d_\nu(x)\right)$$

with $A^d_\delta(x) \sim 1/g$ background field

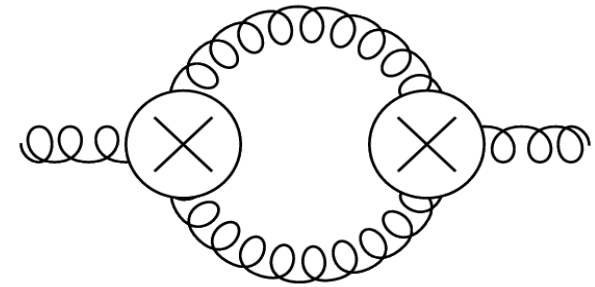
$$C_{ab,cd} = \left(f^{abe} f^{cde} + f^{ade} f^{cbe}\right)$$
$$g^2 \delta^{d+1}(x-y)\,\delta^{d+1}(x-z)$$

V_0 Kinematically forbidden on shell

Stationarity condition:

$$\Pi_F(p)\rho(p) - \Pi_\rho(p) F(p) = 0$$

Classical part of stationarity condinition:



$$\Pi_F \rho - \Pi_\rho F = \int_{pqk} (2\pi)^4 \delta^{(4)}(p+q+k) V^2 \left(\rho(p) F(q) F(k) + F(p)\rho(q) F(k) + F(p) F(q) \rho(k)\right)$$

Scaling ansatz:

$$F(sp) = |s|^{-(2+\kappa)} F(p) \qquad \rho(sp) = |s|^{-2} \mathrm{sgn}(s)\rho(p) \qquad V(sp, sq, sk) = s^{v} V(p, q, k)$$

Transformation (swapping and rescaling)

$$q \to \frac{p_0}{k_0} q, \qquad k \to \frac{p_0}{k_0} p, \qquad p \to \frac{p_0}{k_0} k$$

$$\Pi_F \rho - \Pi_\rho F = \int_{pqk} (2\pi)^4 \delta^{(4)}(p+q+k) V^2 \rho(p) F(q) F(k) \left(1 + \left|\frac{p_0}{k_0}\right|^{\Delta} \mathrm{sgn}\left(\frac{p_0}{k_0}\right) + \left|\frac{p_0}{q_0}\right|^{\Delta} \mathrm{sgn}\left(\frac{p_0}{q_0}\right)\right)$$

Solution:

$$\Delta = -1 \qquad \kappa = \frac{3}{2}$$

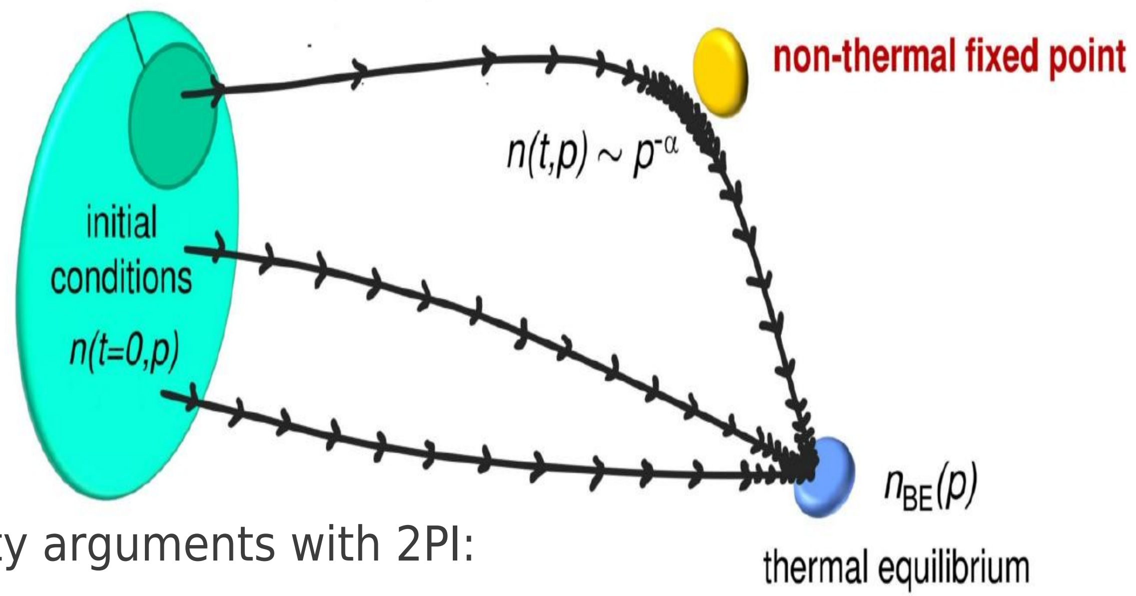
Scaling exponents from constant cascade

Cascades in O(N) scalar fields

Weak turbulence $1 \ll n_p \ll \frac{1}{\lambda}$ $\kappa_W^{ener} = d - \frac{4}{3}$ $\kappa_W^{part} = d - \frac{5}{3}$

Strong turbulence $n_p > \frac{1}{\lambda}$ $\kappa_S^{ener} = d + 2z - \eta$ $\kappa_S^{part} = d + z - \eta$

Berges,Sexty (2010)



