Using the Complex Langevin equation to map out the phase diagram of QCD

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1. Introduction to the sign problem and CLE
2. Review of results so far
3. full QCD
    - challenges of a full QCD simulation with CLE
    - pressure
    - Improved actions

We are interested in a system
Described with the partition sum:

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \sum_C W[C]$$

Typically exponentially many configurations,
no direct summation possible.

If the Weight is positive, build a Markov chain with the Metropolis alg.

$$\ldots \rightarrow C_{i-1} \rightarrow C_i \rightarrow C_{i+1} \rightarrow \ldots$$

Probability of visiting C $p(C) = \frac{1}{N_W} W[C]$

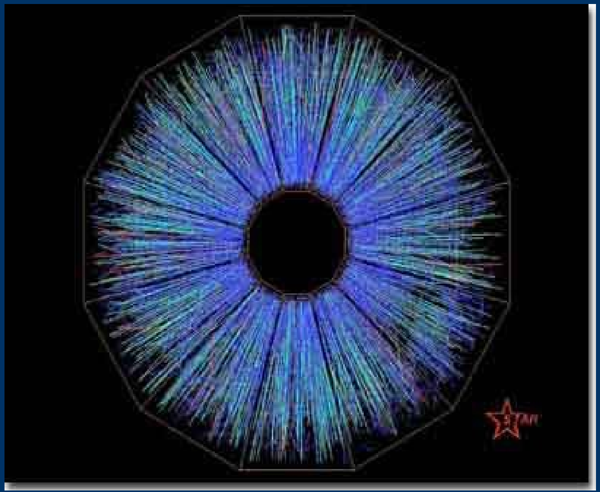
Importance sampling

$$\langle X \rangle = \frac{1}{Z} \mathrm{Tr}\, X e^{-\beta(H-\mu N)} = \frac{1}{N_W} \sum_C W[C] X[C] = \frac{1}{N} \sum_i X[C_i]$$

This works if we have $W[C] \geq 0$

Otherwise we have a Sign problem

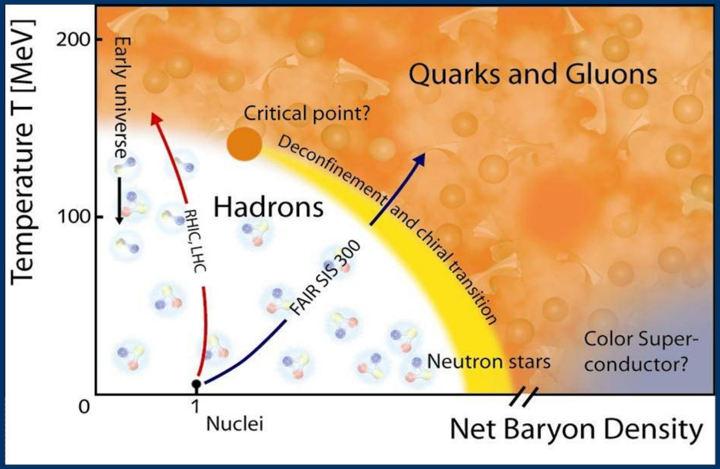
Sign problems in high energy physics



Real-time evolution in QFT

"strongest" sign problem e^{iS_M}

Non-zero density (and fermionic systems)



$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int DU\, e^{-S[U]} \det(M[U])$$

Many systems: Bose gas
XY model
SU(3) spin model
Random matrix theory
QCD

Theta therm

$$S = F_{\mu\nu} F^{\mu\nu} + i\Theta\, \epsilon^{\mu\nu\theta\rho} F_{\mu\nu} F_{\theta\rho}$$

And everything else with complex action

$w[C] = e^{-S[C]}$ $\qquad w[C]$ is positive $\leftarrow\rightarrow$ $S[C]$ is real

How to solve the sign problem?

Probably no general solution – There are sign problems which are NP hard

[Troyer Wiese (2004)]

Many solutions for particular models with sign problem exist

Transforming the problem to one with positive weights

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \sum_C W[C] = \sum_D W'[D]$$

Dual variables
Worldlines

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \sum_n Z_n e^{\beta\mu n}$$

Canonical ensemble

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int dE\, \rho_\mu(E)\, e^{-\beta E}$$

Density of states

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \sum_C W[C] = \sum_S \left(\sum_{C\in S} W[C] \right)$$

Subsets

How to solve the sign problem?

Extrapolation from a positive ensemble

Reweighting $\langle X \rangle_W = \frac{\sum_c W_c X_c}{\sum_c W_c} = \frac{\sum_c W'_c (W_c / W'_c) X_c}{\sum_c W'_c (W_c / W'_c)} = \frac{\langle (W/W') X \rangle_{W'}}{\langle W/W' \rangle_{W'}}$

Taylor expansion $Z(\mu) = Z(\mu=0) + \frac{1}{2}\mu^2 \partial_\mu^2 Z(\mu=0) + \ldots$

Analytic continuation from imaginary sources
(chemical potentials, theta angle,..)

Using analyticity (for complexified variables)

Complex Langevin

Complexified variables – enlarged manifolds

Lefschetz thimble

Integration path shifted onto complex plane

Complex Langevin Equation

Given an action $S(x)$

Stochastic process for x: $$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

Gaussian noise

$$\langle \eta(\tau) \rangle = 0$$
$$\langle \eta(\tau)\eta(\tau') \rangle = \delta(\tau - \tau')$$

Averages are calculated along the trajectories:

$$\langle O \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T O(x(\tau))\, d\tau = \frac{\int e^{-S(x)} O(x)\, dx}{\int e^{-S(x)}\, dx}$$

$$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

The field is complexified

real scalar → complex scalar

link variables: SU(N) → SL(N,C)

compact → non-compact

$$det(U) = 1, \quad U^{+} \neq U^{-1}$$

Analytically continued observables

$$\frac{1}{Z} \int P_{comp}(x) O(x)\, dx = \frac{1}{Z} \int P_{real}(x, y) O(x + iy)\, dx\, dy$$

$$\langle x^2 \rangle_{real} \rightarrow \langle x^2 - y^2 \rangle_{complexified}$$

Status of Complex Langevin

Theoretically

- Good understanding of the failure modes (boundary terms, poles)
- Monitoring prescriptions allow for independent detection of failure
  - unitarity norm, eigenspectrum, histograms, boundary terms
- Is a cutoff allowed? (Dynamical stabilization)
- How to cure problems? – No general answer, hit and miss

In practice

- Many lattice models solved, crosschecked with alternative methods
  - (Bose gas, SU(3) Spin model, HDQCD, kappa exp., cond. mat. systems...)
- Some remain unsolved (xy model, Thirring,...)

Full QCD

- High temperatures seem to be unproblematic
  - checks with reweighting, Taylor expansion
- Status of low T and near T_c is unclear – more work needed

See below for ongoing work concerning phase diag, EOS and improved actions

Proof of convergence for CLE results

If there is fast decay $P(x,y) \to 0$ as $x,y \to \infty$

and a holomorphic action $S(x)$

then CLE converges to the correct result

[Aarts, Seiler, Stamatescu (2009)
Aarts, James, Seiler, Stamatescu (2011)]

Loophole 1: Non-holomorphic action for nonzero density

$$S = S_W[U_\mu] + \ln \mathrm{Det}\, M(\mu)$$

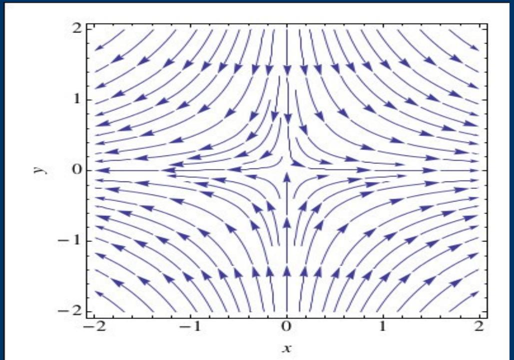
measure has zeros $(Det\ M = 0)$
complex logarithm has a branch cut
→ meromorphic drift

[Mollgaard, Splittorff (2013), Greensite(2014)]

Drift around a pole:

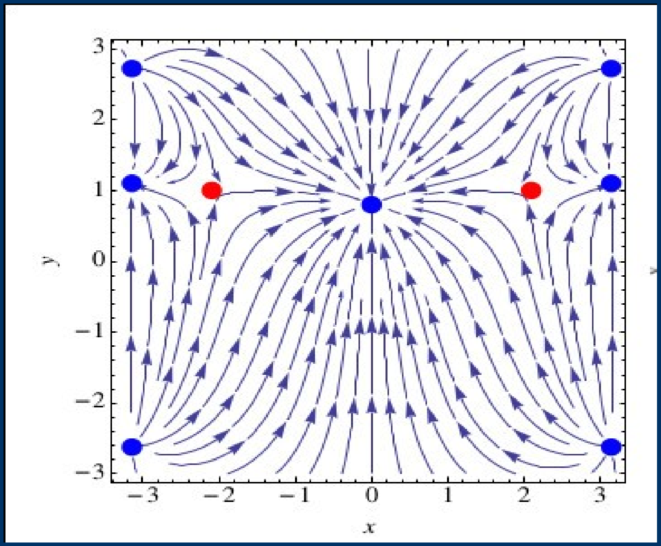
$$\rho(x) = (x - z_p)^{n_f} e^{-S(x)}$$

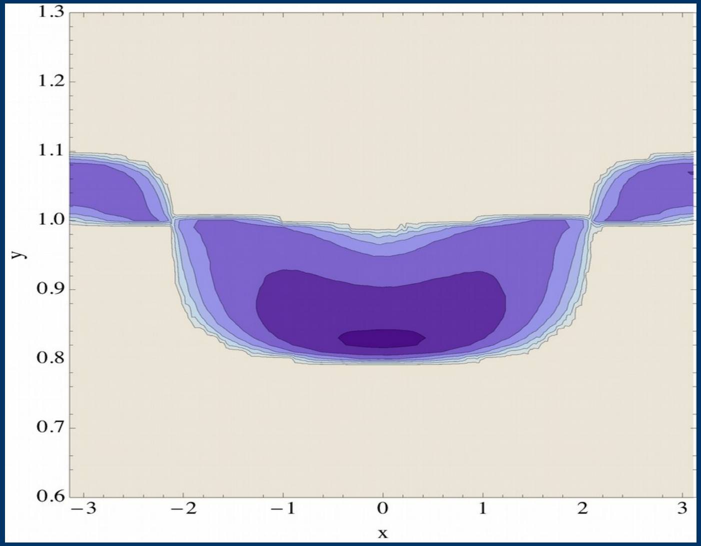
$$K(z) = \frac{\partial_z \rho(z)}{\rho(z)} = \frac{n_f}{x - z_p} + K_S(z)$$



Poles can be

inside the distribution





Pole pinches distribution
Acts as a bottleneck
might cause “separation phenomenon”
(potentially) wrong results

$$\rho(x) = (1 + \kappa \cos(x - i\mu))^{n_f} e^{-\beta \cos(x)}$$

outside of the distribution

Langevin time evolved observables get singularities around pole
Zero of the distribution counteracts that
Proof goes through
correct results

For HDQCD and full QCD at high temperatures this is satisfied
[Aarts, Seiler, Sexty, Stamatescu ‘17]

Loophole 2: decay not fast enough

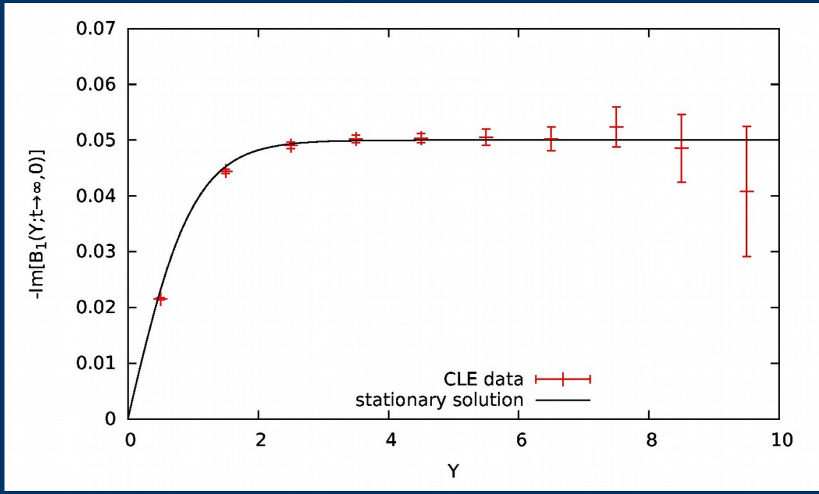
What we want — What we get with CLE

$$\int dx\, \rho(x) O(x) = \int dx\, dy\, P(x,y) O(x+iy)$$

Using analyticity
and partial integrations

boundary terms can be nonzero
explicit calculation of boundary terms

[Scherzer, Seiler, Sexty, Stamatescu (2018)]



See talk by Stamatescu

Gaussian Example

$$S[x]=\sigma x^2+i\lambda x$$

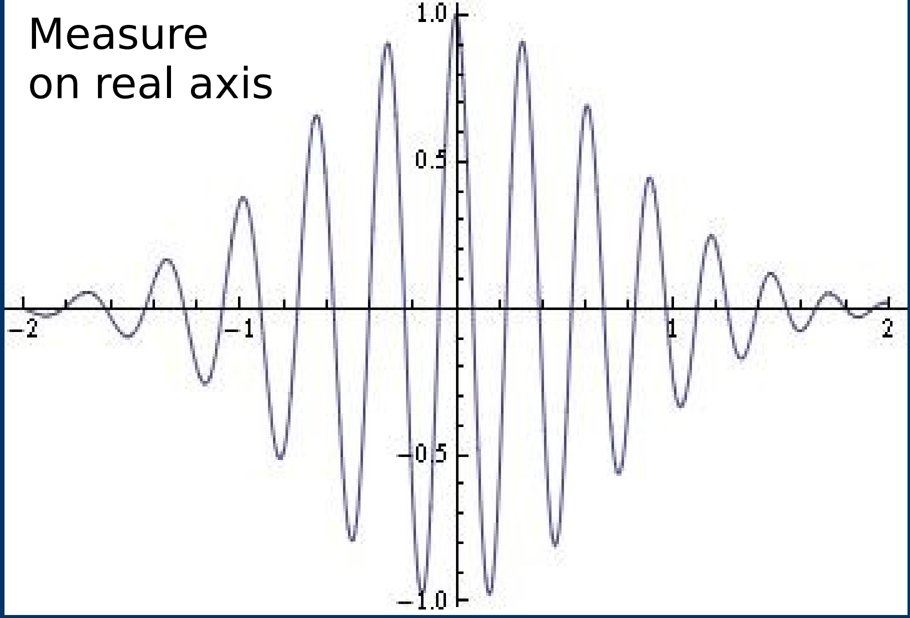
CLE

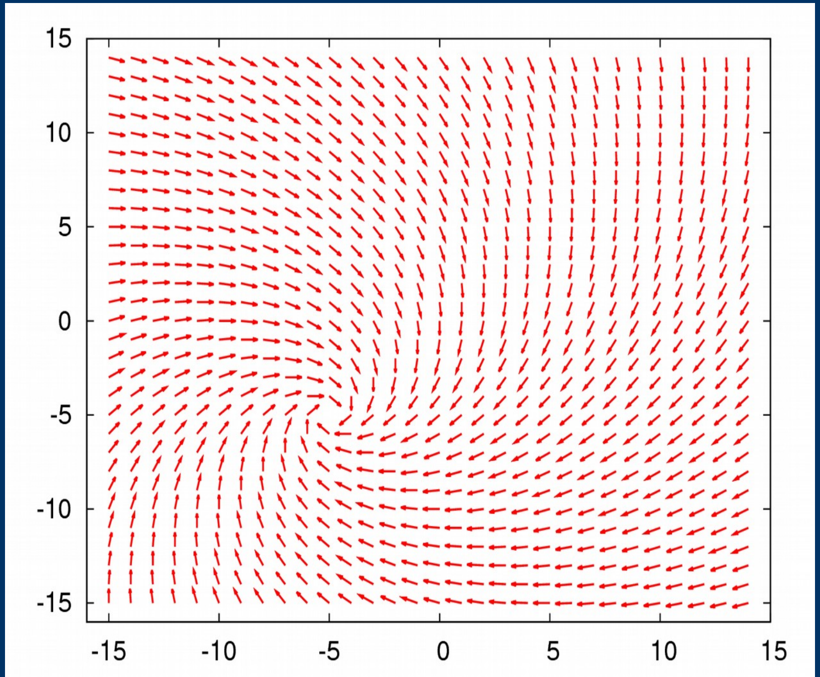
$$\frac{d}{d\tau}(x+iy)=-2\sigma(x+iy)-i\lambda+\eta$$

