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An Ergonomic Model of Keyboard Fingering for Melodic Fragments

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The fingerings used by keyboard players are determined by a range of ergonomic (anatomic/motor), cognitive, and music-interpretive constraints. We have attempted to encapsulate the most important ergonomic constraints in a model. The model, which is presently limited to isolated melodic fragments, begins by generating all possible fingerings, limited only by maximum practical spans between finger pairs. Many of the fingerings generated in this way seldom occur in piano performance. In the next stage of the model, the difficulty of each fingering is estimated according to a system of rules. Each rule represents a specific ergonomic source of difficulty. The model was subjected to a preliminary test by comparing its output with fingerings written by pianists on the scores of a selection of short Czerny studies. Most fingerings recommended by pianists were among those fingerings predicted by the model to be least difficult; but the model also predicted numerous fingerings that were not recommended by pianists. A variety of suggestions for improving the predictive power of the model are explored.

A significant upsurge in psychological studies of performance in the past 15 years (e.g., Clarke, 1988; Drake & Palmer, 1993; Palmer 1989; Repp, 1992; Shaffer 1981; Sloboda, 1983; Sundberg, Friberg, & Frydén,

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1991; Todd, 1985) has provided a wealth of information on musical expression and its relationship to structure, but, with few exceptions,[1] has largely ignored what is perhaps the most concrete and tangible element in the process—the ways in which performers actually manipulate their instruments physically in order to produce sound. The various roles that physical movements play in the making of music have been addressed and surveyed by Repp (1993), in his synopsis of Truslit, and by Shove and Repp (1995). Studies by Shaffer (1981), Palmer (1989), and Palmer and van de Sande (1993) have focused on the relatively abstract level of motor programming rather than the physical level of the body part and its contact with the instrument. Baily (1985) argued that the musical performer must "operate within a spatial framework and, at some level, represent the task as a spatial one. A question is now whether this representation is a low-level cognitive process or whether it is, or can be, a high-level process at which the conscious planning of performance takes place" (p. 256). In spite of insights of this kind, there still exists no systematic work that examines the transduction of cognitive representations into physical movements.

For many musical instruments, of course, the principal interface between the musician and the instrument is the hand and fingers. Piano performance represents a particularly interesting case in the physical domain of performance because very large numbers of alternative fingerings may exist for virtually any passage of music. This contrasts with many other instruments (e.g., most wind instruments) where there is generally something close to a one-to-one correspondence between a key and a finger normally used to activate that key. In the case of the piano, a sounded note is defined by a particular key to be struck, but any of the 10 available fingers may be considered as potential candidates to execute the note: there is no such thing as the "standard" fingering for a note (although even isolated notes are more likely to be played by certain fingers (*2* or *3*) than by others). The optimal fingering of a note depends thus almost entirely on the context—both physical (on the keyboard) and musical (as expressed in the score)—in which it appears.

Fingering strategies have always been of intense interest to keyboard players, because fingering can significantly affect the technical and expressive qualities of a performance. There is rarely one "best" fingering in any situation, some fingerings assisting precision and speed, others phrasing and dynamic articulation, and yet others memorization; master pianists tend to emphasize the role of musical interpretation in their choice of fingerings (Bamberger, 1976; Clarke, Parncutt, Raekallio, & Sloboda, 1997; Neuhaus, 1958/1973). According to pianists' own accounts, it is not usual to decide on a fingering on purely technical or ergonomic grounds and then, when the notes are under the fingers, to work on musical interpreta-

1. Davies, Kenny, and Barbenel (1989) investigated the interface between the trumpet mouthpiece and the mouth.

tion; instead, a final fingering is typically decided on only after considerable interpretive groundwork has already been completed. A significant component in piano instruction and pedagogy is concerned with assisting the learner to choose fingerings that will best achieve an intended performance. There is still, however, virtually nothing in the way of direct empirical evidence to confirm or refute what pianists say.

This paper is part of a larger project whose aim is investigate the fingering strategies of pianists of differing degrees of expertise and in different performing contexts. In the absence of previous research on fingering, an important requirement is to establish some background predictions—a baseline against which our empirical findings can be measured. The purpose of this paper is to present a model of fingering options based on a set of simple anatomic and motor constraints and to compare the predictions of this model with fingerings written onto scores by skilled pianists; the music is limited to right-hand melodic figurations in a familiar tonal style. By exploring differences between predictions and data, we hope to shed light on cognitive and music-interpretive constraints not accounted for by the model.

Model

In its present form, the model is restricted to the right hand and deals only with the fingering of consecutive notes in melodic fragments of limited length. An appropriate length of a fragment is taken to be the distance that pianists typically look ahead as they sight-read. Previous research has indicated that the number of notes held in working memory in advance of the note currently being played (the *eye-hand span*, Sloboda, 1974) is no more than about six and that more skilled music readers look farther ahead than less skilled readers. Sloboda (1982) reported that professional musicians typically read about 2 s ahead of the music they are playing, a finding broadly confirmed by Goolsby (1994).

Specifically, the input to the model is a series of pitches, which are entered as intervals in semitones above any C on the keyboard. Variations in duration, articulation, dynamics, and other expressive features are ignored; in effect, we assume legato isochrony at a moderate (mezzoforte) dynamic level. The output is a list of possible fingerings, ranked from easiest to hardest, along with estimates of their difficulty relative to each other.

TERMINOLOGY AND ESTIMATION OF FINGER SPANS[2]

The maximum spans that are available between pairs of fingers represent a fixed constraint on the composition and performance of piano mu-

2. We number the fingers according to standard keyboard practice: *1* = thumb, *2* = index finger, ..., and *5* = little finger. Italics distinguish finger numbers from other numbers in the text.

sic. The central importance of stretch in piano technique is reflected by the attention given to the subject by published piano methods. At the simplest level, pieces for beginning pianists generally avoid octaves. At the other extreme, wide stretches between all pairs of fingers contribute to technical difficulty of virtuosic piano music. In the music of the late Romantics (Brahms, Liszt, Rachmaninoff, Skriabin), wide stretches are demanded both for sonorities of simultaneous notes and for broken chords (that can cover even wider intervals).

The furthest distance that two fingers can stretch within the limitations of the hand's anatomy is here termed their *maximum possible span* and abbreviated to MaxPoss. MaxPoss has been measured for all finger pairs of more than 200 pianists by Wagner (1988), who called the parameter *active span,* and for selected finger pairs of more than 700 nonpianists by Matzdorff (1968). Wagner's span data exceeded Matzdorff's for all finger pairs; the difference suggests that MaxPoss increases with piano practice (although it is also possible that people with larger hands are more likely to become pianists). The virtuosic stretching exercises of Cortot (1926, 1928) may be regarded as involving MaxPoss.[3]

We do not make use of MaxPoss in the present model. Instead, we refer to the *maximum practical span* (MaxPrac). MaxPrac is understood as the maximum stretch that a pianist would actually use in a musical performance to span two notes played either simultaneously or consecutively in finger legato. In the model, it is the biggest permissible span between the consecutive notes of a melodic fragment.[4]

MaxPrac is typically somewhat smaller than MaxPoss, for two reasons. First, other fingering options are usually available, so that no benefit is to be gained from stretching the hand to its natural limit. Second, MaxPoss may require an unacceptably awkward rotation of the hand and wrist relative to the keyboard, which could adversely affect previous and following finger articulations.[5]

Minimum practical spans (MinPrac) are defined by analogy to MaxPrac as the smallest spans a pianist would actually use in a musical performance.

3. Cortot (1928, Chapter 6 "The technique of extension") encouraged the student gradually to increase the spans between the fingers to 11 semitones (a major 7th) for *1-2*, 13 semitones for *1-3*, 15 for *1-4*, 17 for *1-5*, 8 for *2-3*, 11 for *2-4*, 12 for *2-5*, 7 for *3-4*, 10 for *3-5*, and 8 for *4-5*. Cortot (1926) advocated somewhat smaller spans, but still greater than the MacPrac values used in the present model.

4. We assume that the hand remains close to a horizontal plane. We do not consider rotations of the hand, elbow, and arm that may increase spans.

5. Unusual wrist and forearm movements are sometimes inevitable if finger legato is to be maintained, for example, in the performance of double sixths. (Finger legato is a technique by which the fingers rather than the pedal are used to connect the sound of successive notes; Repp, 1995).

In the model, MinPrac is the smallest permissible span between two consecutive notes. Particular attention is paid to the thumb in formulating MinPrac values, for it is the lateral movement of the thumb that allows pianists to play relatively complex or awkward melodic patterns in finger legato, by placing the thumb on almost any white key (cf. Czerny, 1839). For finger pairs involving the thumb, MinPrac is the maximum distance that a finger can pass over the thumb (or the thumb under the finger) and is expressed as a negative number. For fingers other than the thumb, MinPrac is supposed to represent the closest distance between two fingers that does not involve undue or unnecessary tension; assuming that fingers other than the thumb do not cross over each other,[6] this usually means that the fingers concerned are either touching or almost touching.

These and other spans used in the model are set out in Table 1. All values were first estimated by the first author and then adjusted in consultation with the others. The approach was to adjust all MaxPrac spans until they felt about equally difficult to stretch or seemed about equally (un)likely to occur in piano performance. This approach has the following limitations. First, it neglects variations in the size and shape of the hand. Some pianists, for example, may have relatively long middle fingers, so that their thumb and fifth fingers are relatively short; some may have *4* shorter than *2*; and so on. The design of the model is such that differences of this kind between pianists may be accounted for at least partially by adjusting the values in Table 1; for example, the table might be adjusted for smaller hands simply by multiplying all values by about 0.9. In principle, therefore, the model should be able to account for the fingerings of both female and male pianists.[7] Second, the model neglects variations with transposition of the physical size of chromatic intervals at the keyboard. For example, the major third from C to E is physically smaller on the keyboard than the major third from D♭ to F. In the model, spans are estimated relative to *average* physical sizes of chromatic intervals at the keyboard, where all 12 transpo-

6. It was common practice in the 17th and 18th centuries to turn fingers other than the thumb over each other (Bach, 1753/1957; Couperin, 1717/1933). The practice became less widespread as the pianoforte took over from the harpsichord as the dominant keyboard instrument (see Türk, 1789/1962); but some later pianists continued to favor the technique, the most famous examples being Chopin (Eigeldinger, 1986) and Busoni (Raekallio, 1996). For practical reasons we have chosen not to permit this in the current implementation of the model. It could easily be introduced by appropriate adjustment of the spans in Table 1.

7. It may be argued that the model is limited as a consequence of all authors of the study being men and that other principles may be at work that might have been identified if the model had been developed by women. We think that this is unlikely, given the ergonomic focus of the principles and the adjustability of all aspects of the model based on physical distance to account for different hand sizes and shapes.