Universality:
no dependence on coupling, and N

Agrees with calculations using stationarity arguments with 2PI:

Berges, Hoffmeister (2008)
Scheppach, Berges, Gasenzer (2009)

Bose-Einstein Condensation and Thermalization of the Quark Gluon Plasma

[Blaziot et al, 2011]

Initial CGC: $\epsilon_0 \sim \frac{Q_s^4}{\alpha_s} \qquad n_0 \sim \frac{Q_s^3}{\alpha_s} \quad \rightarrow \quad n_0 e_0^{-3/4} \sim \alpha_s^{-1/4}$

Thermal eq: $\epsilon_{eq} \sim T^4 \qquad n_{eq} \sim T^3 \quad \rightarrow \quad n_0 e_0^{-3/4} \sim 1$

Elastic processes dominate

Particles pile up in the IR

Overpopulation leads to emergence of condensate

What is a condensate?

In equilibrium: $N = V \int \frac{d^3k}{2\pi^3} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$ Maximum at $\mu = \epsilon_0$

Condensation: $N > N_{max}$

Macroscopic occupation of the zero mode

Condensate fraction $\frac{N_0}{N}$

Particle distribution: $n_k = \delta^3(k) n_0 + n'_k$

In terms of 2point function $F(x,y) = \{\phi(x), \phi(y)\}$

$$n_k = F(k)\omega_k \Rightarrow \quad F(k=0) \sim V$$

condensate $= \left(\frac{\int d^3x \, \phi(x)}{V} \right)^2 = \frac{F(k=0)}{V}$ Independent of the volume

Condensation in bose gas

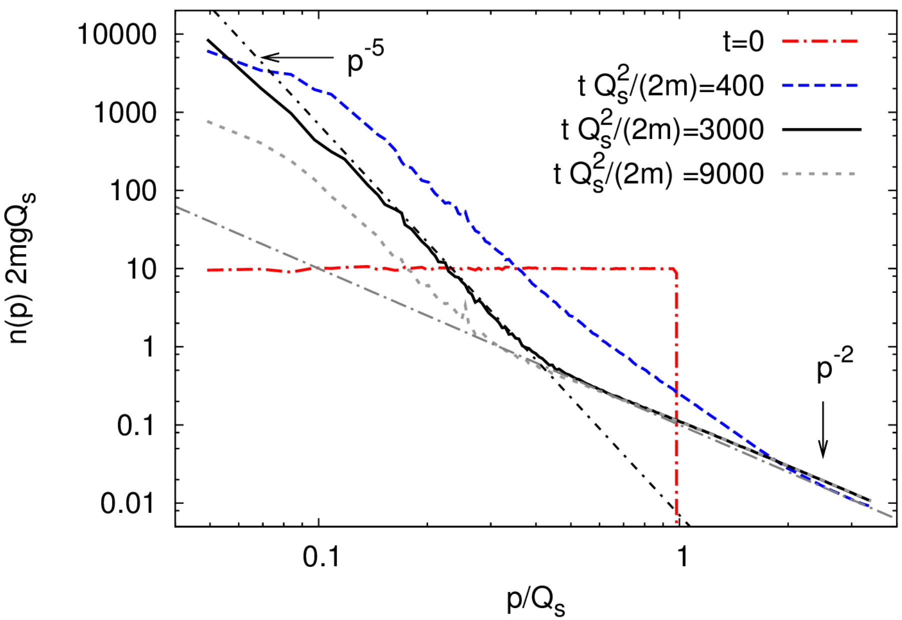
[Berges, Sexty (2012)]

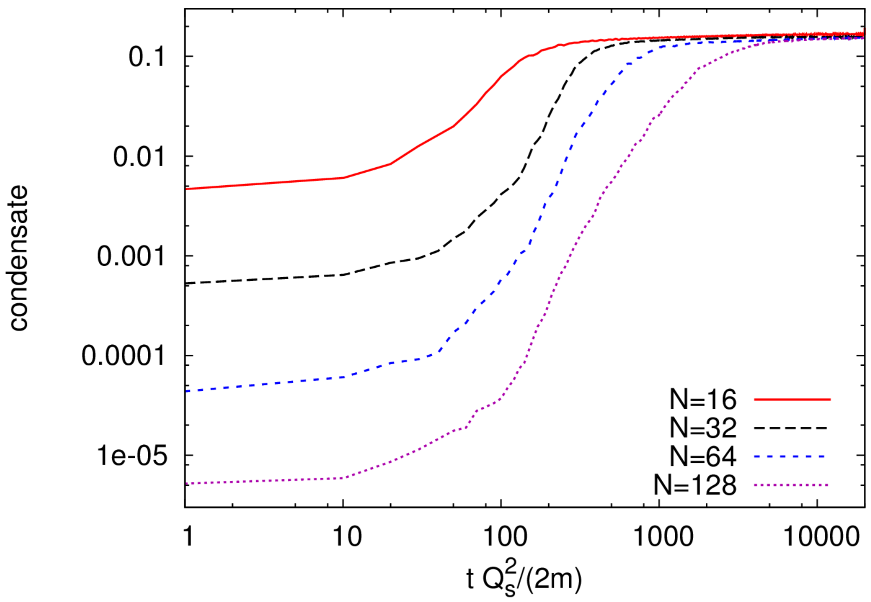
Non relativistic scalars described by complex field $\Psi(x,t)$