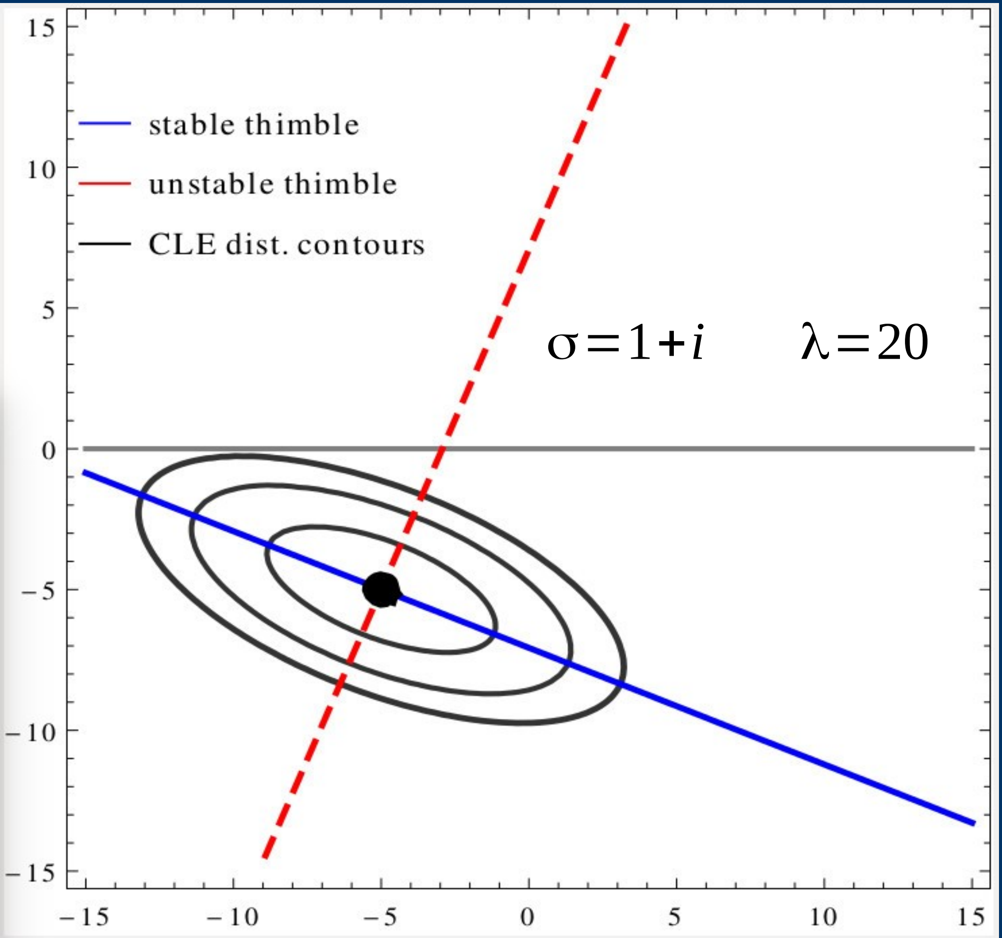
$$P(x,y)=e^{-a(x-x_0)^2-b(y-y_0)^2-c(x-x_0)(y-y_0)}$$

Gaussian distribution around critical point

$$\left.\frac{\partial S(z)}{\partial z}\right|_{z_0}=0$$







QCD sign problem

Euclidean SU(3) gauge theory with fermions:

$$Z=\int DU \exp(-S_E[U]) det(M(U))$$

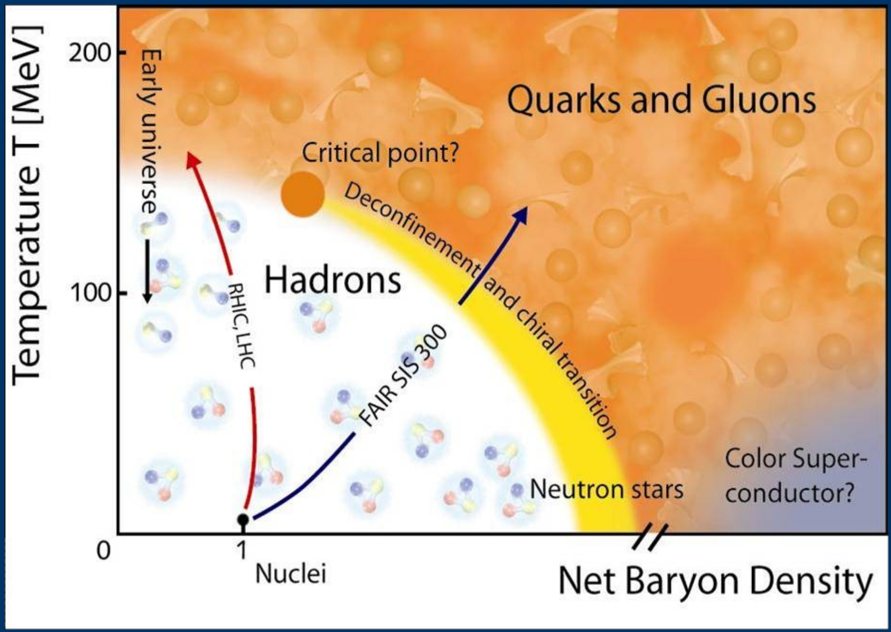
for $\det(M(U))>0$ Importance sampling is possible → Hadron masses, EOS, ...

Non-zero chemical potential

For nonzero chemical potential, the fermion determinant is complex

$$\det(M(U,-\mu^*))=(\det(M(U),\mu))^*$$

Sign problem → Naive Monte-Carlo breaks down



In QCD direct simulation only possible at $\mu = 0$

Taylor extrapolation, Reweighting, continuation from imaginary μ, canonical ens.
all break down around

$$\frac{\mu_q}{T} \approx 1 - 1.5 \qquad \frac{\mu_B}{T} \approx 3 - 4.5$$

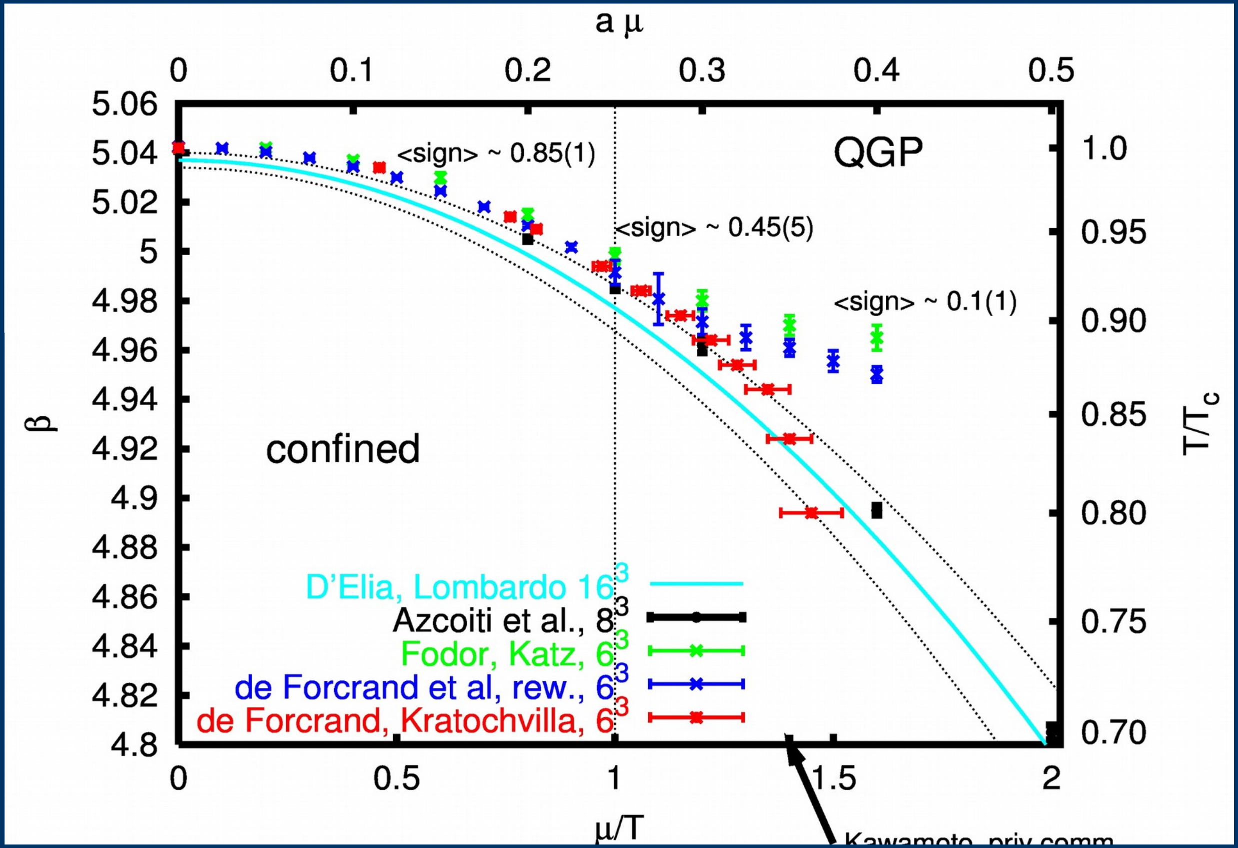
Around the transition temperature
Breakdown at

$$\mu_q \approx 150 - 200\,\mathrm{MeV} \qquad \mu_B \approx 450 - 600\,\mathrm{MeV}$$

Results on

$$N_T = 4, N_F = 4, ma = 0.05$$

using
Imaginary mu,
Reweighting,
Canonical ensemble



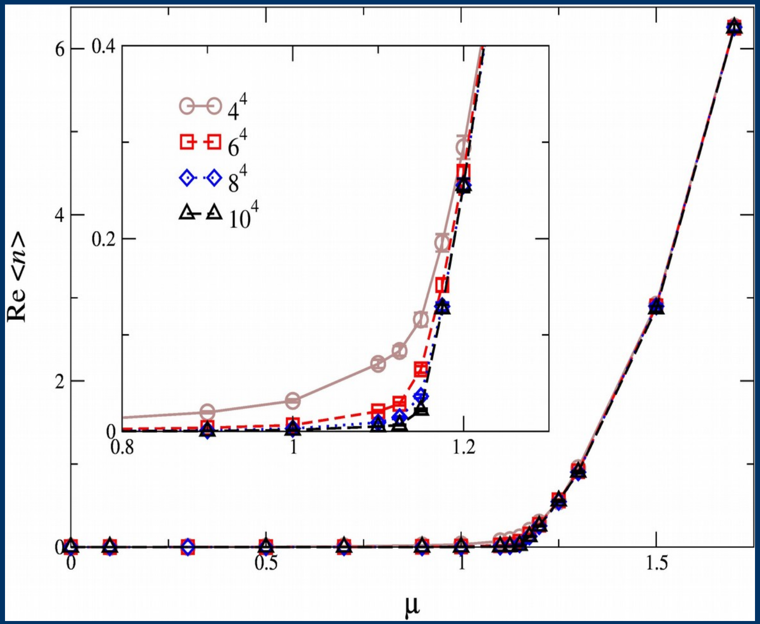
Agreement only at $\mu/T < 1$

Some results of CLE so far

Bose Gas at zero temperature [Aarts '08]

Silver Blaze problem:

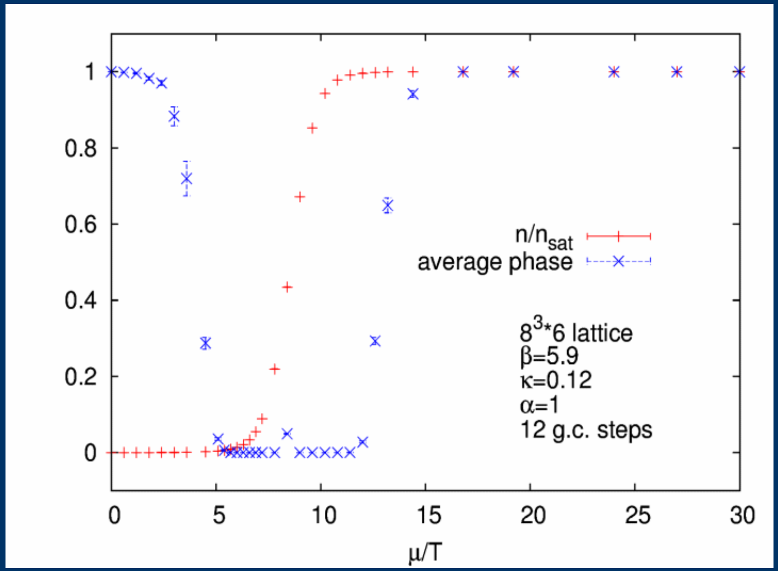
At zero temperature, nothing happens
until first excited state (=1particle) contributes



First spectacular success of complex Langevin

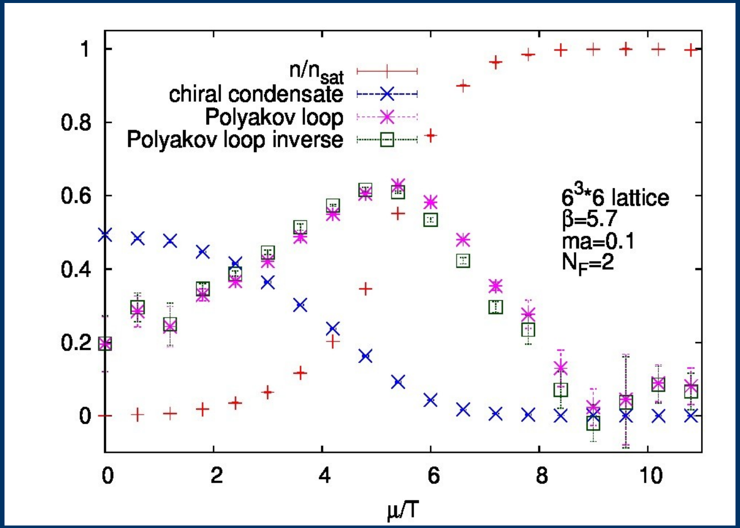
Gaugecooling and study of HDQCD

[Seiler, Sexty, Stamatescu '13]



Full QCD with light quarks

[Sexty '14]



Gauge cooling

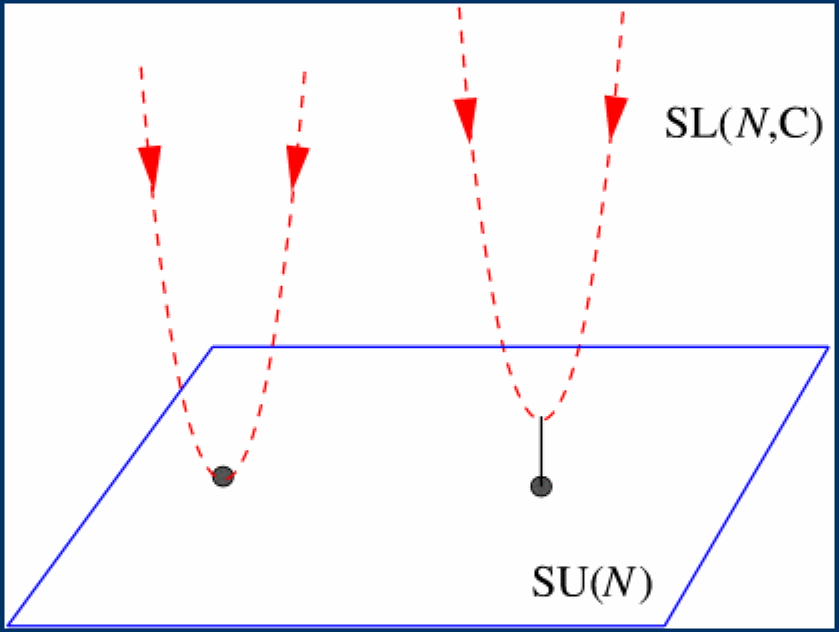
[Seiler, Sexty, Stamatescu (2012)]

complexified distribution with slow decay → convergence to wrong results

Keep the system from trying to explore the complexified gauge degrees of freedom

Minimize unitarity norm $\sum_i Tr\left(U_i U_i^+ - 1\right)$

Distance from SU(N)



Dynamical steps are interspersed with several gauge cooling steps

Empirical observation:
Cooling is effective for $\beta > \beta_{\min}$

but remember, $\beta \rightarrow \infty$ in cont. limit

$a < a_{max} \approx 0.1 - 0.2\, fm$

Can we do more?