TABLE 1
Assumptions of the Model: Minimum and Maximum Practical, Comfortable, and Relaxed Spans (in Semitones) for All Pairs of Right-Hand Fingers

Finger Pairs	MinPrac	MinComf	MinRel	MaxRel	MaxComf	MaxPrac
1-2	–5	–3	1	5	8	10
1-3	–4	–2	3	7	10	12
1-4	–3	–1	5	9	12	14
1-5	–1	1	7	10	13	15
2-3	1	1	1	2	3	5
2-4	1	1	3	4	5	7
2-5	2	2	5	6	8	10
3-4	1	1	1	2	2	4
3-5	1	1	3	4	5	7
4-5	1	1	1	2	3	5

Leftmost column: for each entry *f-g*, *f* is the finger number (thumb = *1*, index finger = *2*, etc.) of the first note, and *g* is the finger number of the second note. Positive entries in the body of the table represent rising intervals, and negative, falling. Because the pairs of finger numbers in the leftmost column are always increasing, and the table is for the right hand, negative entries in the table refer to passing of the fingers over the thumb, or the thumb under the fingers. Finger pairs *1-1, 2-2, 3-3, 4-4,* and *5-5* are omitted; if they were included, all corresponding values in the table would be zero, because the current model does not allow skipping from note to note using the same finger (see Appendix).

sitions of each interval are taken into account. For simplicity, we have rounded all spans to whole numbers of semitones.[8]

Table 1 applies to the right hand. Assuming that the left and right hands are about equal in size,[9] the table may be adapted for the left hand either by reversing the finger numbers in the left column (e.g., by changing *1-2* to *2-1*), or by reversing the sign of the entries (spans in semitones) in the body of the table and interchanging "Max" and Min" in the headings. Similarly, spans for reverse finger pairs *y-x* within the right hand may be found by changing the sign of the entries in the table and interchanging "Max" and Min" in the headings (e.g., MaxPrac for *2-1* is *5* semitones).

Note that only spans between different fingers are listed. The repeated use of the same finger, either on the same note (note repetitions) or on

8. An octave on a modern keyboard spans about 165 mm, so the average size of chromatic intervals may be calculated on the basis of about 13.7 mm per semitone. A future version of the model might benefit by representing distances between fingers in millimeters instead of semitones and comparing them with distances on the keyboard between specific keys, also expressed in millimeters.

9. Cortot (1926; 1928, p. 61) assumes that the left and right hands span, or can span, about the same intervals. But the data of Wagner (1988) show significant differences between the sizes of left and right hands.

different notes (e.g., sliding from black to white keys) is not accounted for in this version of the model (although it would not be difficult to do so), nor does the model allow for mute finger changes (changing fingers on the same note without restriking).

The values for MinPrac set out in the leftmost column in the body of the table represent distances that the fingers are allowed to pass over the thumb, or the thumb under the fingers. The values are deliberately not quite high enough to permit the legato performance of arpeggiated triads. Consider, for example, a C-major arpeggio. To play the arpeggio with the right hand ascending from C_3 with the conventional fingering *1-2-3-1-2-3*..., it is necessary to stretch 5 semitones (a perfect fourth) when passing *1* under *3* (right hand ascending), or *3* over *1* (right hand descending). According to the table, MinPrac for *1-3* is –4 semitones, implying that these changes of hand position cannot be played legato. Experimental evidence confirms that arpeggios are seldom played finger legato in rapid performance (Gát, 1958/1965; Ortmann, 1929, pp. 283–284), even though it may be physically possible for most pianists to play them finger legato at a slow tempo. Here, as before, we are distinguishing here between *practical* and *possible* spans, with the aim of modeling performance practice rather than reflecting the absolute anatomic limits of the hand.

We regard a *relaxed span* as the interval between two fingers when they fall without tension onto the keys. The hand position is optimal in the sense that the fingers are maximally free to move. For modeling purposes, we define a span as "relaxed" when the number of diatonic (or white-key) scale steps spanned corresponds to the difference between the finger numbers, so that consecutive fingers fall on the consecutive keys of a diatonic (major or minor) scale.[10] The hand is assumed to remain relaxed over a small range, limited by the maximum and the minimum relaxed spans, MaxRel and MinRel (see Table 1). For finger pairs not including the thumb, MaxRel is expressed in semitones (i.e., as the number of keys spanned in the chromatic scale) as twice the difference between the finger numbers, and MinRel as an interval 1 semitone smaller than MaxRel. For finger pairs including the thumb, the difference between the MaxRel and MinRel is increased to account for the thumb's lateral mobility: MaxRel is set at 3 semitones less than the *maximum comfortable span* (MaxComf, introduced later). MinRel remains equal to 1 semitone less than twice the difference between the finger numbers.

The *comfortable* spans listed in Table 1 lie between relaxed and extreme spans. MaxComf is defined as 2 semitones smaller the MaxPrac for all

10. This definition of "relaxed" span has two separate aspects, the one anatomic (for detailed data, see the passive spans of Wagner, 1988) and the other cognitive (a kind of "next-note-next-finger" rule, see for example Türk, 1789/1962). It may be useful to distinguish these from each other in a future version of the model.

finger pairs.[11] The *minimum comfortable span* (MinComf) is 2 semitones smaller than MinPrac for all finger pairs including the thumb and equal to MinPrac for other finger pairs.[12]

Summarizing this section, we have proposed tentative values for six different spans between all pairs of fingers in each hand. The proposed span sizes are intended to apply to an "average" adult male pianist and may be adjusted to suit different hand sizes and levels of expertise. Values have been determined in an ad hoc manner, drawing on the intuition and musical experience of the authors, and on a range of scientific, musical, and historical-pedagogical literature sources.

ENUMERATION OF ALL POSSIBLE FINGERINGS

Having accepted the pitches of a melodic fragment as input, the model begins by enumerating all possible fingerings for the fragment. For example, the MaxPrac and MinPrac values in Table 1 would allow the sequence C_4-E_4-G_4 to be fingered in 24 different ways: *121, 123, 124, 125, 131, 134, 135, 141, 145, 212, 213, 214, 215, 231, 234, 235, 241, 245, 312, 313, 314, 315, 341, 345*. The procedure for enumerating all possible fingerings is as follows. First, all possible sequences of the integers 1 to 5 are generated, where the length of the sequence corresponds to the length of the melodic fragment at the input to the model, and repeated integers are avoided. If the input fragment is three notes long, there are 5 x 4 x 4 = 80 sequences (*121, 123, 124, 125, 131, 132, 134, ... 345*) to consider. In general, the number of sequences to be considered is $5 \times 4^{n-1}$, where n is the number of notes in the fragment. Second, only those sequences in which all consecutive pairs of elements (in a three-note sequence, there are two consecutive pairs) represent playable intervals (lying between MinPrac and MaxPrac) are retained.

WEIGHTING FINGERINGS ACCORDING TO DEGREE OF DIFFICULTY

Treatises on piano technique have often compared the difficulty of different fingerings. Kullak (1876, pp. 144–145), for example, considered technical difficulties associated with five different positions of the hand relative to black and white keys: (1) all fingers on white keys; (2) *1* and *5* on white,

11. In our experience as pianists, the difference between MaxComf and MaxPrac depends to a small extent on the fingers involved. For example, the difference is smaller for *3-4* than for *1-5*. This dependency is neglected in the present version of the model.

12. The term "comfortable" is the best we could come up with, but it is not without problems. For example, Gieseking used *5-1* in the left hand for slowly alternating white keys at an interval of a major second (C-D-C-D...) in Debussy's *Des pas sur la neige* (Elder, 1988, p.226), claiming that this fingering allowed better nuance control. He would presumably have disagreed with the model's description of this fingering as "uncomfortable."

the others on black or white; (3) either *1* or *5* on white, the others on black or white; (4) *1* and *5* on black, the others on white; and (5) all on black. He concluded that (5) was slightly more difficult than (1), that (4) was the most difficult of all, and that (2) and (3) had intermediate degrees of difficulty.

In the present model, the difficulty of a fingering is estimated quantitatively by linear addition of contributions that are assumed to be independent of one another. Each contribution is determined algorithmically by a "rule." Presently, there are 12 rules in the model. Each is named after a technical difficulty that a pianist might seek to avoid; the "Weak-Finger Rule," for example, attempts to account for a pianist's tendency to avoid the use of weak fingers, and the "Stretch Rule" accounts for the avoidance of unnecessary stretches between fingers. The sum of all contributions is the predicted difficulty or "cost" of a specific fingering.

The model allows the relative importance of each rule to be adjusted by applying a weight (or linear coefficient) to it. It is anticipated that different pianists and different musical styles may require different relative weightings; for the purposes of this paper, however, the weightings have been fixed. Like the rules themselves, the weights have been determined in a rather intuitive, ad hoc manner, based on the musical experience and intuitions of the authors, in combination with informal comparisons between predictions and data. The fingering with the lowest calculated difficulty is the one that we predict will be used most often in performance.

The rules are largely inspired by historical precedents. The fingering systems of Couperin (1717/1933), Bach (1753/1957), Türk (1789/1962), Czerny (1839), and Kullak (1876) all incorporate rules that are related to those spelled out in the present paper. Here, aided by technology that was of course unavailable to historic writers, we attempt to develop a rule system that is both more complete and more systematic—at least at the level of ergonomics—than previous systems.

Spans Between Consecutive Notes

1. Stretch Rule: Assign 2 points for each semitone that an interval exceeds MaxComf or is less than MinComf.

This rule accounts for the difficulty associated with stretching from MaxComf to MaxPrac spans and from MinComf to MinPrac for finger pairs including the thumb (passing the thumb under the fingers; see the Appendix for a mathematically concise and complete statement of this and the next two rules in the model). As noted earlier, the difference between "maximum comfortable" and "maximum practical" for these finger pairs is set at 2 semitones. Consider, for example, the interval D_4-C_5 (10 semitones) played *2-5*. According to Table 1, MaxComf for fingers *2-5* is 8 semitones. D_4-C_5 is 2 semitones larger, so the Stretch Rule assigns 2 x 2 = 4 points. If, instead, the interval E♭$_4$-C_5 (9 semitones) is played *2-5*, the rule contribution is reduced to 2 points.[13]

2. Small-Span Rule: For finger pairs including the thumb, assign 1 point for each semitone that an interval is less than MinRel. For finger pairs not including the thumb, assign 2 points per semitone.

Pianists tend to prefer fingerings in which neighboring fingers map to neighboring notes in a diatonic scale. We refer to this as the "next-note-next-finger" principle. The principle was applied to keyboard performance long before the advent of the pianoforte (Bach, 1753/1957; Sancta Maria, 1565/1972; see also Lohmann, 1990) and was widely espoused by writers on piano technique, notably Türk (1789/1962). Kullak (1876, p. 154) distinguished smaller hand positions (*engere Lage*) from larger (*weitere Lage*) by reference to the distances between white keys: smaller positions involve smaller distances (his example: C-C♯-D-D♯-E played *1-2-3-4-5*); larger positions, larger distances (his example: C-D-F♯-G♯-A♯ played *1-2-3-4-5*).

