Numerical calculation of the quark energy spectrum in a mesonic background field

π-Meson: Chiral Hedgehog field

W. Kern P. Kratochwill M. Mandl
B. Riederer M. Schröfl

Project in: "Foundations of Particle Physics"

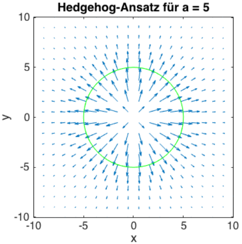
June 2018

DE

Task

1. Calculation of the total energy E_{tot} as a function of the parameter a of the hedgehog field $\boldsymbol{\Theta}(\boldsymbol{x}) = \hat{\boldsymbol{r}}\,\Theta(r)$

$$\Theta(r) = -\pi \begin{cases} 1 - \frac{2r}{3a} & \text{r} \leq \text{a} \\ \frac{a^2}{3r^2} & \text{r} \geq \text{a} \end{cases} \quad (1)$$



2. Diagonalization of the one-particle dirac hamiltonian h

$$h = \boldsymbol{a} \cdot \boldsymbol{p} + \beta m \left(\cos\Theta(r) + i\gamma_5\, \boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \sin\Theta(r)\right) \quad (2)$$

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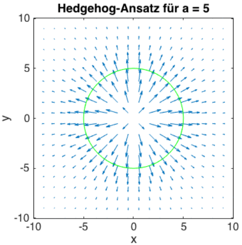

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2. Eigenstates: Grand-Spin operator $\boldsymbol{G} = \boldsymbol{J} + \frac{\tau}{2} = \boldsymbol{l} + \frac{\sigma}{2} + \frac{\tau}{2}$

$$J = \begin{cases} G + 1/2, & l = \begin{cases} G+1 \\ G \end{cases} \\ G - 1/2, & l = \begin{cases} G \\ G-1 \end{cases} \end{cases} \qquad |i, n, J, M\rangle \sim j_l(k_n r)\,|lJGM\rangle \tag{4}$$

3. Problems to solve:
   - Numerical integration (matrix elements: $\int drr^2 f(\Theta) j_l(k_n r) j_{l'}(k_m r)$)
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$$\text{e.g.} \quad |1, n, J = G + 1/2, M\rangle \sim \begin{pmatrix} j_G(k_n r)\,|GG + 1/2GM\rangle \\ j_{G+1}(k_n r)\,|G + 1G + 1/2GM\rangle \end{pmatrix} \qquad (4)$$

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Solving order ↑

Recursion relation

$$\frac{2\mu+1}{x}j_\mu(x) = j_{\mu-1}(x) + j_{\mu+1}(x) \tag{5}$$

Upward recurrence

$$j_\mu(x) = \frac{2\mu-1}{x}j_{\mu-1}(x) - j_{\mu-2}(x) \tag{5a}$$

Starting values:

$$j_0(x) = \frac{\sin x}{x}$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Downward recurrence

$$j_{\mu-1}(x) = \frac{2\mu+1}{x}j_\mu(x) - j_{\mu+1}(x) \tag{5b}$$

Starting values ($\nu >> \mu$):

Arbitrary

$$j_\nu(x) = 0$$
$$j_{\nu+1}(x) = 1$$

Recursion relation

$$\frac{2\mu+1}{x} j_\mu(x) = j_{\mu-1}(x) + j_{\mu+1}(x) \tag{5}$$

Upward recurrence $(x > \mu)$

$$j_\mu(x) = \frac{2\mu-1}{x} j_{\mu-1}(x) - j_{\mu-2}(x) \tag{5a}$$

Starting values:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Downward recurrence $(x \leq \mu)$

$$j_{\mu-1}(x) = \frac{2\mu+1}{x} j_\mu(x) - j_{\mu+1}(x) \tag{5b}$$

Starting values $(\nu >> \mu)$:

Arbitrary

$$j_\nu(x) = 0$$

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Spherical bessel functions

Recursion relation

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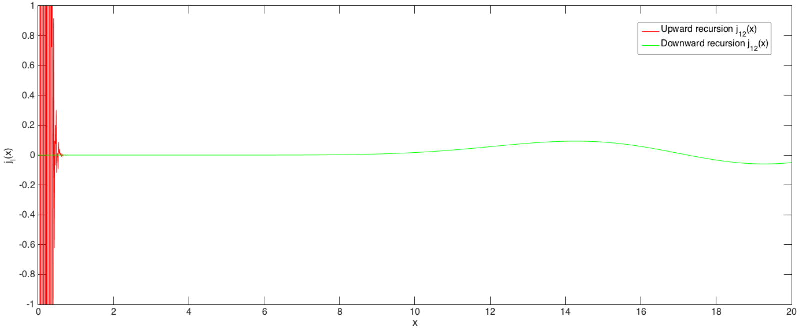

