5

Basic Algebraic Topology

A grand theme in any mathematical discipline is the classification of its objects: When are two such objects "essentially the same"?

In linear algebra, for example, the objects of study are the finite-dimensional vector spaces. One can agree that two such vector spaces are "essentially the same" if they are isomorphic as linear spaces, and one learns in any introduction to the subject that two finite-dimensional vector spaces (over the same field) are isomorphic if and only if their dimensions coincide. For example, $\mathbb{R}^3$ and the vector space of all real polynomials of degree at most two are isomorphic because they are both three-dimensional, but $\mathbb{R}^3$ and $\mathbb{R}^2$ are not isomorphic because their respective dimensions are different.

The dimension of a finite-dimensional vector space is what's called a numerical *invariant*: a number assigned to each such vector space, which can be used to tell different spaces apart.

It would be nice if a classification of topological spaces could be accomplished with equal simplicity, but this is too much to expect. There is a notion of dimension for topological spaces (we have only encountered zero-dimensional spaces in this book; see Exercise 3.4.10), and there are other numerical invariants for (at least certain) topological spaces. In general, however, mere numbers are far too unstructured to classify objects as diverse as topological spaces.

In algebraic topology, one therefore often does not use numbers, but algebraic objects, mostly groups, as invariants. To each topological space, particular groups are assigned in such a way that, if the spaces are "essentially the same", then so are the associated groups.

5.1 Homotopy and the Fundamental Group

If two topological spaces are homeomorphic, then they are "the same" in the sense that they are indistinguishable as far as every property is concerned that can be formulated in terms of their topologies. Hence, for example, the closed unit disc in $\mathbb{R}^n$, which is compact, cannot be homeomorphic to the open unit disc, which isn't. Very often, however, it is not so straightforward to decide whether two spaces are homeomorphic.

The closed unit disc in $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ are both compact, connected, and metrizable spaces. Why shouldn't they be homeomorphic? One can show by elementary means that they aren't (see Exercise 1 below), but the argument requires a little trick. And what about the closed unit disc in $\mathbb{R}^2$ and a closed annulus? Again, both spaces are compact, connected, and metrizable, but—unless the annulus is a circle—the trick from Exercise 1 is useless.

We show in this section that the closed unit disc in $\mathbb{R}^2$ cannot be homeomorphic to an annulus (Example 5.1.27 below), but for this purpose we require new and more powerful tools than we have developed so far. As can be expected, developing those tools requires new definitions.

Definition 5.1.1. *Let* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *be topological spaces. Two continuous maps* $f, g\colon X \to Y$ *are called* homotopic, $f \sim g$ *in symbols, if there is a continuous map* $F\colon [0,1] \times X \to Y$ *such that*

$$F(0,x) = f(x) \quad \text{and} \quad F(1,x) = g(x) \qquad (x \in X).$$

The map F *is called a* homotopy *between* f *and* g.

Intuitively, one can think of a homotopy as a way of "morphing" one function into another.

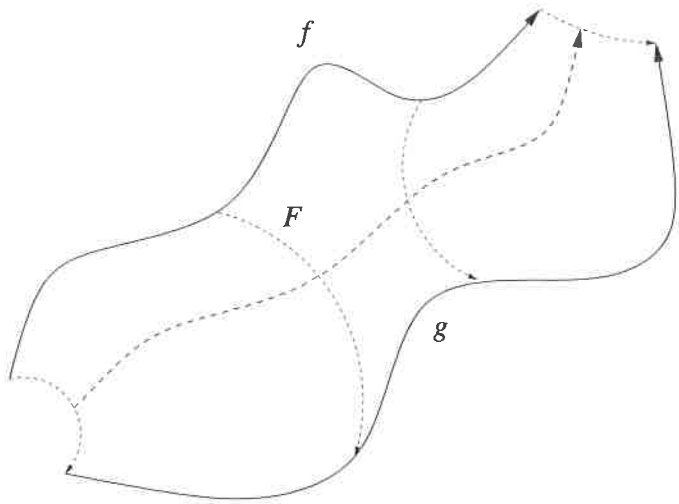

Fig. 5.1: Homotopy

Exercises

1. Prove by elementary means (i.e., without involving any notion of homotopy) that B_2 and $\mathbb{S}^1$ are not homeomorphic. (*Hint*: What can you say about the connectedness of B_2 and $\mathbb{S}^1$ if two distinct points have been removed from both spaces?)