The Small-Span Rule applies whenever the span between two fingers is smaller than a relaxed span, making the hand position feel cramped.[14] This includes passing the thumb under another finger or a finger over the thumb. Consider, for example, the interval E_4-G_4 played *3-1* (passing the thumb). MinRel for *1-3* (without passing the thumb) is 3 semitones. Passing *1* under *3* through an interval of 3 semitones involves a span that is smaller than MinRel by 3 + 3 = 6 semitones. So the Small-Span Rule contributes 6 points to the estimated difficulty of this fingering.[15]

It is easier to move the thumb sideways relative to the other fingers than to move the fingers sideways relative to each other. The Small-Span Rule consequently assigns more points to intervals not involving the thumb. This only affects finger pairs for which MinComf and MinRel are different: *2-4*, *2-5*, and *3-5*. As an example of the application of the rule, the interval C_4-

13. This procedure, like other procedures in the model, is no more than a first approximation. It is not necessarily true, for example, that 1 semitone less than MaxPrac for *1-5* is just as difficult as 1 semitone less than MaxPrac for *3-4*. Moreover, the difference in tension between a given MaxPrac and a span 1 semitone smaller may seem insignificant to one pianist and profound to another; the relationship between "comfortable" and "maximum" may be a very personal matter. The present model makes no attempt to account for these finer effects, but rather attempts to encapsulate the most important phenomena in the simplest possible way.

14. Of course, pianists will differ on how "cramped" a given position feels. Again, we refer here to an "average" (idealized?) adult male pianist.

15. Two other rules can contribute to the difficulty of passing the thumb: the Stretch Rule, when the interval covered is smaller than MinComf, and the Thumb-Passing Rule (presented later) that accounts for effects of key elevation (black or white).

D_4 played *2-5* is assigned 6 points, because MinRel between fingers *2* and *5* is *5* semitones (3 semitones larger than the interval between C and D) and 3 x 2 = 6.

***3. Large-Span Rule: For finger pairs including the thumb, assign 1 point for each semitone that an interval exceeds MaxRel. For finger pairs not including the thumb, assign 2 points per semitone.*[16]**

Like the Small-Span Rule, this rule is based on the next-note-next-finger principle. Spans are assumed to be difficult if they exceed MaxRel. For spans that also exceed MaxComf, the difficulty is further augmented by the Stretch Rule. For example, the interval C_4-B_4 (11 semitones) played *1-3* (MaxRel = 7 semitones) is assigned 11 – 7 = 4 points by the Large-Span Rule, and 2 additional points by the Stretch Rule (11 semitones is 1 semitone larger than MaxComf for *1-3*), making a total of 6 points for the two rules.

Like the Small-Span Rule, and for the same reasons, the Large-Span Rule assigns more points to intervals not involving the thumb. For example, the interval C_4-G_4 (7 semitones) played *2-4* (MaxRel = 4 semitones) is assigned 2 x (7 – 4) = 6 points by the Large-Span Rule, plus 4 points by the Stretch Rule (7 semitones is 2 semitones larger than MaxComf for *2-4*), making a total of 10 points for the two rules.

Figure 1 illustrates how the Stretch, Large-Span, and Small-Span Rules combine to predict difficulty as a function of span for one finger pair, *2-4*. Intervals smaller than MinPrac (1 semitone) or larger than MaxPrac (7 semitones) are not permitted by the model. Intervals in the range 3 to 4 semitones are regarded as "relaxed." According to the Small-Span Rule, intervals smaller than MinRel become more difficult at the higher rate of 2 points per semitone (because neither of the fingers involved is the laterally mobile thumb). Similarly, the Large-Span Rule adds 2 points to the interval of 5 semitones because it is 1 semitone larger than MaxRel. Beyond 5 semitones, calculated difficulty increases by 4 points for every semitone—2 from the Large-Span Rule and 2 from the Stretch Rule.

The two peaks of the graph in Figure 1 have clearly different heights, that is, they correspond to different calculated degrees of difficulty. MaxPrac is assumed more difficult than MinPrac, because MinPrac is limited by the simple physical limitation of fingers *2* and *4* running into each other, whereas MaxPrac is limited by how far the hand can be stretched. The same applies to other finger pairs not including the thumb, as shown in the lower seg-

16. Ortmann (1929), in a physiological analysis of piano technique, observed that it is more difficult to stretch the hand than to contract it: "Finger adduction is easier than finger abduction because the fully adducted position (fingers side by side) is the position of normal rest" (p. 48). We are not sure of the extent to which this effect might influence fingering, however, and when formulating the model, found it unnecessary to account for it.

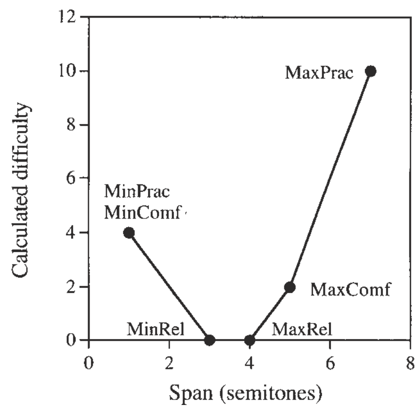


Fig. 1. Calculated difficulty of spans between fingers *2* and *4* found by summing contributions from the Stretch, Large-Span, and Small-Span rules. Contributions from other rules are neglected.

ment of Table 2. When the thumb is involved (upper segment of the table), the model assumes the converse: MinPrac (passing the thumb the maximum practical distance under the finger) is more difficult than MaxPrac. Thus, we assume that the strain involved in passing the thumb under the

Table 2

Combined Effect of the Stretch, Small-Span, and Large-Span Rules for All Pairs of Right-Hand Fingers and All Intervals

Finger	Span (semitones)																				
Pair	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1-2	10	7	**4**	3	2		0	0	0	0	0	1	2	**3**	6	9					
1-3		11	8	**5**	4		2	1	0	0	0	0	0	1	2	**3**	6	9			
1-4			12	9	**6**		4	3	2	1	0	0	0	0	0	1	2	**3**	6	9	
1-5					12		**6**	5	4	3	2	1	0	0	0	0	1	2	**3**	6	9
2-3							0	0	**2**	6	10										
2-4							**4**	2	0	0	**2**	6	10								
2-5								**6**	4	2	0	0	2	**4**	8	12					
3-4							0	**0**	4	8											
3-5							**4**	2	0	0	**2**	6	10								
4-5							0	0	**2**	6	10										

Numbers within the main body of the table are sums of difficulty values assigned by the first three rules of the model. All other rules are ignored. Only fingerings falling within the bold boundaries are allowed by the model. Numbers adjacent to left-hand boundaries correspond to MinPrac intervals; adjacent to right-hand boundaries, MaxPrac. Numbers in bold correspond to MinComf (leftmost) and MaxComf (rightmost) intervals. Underlined numbers correspond to MinRel (leftmost) and MaxRel (rightmost) intervals.

fingers at MinPrac spans is greater than the strain of stretching or broadening the hand at MaxPrac spans.

Table 2.shows the combined effect of the first three rules of the model, for all intervals playable by all permitted finger pairs. For example, a falling interval of *5* semitones (shown as –5) played with the fingering *1-2* produces a calculated difficulty of 10 points, consisting of 4 points from the Stretch Rule (the interval is 2 semitones smaller than MinComf = –3, and 2 points are applied per semitone) and 6 points from the Small-Span Rule (the interval is 6 semitones smaller than MinRel = +1).The row in Table 2 for finger pair *2-4* corresponds to the data plotted in Figure 1.

Spans Between Next-to-Consecutive Notes

4. Position-Change-Count Rule: Assign 2 points for every full change of hand position and 1 point for every half change. A change of hand position occurs whenever the first and third notes in a consecutive group of three span an interval that is greater than MaxComf or less than MinComf for the corresponding fingers. In a full change, three conditions are satisfied simultaneously: The finger on the second of the three notes is the thumb; the second pitch lies between the first and third pitches; and the interval between the first and third pitches is greater than MaxPrac or less than MinPrac. All other changes are half changes.[17]

If the interval between the first and third notes in a group of three is too wide or narrow to have been played comfortably without the aid of an intervening finger, a change of hand position may be said to have occurred. For example, if the first note in a melodic fragment is played by finger *2* and the third note by finger *3*, a position change occurs if the interval between the first and third notes is greater than 3 semitones (MaxComf) or less than 1 semitone (MinComf). The rule refers to MaxComf, rather than MaxPrac or MaxRel, on the assumption that the hand is comfortably extended during changes of position, but not stretched to its practical limit.

The intervening finger may be called a *pivot*. Some examples are presented in Figure 2. In a full change, the pivot is the thumb *and* lies between the first and third notes in pitch (Figure 2, i, ii). Otherwise, we call the change of hand position a *half change* (Figure 2, iii, iv, v).[18] The example in

17. The terms *full* and *half* change of hand position have been coined for the purpose of this paper. We could not find equivalents in the English language literature on piano fingering.

18. The word "half" is not intended to mean that the change is half as difficult as a full change (although in the current version of the model it is in fact assigned half the number of points), but that it is smaller than a full change.

Fig. 2. Some examples of changes in hand position.

Figure 2iii, a rotation, involves pivoting around a finger other than the thumb (by rotation of the wrist and arm). Parts (iv) and (v) may be called insertions (or Einsetzen: Türk, 1789/1962, p. 140).[19]

The effect of the Position-Change-Count Rule within the algorithm as a whole is to favor fingerings in which the number of changes of hand position is minimized or fingerings in which relatively long sequences of notes are played within a single hand position. To our knowledge, the earliest documentation of the idea that the number of changes of hand position should be kept to a minimum dates to the Renaissance (Diruta, 1593–1609/1983; Sancta Maria, 1565/1972; see also Lohmann, 1990).

5. Position-Change-Size Rule: If the interval spanned by the first and third notes in a group of three is less than MinComf, assign the difference between the interval and MinComf (expressed in semitones). Conversely, if the interval is greater than MaxComf, assign the difference between the interval and MaxComf.

For examples of this rule, consider again Figure 2. In part (iii) of the figure, the interval between the first and third notes (17 semitones) is greater than MaxComf for *1-2* (13 semitones), so the difficulty contribution from the Position-Change-Size Rule is 17 – 15 = 2 points. The intervals in parts (i), (ii), (iv), and (v) are smaller than MinPrac. In (i), for example, the rule's contribution is 4 – (–1) = 5 points.

The Position-Change-Size Rule is an expression of the principle of economy of movement in piano technique: changes in hand position should

19. Türk wrote that "Even when fingers are not passed over the thumb (*Überschlagen*) or the thumb under the fingers (*Untersetzen*), one very often can and should insert a finger out of sequence so as to allow the following tones to be played more comfortably. . . . This so-called insertion (or pulling the fingers together) should not be confused with passing the thumb or the fingers. For here, with an insertion, the finger that does not follow the established sequence is placed *beside* (not under or over) another..." [original text: "*Man kann und muß sehr oft ... wenn das Ueberschlagen und Untersetzen nicht statt findet, außer der Reihe einen noch weit entfernten Finger auf eine Taste einsetzen, (nachziehen,) um die folgenden Töne bequemer heraus zu bringen ... Dies so genannte Einsetzen (Nach- oder Aneinanderziehen der Finger) muß man nicht mit dem Untersetzen oder Ueberschlagen verwechseln; denn hier, beim Einsetzten, wird der Finger, welcher der Reihe nach nicht folge, weder unter noch über, sondern* neben *einen andern gesetzt...*"]

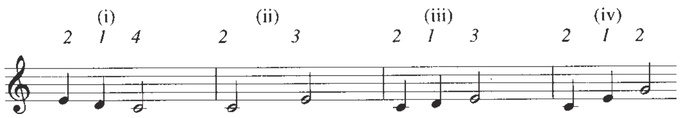


Fig. 3. Further examples of changes in hand position.

involve as little hand and finger movement as possible (Bach, 1753/1957).[20] We have assumed, when formulating this rule, that the distance in semitones traveled by the hand's center of gravity during a change of hand position is a reasonable measure of the difficulty of the change.