Pseudocode:

```
function root = bisec(f,a,b,eps)

if f(a) * f(b) > 0, error('Too few or
     too many roots in [a,b]')
elseif f(a) == 0, root = a
elseif f(b) == 0, root = b
else
  while eps <= abs(a-b)/4 % error
    c = (a + b) / 2;
          root = c;
          if f(a) * f(c) < 0, b = c;
          elseif f(c) * f(b) < 0, a = c;
          else , break
          end
  end
end
```

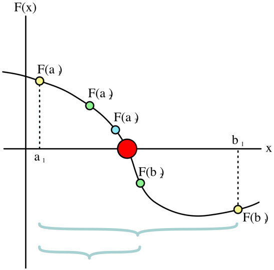

Figure: Bisection method [1]

Error estimation: $$\epsilon^{(n)}_{Bisec} = \frac{a^{(n)} - b^{(n)}}{4} \quad (6)$$

[1] Source: https://commons.wikimedia.org/wiki/File:Bisection_method.svg

- Problem: Solve $S = \int_a^b f(x)dx$
- Idea: approximate integrand $f(x)$ by an integrable quadratic function $P(x)$. $P(x)$ is the interpolant through the points a, b and $(a+b)/2$.
- Improvement: Divide interval $[a, b]$ into N subintervals for good approximation.

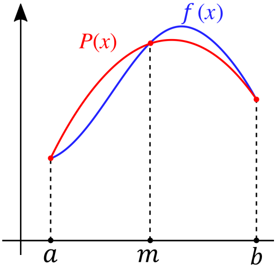

Figure: Simpson's rule [1]

$$S^{(n)} = \frac{h}{3}\left(f(a) + f(b) + 2\sum_{k=1}^{n-1} f\left(a + 2kh\right) + 4\sum_{k=1}^{n} f\left(a + (2k-1)h\right)\right), \quad h = \frac{b-a}{2n} \tag{7}$$

[1] Source: https://commons.wikimedia.org/wiki/File:Simpsons_method_illustration.svg

- Wanted: $D = U^{\dagger}AU$ with $D = diag(\lambda_1, \ldots, \lambda_n)$ and a symmetric
- Set iteratively matrix elements to zero by applying orthogonal rotation matrices $S^{(n)}$: $U = S^{(k)} \cdot \ldots S^{(0)}$
- Two different versions:
  1. Cyclic: Sets each element of the strict upper triangle to zero
  2. Classic: Sets the element with the largest absolute value to zero

Hamiltonian-matrix I

Momentum eigenstates and matrix dimension

- Momentum values k_n are restricted by two conditions:
  1. Momentum Cutoff (UV-Cutoff): $k_{max} = \Lambda \hat{=} \mathcal{O}(10^2) - \mathcal{O}(10^3) MeV$
  2. Box size D (Infrared-Cutoff): $j_l(k_n D) \stackrel{!}{=} 0$

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- Dimension $4N(G)x4N(G)$ for $G > 0$ and $2N(G)x2N(G)$ for $G = 0$

Hamilton matrix

$$h^{int}_{nm} = N_{G+1n} N_{G+1m} \begin{pmatrix} \begin{matrix} a^G_{nm} - b^{G-1}_{mn} + \\ + \frac{1}{2G+1}\left(c^{GG-1}_{nm} + c^{GG-1}_{mn}\right) \end{matrix} & \frac{2\sqrt{G(G+1)}}{2G+1}\left(c^{GG-1}_{nm} - c^{GG+1}_{mn}\right) \\ \frac{2\sqrt{G(G+1)}}{2G+1}\left(c^{GG-1}_{nm} - c^{GG+1}_{mn}\right) & \begin{matrix} a^G_{nm} - b^{G+1}_{mn} + \\ + \frac{1}{2G+1}\left(c^{GG+1}_{nm} + c^{GG+1}_{mn}\right) \end{matrix} \end{pmatrix} \tag{8}$$

$$a^l_{nm} \sim b^l_{nm} \sim \int_0^D drr^2 \left(\cos\Theta(r) - 1\right) j_l(k_n r) j_l(k_m r) \tag{9}$$

$$c^{ll'}_{nm} \sim \int_0^D drr^2 \sin\Theta(r) j_l(k_n r) j_{l'}(k_m r) \tag{10}$$