Dynamical Stabilization → soft cutoff in imaginary directions

[Attanasio, Jäger (2018)]

Chiral random matrix theory
[Mollgaard, Splittorff '13+'14]
Poles can be problematic

Study of the pole problem
[Nishimura, Shimasaki '15]
[Aarts, Seiler,Sexty, Stamatescu '17]
Distribution at poles (spectrum) should be monitored

Hopping parameter expansion
[Aarts, Seiler, Sexty, Stamatescu '15]
Very high orders easily calculated

Investigating Silver Blaze for QCD
[Kogut, Sinclair '16]
[Ito, Nishimura '16]
[Tsutsui, Ito, Matsufuru, Nishimura, Shimasaki, Tsuchiya '18]
Jury still out

0+1 dim Thirring model
[Fujii, Kamata, Kikukawa '17]
Reweighting or deformation makes CLE ok

Gauge cooling for eigenvalues
[Nagata, Nishimura, Shimasaki '16]
Shifts e.v.s away from origin in RMT

Gauge cooling for Random Matrix models
[Bloch, Glessaaen, Verbaarschot, Zafeiropoulos '18]
1 hit 1 miss

Exact drift terms with selected inverse

[Bloch, Schenk '17]

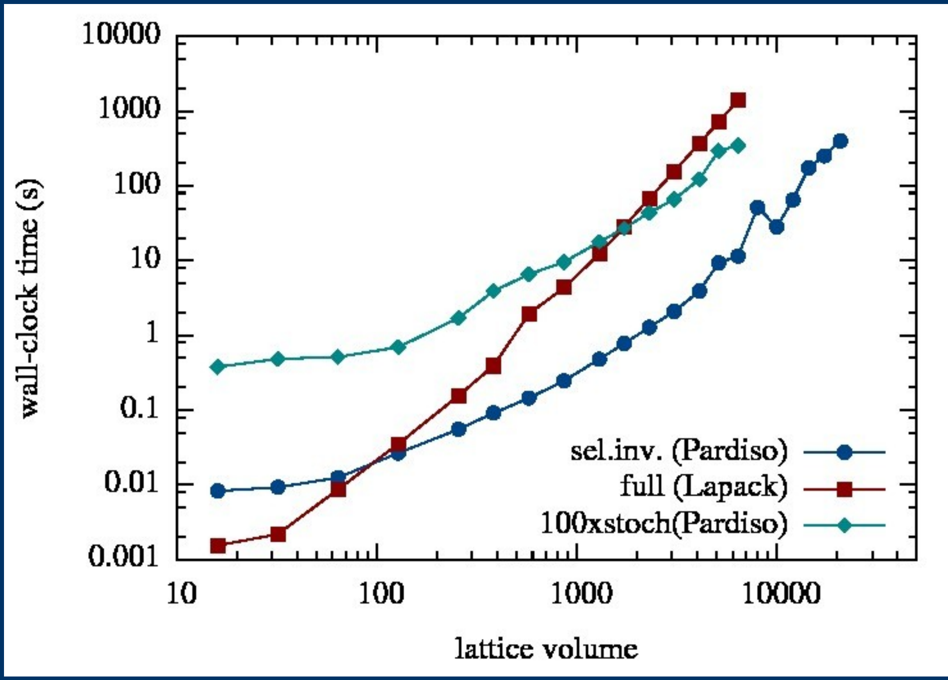
Fermionic drift term: $\mathrm{Tr}\left(M^{-1} D_{a\mu x} M\right)$

with sparse Dirac Matrix M

Use sparse LU decomposition to calculate inverse

No additional noise from stochastic estimator

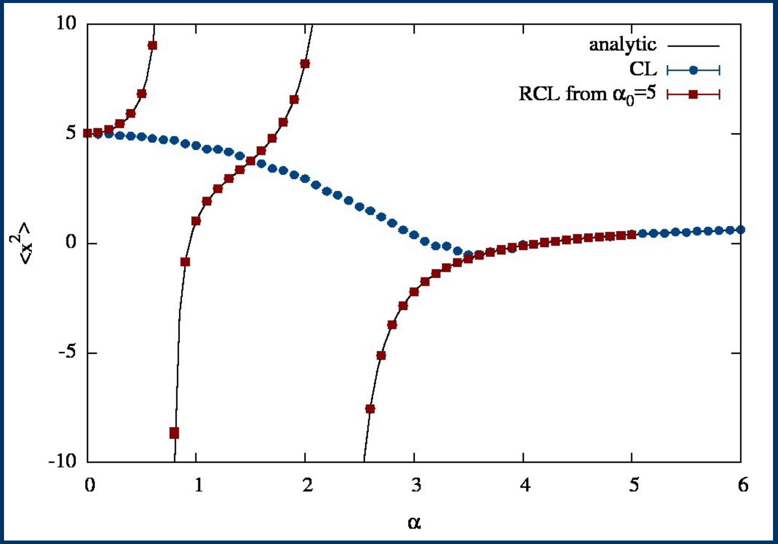
Unitarity norm is better controlled



Reweighting complex Langevin trajectories

[Bloch '17]

Reweighting from one non-positive ensemble to another



Equation of state for 1D non-relativistic fermions

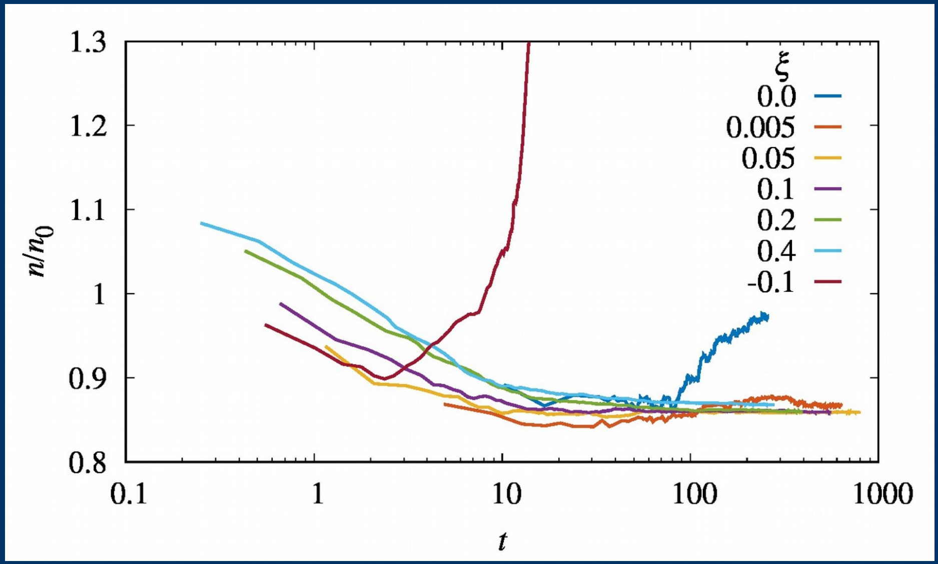
[Loheac, Drut '18]

$$Z = \int d\sigma \det M_{\text{up}}(\sigma) \det M_{\text{down}}(\sigma)$$

$\sigma =$ Hubbard-Stratonovich field

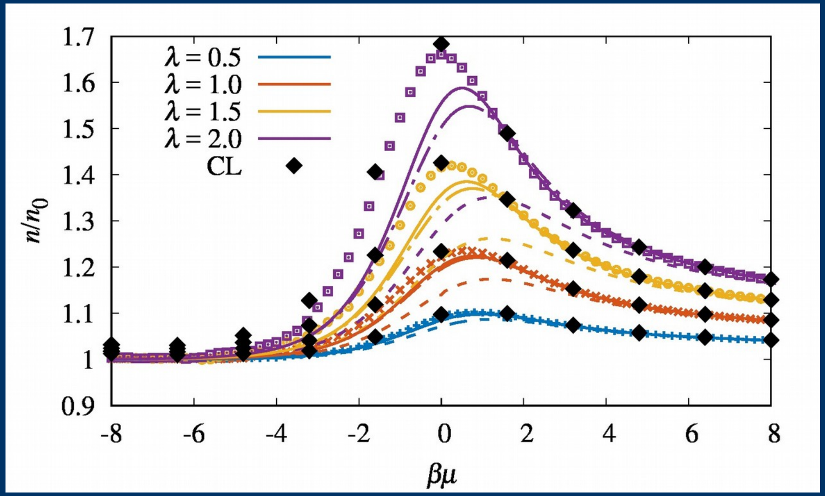
Modify action to add an attractive force

$$S(\sigma) = S_{old}(\sigma) + \xi \sigma^2$$

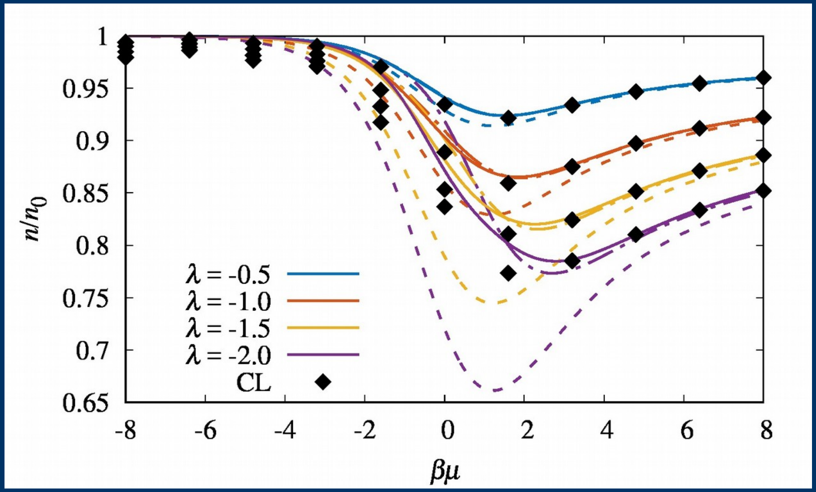


Local interactions

2 parameters: coupling λ, chemical pot. μ



Attractive – pos. det



Repulsive – non pos. det

Mapping the phase diagram of HDQCD

[Aarts, Attanasio, Jäger, Sexty (2016)]

Hopping parameter expansion of the fermion determinant
Spatial fermionic hoppings are dropped
Full gauge action

$$\mathrm{Det}\, M(\mu) = \prod_x \det(1 + C P_x)^2 \det(1 + C' P_x^{-1})^2$$

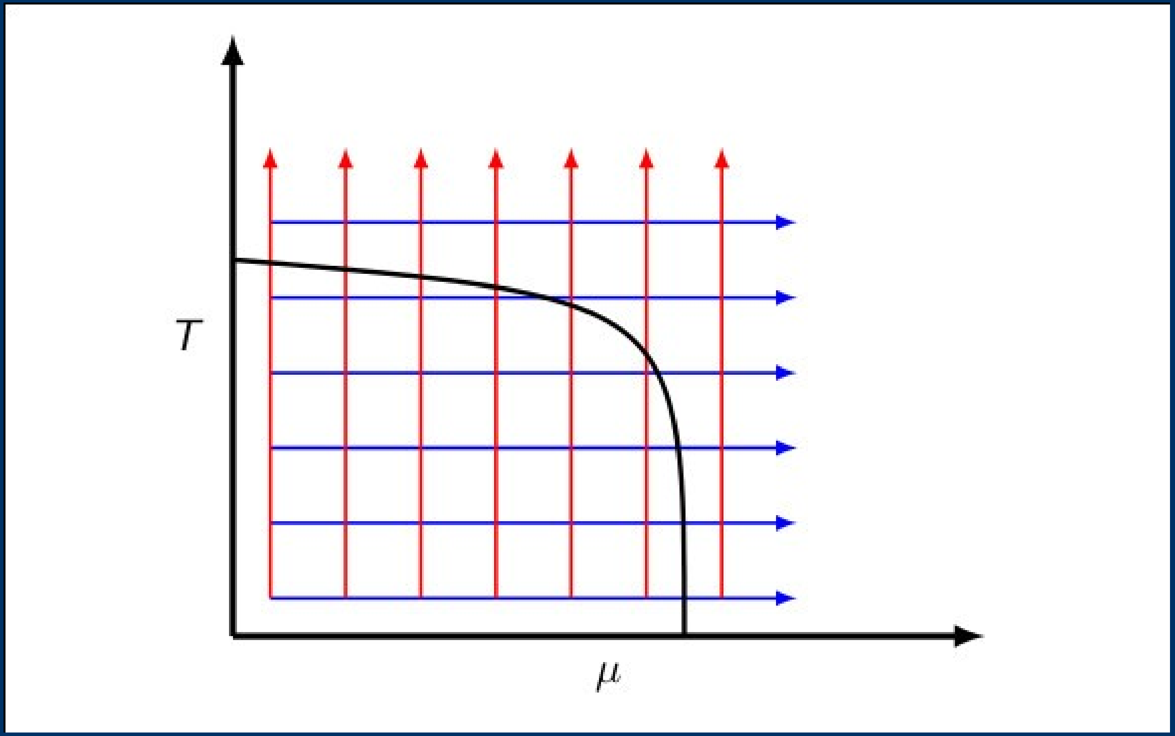
Strategy to map $T-\mu$ plane

fixed $\beta = 5.8$ → $a \approx 0.15\,\mathrm{fm}$ Unitarity norm is mostly under control

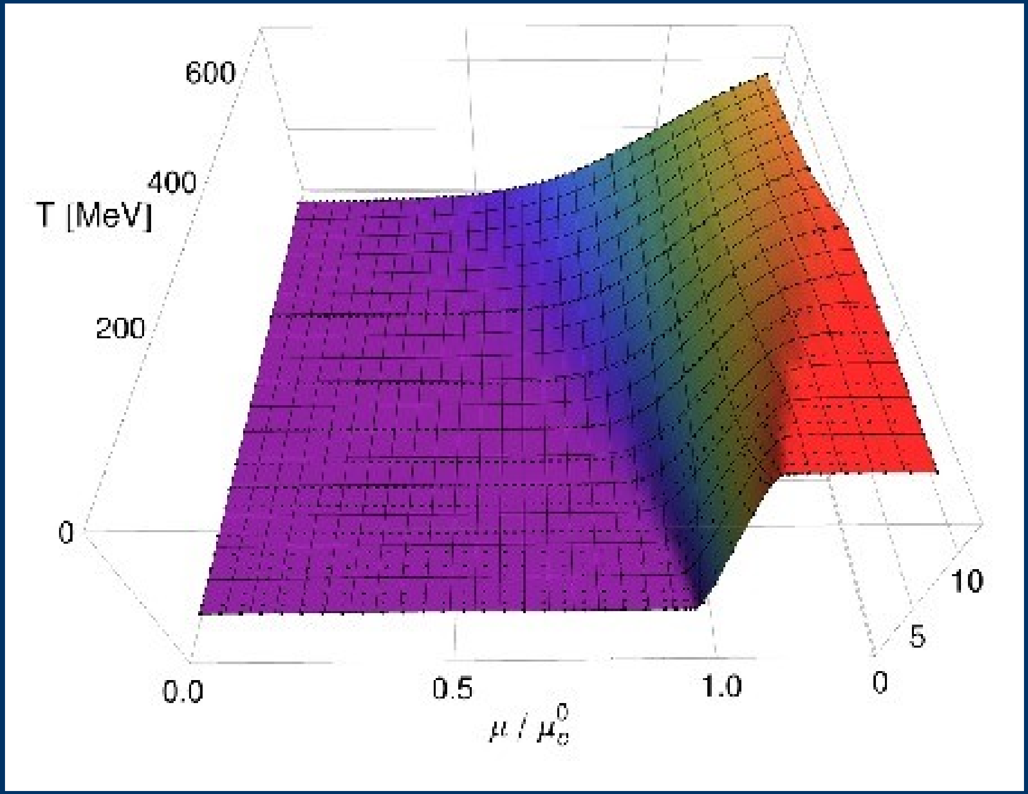
$\kappa = 0.04$
onset transition at $\mu = -\ln(2\kappa)$