Gross-Pitaevski equation: $i\partial_t \Psi(x,t) = \left(-\frac{\partial_i^2}{2m} + g|\Psi(x,t)|^2\right)\Psi(x,t)$

conserved particle number $n_{tot} = \int d^3x |\Psi(x,t)|^2$

occupation in zero mode: condensate = $\left|\frac{\int d^3x \Psi(x,t)}{V}\right|^2$





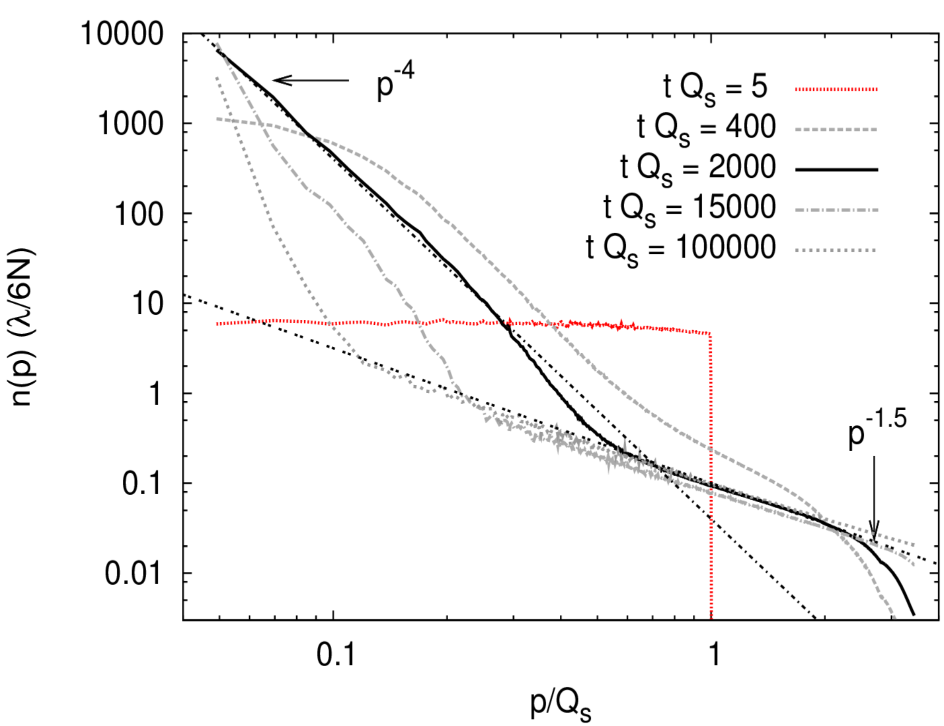
Non-equilibrium Bose condensation

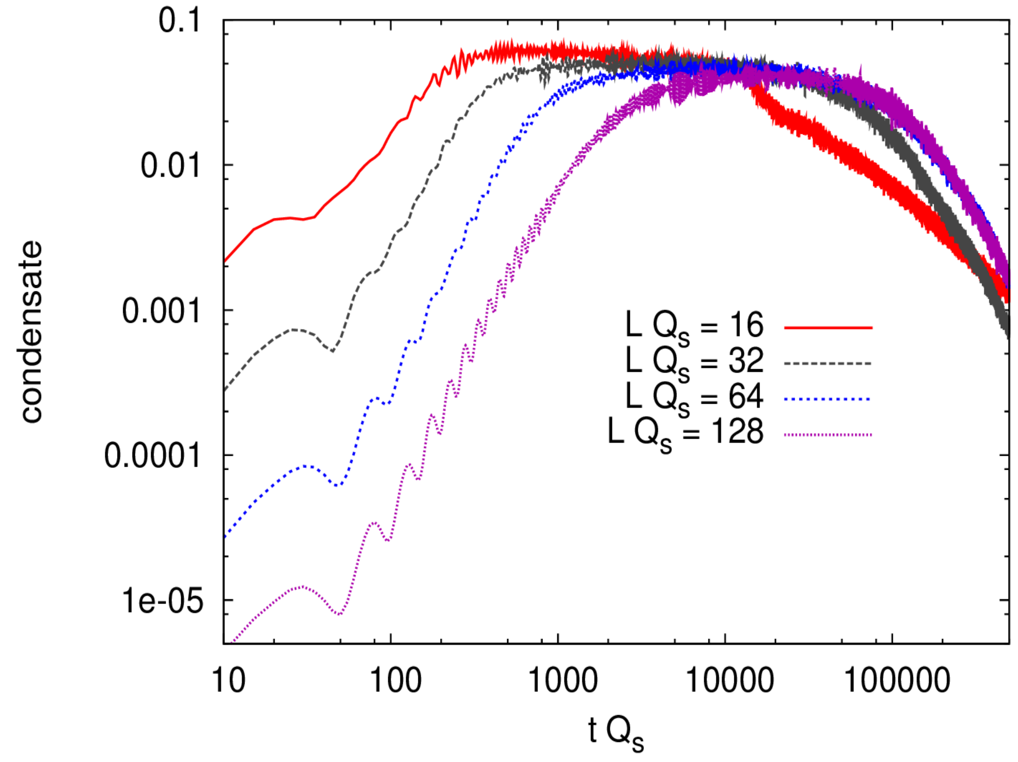
[Berges, Sexty (2012)]