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Hamiltonian-matrix II

Putting parts together and diagonalization

- Define mass of the Valence Quark $m_{Vq} = 400 MeV$
- Varying parameter am_{Vq} in $[0, 20]$
- Boxsize $D >> a \Rightarrow D = 5 \max(a) \hat{=} 0.25 MeV^{-1}$
- For each a:
  1. Calculate k_n in $[0, \Lambda]$ using $j_G(k_n D) \overset{!}{=} 0$ (Bisec method) and corresponding energies
  2. Calculate a^l_{nm}, b^l_{nm} and $c^{ll'}_{nm}$ (Simpson's rule)
  3. Calculate matrix $h_{nm} = h^D_{nm} + h^{int}_{nm}$
  4. Calculate eigenvalues ϵ_μ (Jacobi rotations)

Calculating E_{tot}

E_{tot}

$$E_{tot} = \underbrace{E_{vac} - E_{vac}(\mathbf{\Theta} = 0)}_{1} + \underbrace{E_{val}}_{2} + \underbrace{E_m}_{3} \tag{11}$$

1. Describes the energy variation of the vacuum from the trivial vacuum

$$E_{vac} = \frac{N_C}{2} \sum_{\mu} \left\{ \frac{\Lambda}{\sqrt{\pi}} \exp\left(-(\epsilon_\mu/\Lambda)^2\right) - |\epsilon_\mu| \mathrm{erfc}(|\epsilon_m u/\Lambda|) \right\} \tag{12}$$

2. Describes the energy of the Valence quark (Here: $\Theta(x)$ stepfunction)

$$E_{val} = N_C |\epsilon_\mu| \Theta(\epsilon_\mu) \tag{13}$$

3. Describes the energy of the mesonic part for the hedgehog ansatz

$$E_m = 4\pi m_\pi^2 f_\pi^2 \int drr^2 \left(1 - \cos\left(\Theta(r)\right)\right) \tag{14}$$

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[6] R. Alkofer, H. Reinhardt, H. Weigel *Baryons as Chiral Solitons in the Nambu-Jona-Lasinio Model*, [arXiv:hep-ph/9501213], 1994

Numerische Berechnung eines Quarkspektrums in einem mesonischen Hintergrundfeld

π-Meson: Chiraler Hedgehog-Ansatz

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Juni 2018

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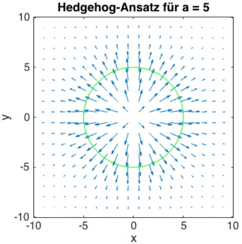

2. Diagonalisieren des Ein-Teilchen Dirac Hamiltonoperators h

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EN

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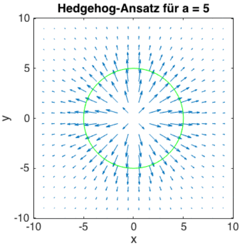

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   - Numerische Integration (Matrixelement: $\int drr^2 f(\Theta) j_l(k_n r) j_{l'}(k_m r)$)
   - Nullstellensuche (Boxgröße und Impulseigenzustände: $j_l(k_n D) \stackrel{!}{=} 0$)
   - Sphärische Besselfunktion (Eigenzustände: $\sim j_l(x)$)

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   - Nullstellensuche (Boxgröße und Impulseigenzustände: $j_l(k_n D) \stackrel{!}{=} 0$)
   - Sphärische Besselfunktion (Eigenzustände: $\sim j_l(x)$)

Aufgabenstellung

1. Hamiltonoperator

$$h = \boldsymbol{a} \cdot \boldsymbol{p} + \beta m + \beta m \left(\cos\Theta(r) - 1 + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \sin\Theta(r)\right) \qquad (3)$$

2. Eigenzustände: Grand-Spin Operator $\boldsymbol{G} = \boldsymbol{J} + \frac{\boldsymbol{\tau}}{2} = \boldsymbol{l} + \frac{\boldsymbol{\sigma}}{2} + \frac{\boldsymbol{\tau}}{2}$

$$J = \begin{cases} G + 1/2, & l = \begin{cases} G+1 \\ G \end{cases} \\ G - 1/2, & l = \begin{cases} G \\ G-1 \end{cases} \end{cases} \qquad |i, n, J, M\rangle \sim j_l(k_n r)\,|lJGM\rangle \qquad (4)$$

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$$\text{e.g.} \quad |1, n, J = G + 1/2, M\rangle \sim \begin{pmatrix} j_G(k_n r)\,|GG + 1/2GM\rangle \\ j_{G+1}(k_n r)\,|G + 1G + 1/2GM\rangle \end{pmatrix} \quad (4)$$