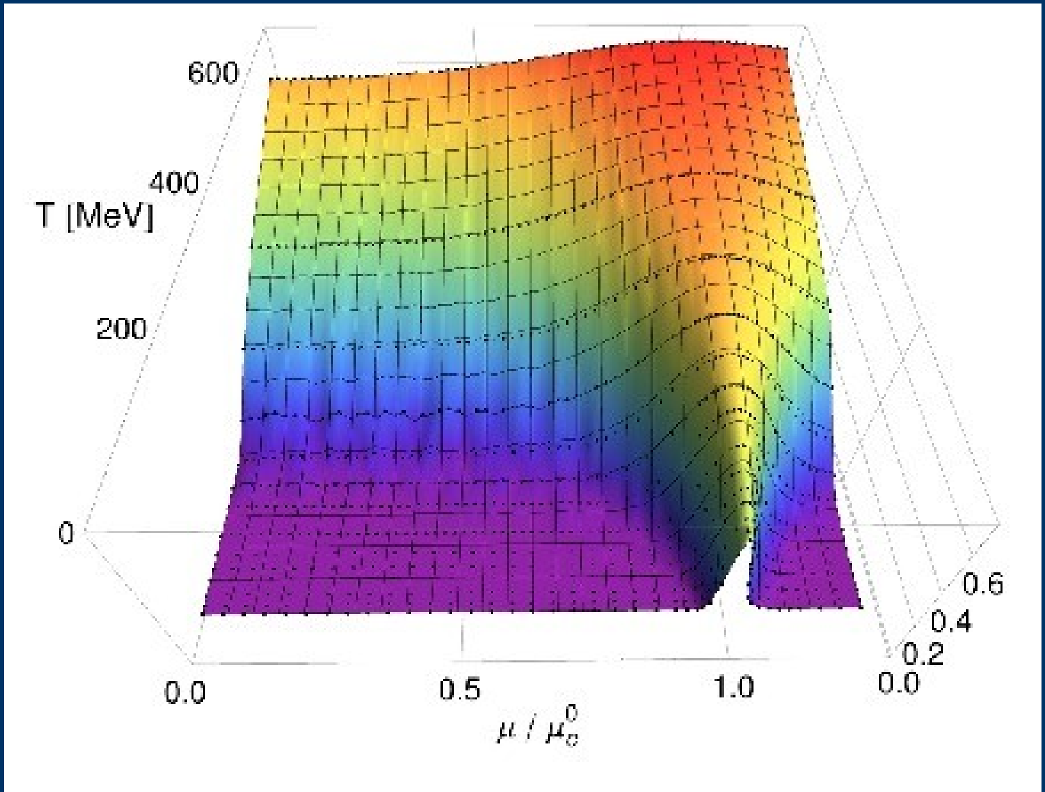
$N_t * (6^3, 8^3, 10^3)$ lattice
$N_t = 2..28$

Temperature scanning
$T = 48 - 671\,\mathrm{MeV}$



Mapping the phase diagram of HDQCD

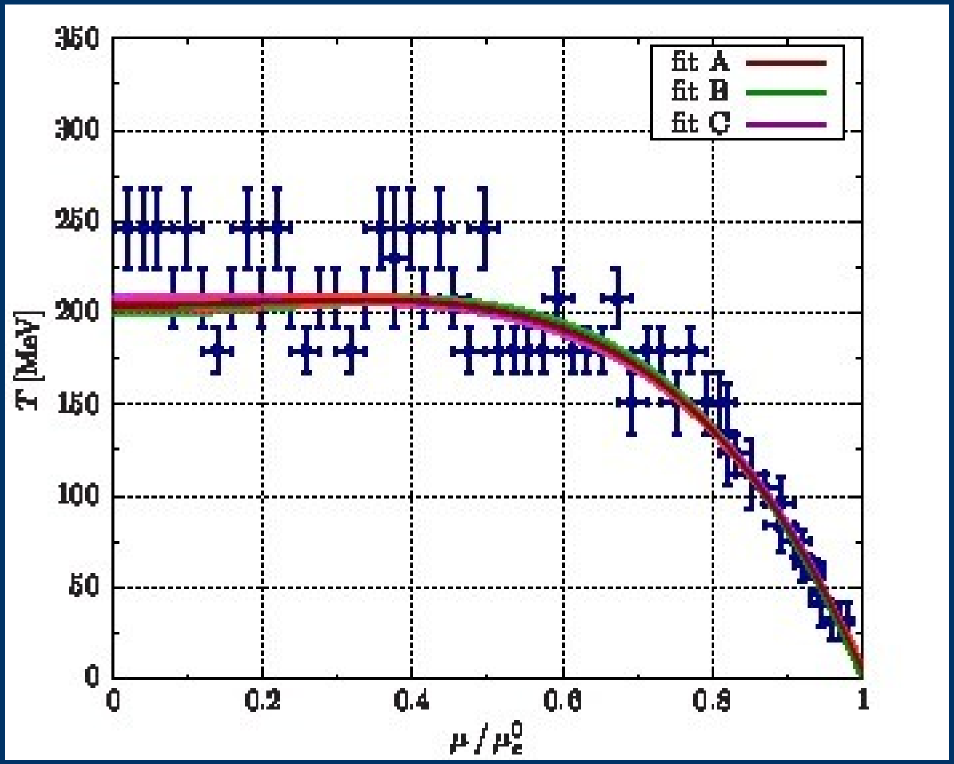


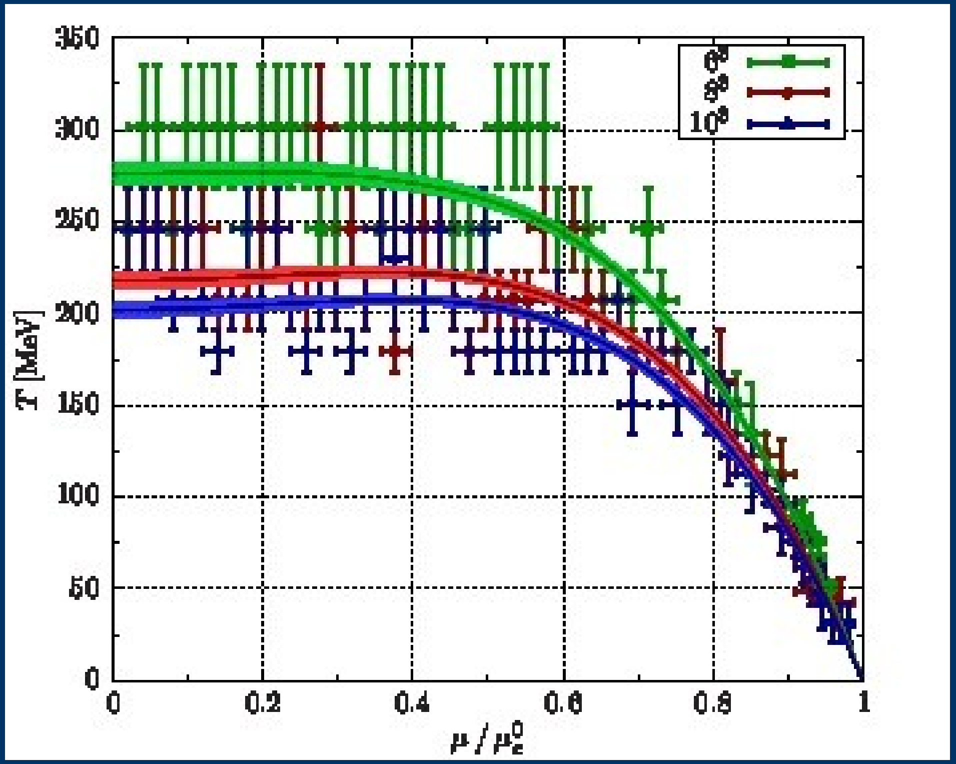


Onset in fermionic density
Silver blaze phenomenon

Polyakov loop
Transition to deconfined state

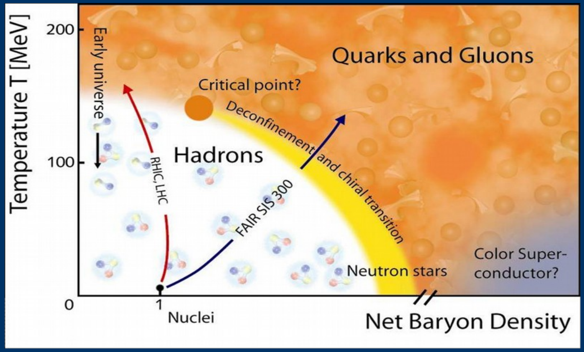
Fits of the phase transition line





Deconfinement transition and onset transition meet in the middle
Errors from discretisation scheme
Volume dependence under control

much simpler phase diagram than full QCD



Reweighting

$$\langle F \rangle_\mu = \frac{\int DU\, e^{-S_E} \det M(\mu)\, F}{\int DU\, e^{-S_E} \det M(\mu)} = \frac{\int DU\, e^{-S_E} R \frac{\det M(\mu)}{R} F}{\int DU\, e^{-S_E} R \frac{\det M(\mu)}{R}}$$

$$= \frac{\langle F \det M(\mu)/R \rangle_R}{\langle \det M(\mu)/R \rangle_R} \qquad R = \det M(\mu = 0),\ |\det M(\mu)|, \text{ etc.}$$

$$\left\langle \frac{\det M(\mu)}{R} \right\rangle_R = \frac{Z(\mu)}{Z_R} = \exp\left(-\frac{V}{T} \Delta f(\mu, T)\right)$$

$\Delta f(\mu, T)$ =free energy difference

Exponentially small as the volume increases $\langle F \rangle_\mu$ → $0/0$

Reweighting works for large temperatures and small volumes

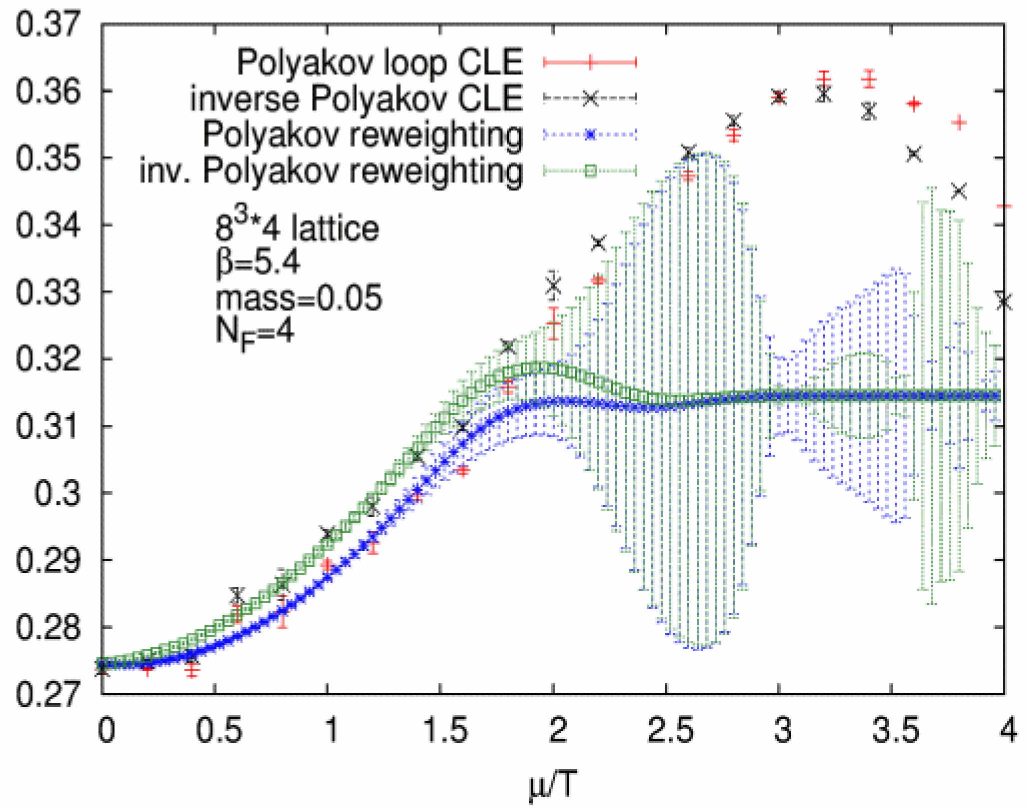
Sign problem gets hard at $\mu/T \approx 1$

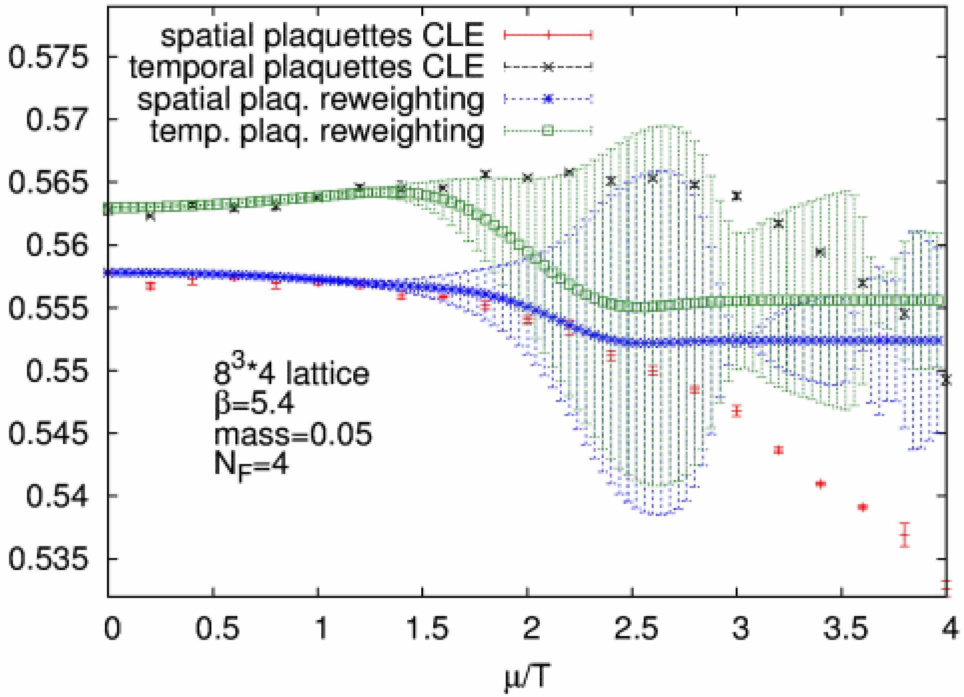
Comparison with reweighting for full QCD

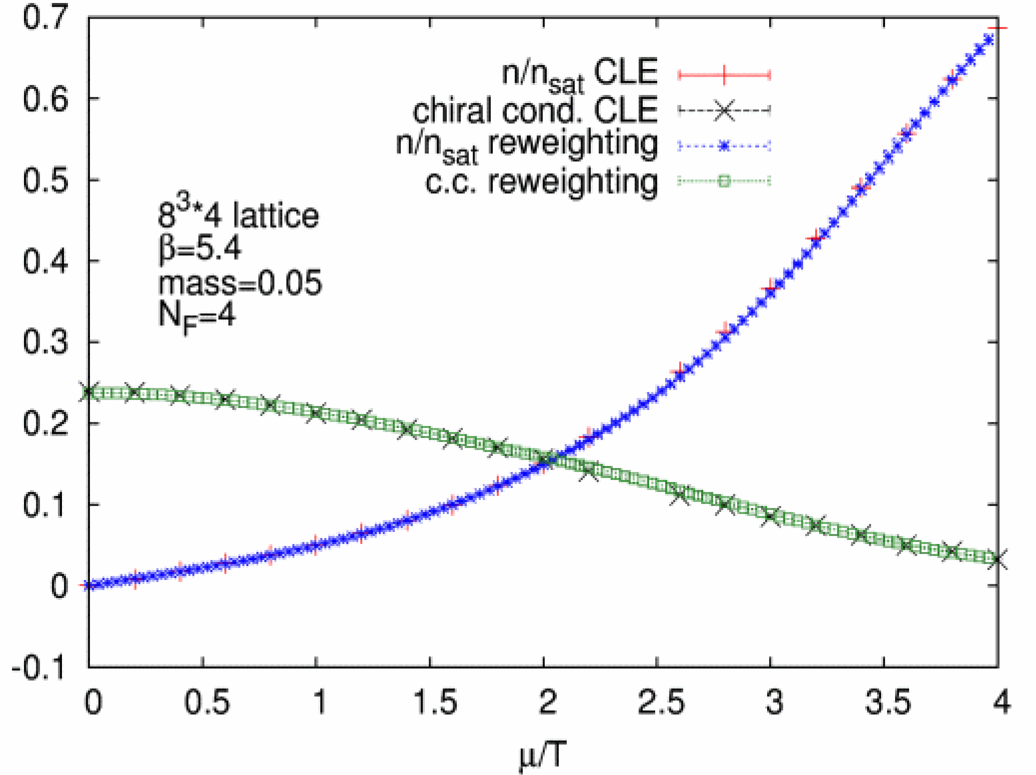
[Fodor, Katz, Sexty, Török 2015]