Figure 3 (i) shows a right-hand fingering sequence that might happen in the context of a descending C-major scale. We assume that an experienced pianist will facilitate the change of hand position by minimizing the overall amount of finger and hand movement required to execute the transition. As the fingers together move across the thumb to their new position, overall movement may be minimized by keeping fingers *2* and *4* close together. In the model, the optimal distance between these fingers during the position change is identified as MinComf. According to the Position-Change-Size Rule, the overall difficulty of this change in hand position is calculated as the difference between MinComf for *2-4* (1 semitone) and the physical distance between the notes E and C (–4 semitones): 1 – (–4) = 5 points.

Figure 3 shows three further examples of changes of hand position that clarify the relationship between the Stretch, Span, and Position-Change Rules.

(ii) C_4-E_4 played *2-3* is assigned 2 points according to the Stretch Rule, because the interval of 4 semitones is 1 semitone larger than MaxComf for *2-3*. A further 4 points are assigned by the Large-Span Rule, because the interval is 2 semitones larger than MaxRel, and the thumb is not involved.

(iii) If a pivot note D_4, played with the thumb, is inserted between C_4 and E_4 in (ii), the Stretch and Large-Span contributions vanish, because no span between consecutive notes exceeds MaxRel. But the fingering is assigned 4 points according to the Small-Span Rule: 3 points for fingers *1-2* (MinRel for this finger-pair is 1 semitone, which is 3 semitones larger than the first interval of effectively –2 semitones); and 1 point for *1-3* (for which MinRel

20. Here we assume legato performance. If the intention is to play more detached or lightly, changes of hand position are less problematic and may even be desirable, if not too technically difficult. Consider, for example, the repeating pattern $C\sharp_4$-E4-$B\flat_4$-E_4. An optimal legato fingering might be *1-2-5-2*, with no change of hand position. In nonlegato, the best fingering might be *2-1-4-1*. According to the present model, the latter fingering involves two half changes of hand position in each cycle, because the interval between $C\sharp_4$ and $B\flat_4$ (9 semitones) is greater than MaxComf for *2-4* (7 semitones).

is 3 semitones, which is 1 semitone larger than the second interval of +2 semitones).[21] The model registers a half change of hand position, because the interval of 4 semitones exceeds MaxPrac for *2-3*; so the Position-Change-Count Rule assigns 1 point, and the Position-Change-Size Rule, 1 point.

(iv) In the sequence C_4-E_4-G_4 played *2-1-2*, finger *2* jumps 7 semitones, so the Position-Change-Count Rule accumulates 2 points for a full change and the Position-Change-Size Rule 7 points. The model is effectively assuming that the center of gravity of the hand (or at least, the group of fingers *2-5*) moves 7 semitones. An additional 2 points are assigned by the Stretch Rule for C_4-E_4 played *2-1*, because 4 semitones is 1 semitone greater than MaxComf for *2-1*, 3 semitones (which is the negative of MinComf for *1-2*, –3 semitones). The Small-Span Rule assigns 5 points for C-E played *2-1*, because C-E (4 semitones) is 5 semitones smaller than the MinRel of 1 semitone in the opposite direction.

Consider now C_4-E_4-G_4 played *2-1-3*. The center of gravity of the hand moves a shorter distance than it did for *2-1-2*. The difference corresponds to MaxComf between fingers *2* and *3*, or 3 semitones. The contribution of the Position-Change-Size Rule is thus reduced from 7 to 4 points. The Stretch Rule independently adds 2 points, the Small-Span Rule 5 points, and the Position-Change-Count Rule 2 points.

When the same notes are played *3-1-2*, the chances of missing the G are considerably higher, because the fingers have to travel farther (i.e., there is a bigger change of hand position). By comparison to *2-1-2*, the increase in distance traveled by the fingers is given by MinComf between *2* and *3* (1 semitone). The addition to estimated difficulty according to the Position-Change Size Rule is thus 7 + 1 = 8. Other contributions are Stretch, 4 points (C_4-E_4 played *3-1* is 2 semitones smaller than MinComf); Small Span, 7 points; Position-Change Count, 2 points.

Changing Fingers on Next-to-Successive Notes

According to the above definitions, the Position-Change-Count Rule generates points only when the Position-Change-Size Rule does, and vice versa. An exception is made for the case of changing fingers on repeated notes with one intervening note. As an example, consider C_4-F_4-C_4 played *2-5-1*. According to the above procedures, this is not a change in hand position. A special addition to the model regards this as a half change of hand position of zero size, so that the Position-Change-Count Rule generates 1 point, and the Position-Change-Size Rule, 0 points.

21. An additional contribution to the difficulty of thumb passes is made by the Thumb-Passing Rule, to be described later.

Variations in Finger Strength and Agility

An explicit avoidance of weak fingers can be observed in the fingerings of certain pianists and pedagogues (e.g., Beethoven: Hiebert, 1986, Newman, 1982; Chopin: Eigeldinger, 1986) but not others (e.g., Liszt: Walker, 1983). From an anatomic viewpoint, the fingers of the hand divide most naturally into the thumb and the other fingers (enabling grasping). The other fingers in turn divide most readily into finger 2 (which is independently extensible, hence the name "index"; Ortmann, 1929) and the other three fingers, whose independence is inhibited by the interconnecting *juncturae tendinum* (Green, 1988; Kaplan, 1959). These tendons limit the extension of the fourth finger, but not its flexion, and although countless exercises have been devised by pianists to increase the fourth finger's independence (Gellrich, 1992), "no amount of practice can overcome entirely this physiological limitation" (Ortmann, 1929, p. 45). The next three rules in the model account for the relative nonindependence of the fourth finger from its immediate neighbors.

6. Weak-Finger Rule: Assign 1 point every time finger 4 or finger 5 is used.

Here, we assume simply that fingers *4* and *5* are less strong and agile than fingers *1*, *2*, and *3*. For example, the repeating sequence in Figure 4 is easier to play with the fingering in (i), with no points assigned per cycle, than with that in (ii), with 3 points per cycle.

7. Three-Four-Five Rule: Assign 1 point every time fingers 3, 4, and 5 occur consecutively in any order, even when groups overlap.

The Three-Four-Five Rule acknowledges the difficulty of coordinating runs of notes played by fingers on the weak side of the hand. Points assigned according to the rule are added to points already assigned by the Weak-Finger Rule.

As an example, consider the first eight notes of Piece A in Figure 7 played with Czerny's fingering, *3-5-4-5-3-4-2-3*. The fingering includes two instances of finger *4* and two of finger *5*, so the Weak-Finger Rule generates

Fig. 4. Some examples of the use or avoidance of weaker fingers.

4 points. Embedded within the fingering are the consecutive sequences *3-5-4, 4-5-3,* and *5-3-4,* so the Three-Four-Five Rule assigns an extra 3 points.

8. Three-to-Four Rule: Assign 1 point each time finger 3 is immediately followed by finger 4.
9. Four-on-Black Rule: Assign 1 point each time fingers 3 and 4 occur consecutively in any order with 3 on white and 4 on black.

The transition from finger *3* to finger *4* is difficult because it involves raising the fourth finger while the third finger is held down. We have observed that pianists sight-reading a set of Czerny studies use the fingering *4-3* significantly more often than *3-4* on consecutive notes of a single right-hand line.[22] Consistent with these observations, the Three-to-Four Rule accounts for the difficulty of playing finger *4* after finger *3*, regardless of whether the notes involved are black or white.[23] The Four-on-Black Rule accounts for the additional difficulty that arises if finger *4* plays a black key and finger *3* a white key. Unlike the Three-to-Four Rule, the Four-on-Black Rule involves fingers *3* and *4* played in succession in either order. Both rules are independent of the interval between the two notes (which, of course, is limited by the matrix of maximum spans).

Black and White Keys

10. Thumb-on-Black Rule: Assign 1 point whenever the thumb plays a black key. If the immediately preceding note is white, assign a further 2 points. If the immediately following note is white, assign a further 2 points.
11. Five-on-Black Rule: If the fifth finger plays a black key and the immediately preceding and following notes are also black, assign 0 points. If the immediately preceding note is white, assign 2 points. If the immediately following key is white, assign 2 further points.

22. Sloboda et al. (in press) obtained fingerings for sight-read performances by 17 pianists of 7 pieces, selected from the 10 Czerny studies analyzed here. Each piece contained more than 120 right-hand notes, and each pianist played each piece twice. Within this data base of some 30,000 fingered notes, the transition *4-3* occurred 1663 times, whereas the reverse sequence *3-4* occurred 1142 times, or significantly less often (binomial distribution: $p<.01$). Expressed as a ratio, *4-3* occurred 45% more often than *3-4* did. In the right hand of these 7 Czerny studies, there are 397 rising and 441 falling intervals; falling intervals outnumber rising ($p < .01$)—consistent with Vos and Troost (1989)—but by only 11%. The difference between these two percentages is evidence for the reality of the Three-to-Four Rule. Other significant asymmetric finger transitions observed in the same data (*3-1* > *1-3*, *1-4* > *4-1*, *3-2* > *2-3*, *2-4* > *4-2*, *5-4* > *4-5*) were less pronounced (again, when measured as percentages) and are not accounted for by the present model.

23. We apply the Three-to-Four Rule even if the third finger plays a black key and the fourth a white. Two reasons: (a) experience suggests that even this combination is hard to play repeatedly *3-4-3-4-3-4*, and (b) the difficulty involved with placing *4* on a black key is accounted for elsewhere in the model.

Historically, pianists have mostly avoided using the thumb and (to a lesser extent) the fifth finger on black keys (Bach, 1753/1957; Türk, 1789/1962). We consider two explanations. First, fingers *1* and *5* are the shortest of the five fingers, and the black keys are shorter than the white; so placing *1* or *5* on a black key tends to displace the hand away from a comfortable position, and avoiding *1* or *5* on black is a way of avoiding arm movements. Second, the force required to depress a piano key depends on the distance of the point of contact from the pivot point (fulcrum); the fulcrums of black and white keys are offset from each other so that adjacent black and white keys produce notes of equal loudness if the keys are depressed with equal force, provided that each black key is played near its proximal end[24] and each white key is played roughly midway between its proximal end and the proximal ends of neighboring black keys. Thus, evenness of touch may be achieved either by placing fingers at optimal positions—which necessarily means avoiding placing *1* or *5* on black—or by adjusting the force of finger strikes appropriately. The effect is most critical in the bass registers, where considerably more physical force is required to produce a given dynamic level than in the treble registers.[25]

As piano technique developed during the 18th and 19th centuries, the structure of the keyboard adapted to the demands of an increasingly virtuosic piano repertoire: the black keys became longer, and the spans between the keys become wider (Grundmann & Mies, 1966). Consequently, the rule that *1* and *5* be kept off black keys became less important. In some instances, the rule was deliberately ignored, for example, when pianists such as von Bülow (1869) and Busoni (Ley, 1957/1965) recommended practicing a piece in several different keys without changing the fingering.[26]

The difficulty involved in placing short fingers on black keys depends on the context in which a note appears, for context can change the position of the hand relative to the keyboard. It is easier to play a black key with *1* or *5* if the hand is already relatively distal ("into the keys"), which is the case if the preceding or following key is also black. The context dependency of

24. By *proximal* we mean at the front of the keyboard, "out of the keys," close to the pianist. By *distal,* we mean at the back of the keyboard, "into the keys," away from the pianist.