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Arbeitsreihenfolge ↑

EN

Sphärische Besselfunktionen

Rekursionsrelation

$$\frac{2\mu+1}{x} j_\mu(x) = j_{\mu-1}(x) + j_{\mu+1}(x) \tag{5}$$

Vorwärts-Rekursion

$$j_\mu(x) = \frac{2\mu-1}{x} j_{\mu-1}(x) - j_{\mu-2}(x) \tag{5a}$$

Startwerte:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Rückwärts-Rekursion

$$j_{\mu-1}(x) = \frac{2\mu+1}{x} j_\mu(x) - j_{\mu+1}(x) \tag{5b}$$

Startwerte ($\nu >> \mu$):

Willkürlich

$$j_\nu(x) = 0$$

$$j_{\nu+1}(x) = 1$$

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Rekursionsrelation

$$\frac{2\mu+1}{x} j_\mu(x) = j_{\mu-1}(x) + j_{\mu+1}(x) \tag{5}$$

Vorwärts-Rekursion $(x > \mu)$

$$j_\mu(x) = \frac{2\mu-1}{x} j_{\mu-1}(x) - j_{\mu-2}(x) \tag{5a}$$

Startwerte:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Rückwärts-Rekursion $(x \leq \mu)$

$$j_{\mu-1}(x) = \frac{2\mu+1}{x} j_\mu(x) - j_{\mu+1}(x) \tag{5b}$$

Startwerte $(\nu >> \mu)$:

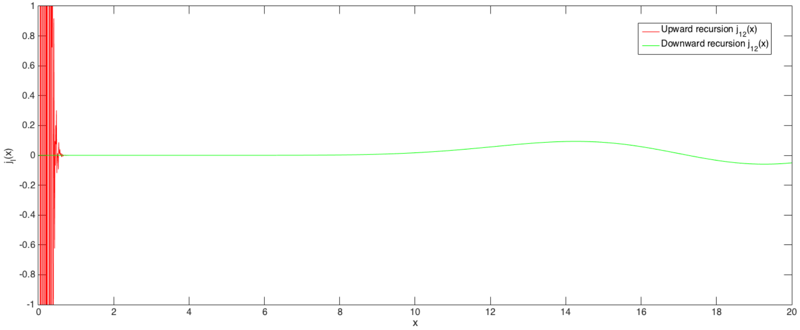
Willkürlich

$$j_\nu(x) = 0$$

$$j_{\nu+1}(x) = 1$$

Rekursionsrelation

$$\frac{2\mu+1}{x} j_\mu(x) = j_{\mu-1}(x) + j_{\mu+1}(x) \tag{5}$$



Pseudocode:

```
function root = bisec(f,a,b,eps)

if f(a) * f(b) > 0, error('Too few or
    too many roots in [a,b]')
elseif f(a) == 0, root = a
elseif f(b) == 0, root = b
else
  while eps <= abs(a-b)/4 % error
    c = (a + b) / 2;
        root = c;
        if f(a) * f(c) < 0, b = c;
        elseif f(c) * f(b) < 0, a = c;
        else , break
        end
  end
end
```

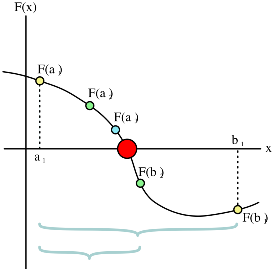


Abbildung: Bisektion-Methode [1]

Fehlerabschätzung:

$$\epsilon^{(n)}_{Bisec} = \frac{a^{(n)} - b^{(n)}}{4} \qquad (6)$$

[1] Source: https://commons.wikimedia.org/wiki/File:Bisection_method.svg

- Problem: Löse $S = \int_a^b f(x)dx$
- Idee: ersetze zu integrierende Funktion $f(x)$ durch integrierbare Parabel $P(x)$. $P(x)$ ist hierbei ein Interpolationspolynom durch die Funktionswerte a, b und $(a+b)/2$.
- Verbesserung: Teile das Interval $[a, b]$ in N Teilintervalle für möglichst gute Näherung.