Reweighting from ensemble at

$$R = \mathrm{Det}\, M(\mu = 0)$$





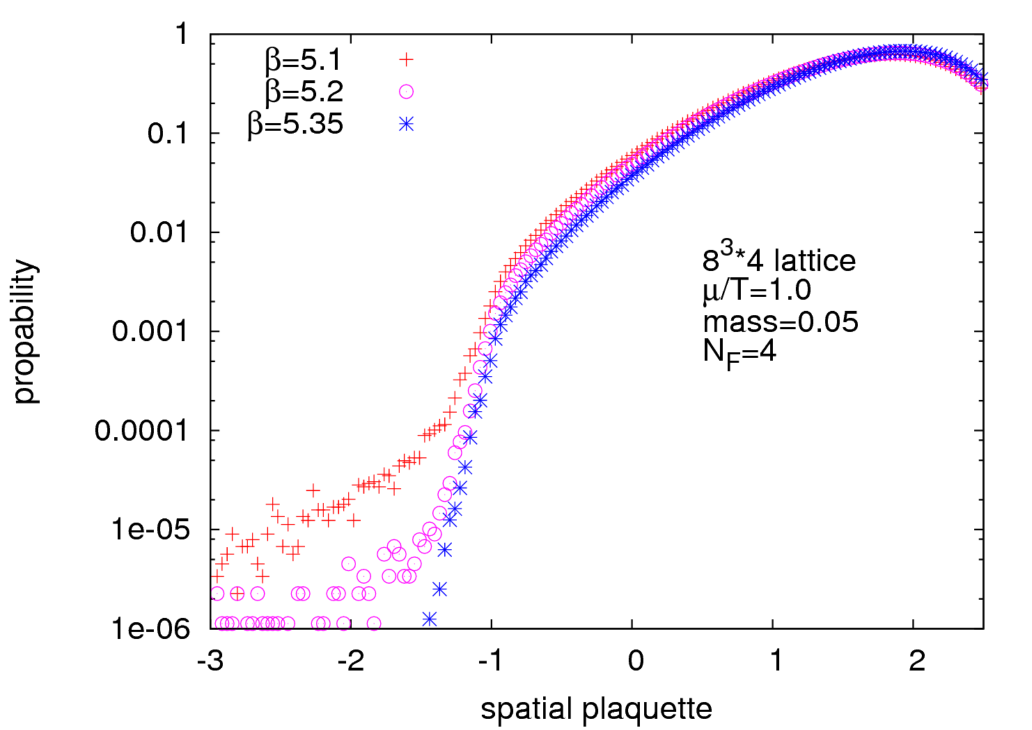


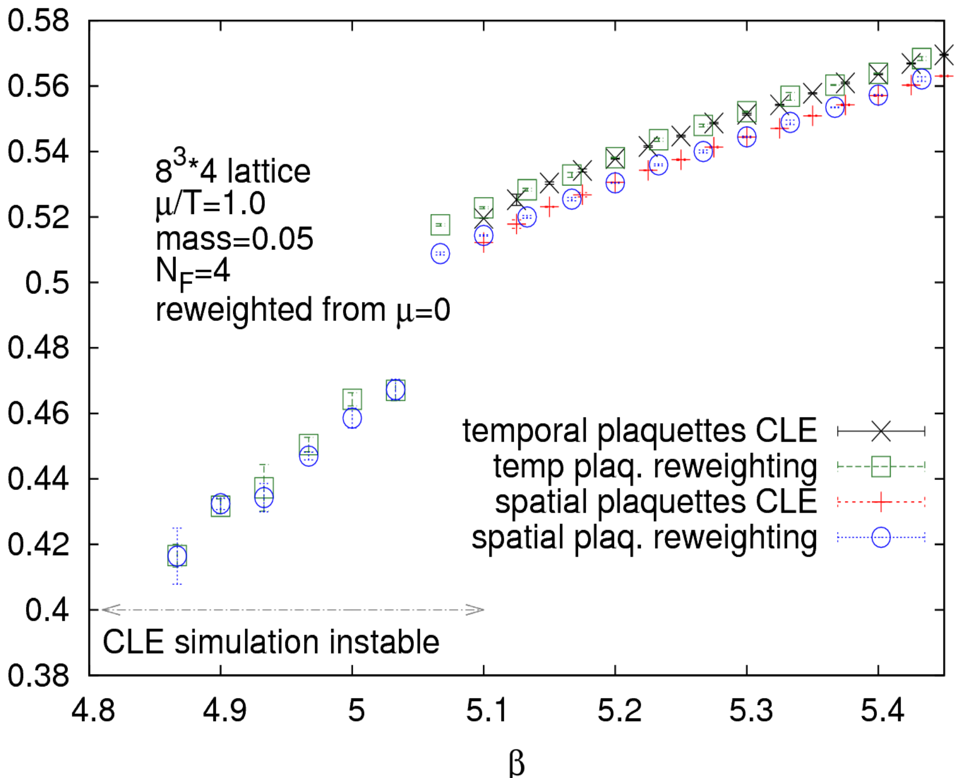
Comparisons as a function of beta

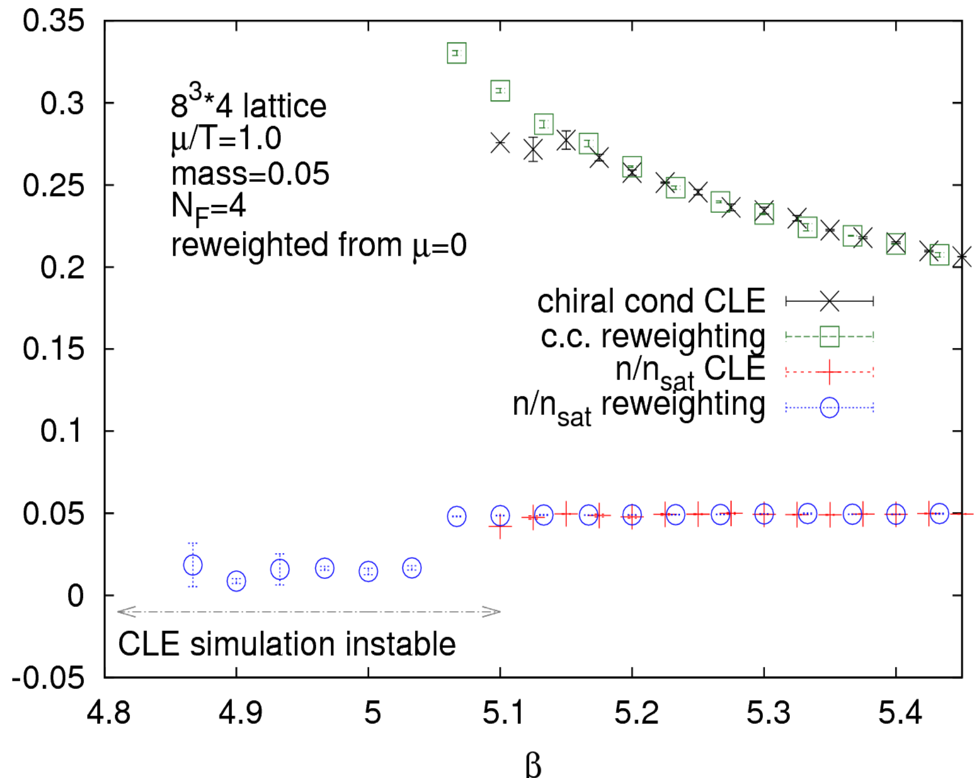
Similarly to HDQCD
Cooling breaks down at small beta

at $N_T = 4$ breakdown at $\beta = 5.1 - 5.2$

At larger N_T?

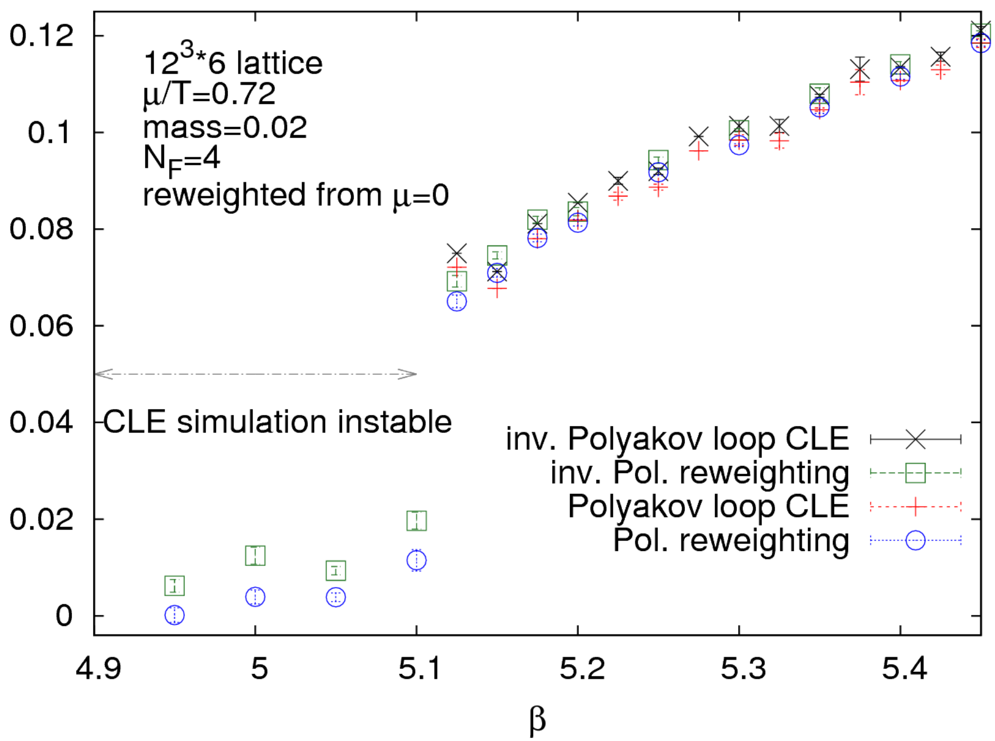


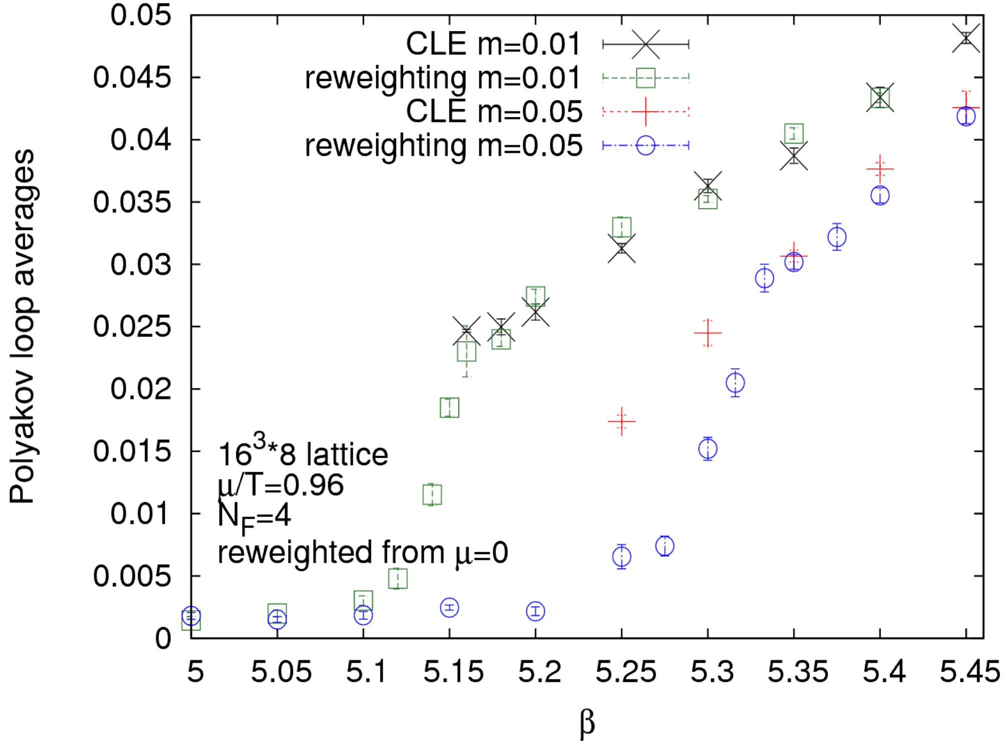




Comparisons as a function of beta

$N_T = 6$ $N_T = 8$





Breakdown prevents simulations in the confined phase for staggered fermions with $N_T = 4,6,8$

Two ensembles: $m_\pi \approx 4.8\, T_c$

$m_\pi \approx 2.3\, T_c$

Ongoing efforts concerning the QCD phase diag

with Manuel Scherzer and Nucu Stamatescu

1. Following phase transition line
   Do we meet a critical point?

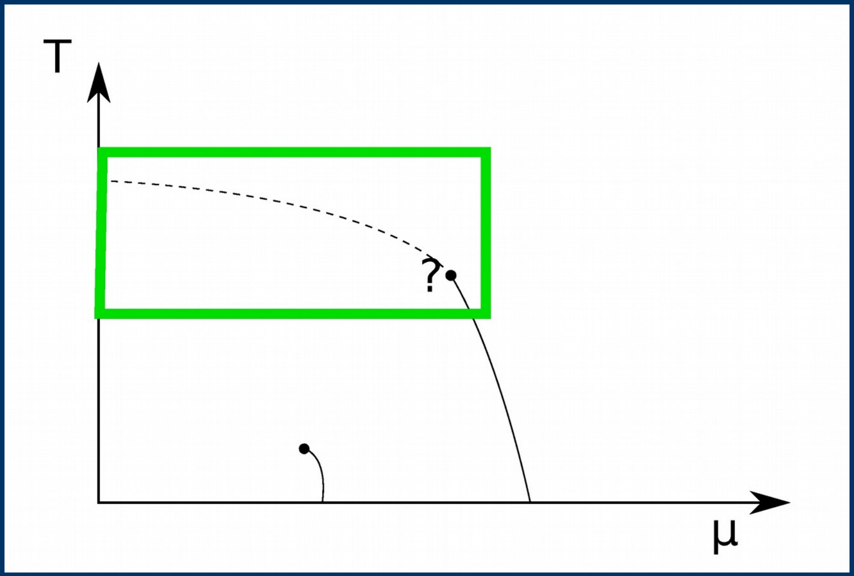
2. Onset transition at small temperatures

$$\frac{m_\pi}{2} \text{ vs. } \frac{m_N}{3}$$

3. Calculating the pressure at high temperatures
   compare with know results

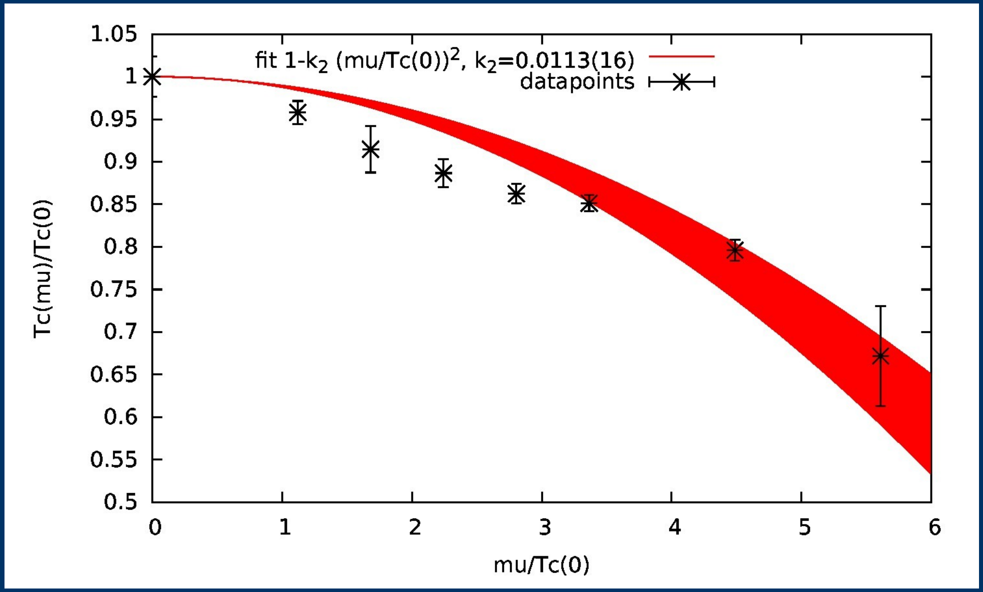
4. Implementing improved actions
   also for fermions

Mapping out the phase transition line



Follow the phase transition line
starting from $\mu = 0$

Using Wilson fermions



Can follow the line to
quite high μ/T

Compatible with expected behavior
at small chemical pot.