25. We thank Diana Weekes and Anders Askenfelt for these insights. The problem of unevenness resulting from placing thumb or fifth finger on blacks and other fingers on whites was mentioned by Kullak (1876/1994, p. 145): "*Die Mittelfinger haben durch ein tieferes Herabfallen ihre Kraft behufs der Egalität in einem neuen Maaße zu studiren*". Discussions of this matter in the piano-pedagogical literature (see also Bach, 1753/1957; Türk, 1789/1962) imply that such unevenness can be overcome with practice, which in turn implies that pianists are good at adjusting the force of fingerstrikes, manipulating the key from either its proximal or its distal end to create a controlled tone. Exactly how well pianists are able to do this has not yet been investigated systematically.

26. From the point of view of the present model, these pianists attached less weight to the Thumb-on-Black Rule than did their predecessors. It is unlikely that the rule altogether ceased to affect their actual fingerings in performance.

the short-finger-on-black rules is illustrated by the following examples. When the interval $D♭_4$-$B♭_4$ is played *1-5*, the Thumb-on-Black Rule generates 1 point for *1* on D♭, but the Five-on-Black Rule does not assign points for *5* on B♭, because both keys are black. If instead the interval D_4-$B♭_4$ is played *1-5*, no points are assigned for *1* on D, but 2 points are assigned for *5* on B♭. If $D♭_4$-B_4 is played *1-5*, 3 points are assigned for *1* on D♭, and 0 for *5* on B.

12. Thumb-Passing Rule: Assign 1 point for each thumb- or finger-pass on the same level (from white to white or black to black).[27] Assign 3 points if the lower note is white, played by a finger other than the thumb, and the upper is black, played by the thumb.[28]

The difficulty associated with passing the thumb under a finger, or a finger over the thumb, depends on the relative elevation of the keys before and after the pass. Four possibilities may be distinguished. The easiest example—reflected by standard scale-book fingerings, as set out most clearly by Clementi (1801/1974), and earlier by Marpurg (1755/1969) and Bach (1753/1957)—proceeds from nonthumb on black to thumb on white, or from thumb on white to nonthumb on black (Lohmann, 1990; Rosenblum, 1988). This sequence allows plenty of room for the thumb to pass under the other finger, or vice versa. Next in order of difficulty is the pass from white to white, closely followed by black to black. The pass from nonthumb on white to thumb on black (or vice versa) is by far the most difficult of the four kinds and occurs only in the more technically difficult pieces of the piano repertoire (e.g., Chopin Study op. 25 no. 11 in A minor). In these cases, legato at changes of hand position is overwhelmingly produced by the pedal rather than the fingers. (Note that in the Chopin example the pedal is typically held down not just for the change of hand position but for two or three positions in succession.)

The Thumb-Passing and Thumb-on-Black rules combine to account for the difficulty of thumb passes that involve placing the thumb on a black key. For example, according to the Thumb-on-Black Rule, the fragment $C♯_4$-E_4 played *1-2* is assigned 3 points for thumb on black followed by nonthumb on white. Adding an extra note to the beginning of the fragment to give $A♯_3$-$C♯_4$-E_4 played *2-1-2,* the Thumb-on-Black rule still assigns 3 points, and the Thumb-Passing Rule assigns an extra point for passing from nonthumb on black to thumb on black (i.e., passing on same level). A more difficult fragment, A_3-$C♯_4$-E_4 played *2-1-2*, now scores 5 points for thumb

27. The extra difficulty of the black-to-black pass over the white-to-white pass is accounted for elsewhere, by the Thumb-on-Black Rule.

28. In this case, 3 points will already have been allocated according to the Thumb-on-Black Rule, making a total of 6.

on black, because the black note is both preceded and followed by a white note; the Thumb-Passing Rule assigns 3 additional points for passing from nonthumb on white to thumb on black.[29]

RANK ORDERING

Once they have been assigned difficulty values by the above rules, the fingerings are ranked in order of calculated difficulty. The fingerings that pianists actually use or recommend should then appear among the least difficult calculated fingerings.[30] Examples of this procedure will be given later in this paper, when experimental data are compared with model predictions.

OPTIMAL FINGERING OF LONGER SEQUENCES

The algorithm described earlier can only be applied to relatively short fragments (our implementation goes to eight notes), because the number of possible fingerings for a fragment varies exponentially with the length of the fragment. It is nonetheless possible to calculate a single optimal fingering of a longer fragment by exploiting commonalities between different fingerings: for example, two fingerings for a given sequence may begin differently, but converge for the last few notes. In this case, calculations of fingering difficulty for the shared part need not be duplicated.

The method involves constructing and processing a representation (a network) of all possible fingerings at once, using the standard technique of *dynamic programming* (Cormen, Leiserson, & Rivest, 1990). The optimal coding search is converted into the problem of finding a shortest route through a network. For this problem, a standard, nonexponential algorithm exists. van der Helm and Leeuwenberg (1991) use the same approach to account for regularity and symmetry in mental codes for visual perception. Shortest route finding has been applied to the problem of string fingering as well, using neural networks instead of a symbolic algorithm (Sayegh, 1989).

The method may best be understood from a specific example. Consider the sequence A_4-C_4-A_4-E_5. We start by constructing a network with nodes (cells) for each fingering of each note, as shown in Figure 5A. The network

29. In all cases of thumb passing, there will also be points added according to the Small-Span Rule, the number of points depending on which other finger is involved.

30. We offer no strict definition of "least difficult" fingerings because of the impractability of defining a threshold of difficulty beyond which fingerings are not playable. Such a threshold would depend primarily on the number of notes in the group (the higher the number of notes, the greater the calculated difficulty). It would also depend on less tangible factors, such as the general difficulty of the context in which the passage appears and the speed with which it is played.

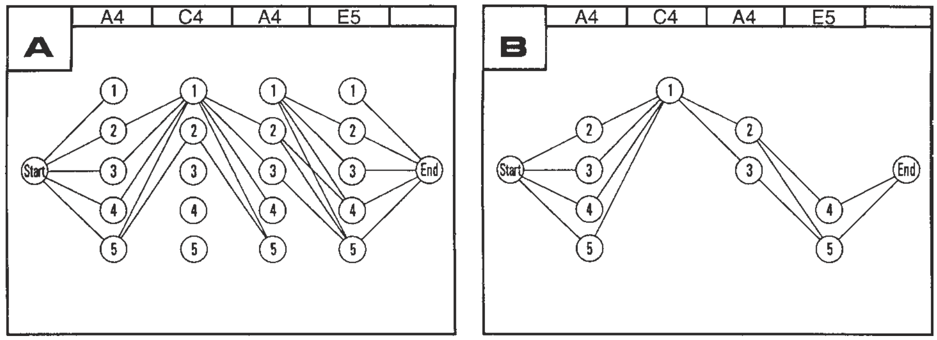


Fig. 5. (A) Fingering network for the fragment A_4-C_4-A_4-E_5. Each note is represented by fivenodes, labeled with all five possible fingers. Links (lines) between nodes represent playable intervals. Each complete path from the start to the end is a possible fingering. (B) The same network with dead ends and unreachable goals, and associated paths, eliminated.

contains 5 x 4 = 20 nodes, plus a start and end node to represent initial and final states (where the hand is away from the keyboard). The nodes are labeled according to the finger playing the key. Each node has incoming links from the left representing connections to fingerings that may have happened before and outgoing links to the right representing allowed fingerings for the next note. Neighboring nodes are connected only where the interval is less than or equal to MaxPrac (see Table 1) for the finger pair in question.

Any path in the network from the leftmost start node to the rightmost end node represents a possible fingering of the complete fragment.

Some of the nodes in Figure 5A lack a connection either to the right or to the left (or both). Nodes without right-hand (outgoing) links are dead ends from which no fingering for the next note is allowed, and nodes without left-hand (incoming) links are states that can never be reached from the previous context. An example of the former is the node for finger *1* on the first note (A_4). The thumb is not a candidate for this note, because with the thumb on A_4, it is impossible to play the second note, C_4 (whichever finger is chosen, the interval exceeds MaxPrac) and hence to proceed further through the network. An example of a node that lacks a connection to the left is *1* on the last note: in legato playing, it is impossible to reach the last note with the thumb, regardless of which finger plays the second-last note.

In the next stage, all nodes representing dead ends or unreachable goals are trimmed from the network. The result (Figure 5B) still includes all possible fingerings. In our example, any one of the 12 possible paths through the graph from the start to the end represents a playable fingering.

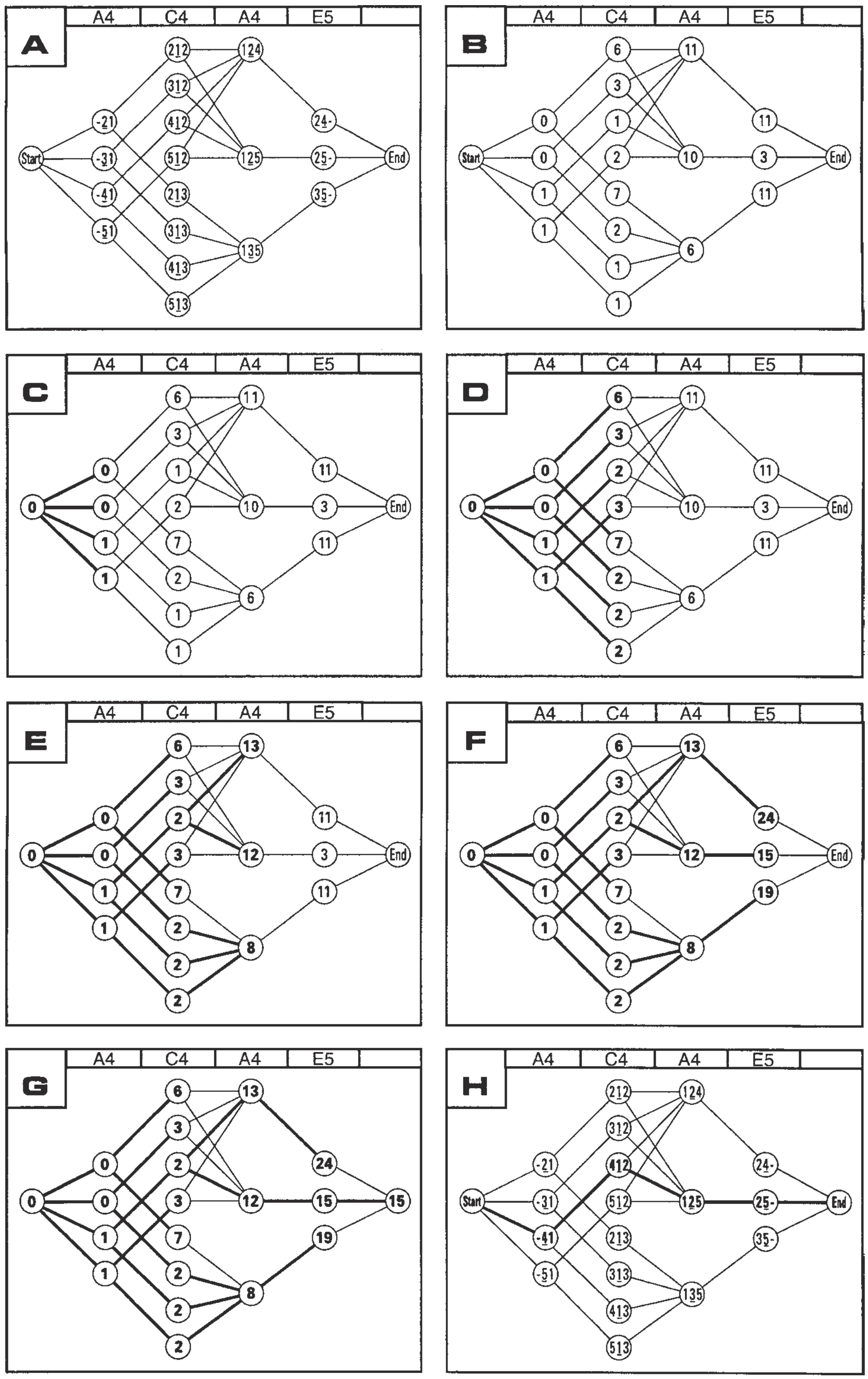
After this, it is necessary to evaluate the difficulty associated with each state. This is not straightforward, as some fingering rules apply to individual notes (e.g., the Weak-Finger Rule), some to pairs of notes (e.g., the Large-Span Rule), and some to sequences of three notes (e.g., the Position-Change rules). We solve this problem by making the context information implicit in the nodes of the graph. In Figure 6A, a single finger on a single key is no longer considered a basic fingering unit. Instead, each node represents the current note played with a specified finger, plus the preceding and following fingerings. All involved fingers are marked inside the node; the underlined number represents the fingering of the current note. For example, the node representing finger *2* on the second A_4 is exploded into two nodes representing fingerings *1-2-4* and *1-2-5* on C_4-A_4-E_5. Note that this representation still contains the same information as Figure 6B.