O(4) massless relativistic scalars

Initial conditions: overpopulation

condensate $= \left\langle \left| \frac{\int d^3x\, \phi_a(x)}{V} \right|^2 \right\rangle_{ens}$





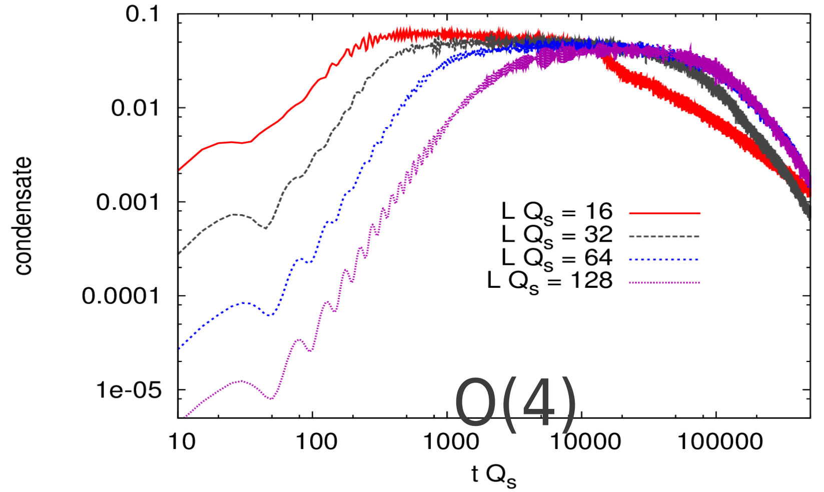
Decay of the condensate

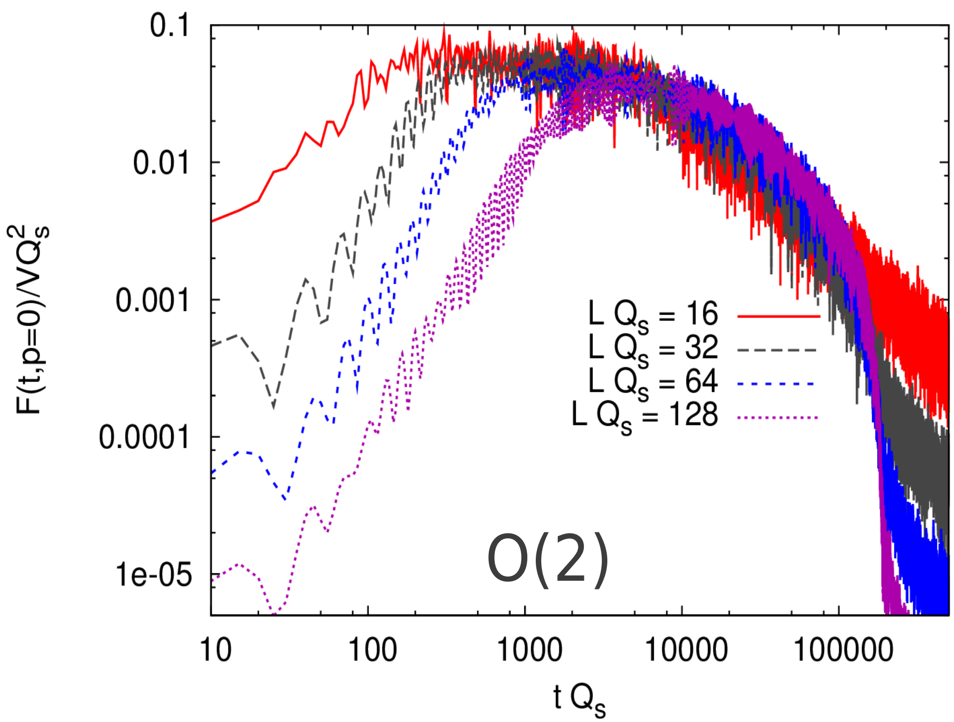
Particle number changing processes suppressed ⟶ condensation