See talk by Scherzer

Onset transition in QCD

Low temperature, chemicial potential is increased

Nuclear matter onset at $\mu_c = m_N/3$

"benchmark:" Phasequenched theory (equivalent to isospin chem. pot.)

$\det M(\mu) \rightarrow |\det M(\mu)|$ Simulation with ordinary importance sampling

Pion condensation onset at $\mu_{c,PQ} = m_\pi/2$

Can we see the difference?

Hard problem:

For large quark masses $m_\pi/2 \approx m_N/3$

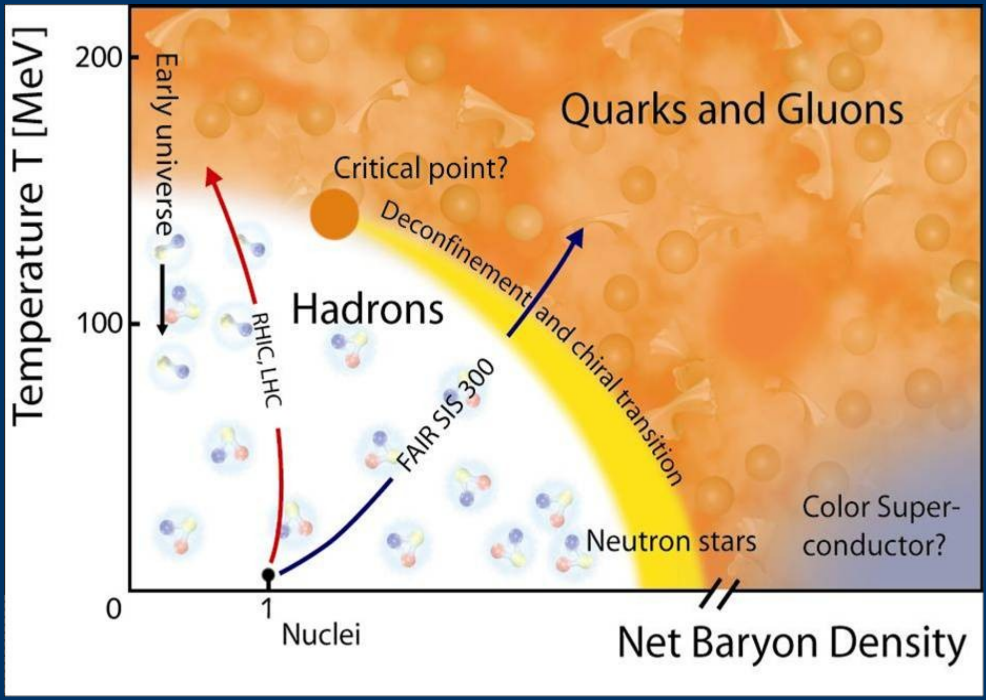
Low quark masses are expensive

Temperature effects might shift μ_c

Low temperature is expensive

Huge finite size effects

Thermalization potentially slow

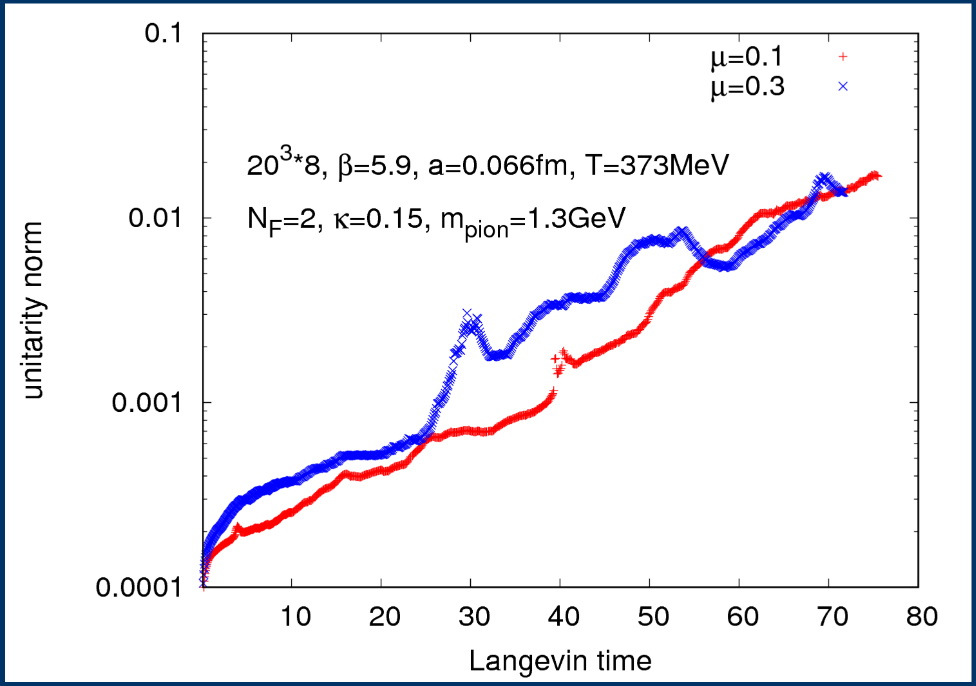


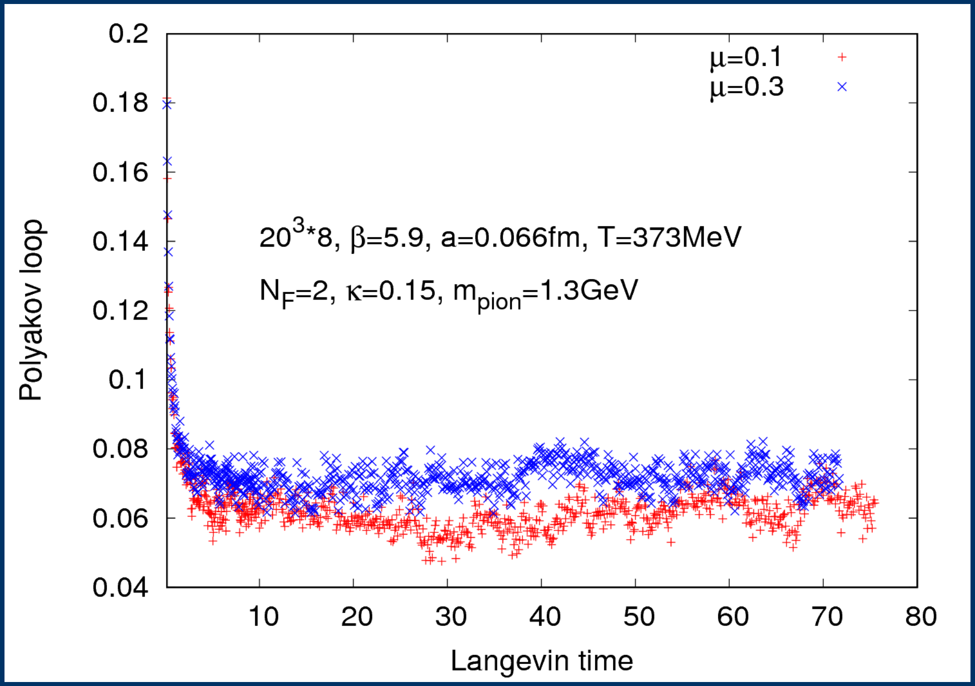
Long runs with CLE

Unitarity norm has a tendency to grow slowly (even with gauge cooling)

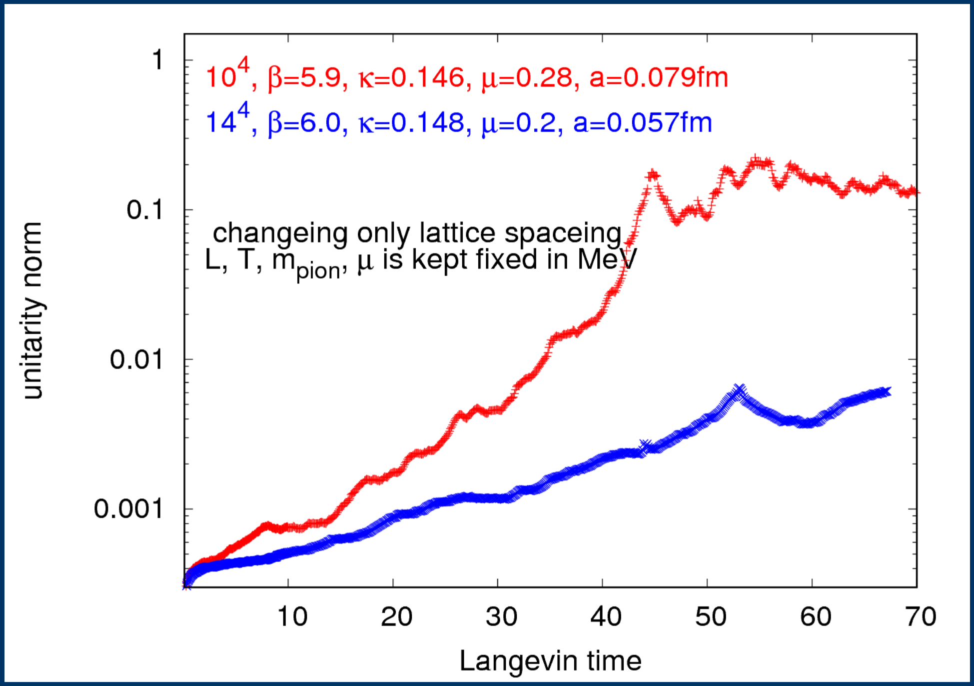
Runs are cut if it reaches ~ 0.1

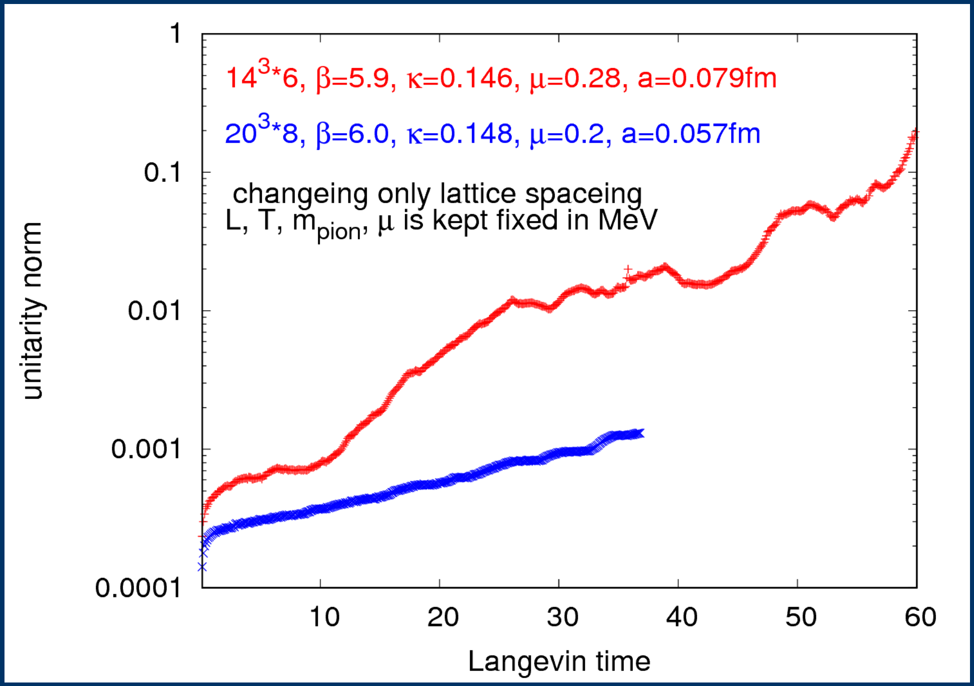
Thermalization usually fast
– might be problematic close to critical point or at low T





Getting closer to continuum limit





Test with Wilson fermions
Increase β by 0.1 – reduces lattice spacing by 30%
change everything else to stay on LCP

behavior of Unitarity norm improves

Pressure of the QCD Plasma at non-zero density

$$\frac{p}{T^4} = \frac{\ln Z}{V T^3}$$

→ Derivatives of the pressure are directly measureable

→ Integrate from T=0

Other strategies:

Measure the Stress-momentum tensor using gradient flow

[Suzuki, Makino (2013-)]

Shifted boundary conditions

[Giusti, Pepe, Meyer (2011-)]

Non-equilibrium quench

[Caselle, Nada, Panero (2018)]

First integrate along the temperature axis, then explore $\mu > 0$

Taylor expansion [Allton et. al. (2002-), ...]

Simulating at imaginary μ to calculate susceptibilities

[Bud.-Wupp. Group (2018)]

Pressure of the QCD Plasma at non-zero density

$$\Delta\left(\frac{p}{T^4}\right)=\frac{p}{T^4}(\mu=\mu_q)-\frac{p}{T^4}(\mu=0)$$

If we want to stay at $\mu=0$

$$\Delta\left(\frac{p}{T^4}\right)=\sum_{n>0,even} c_n(T)\left(\frac{\mu}{T}\right)^n$$

$$c_2=\frac{1}{2}\frac{N_T}{N_s^3}\frac{\partial^2 \ln Z}{\partial\mu^2}$$

$$c_4=\frac{1}{24}\frac{1}{N_s^3 N_T}\frac{\partial^4 \ln Z}{\partial\mu^4}$$

Measuring the coefficients of the Taylor expansion

$$\frac{\partial^2 \ln Z}{\partial\mu^2}=N_F^2\langle T_1^2\rangle+N_F\langle T_2\rangle$$

$$\frac{\partial^4 \ln Z}{\partial\mu^4}=-3\left(\langle T_2\rangle+\langle T_1^2\rangle\right)^2+3\langle T_2^2\rangle+\langle T_4\rangle$$
$$+\langle T_1^4\rangle+4\langle T_3 T_1\rangle+6\langle T_1^2 T_2\rangle$$

$$T_1/N_F=\mathrm{Tr}\left(M^{-1}\partial_\mu M\right)$$
$$T_{i+1}=\partial_\mu T_i$$
$$T_2/N_F=\mathrm{Tr}\left(M^{-1}\partial_\mu^2 M\right)-\mathrm{Tr}\left(\left(M^{-1}\partial_\mu M\right)^2\right)$$
$$T_3/N_F=\mathrm{Tr}\left(M^{-1}\partial_\mu^3 M\right)-3\,\mathrm{Tr}\left(M^{-1}\partial_\mu M\, M^{-1}\partial_\mu^2 M\right)$$
$$+2\,\mathrm{Tr}\left(\left(M^{-1}\partial_\mu M\right)^3\right)$$
$$T_4/N_F=\mathrm{Tr}\left(M^{-1}\partial_\mu^2 M\right)-4\,\mathrm{Tr}\left(M^{-1}\partial_\mu M\, M^{-1}\partial_\mu^3 M\right)$$
$$-3\,\mathrm{Tr}\left(M^{-1}\partial_\mu^2 M\, M^{-1}\partial_\mu^2 M\right)-6\,\mathrm{Tr}\left(\left(M^{-1}\partial\mu\, M\right)^4\right)$$
$$+12\,\mathrm{Tr}\left(\left(M^{-1}\partial_\mu M\right)^2 M^{-1}\partial_\mu^2 M\right)$$

Pressure of the QCD Plasma using CLE

[Sexty (in prep.)]