In Figure 6B, the difficulty associated with each node is calculated. The rule involving only one finger (Weak-Finger Rule) is applied only to the current note; rules involving two fingers (Stretch, Small-Span, Large-Span, Three-to-Four, and Four-on-Black rules) are applied only to the current and previous notes; and rules involving three notes (Position-Change-Count, Position-Change-Size, Three-Four-Five, Thumb-on-Black, Five-on-Black, and Thumb-Passing rules) are applied to current, previous, and following notes. For clarity, only the result of this calculation is shown; the fingering on which it is based may be seen by referring back to Figure 6A.

The difficulty of a complete fingering can now be calculated by following a path through the graph and accumulating difficulties associated with each node visited. The optimal fingering(s) is/are the least difficult path(s) through the net. This is an example of the well-known problem of finding the shortest route through a network. An algorithm for a general solution of this problem was described by Dijkstra (1959). It involves progressing through the network (here, sweeping from left to right) and annotating each node with the accumulated cost (here, total difficulty of the cheapest fingering to reach this state from the start).

Let us assume that the sweep is at a stage in which all the nodes representing the fingering of a certain note have been annotated with the cost of the cheapest arrival route. Then each node for the next note can "look" at its incoming links and select as best previous fingering the one with the minimal accumulated cost. After that, it can add its own cost to this number, yielding the cheapest way to get to this node from the start.

Parts (C) to (G) of Figure 6 illustrate, step by step, the sweep through the network for the fragment A_4-C_4-A_4-E_5. Optimal links from the beginning to the current note are shown by bold lines, and the accumulated costs of optimal routes are shown by bold numbers. For clarity, cumulative difficulty values (in bold) replace the individual difficulties of nodes in each successive step. At the final step (Figure 6H), a single optimal route has been identified (fingering *4-1-2-5*) and the associated difficulty score determined (15 points).


A
A4 C4 A4 E5
Start
End
B
A4 C4 A4 E5
Start
End
C
A4 C4 A4 E5
End
D
A4 C4 A4 E5
End
E
A4 C4 A4 E5
End
F
A4 C4 A4 E5
End
G
A4 C4 A4 E5
H
A4 C4 A4 E5
Start
End

It can be proven that the computational cost of the algorithm increases polynomially with the size of the network. In contrast with a method that enumerates and evaluates all possible paths individually and takes exponential time, this algorithm thus makes calculating a shortest-route feasible. Here, we can do even better, because in our case a given node is only connected to nodes in immediately preceding and following columns. In this special case, our trials have suggested that computation time increases approximately linearly with length of fragment. The determination of the optimal fingering of a melody of length n notes requires an exploded network (cf. Figure 6A) of no more than n x 5 x 4 x 4 + 2 nodes. Computational memory needs thus increase linearly with respect to fragment lengths. This estimate applies to the worst case in which all five fingers are allowed on every key, and repeated use of the same finger is not permitted (as in the present algorithm); normally, this number is reduced considerably by the requirement that successive finger spans not exceed MaxPrac. In sum, the problem of finding the optimal fingering of a long melody is not computationally difficult.

Experiment

The literature on piano performance abounds with pedagogical works by pianists that refer to principles of fingering, and the model just set out attempts to capture some of this information. Yet to our knowledge no one has ever published the results of an objective comparison between the preferred fingerings of a number of different pianists for a given piece or corpus of music. The primary aim of the present experiment is to establish an initial data set of this kind. The experiment was designed so that the data could easily be compared with the predictions of the model.

METHOD

Participants

Written fingerings of 28 pianists were considered. Among them were 13 professionals (mean total number of years practicing and/or performing = 31) and 3 pianists in the undergraduate music program at the Department of Music, Keele University; all 15 had recently participated in a companion study (Sloboda, Clarke, Parncutt, & Raekallio, 1997). Of the remainder, 8 were professional pianists and piano teachers (personal friends and colleagues

Fig. 6 (facing page). (A) Exploded version of Figure 5B that includes the fingering of immediately preceding and following notes within each node (the absence of a preceding or following note is indicated by a dash). (B) The same network annotated with the calculated difficulty of each node. (C–G) Successive stages of a shortest-route sweep. In each stage, each node is connected to predecessors with the lowest total cost (difficulty) and annotated (bold) with the total calculated difficulty for arriving at that state from the start. (H) Any bold path from start to end (fingering *4125*; calculated difficulty, 15) represents an optimal fingering.

of the first author who had had no other contact with the piano fingering project); 3 were the authors of this paper;[31] and 1 was Czerny himself.[32] All but 2 of the 28 participants were classically trained (the two in question played mainly popular music and jazz), and all but the 3 students performed regularly on a professional basis.

Materials

Seven short pieces were selected from the 160 *Kurze Übungen* op. 821 of Carl Czerny, as published by Peters (Edition Nr. 2405, 1920). The criterion for choosing the pieces was that at least two distinct, but arguably equally good, fingerings existed for the opening of each piece. We therefore expected each of these fingerings to be preferred by a significant number of pianists. The first two measures of the right hand of each piece are given in Figure 7.

Procedure

Each pianist received the complete score of the seven pieces. The scores included tempo and dynamic markings, but not fingerings, which had been carefully erased (by whiting out and recopying). The 12 professional pianists and 3 students received the scores after participating in a companion study that involved sight-reading each piece twice. They took the scores home before writing down their preferred fingerings. The other participants received the scores by mail. None of the participants were directly paid for their services; however, the pianists who had participated in the sight-reading study were reimbursed (£25 for professionals and £5 per hour for students).

Pianists were informed that the pieces were by Czerny, but were not told the title or opus number of the collection. They were asked to write on the score their "preferred fingerings" for the right hand of the first two measures of each piece. In order to avoid ambiguity, they were asked to write a finger number against each and every note. A preferred fingering was defined as "the fingering that you would probably use in performance." They were asked to disregard any fingerings that they thought or knew that the composer intended for the pieces; retrospectively, we found no reason to believe that any of them were influenced by such knowledge or belief. They were also asked to write brief explanations for their choice of fingerings directly on the scores; the aim of asking for this commentary was primarily to ensure that the pianists had given the matter some thought and were sure of their fingering solutions.

RESULTS

For the purpose of analysis, we considered only the first few notes of the right hand of each piece. The length of each analyzed fragment was deter-

31. We arrived at fingerings independently of each other and before any fingerings had been collected from other participants.

32. Czerny's (1839) fingerings tend to minimize difficulty and maximize the tempo with which the music could be played. In this respect, Czerny's thinking corresponds to that of the present ergonomic model—but differs from that of previous pianists such as C. P. E. Bach (1753/1957), who advocated different fingerings for the same passage depending on the desired musical effect. Another way in which Czerny's concept matches that of the present model is his avoidance of the older practice of turning the longer fingers over each other. The fingerings of op. 821 are generally consistent with the idea of minimizing ergonomic difficulty. Incidentally, we presume that the fingerings in our scores are indeed Czerny's and not an editor's, as no editor is mentioned in the edition. But given that the edition is not explicitly Urtext we cannot be certain of this assumption.

Fig. 7. The first two measures of each of the seven Czerny studies used in the experiment.

mined by three criteria. (1) Fragments were constrained to be no longer than eight notes in length. Since the hand-eye span in sight-reading is limited to roughly six notes (Sloboda, 1974), we suppose that fingering is processed in "chunks" of about the same size. This implies that the input to the model described earlier should also be appropriately "chunked."[33] (2) The last note of the fragment was chosen such that there was considerable agreement among the pianists in this study regarding the fingering of that note. The idea here is that the fingering of any fragment is influenced by the notes preceding and following that fragment; but those notes are not input to the model. The simplest way to solve this problem is to specify the fingering of the initial and final note as a constraint at the input to the model. The output of the model may then be compared only with written fingerings conforming with this constraint. (3) For the cyclically repeating pattern of notes at the start of Piece B, the fingerings that were accepted as data were those that were used consistently in successive cycles. The predictions with which these data were compared similarly took into account the cyclic nature of the input.

The distribution of written fingerings for each of these fragments is shown in Table 3. For all but Piece F, at least two different fingerings were chosen three times or more, confirming that the fingering of these fragments is sufficiently ambiguous to warrant systematic investigation. Subjectively, the choice between the two main fingerings of the opening of Piece A appears to have been primarily determined by pitting the Position-Change Rules against the Weak-Finger rules; and for Piece B, by pitting the Thumb-on-Black against the Small-Span rules. For the other pieces, the Thumb-on-Black and Position-Change rules appear to have been predominant in determining the most popular fingerings.

Table 4 compares the fingerings chosen by the pianists with the fingerings predicted by the model. For each melodic fragment, we have listed the 10 fingerings with the lowest calculated difficulty. Where fingerings chosen in the experiment are not listed among the top 10, we have appended them along with their rank positions in the original list output by the model.