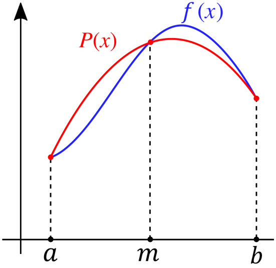


Abbildung: Simpson Regel [1]

$$S^{(n)} = \frac{h}{3}\left(f(a) + f(b) + 2\sum_{k=1}^{n-1} f\left(a + 2kh\right) + 4\sum_{k=1}^{n} f\left(a + (2k-1)h\right)\right), \quad h = \frac{b-a}{2n} \tag{7}$$

[1] Source: https://commons.wikimedia.org/wiki/File:Simpsons_method_illustration.svg

Diagonalisierung

Jacobi-Verfahren

- Gesucht: $D = U^\dagger A U$ mit $D = diag(\lambda_1, \ldots, \lambda_n)$ und a symmetrisch
- Setze iterativ Matrixelemente auf 0 durch Multiplikation mit Rotationsmatrizen $S^{(n)}$: $U = S^{(k)} \cdot \ldots S^{(0)}$
- Zwei Varianten:
  1. Zyklisch: Setzt jedes Element der strikten oberen Dreiecksmatrix auf 0
  2. Klassisch: Setzt immer das Element mit dem größten Absolutwert auf 0

Hamiltonmatrix I

Impulseigenzustände und Matrixdimension

- Impulszustände k_n sind durch zwei Bedingungen eingeschränkt:
  1. Impuls Cutoff (UV-Cutoff): $k_{max} = \Lambda \hat{=} \mathcal{O}(10^2) - \mathcal{O}(10^3) MeV$
  2. Boxgröße D (Infrarot-Cutoff): $j_l(k_n D) \stackrel{!}{=} 0$

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- Dimension: $4N(G)x4N(G)$ für $G > 0$ und $2N(G)x2N(G)$ für $G = 0$

Hamilton matrix

$$h^{int}_{nm} = N_{G+1n} N_{G+1m} \begin{pmatrix} \begin{matrix} a^G_{nm} - b^{G-1}_{mn} + \\ + \frac{1}{2G+1}\left(c^{GG-1}_{nm} + c^{GG-1}_{mn}\right) \end{matrix} & \frac{2\sqrt{G(G+1)}}{2G+1}\left(c^{GG-1}_{nm} - c^{GG+1}_{mn}\right) \\ \frac{2\sqrt{G(G+1)}}{2G+1}\left(c^{GG-1}_{nm} - c^{GG+1}_{mn}\right) & \begin{matrix} a^G_{nm} - b^{G+1}_{mn} + \\ + \frac{1}{2G+1}\left(c^{GG+1}_{nm} + c^{GG+1}_{mn}\right) \end{matrix} \end{pmatrix} \tag{8}$$

$$a^l_{nm} \sim b^l_{nm} \sim \int_0^D drr^2 \left(\cos\Theta(r) - 1\right) j_l(k_n r) j_l(k_m r) \tag{9}$$

$$c^{ll'}_{nm} \sim \int_0^D drr^2 \sin\Theta(r) j_l(k_n r) j_{l'}(k_m r) \tag{10}$$

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$$c^{ll'}_{nm} \sim \int_0^D drr^2 \sin\Theta(r) j_l(k_n r) j_{l'}(k_m r) \quad (10)$$

- Definiere Masse des Valenz-Quarks $m_{Vq} = 400 MeV$
- Variiere Parameter am_{Vq} in $[0, 20]$
- Boxgröße $D >> a \Rightarrow D = 5 \max(a) \hat{=} 0.25 MeV^{-1}$
- Für jedes a:
  1. Berechne k_n in $[0, \Lambda]$ mittels $j_G(k_n D) \stackrel{!}{=} 0$ (Bisektion) und die zugehörigen Energien
  2. Berechne a^l_{nm}, b^l_{nm} und $c^{ll'}_{nm}$ (Simpson Regel)
  3. Berechne Matrix $h_{nm} = h^D_{nm} + h^{int}_{nm}$
  4. Berechne Eigenwerte ϵ_μ (Jacobi-Verfahren)

E_{tot}

$$E_{tot} = \underbrace{E_{vac} - E_{vac}(\mathbf{\Theta} = 0)}_{1} + \underbrace{E_{val}}_{2} + \underbrace{E_m}_{3} \tag{11}$$

1. Änderung der Vakuumsenergie im vergleich zum trivialen Vakuum

$$E_{vac} = \frac{N_C}{2} \sum_{\mu} \left\{ \frac{\Lambda}{\sqrt{\pi}} \exp\left(-(\epsilon_\mu/\Lambda)^2\right) - |\epsilon_\mu| \mathrm{erfc}(|\epsilon_m u/\Lambda|) \right\} \tag{12}$$

2. Energie des Valenz-Quarks (Here: $\Theta(x)$ Stepfunction)

$$E_{val} = N_C |\epsilon_\mu| \Theta(\epsilon_\mu) \tag{13}$$

3. Energie des mesonischen Parts für den Hedgehog-Ansatz

$$E_m = 4\pi m_\pi^2 f_\pi^2 \int drr^2 \left(1 - \cos\left(\Theta(r)\right)\right) \tag{14}$$

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