If we can simulate at $\mu > 0$

$$\Delta\left(\frac{p}{T^4}\right) = \frac{p}{T^4}(\mu=\mu_q) - \frac{p}{T^4}(\mu=0) = \frac{1}{V T^3}\left(\ln Z(\mu) - \ln Z(0)\right)$$

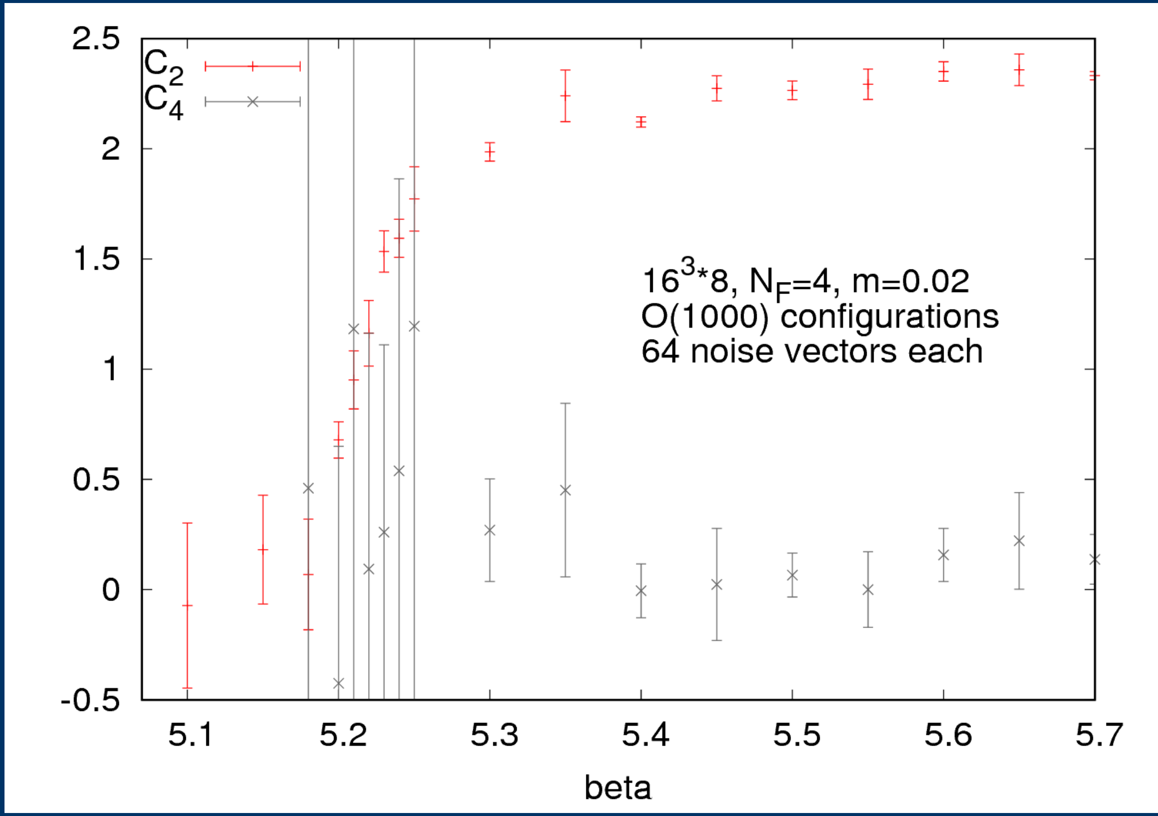
$$\ln Z(\mu) - \ln Z(0) = \int_0^\mu d\mu \frac{\partial \ln Z(\mu)}{\partial \mu} = \int_0^\mu d\mu\, \Omega\, n(\mu)$$

$$n(\mu) = \langle \mathrm{Tr}\left(M^{-1}(\mu)\, \partial_\mu M(\mu)\right)\rangle$$

Using CLE it’s enough to measure the density – much cheaper

Taylor expansion

Using naiv staggered action with $N_F = 4$



Observables very noisy

state of the art calculations barely see a signal at 8th order

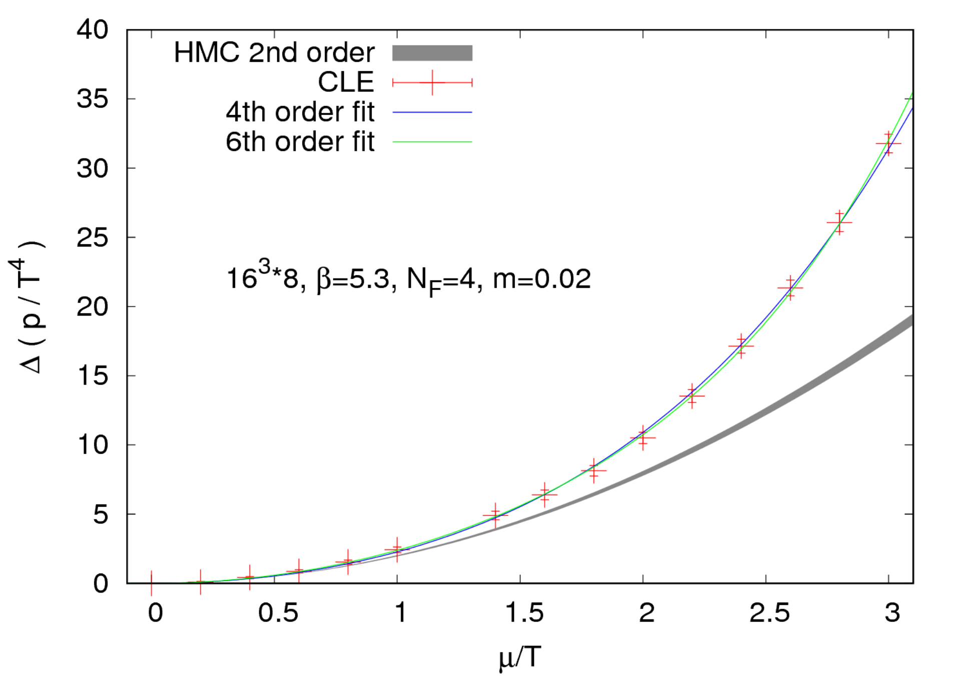
Disconnected terms

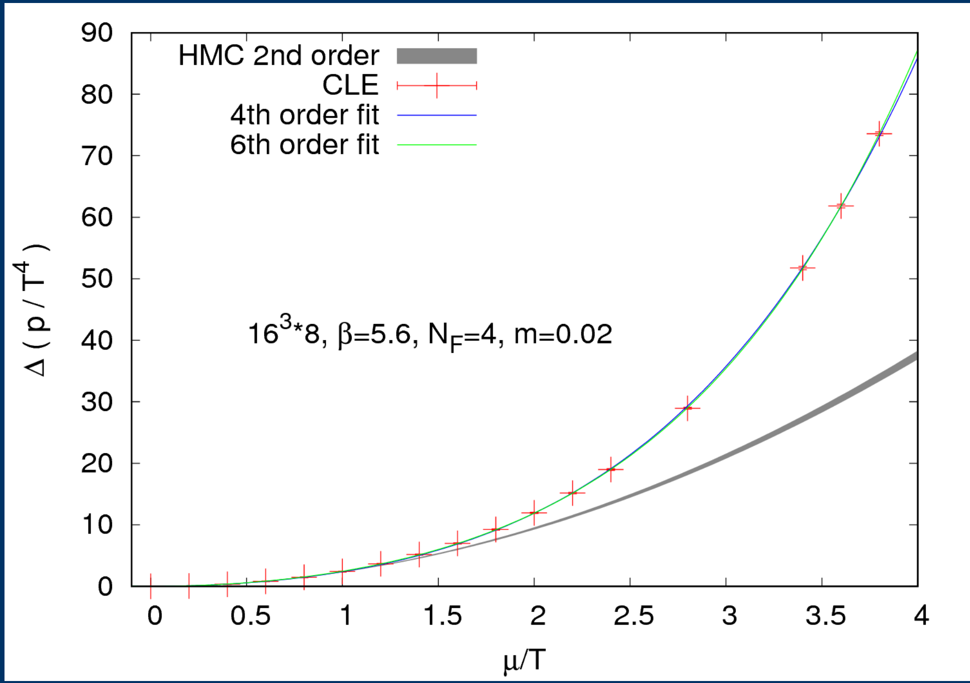
e.g. $\langle T_1^2 T_2 \rangle$

contribute most of the nosie

Pressure calculated with CLE

Integration performed numerically
Jackknife error estimates



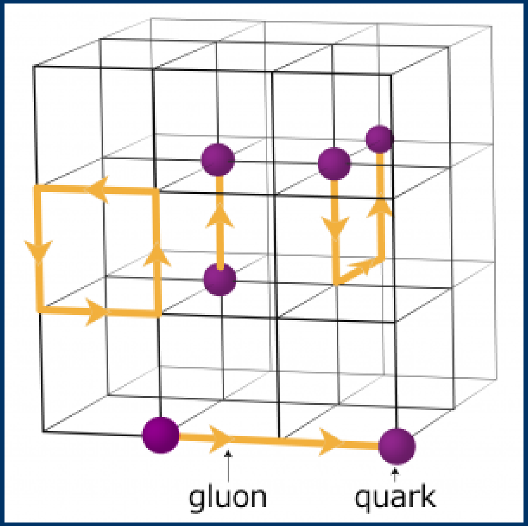


$T = 250\,\text{MeV}\,,\quad T_c \approx 190\,\text{MeV}$

$T = 475\,\text{MeV}$

β	a (fm)	c_2 HMC	c_4 HMC	c_2 CLE	c_4 CLE
5.3	0.099 ± 0.001	1.986 ± 0.042	0.27 ± 0.23	2.117 ± 0.1	0.152 ± 0.05
5.6	0.052 ± 0.0013	2.351 ± 0.044	0.16 ± 0.12	2.168 ± 0.1	0.200 ± 0.05

Improved actions for lattice QCD



Carrying out continuum extrapolation $a \to 0$

Simulate at multiple lattice spacings

Fitting some observable

$$O(a) = O_0 + O_1 a + O_2 a^2 + \dots$$

Change action such that O_1 is eliminated

Gauge improvement

Include larger loops in action

Symanzik action: $S = -\beta \left(\frac{5}{3} \sum \mathrm{Re\,Tr} \, \square \; - \frac{1}{12} \sum \mathrm{Re\,Tr} \, \square\square \right)$

Straightforwardly implemented in CLE

Analyticity must be preserved: $2\,\mathrm{Re\,Tr}\, U = \mathrm{Tr}\, U + \mathrm{Tr}\, U^{+} \;\to\; \mathrm{Tr}\, U + \mathrm{Tr}\, U^{-1}$

Improved fermion actions

Changeing the Dirac operator

Wilson fermions: clover improvement adds a clover term

Staggered fermions: naik or p4 take into account 3-link terms

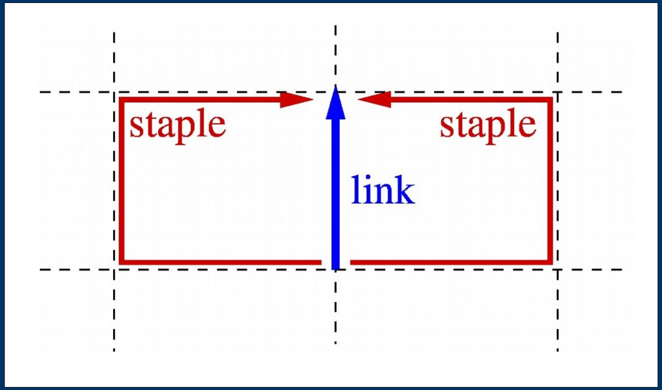
Fat links

Smear the gauge fields inside the Dirac operator

APE, HYP

$$V_\mu = (1-\alpha) U_\mu + \alpha \sum staples$$

$$U'_\mu = \mathrm{Proj}_{SU(3)} V_\mu$$



Stout

$$U'_\mu = e^{i Q_\mu} U_\mu \qquad Q_\mu = \rho \sum \text{staples}$$

essentially one step of gradient flow
with stepsize ρ

Stout smearing

$$U'_\mu = e^{iQ_\mu} U_\mu \qquad Q_\mu = \rho \sum \text{staples}$$

Usually multiple steps: $U \rightarrow U^{(1)} \rightarrow U^{(2)} \rightarrow \ldots \rightarrow U^{(n)}$

Replace gauge fields in Dirac matrix $\det M(U) \rightarrow \det M(U^{(n)})$

For the Langevin eq. we need drift terms: $\dfrac{\partial S_{\text{eff}}}{\partial U}$ with $S_{\text{eff}} = S_g + \ln \det M(U^{(n)})$

Calculated by “going backwards” $\dfrac{\partial S_{\text{eff}}}{\partial U} = \dfrac{\partial S_{eff}}{\partial U^{(n)}} \dfrac{\partial U^{(n)}}{\partial U^{(n-1)}} \ldots \dfrac{\partial U^{(1)}}{\partial U}$

One iteration: $\dfrac{\partial U'}{\partial U} = \dfrac{\partial e^{iQ}}{\partial U} U + e^{iQ}$

local terms
+ nonlocal terms from staples

Stout smearing and complex Langevin

[Sexty (in prep.)]

$$U'_\mu = e^{iQ_\mu} U_\mu \qquad Q_\mu = \rho \sum \text{staples}$$

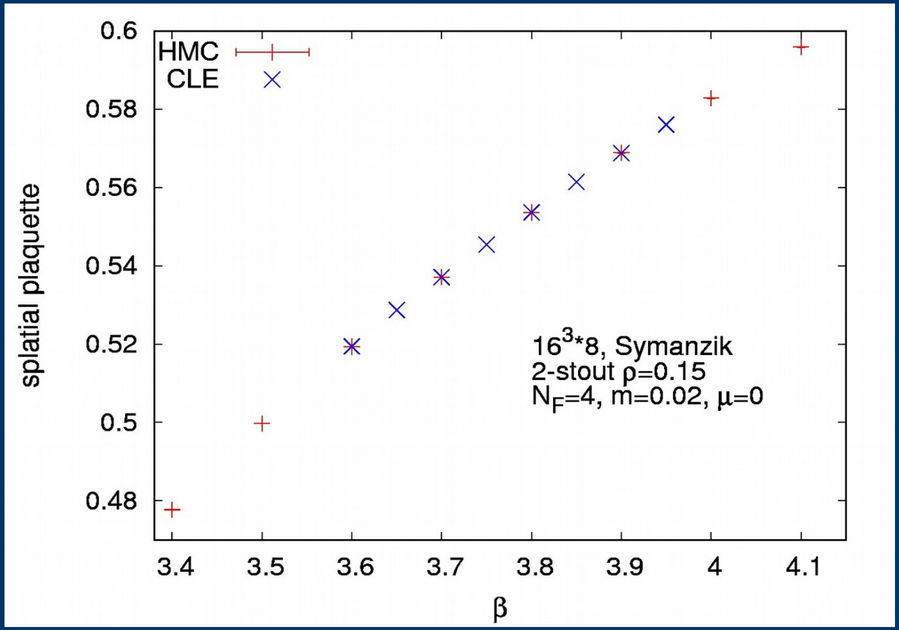
Adjungate is replaced with inverse for links

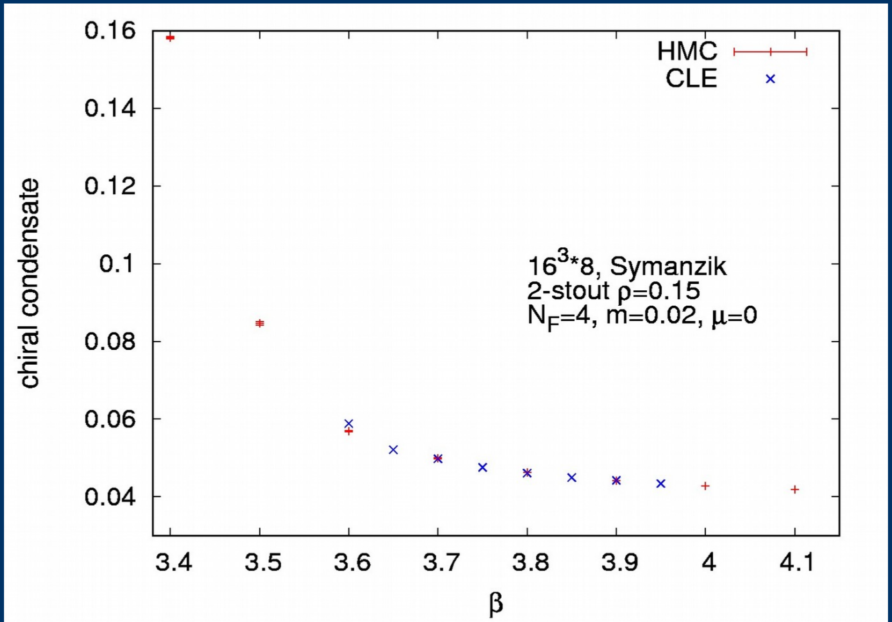
Q^+ is not replaced with Q^{-1} (because its a sum)

Q is no longer hermitian

Calculation of $\frac{\partial e^{iQ}}{\partial U}$ becomes trickier

Benchmarking with HMC at $\mu = 0$





$$a(\beta = 3.6) = 0.12\,\text{fm} \qquad a(\beta = 3.9) = 0.064\,\text{fm}$$

What happens with the configurations?