In general, the model successfully predicts the most commonly selected fingerings out of the typically very large number of possible fingerings. In most cases, the most popular fingering is among the top 10; the only exception is Piece F, where the most popular fingering involves a constantly stretched hand position (14 points assigned by the Large-Span Rule). This example suggests that the model does not account adequately for the pianistic advantage of maintaining a roughly constant stretch between the fingers while playing arpeggio-like figures. For each piece, it is possible to adjust the relative weighting of the different rules to improve predictions for that piece (for example, predictions for Piece A are improved if less weight is given to the Weak-Finger and Small-Span rules, reducing the total calculated difficulty of fingerings a and b), but such adjustments invariably cause the model's predictions to deteriorate for other pieces.

33. We expect this constraint to apply to sight-reading to a greater extent than it applies to premeditated written fingerings.

Table 3
Results: Distribution of Written Fingerings for the Opening Fragments of Seven Czerny Studies

Label	Fingering	Count
Piece A (op. 821 no. 1 in C)		
first eight notes (E G F G E F D E)		
a	***35453423***	**8**
b	*13231423*	7
c	*24342312*	4
d	*13231313*	2
e	*14342312*	2
f	*14341423*	1
g	*14342413*	1
h	*15453423*	1
i	*25352312*	1
j	*25352313*	1
Piece B (op. 821 no. 37 in A)		
opening cyclic pattern (E C♯ D F♯)		
a	*3214*	17
b	***3124***	**8**
c	*4215*	1
d	*1234*	1
e	*1235*	1
Piece C (op. 821 no. 38 in A)		
first five notes (E C♯ E A E)		
a	*32351*	10
b	*21251*	5
c	*21241*	4
d	*12121*	3
e	*32121*	2
f	*21231*	1
g	*21242*	1
h	***31251***	**1**
i	*42121*	1
Piece D (op. 821 no. 54 in C♯)		
first seven notes (C♯ E♯ D♯ F♯ E♯ G♯ F♯)		
a	***1324132***	**15**
b	*2524132*	3
c	*2435132**	3
d	*2423132*	2
e	*2424132*	2
f	*1213243*	1
g	*2414142*	1
h	*2523132*	1
Piece E (op. 821 no. 62 in d)		
first eight notes (A G♯ A E F C♯ D G♯)		
a	*14515352*	4
b	*14515342*	3
c	*14534253*	3
d	*14535342*	3
e	*14524252*	2
f	*13415342*	1
g	*14514231*	1
h	*14514252*	1
i	*14515212*	1
j	***14523152***	**1**
k	*14525352*	1
l	*14534231*	1
m	*14534232*	1
n	*14534242*	1
o	*14534252*	1
p	*14535242*	1
q	*14535352*	1
r	*14535353*	1
Piece F (op. 821 no. 66 in g)		
first six notes (G B♭ D B♭ G D)		
a	***123251***	**23**
b	*124251*	2
c	*123151*	1
d	*125251*	1
e	*135251*	1
Piece G (op. 821 no. 96 in b)		
first seven notes (F♯ G F♯ E F♯ E D)		
a	*3432321*	14
b	***2432432****	**3**
c	*2321321*	2
d	*2432321*	2
e	*3432431*	2
f	*3432432*	2
g	*1321321*	1
h	*3531421*	1
i	*4543432*	1

The labels in the left columns are for reference and link Tables 3 and 4. “Count” denotes the number of pianists who chose each fingering. Entries marked with an asterisk are examples of fingerings that appear to have been determined primarily by cognitive principles not incorporated in the model (see Discussion). Fingerings in bold are Czerny’s.

The table enables systematic identification of the rules that were most influential in determining a given fingering. Take, for example, the most popular fingering for the start of Piece A, fingering a (*35453423*). Accord-

Table 4
First 10 Predicted Fingerings for Opening Fragments of Seven Czerny Studies

No.	Fingering	str	sma	lar	pcc	pcs	wea	345	3t4	bl4	bl1	bl5	pa1	TOT	lb	ct
Piece A (op. 821 no. 1 in C): first eight notes (E G F G E F D E)																
input constraint: last finger = *3*; 2279 playable fingerings																
1	*24342313*	0	1	0	1	1	2	0	1	0	0	0	0	6		
2	*13231213*	0	1	0	2	4	0	0	0	0	0	0	0	7		
3	*14342313*	0	3	0	1	1	2	0	1	0	0	0	0	8		
4	***35453423***	**0**	**0**	**0**	**0**	**0**	**4**	**3**	**1**	**0**	**0**	**0**	**0**	**8**	**a**	**8**
5	*13231323*	0	2	2	2	3	0	0	0	0	0	0	0	9		
6	*13231423*	0	4	0	1	3	1	0	0	0	0	0	0	9	b	7
7	*13242313*	0	3	0	3	2	1	0	0	0	0	0	0	9		
8	*12142313*	0	4	0	3	2	1	0	0	0	0	0	0	10		
9	*13232313*	0	1	2	3	4	0	0	0	0	0	0	0	10		
10	*24342323*	0	0	2	2	3	2	0	1	0	0	0	0	10		
Ranked beyond 10:																
12	*13231313*	0	3	0	3	5	0	0	0	0	0	0	0	11	d	2
13	*15453423*	0	4	0	0	0	4	2	1	0	0	0	0	11	h	1
90	*14341423*	0	8	0	1	2	3	0	1	0	0	0	0	15	f	1
123	*14342413*	0	9	0	1	2	3	0	1	0	0	0	0	16	g	1
181	*25352313*	0	13	0	1	1	2	0	0	0	0	0	0	17	j	1
Excluded by input constraint:																
c 4 (*24342312*), e 2 (*14342312*), and i 1 (*25352312*)																
Piece B (op. 821 no. 37 in A): opening cyclic pattern (E C♯ D F♯)																
input constraint: cyclic pattern; 48 playable fingerings																
1	***3124***	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**5**	**0**	**0**	**7**	**b**	**8**
2	*3214*	0	3	2	0	0	1	0	0	1	0	0	0	7	a	17
3	*4235*	0	0	0	0	0	2	1	0	0	0	4	0	7		
4	*4215*	0	5	0	0	0	2	0	0	0	0	4	0	11	c	1
5	*1235*	0	9	0	0	0	1	0	0	0	0	4	0	14	e	1
6	*2124*	0	2	0	2	4	1	0	0	0	5	0	0	14		
7	*3125*	0	4	0	0	0	1	0	0	0	5	4	0	14		
8	*3215*	0	7	2	0	0	1	0	0	0	0	4	0	14		
9	*3235*	0	2	2	2	4	1	0	0	0	0	4	0	15		
10	*4125*	0	4	0	0	0	2	0	0	0	5	4	0	15		
Ranked beyond 10:																
16	*1234*	4	7	4	0	0	1	0	1	1	0	0	0	18	d	1
Excluded by input constraint: none																
Piece C (op. 821 no. 38 in A): first five notes (E C# E A E)																
input constraint: last finger = *1*; 67 playable fingerings																
1	*21241*	0	0	2	1	0	1	0	0	0	5	0	0	9	c	4
2	*21251*	0	2	0	1	0	1	0	0	0	5	0	0	9	b	5
3	*31241*	0	0	2	2	0	1	0	0	0	5	0	0	10		
4	*31251*	0	2	0	2	0	1	0	0	0	5	0	0	10	h	1
5	*32351*	0	2	6	1	0	1	0	0	0	0	0	0	10	a	10
6	*42351*	0	2	4	2	0	2	0	0	0	0	0	0	10		
7	*31351*	0	2	2	1	0	1	0	0	0	5	0	0	11		
8	*42151*	0	8	0	1	0	2	0	0	0	0	0	0	11		
9	*21351*	0	2	2	2	0	1	0	0	0	5	0	0	12		
10	*32151*	0	8	2	1	0	1	0	0	0	0	0	0	12		

No.	Fingering	str	sma	lar	pcc	pcs	wea	345	3t4	bl4	bl1	bl5	pa1	TOT	lb	ct
Ranked beyond 10:																
26	*21231*	4	0	6	1	0	0	0	0	0	5	0	0	16	f	1
27	***42121***	**0**	**4**	**0**	**3**	**8**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**16**	**i**	**1**
31	*32121*	0	4	2	3	8	0	0	0	0	0	0	0	17	e	2
34	*12121*	0	8	0	2	8	0	0	0	0	0	0	0	18	d	3
Excluded by input constraint: g 1 (*21242*)																
Piece D (op. 821 no. 54 in C#): first seven notes (C♯ E♯ D♯ F♯ E♯ G♯ F♯)																
input constraint: last finger = *2*; 1143 playable fingerings																
1	*1323132*	0	2	2	2	3	0	0	0	0	3	0	0	12		
2	***1324132***	**0**	**4**	**0**	**1**	**3**	**1**	**0**	**0**	**0**	**3**	**0**	**0**	**12**	**a**	**15**
3	*2412132*	0	3	0	2	4	1	0	0	0	3	0	0	13		
4	*1212132*	0	0	0	3	5	0	0	0	0	6	0	0	14		
5	*1213242*	0	2	0	2	3	1	0	0	0	6	0	0	14		
6	*1323142*	0	6	2	1	1	1	0	0	0	3	0	0	14		
7	*1323242*	0	2	2	3	4	1	0	0	0	3	0	0	15		
8	*2413242*	0	5	0	2	3	2	0	0	0	3	0	0	15		
9	*2435132*	0	6	0	1	3	2	1	0	0	0	2	0	15	c	3
10	*1213232*	0	0	2	3	5	0	0	0	0	6	0	0	16		
Ranked beyond 10:																
14	*2423132*	0	4	2	3	6	1	0	0	0	0	0	0	16	d	2
22	*2424132*	0	6	0	3	6	2	0	0	0	0	0	0	17	e	2
113	*2524132*	0	12	0	3	7	2	0	0	0	0	0	0	24	b	3
181	*2414142*	0	13	0	3	5	3	0	0	0	3	0	0	27	g	1
Excluded by input constraint: f 1 (*1213243*), and h 1 (*2523132*)																
Piece E (op. 821 no. 62 in d): first eight notes (A G♯ A E F C♯ D G♯)																
input constraint: last finger = *2*; 239 playable fingerings																
1	***14523152***	**0**	**6**	**2**	**2**	**6**	**3**	**0**	**0**	**0**	**5**	**0**	**0**	**24**	**j**	**1**
2	*14523142*	2	4	6	2	6	3	0	0	0	5	0	0	28		
3	*14512152*	0	8	2	3	10	3	0	0	0	5	0	0	31		
4	*14513152*	0	10	2	3	9	3	0	0	0	5	0	0	32		
5	*14515342*	2	8	6	3	7	4	1	1	0	0	0	0	32	b	3
6	*14515352*	0	12	2	4	10	4	0	0	0	0	0	0	32	a	4
7	*14524152*	0	11	2	3	7	4	0	0	0	5	0	0	32		
8	*13412152*	2	6	4	3	10	2	0	1	0	5	0	0	33		
9	*13523152*	2	10	4	3	7	2	0	0	0	5	0	0	33		
10	*14515452*	2	8	6	3	9	5	0	0	0	0	0	0	33		
Ranked beyond 10:																
13	*13415342*	4	6	8	3	8	3	1	2	0	0	0	0	35	f	1
16	*14514252*	2	14	2	3	10	4	0	0	0	0	0	0	35	h	1
27	*14534242*	2	4	8	4	11	4	2	1	0	0	0	0	36	n	1
28	*14535342*	2	4	8	4	11	4	2	1	0	0	0	0	36	d	3
29	*14535352*	0	8	4	5	14	4	1	0	0	0	0	0	36	q	1
35	*14524252*	2	12	2	4	13	4	0	0	0	0	0	0	37	e	2
37	*14525352*	2	12	2	4	13	4	0	0	0	0	0	0	37	k	1
39	*14534252*	2	8	4	4	12	4	2	1	0	0	0	0	37	o	1
54	*14535242*	2	10	8	3	11	4	1	0	0	0	0	0	39	p	1
Excluded by input constraint:																
c 3 (*14534253*), g 1 (*14514231*), l 1 (*14534231*), r 1 (*14535353*)																
Illegal (last span too big):																
i 1 (*14515212*), m 1 (*14534232*)																