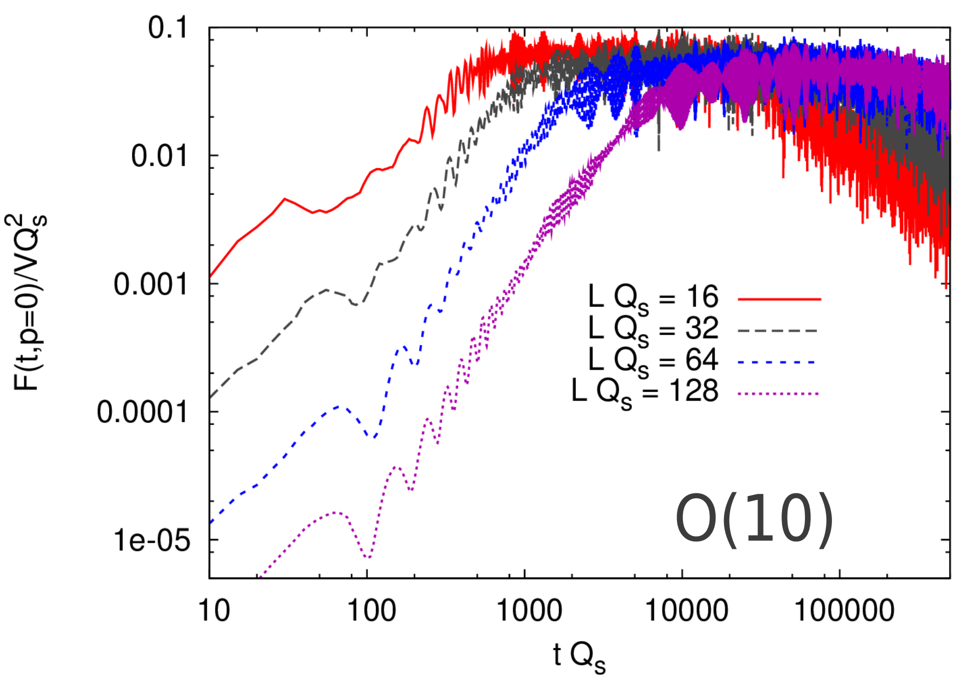
↓

decay of condensate

Suppression compared to 2->2 $\sim \frac{1}{N}$

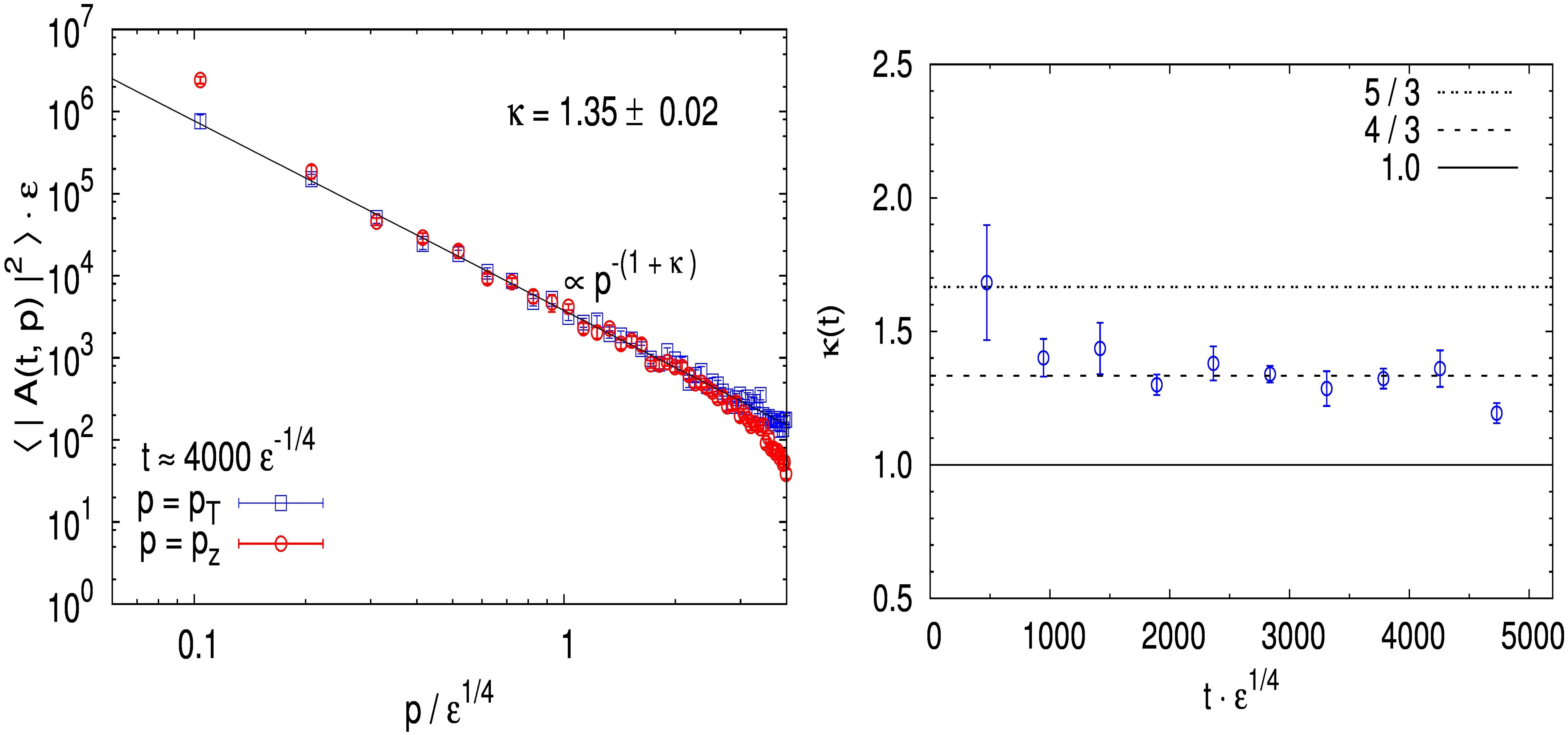






UV cascade in gauge theories

SU(2) gauge theory after Weibel instability and isotropisation