$16^3 \times 8$, stoutn=4, $\rho = 0.125$, $m = 0.02$, $\mu = 0.3$, $\beta = 3.9$

smearing step	plaqavg	unitarity norm
0	0.562948	0.00913145
1	0.837735	0.0108531
2	0.929264	0.0118543
3	0.964314	0.0125011
4	0.97947	0.0129698
5	0.986835	0.0133134
6	0.990828	0.0135655
7	0.993218	0.0137527
8	0.994766	0.0139118
9	0.995831	0.0140539
10	0.996598	0.0141745

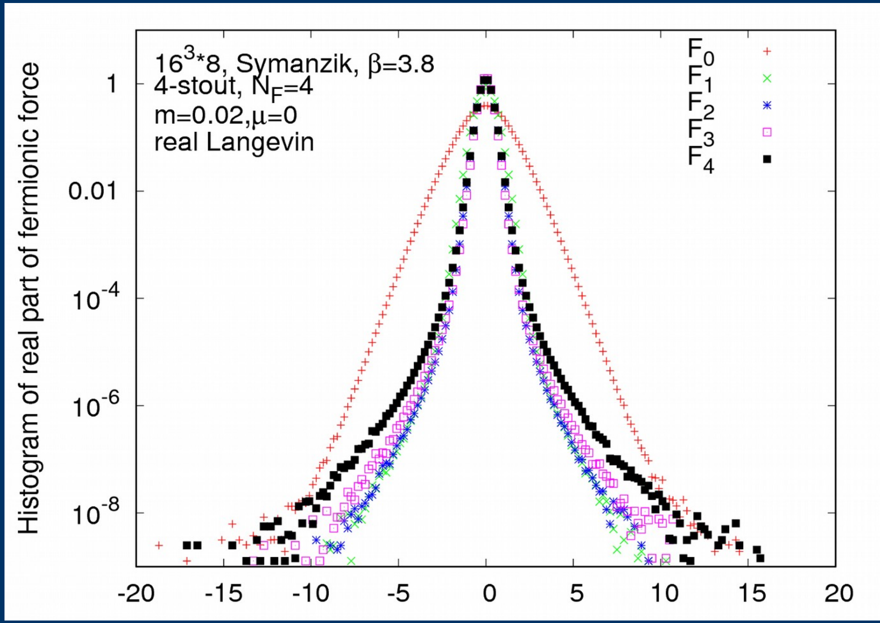
Real part of gauge fields decay

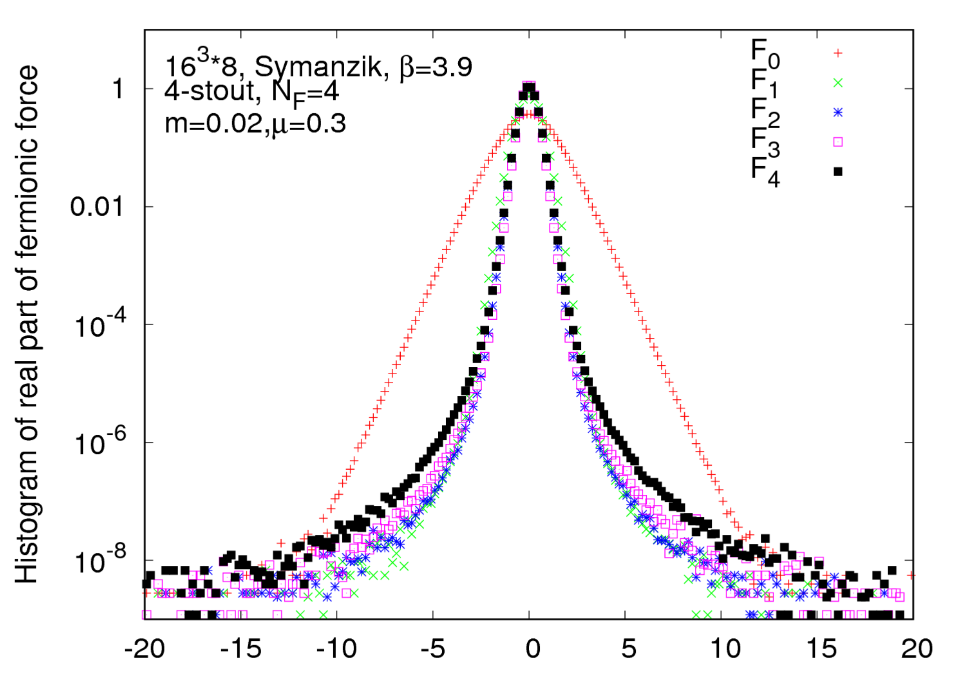
Unitarity norm slightly rises

What happens with the drift terms?

$$\frac{\partial S_{\text{eff}}}{\partial U}=\frac{\partial S_{\text{eff}}}{\partial U^{(n)}}\frac{\partial U^{(n)}}{\partial U^{(n-1)}}\cdots\frac{\partial U^{(1)}}{\partial U}$$

$$\frac{\partial S_{\text{eff}}}{\partial U^{(n)}}=F_0, \quad \frac{\partial S_{\text{eff}}}{\partial U^{(n)}}\frac{\partial U^{(n)}}{\partial U^{(n-1)}}=F_1, \ldots \quad \frac{\partial S_{\text{eff}}}{\partial U}=F_n$$





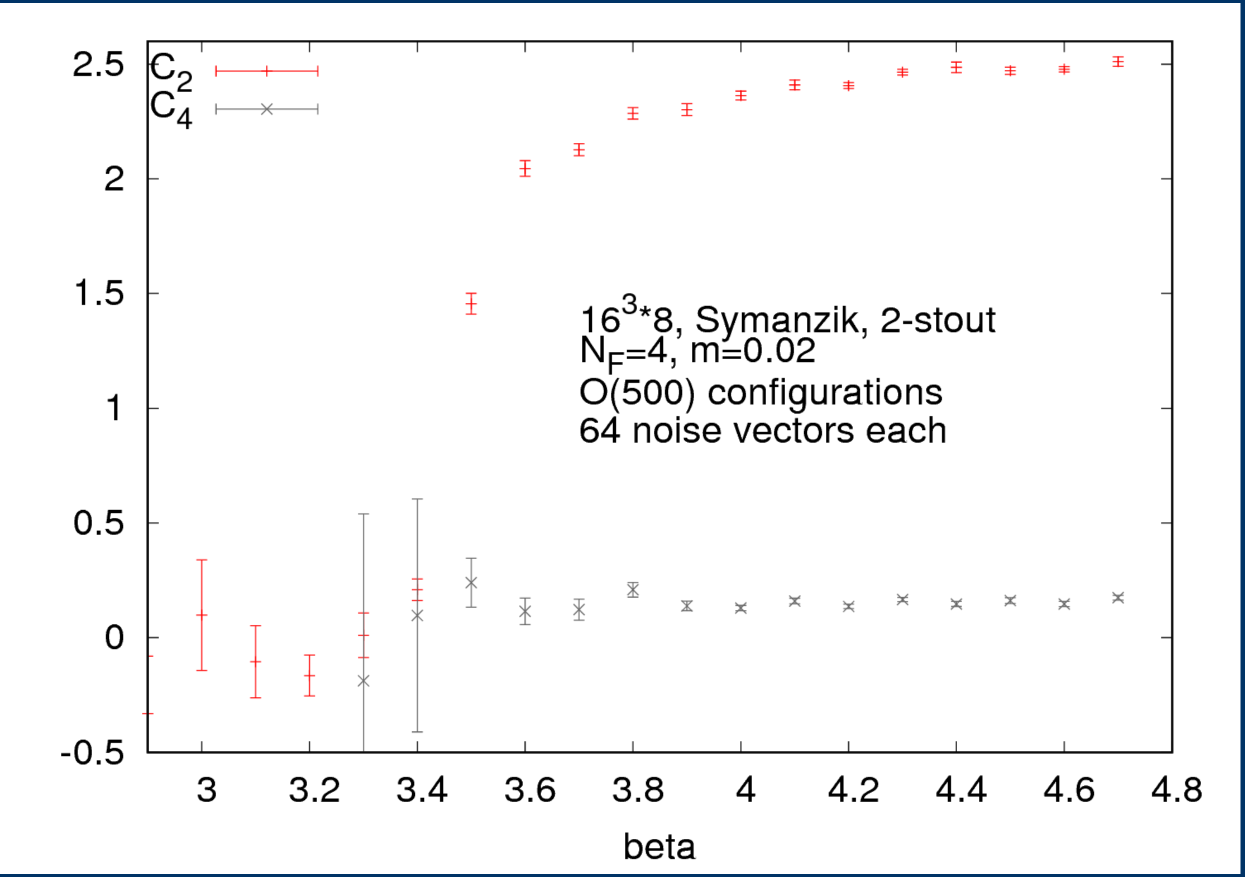
Average drift term is smaller

Long tail

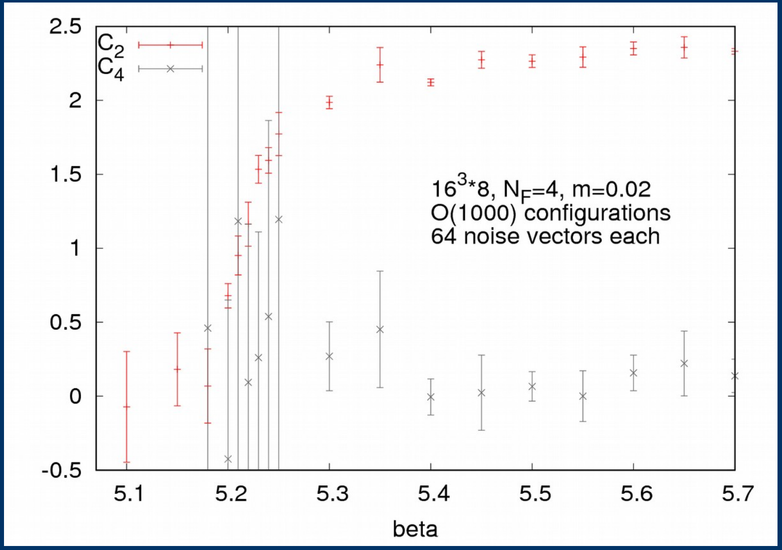
→ More prone to runaways

→ Smaller stepsize needed

Pressure with improved action



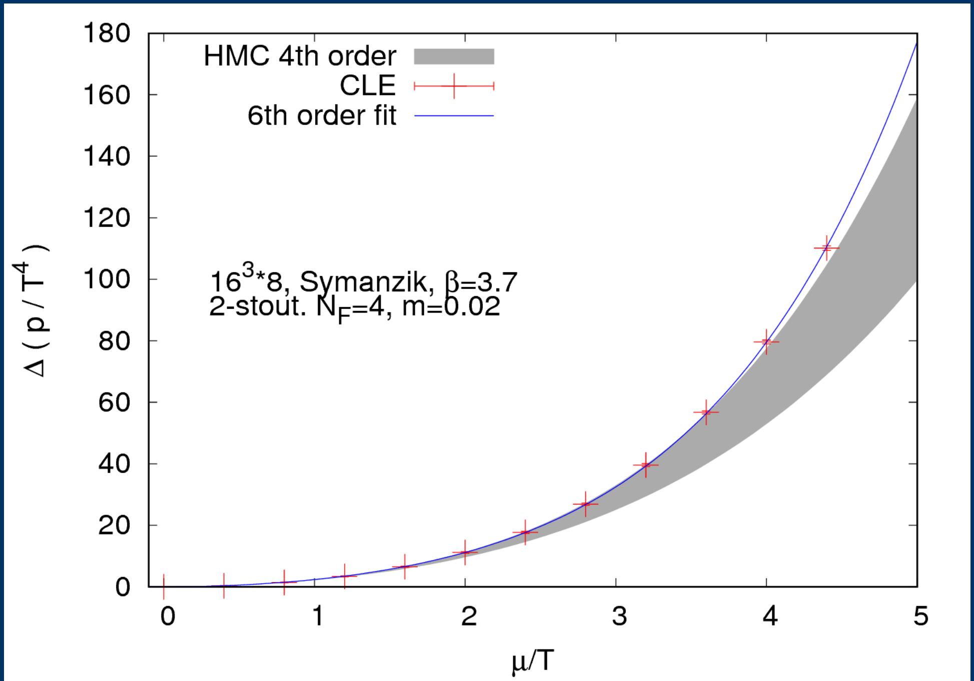
naiv action



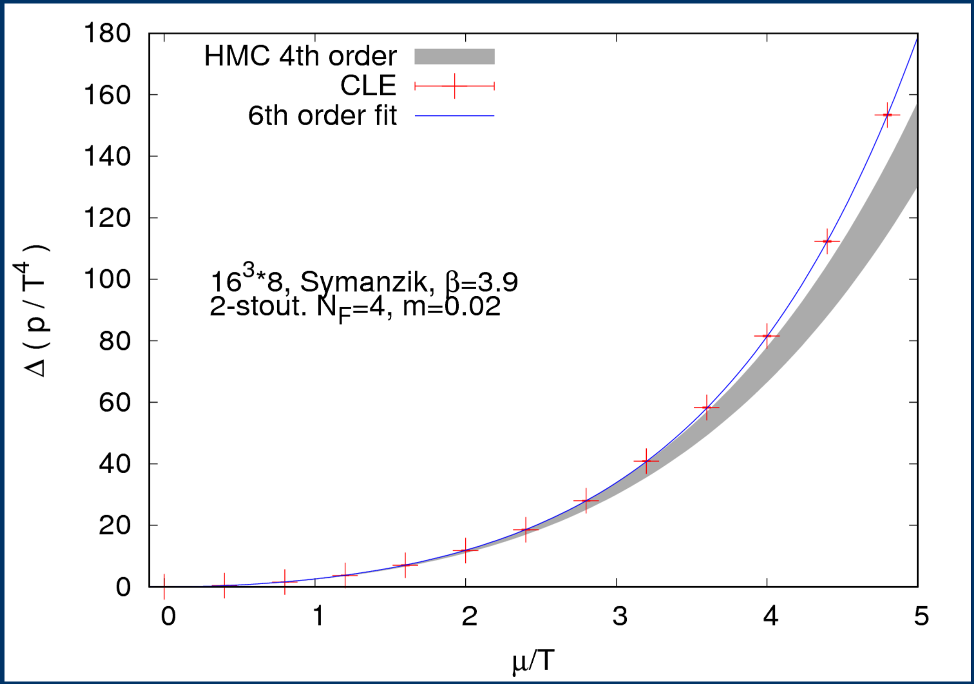
C_4 is measurable with this action at high T (with O(500) configs.)

Pressure with improved action

Symanzik gauge action
stout smeared staggered fermions



$T = 260\,\mathrm{MeV}$



$T = 385\,\mathrm{MeV}$

β	a(fm)	c_2 HMC	c_4 HMC	c_2 CLE	c_4 CLE	c_6 CLE
3.7	0.094 ± 0.001	2.127 ± 0.026	0.122 ± 0.046	2.143 ± 0.07	0.151 ± 0.02	0.0014 ± 0.001
3.9	0.064 ± 0.001	2.302 ± 0.026	0.138 ± 0.021	2.314 ± 0.04	0.143 ± 0.007	0.0018 ± 0.0003

Good agreement
CLE calculation is much cheaper

Summary

CLE is a versatile tool to solve sign problems

Potential problems with boundary terms and poles
- Monitoring of the process is required

Promising results for many systems:
- phase diagram of HDQCD mapped out

Ongoing effort for full QCD to get physical results
- Mapping out phase transition line
- Onset transition at small temperatures
- Calculating the pressure
- Using improved actions