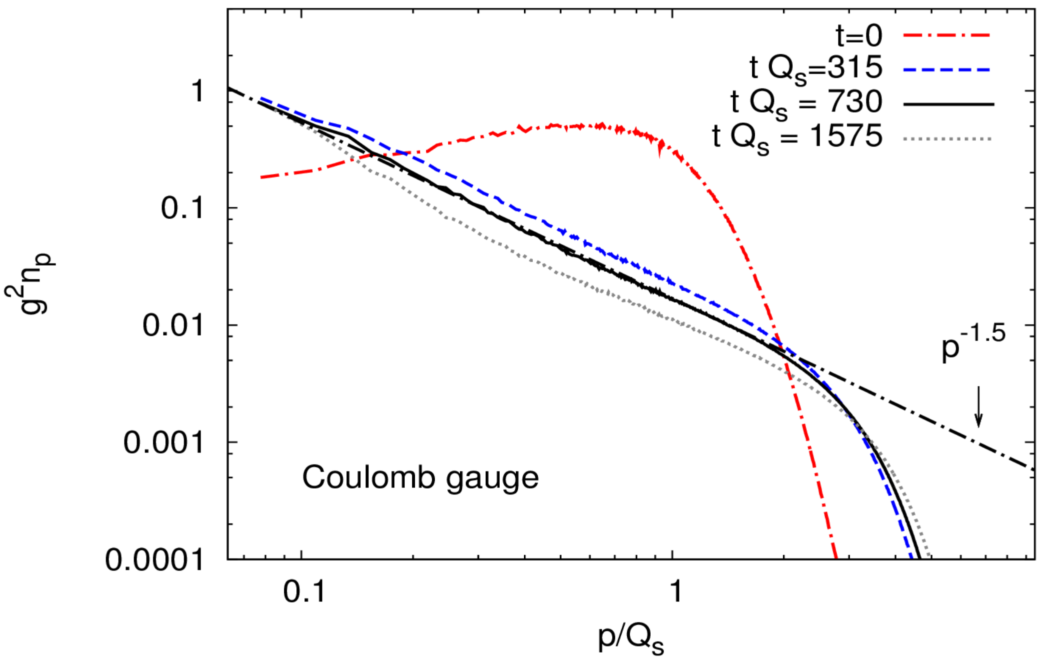


[Berges, Scheffler, Sexty (2009)]

See also [Kurkela, Moore (2012)]
[Schlichting (2012)]

Gauge theory UV turbulence

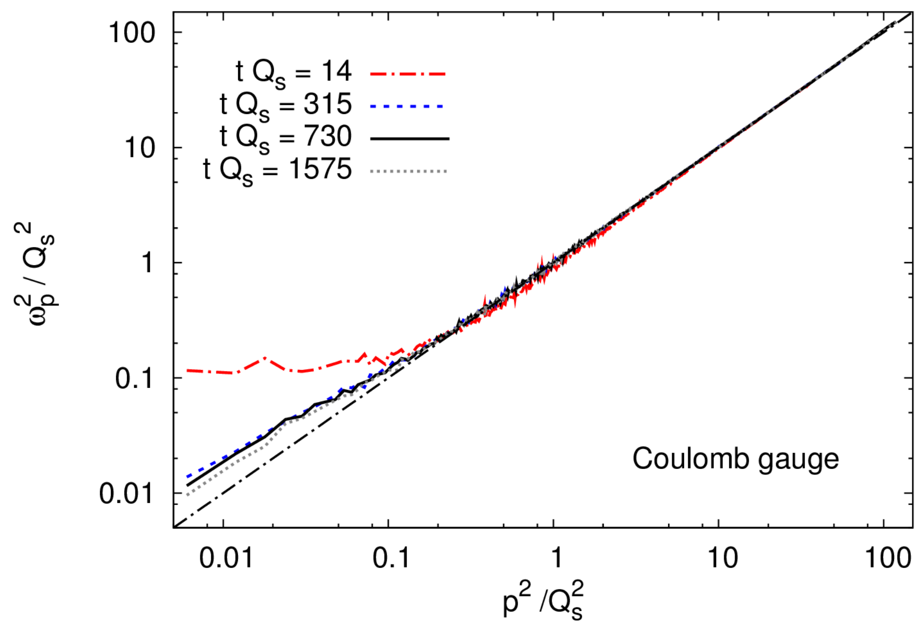
Pure SU(2) gauge theory   overpopulated initial condition



$$n_p \sim p^{-1.5}$$

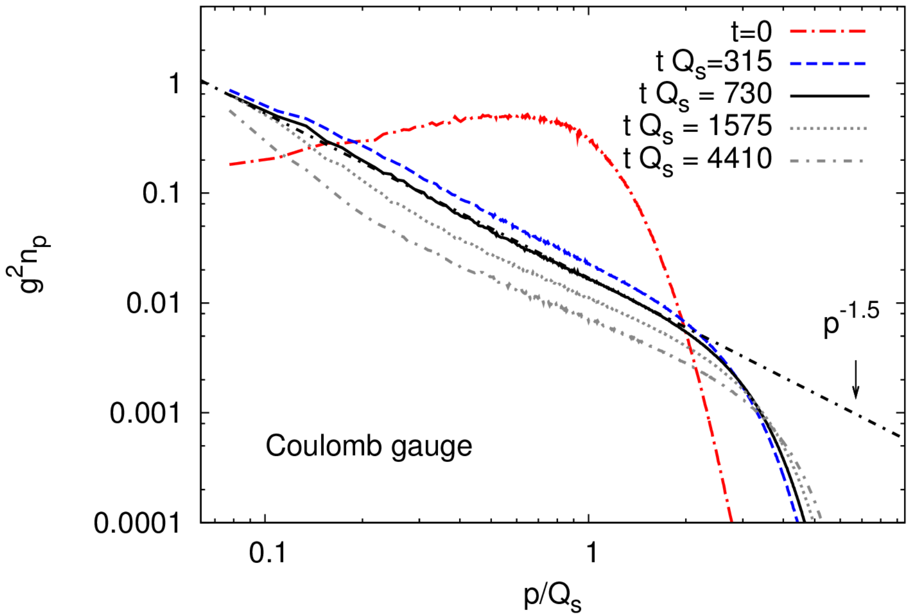
same as scalar UV exponent

Dispersion



[Berges, Schlichting, Sexty (2012)]

Time dependence of gauge theory exponent

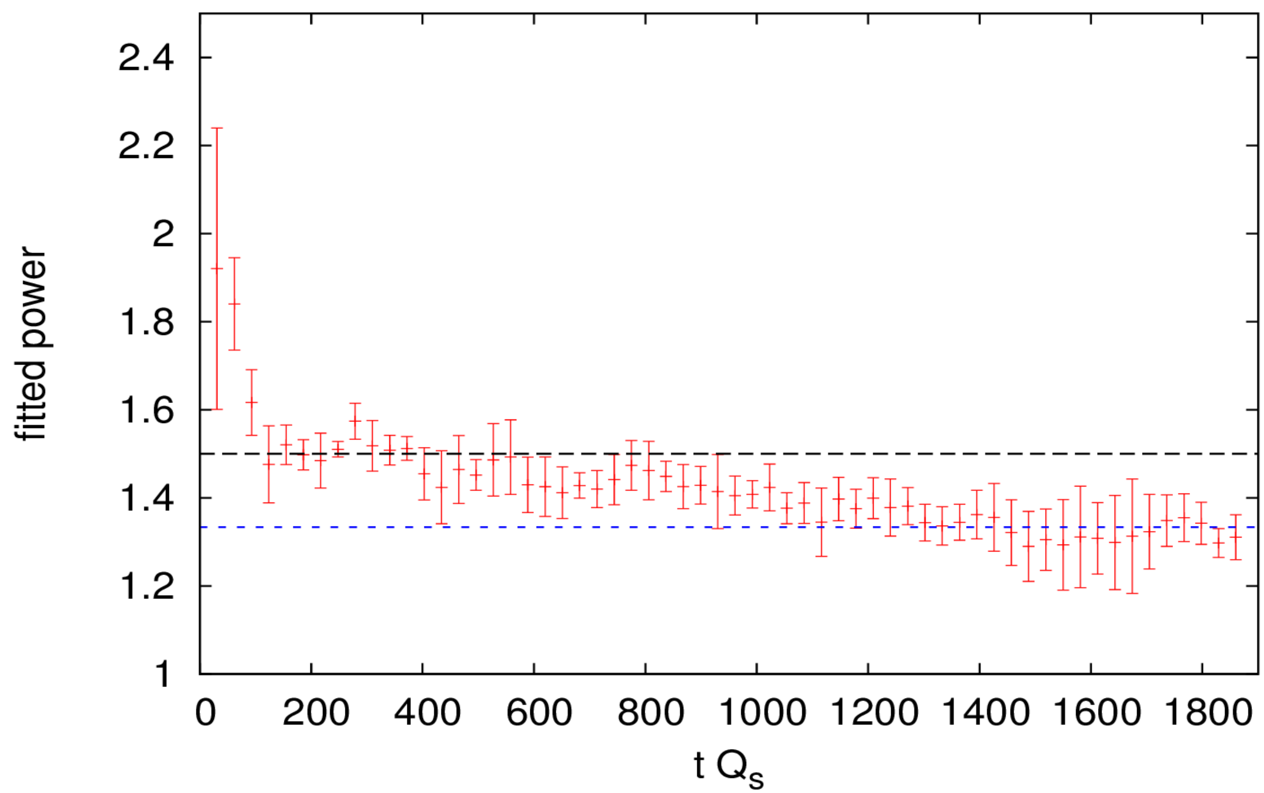


Fit $n(p) \sim p^{-\kappa}$ [0.4: 1.0]

Weak wave turbulence:

$\kappa = 3/2$ With a condensate

$\kappa = 4/3$ Without a condensate



Condensation in Abelian Higgs model

$$S=\int_x \left(\frac{1}{2}D_\mu \phi D^\mu \phi - V(|\phi^2|) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \text{ with } \phi\in\mathbb{C}$$

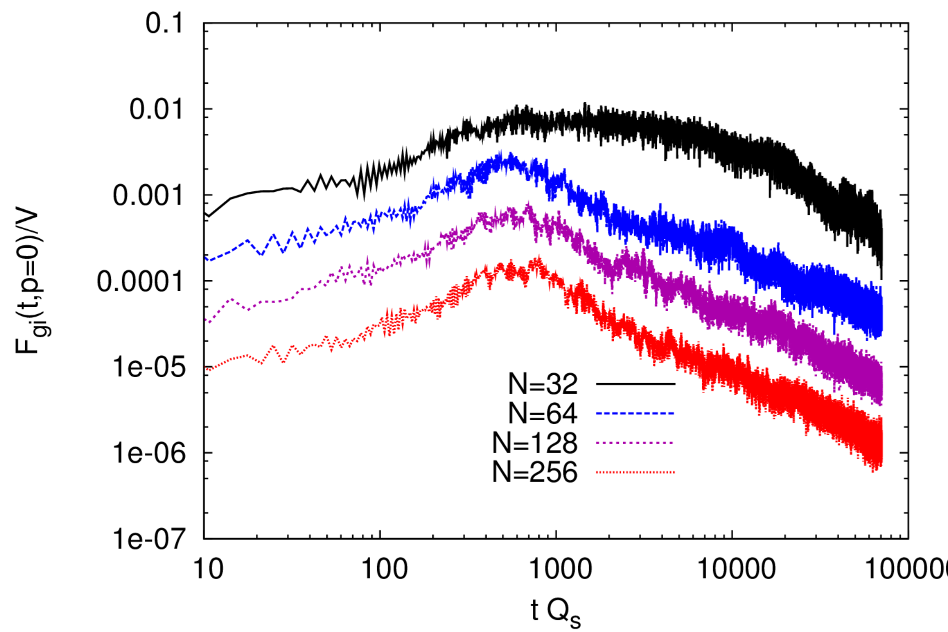
$$\text{condensate} \;= \frac{F(k=0)}{V} = \left(\frac{\int d^3x\, \varphi(x)}{V}\right)^2 = \frac{1}{V^2}\int d^d x\, d^d y\, \phi^+(x)\phi(y)$$

Initial condition: overpopulation in Higgs field $m^2=0$

Local gauge symmetry:

$$\phi(x)\to e^{i\theta(x)\phi(x)}$$

Nonlocal contributions average to zero



Only local contribution

$$\frac{1}{V^2}\int d^d x\, d^d y\, \phi^+(x)\phi(y) \to \frac{1}{V^2}\int d^d x\, \phi^+(x)\phi(x) \sim \frac{1}{V}$$

Make it gauge invariant:

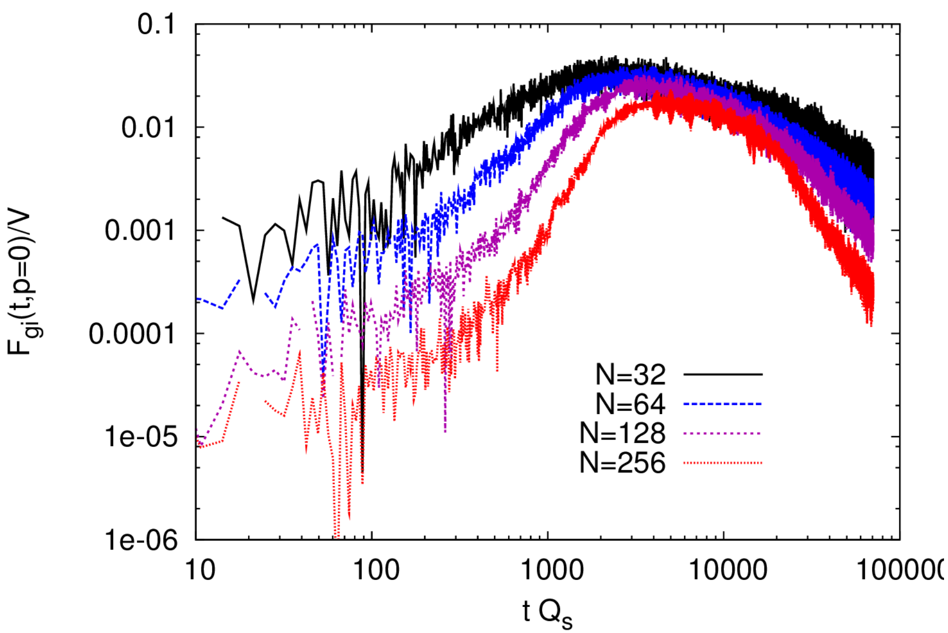
$$\frac{1}{V^2}\int d^d x\, d^d y\, \phi^+(x)\phi(y) \quad \rightarrow \quad \frac{1}{V^2}\int d^d x\, d^d y\, \phi^+(x)U(x,y)\phi(y)$$

Parallel transporter $\quad U(x,y)=\int_y^x \wp\, e^{-i A_\mu(x) dx^\mu}$

$$U(x,y)\rightarrow \exp(i\theta(x))U(x,y)\exp(-i\theta(y)) \qquad \phi(x)\rightarrow e^{i\theta(x)}\phi(x)$$

$\phi^+(x)U(x,y)\phi(y)$ Gauge invariant

Non equilibrium
condensation in gauge theory



[Gasenzer, McLerran, Pawlowski, Sexty in prep.]

In the limit $n \gg 1$ classical physics is exact

Classical statistical field theory includes loop effects

First principle numerical simulations
on real-time lattices are feasible

Describes instabilities, isotropisation, critical exponents, expansion, cold gases, turbulence, condensation …

(but not the final stages of thermalization)