(Continued on next page)

TABLE 4 (CONTINUED)

No.	Fingering	str	sma	lar	pcc	pcs	wea	345	3t4	bl4	bl1	bl5	pa1	TOT	lb	ct
Piece F (op. 821 no. 66 in g): first six notes (G B♭ D B♭ G D)																
input constraint: last finger = *1*; 110 playable fingerings																
1	*124251*	2	2	6	2	3	2	0	0	0	0	0	0	17	b	2
2	*124151*	0	3	0	3	6	2	0	0	0	5	0	0	19		
3	*123151*	2	2	4	2	4	1	0	0	0	5	0	0	20	c	1
4	*124141*	0	1	0	3	9	2	0	0	0	5	0	0	20		
5	*123141*	2	0	4	3	7	1	0	0	0	5	0	0	22		
6	*124131*	0	1	2	3	10	1	0	0	0	5	0	0	22		
7	*135141*	0	3	0	3	10	2	0	0	0	5	0	0	23		
8	*135251*	2	4	6	3	6	2	0	0	0	0	0	0	23	e	1
9	*125251*	2	6	6	2	6	2	0	0	0	0	0	0	24	d	1
10	*135131*	0	3	2	3	10	1	0	0	0	5	0	0	24		
Ranked beyond 10:																
13	***123251***	**6**	**2**	**14**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**25**	**a**	**23**
Piece G (op. 821 no. 96 in b): first seven notes (F♯ G F♯ E F♯ E D)																
input constraint: last finger = *1*; 773 playable fingerings																
1	*3432321*	0	0	0	0	0	1	0	1	0	0	0	0	2	a	14
2	*2321321*	0	1	0	2	0	0	0	0	0	0	0	0	3	c	2
3	*3431321*	0	2	0	1	0	1	0	1	0	0	0	0	5		
4	*2432321*	0	4	0	1	0	1	0	0	0	0	0	0	6	d	2
5	*4542321*	0	2	0	1	0	3	0	0	0	0	0	0	6		
6	*2132321*	0	6	0	1	0	0	0	0	0	0	0	0	7		
7	*2321421*	0	5	0	2	0	1	0	0	0	0	0	0	8		
8	*2321431*	0	4	0	2	0	1	0	0	1	0	0	0	8		
9	*3132321*	0	8	0	0	0	0	0	0	0	0	0	0	8		
10	*3432421*	0	4	0	1	0	2	0	1	0	0	0	0	8		
Ranked beyond 10:																
12	*1321321*	0	3	0	3	0	0	0	0	0	3	0	0	9	g	1
15	*3432431*	0	3	0	2	0	2	0	1	1	0	0	0	9	e	2
90	*3531421*	0	14	0	2	0	2	0	0	0	0	0	0	18	h	1
Excluded by input constraint:																
b 3 (*2432432*), f 2 (*3432432*), i 1 (*4543432*)																

Abbreviations: str = Stretch, sma = Small-Span, lar = Large-Span, pcc = Position-Change-Count, pcs = Position-Change-Size, wea = Weak-Finger, 345 = Three-Four-Five, 3t4 = Three-to-Four, bl4 = Four-on-Black, bl1 = Thumb-on-Black, bl5 = Five-on-Black, pa1 = Thumb-Passing; TOT = total; lb = label; ct = count. Fingerings in bold are Czerny's.

ing to the model, the technical difficulty of this fingering arises from the individual use of weak fingers *4* and *5* (4 points); the use of fingers *3*, *4*, and *5* in succession (3 points); and the transition from *3* to *4* (1 point). The pianists had a choice between this fingering and many others that, to various extents, avoided these difficulties. It follows that the Weak-Finger, Three-Four-Five, and Three-to-Four rules were *not* instrumental in the determination of this fingering. Instead, it was determined by the avoidance of other sources of difficulty—that is, by rules for which no points were assigned. Of these, all those dealing with black keys can be eliminated, because there are no black keys in the fragment. The Stretch and Large-Span rules may also be eliminated, on the grounds that there are no large intervals in the fragment. The fingering was thus determined by a combination of the Small-Span, Position-Change, and Thumb-Passing rules. By the same logic, fingering b (*13231423*) was determined by the Weak-Finger, Three-Four-Five, and Thumb-Passing rules. In general, fingerings in the table may be said to have been determined by those rules for which no points (or relatively few points) were assigned, provided that the notes of the fragment make it reasonably possible to generate points by those rules.

In Table 4, it is often the case that several fingerings have the same total calculated difficulty (e.g., Piece A, fingerings 5–7). The rank position that fingerings are allocated within such a group is arbitrary. Moreover, all total calculated difficulty values are inherently uncertain due to the arbitrary formulation of the rules and the arbitrary way in which the rules are weighted relative to one another. For this reason, fingerings 5–7 for Piece A (total calculated difficulty = 9 points) can hardly be considered to be different in difficulty, according to the model, from fingerings 8–10 (total = 10 points).

Discussion

Table 4 shows that many of the fingerings chosen by the pianists were among the fingerings predicted to be least difficult by the model. This confirms that the model encapsulates at least some of the factors that contribute to the technical difficulty of melodic legato fragments played on keyboard. The model also predicted many fingerings that were not proposed by any of the pianists. These fingerings fall into two groups: fingerings that are interesting and possible pianistically but that our group of subjects did not suggest, and fingerings that are pianistically impractical and reflect deficiencies of the model.[34]

Two possible explanations for the various unproposed fingerings merit consideration. First, the model as it stands may be incorrect or incomplete from the point of ergonomics. The model's effective window length of three notes (according to which only those notes immediately preceding and following a given note are considered) may be too short and could conceivably be extended to include whole hand positions; the rules may be poorly weighted relative to each other; the effects of different rules may not accumulate linearly, as we have assumed; and there may be important ergonomic principles that we have failed to address in the model, such as the following two. First, there may be a tendency to *avoid small hand-position*

34. Clarification and (experimental) investigation of this interesting distinction are beyond the present scope.

changes within large ones; for examples, consider fingerings written by our participants for Piece A (Table 3). Fingerings that avoid small changes within large ones include the most popular fingerings, a, b, and c. Examples of fingerings that include small position changes within large ones are the less popular d (*13231313*) and g (*14342413*). A small position change within a large one may be confusing to the pianist in a similar way that a tongue twister is to a speaker (cf. Rosenbaum, Weber, Hazelett, & Hindorff, 1986); it may be easier to clearly delineate hand positions from each other—to chunk them. The problem could be solved in a future version of the model by assigning extra points for small hand-position changes. Second, pianists may tend in certain cases to *avoid undue acceleration of the hand and fingers*; however it is not yet clear how this idea may be applied generally. Fingerings for Piece A that conform to this criterion include a, c, d, and e; fingerings that do not, b (*13231423*). Yet fingering b is clearly among the best for Piece A. The rule seems to work better for Piece D: in fingering b (*2524132*), there is a clearer benefit from an avoidance of sudden accelerations—the hand glides more smoothly along the keyboard than it does when using (the cognitively easier?) fingering a (*1324132*). It may be possible to make this second proposed rule more general by dividing it into two parts. Regarding the *hand*, the rule could favor fingerings in which the center of gravity of the hand follows a smoothed version of the pitch trajectory of the notes. Regarding the *fingers*, the rule could assign points for sudden changes in finger span (between compact and stretched). Finally, some more straightforward ergonomic issues may have been neglected in the model, such as the difficulty of reaching a white key with a nonthumb when the thumb is on black, that is, of playing white keys close to the back of the keyboard.

A second possibility is that the model's estimations of fingering difficulty are roughly correct from the point of view of ergonomics but not of cognition.[35] Cognitive constraints would facilitate the storage (memory), retrieval, and processing of fingering patterns, by applying similar fingerings to similar musical passages. Four ways of achieving cognitive efficiency are relevant here. All four are discussed or mentioned by Clarke et al. (1997).

1. Pianists may apply standard fingerings[36] for scales and arpeggios, whenever patterns resembling those scales and arpeggios occur in the music. In a sense, this principle is already accounted for by the model, because predictions of the model are consistent

35. Given the limited importance of interpretation in the musical repertoire of this study, we assume that, in this case, interpretive constraints on fingering are relatively unimportant.

36. Standard fingerings for scales and arpeggios are laid down in Britain by the Associated Board and in other countries by similar institutions.

with these well-learned fingering patterns (several of the rules were formulated with the aim of replicating standard fingerings for scales).

2. Pianists may choose fingerings that are familiar from other pieces of music in their repertoire. Having solved a fingering problem in one piece, it is possible to transfer the solution of the problem to a similar passage as it occurs in another piece.
3. The same or a similar fingering may be used when the same or similar passages recur within a single piece of music. This may happen on a relatively large time scale, such as when thematic material in the exposition of a sonata movement recurs in a different key in the development or recapitulation. As already noted, this idea was taken to extremes by pianists such as Busoni and von Bülow, who recommended playing whole pieces in different keys with the same fingering. The principle, sometimes called the "rule of regularity," also played an important role in the system of Kullak (1876) and was central to that of Werkenthin (1888), a pupil of von Bülow.[37] The same principle may be applied on much smaller time scales, when a fast repetitive pattern is fingered by the repetitive use of a specific finger sequence. The fingerings in Table 3 marked with asterisks (*2435132* and *243243*) are good examples. The effect has been investigated for both language (tongue twisters) and tapping (finger fumblers) by Rosenbaum et al. (1986), who found that "motor performance suffers when responses in a repeated sequence have variable rather than fixed parameter mappings" (abstract). Whenever a fingering pattern of brief duration is repeated at high speed, Rosenbaum et al. propose that a motor program is established to coordinate the movements required to execute the pattern and that performance accuracy is enhanced if the program parameters (in this case, the fingering of each note group) remain constant during the course of the repetitions.
4. Fingering patterns may be linked with low-level structural descriptions of the music, a procedure that involves musical interpretation as well as cognition. Sloboda (1974, 1984) demonstrated the existence of such descriptions in piano sight-reading by observing variations in duration of the "eye-hand span" as a function of position within musical phrases. Clarke et al. (1997) reported that pianists tend to use stronger fingers on metrical accents and that changes of hand position are more likely to occur at phrase boundaries, confirming that fingerings, too, are

37. We thank Martin Gellrich for this information.

influenced by phrase boundaries (or grouping accents) as well as other salient structural markers such as metrical, harmonic, and melodic accents (reviewed by Parncutt, 1997).

No attempt has been made to incorporate the above cognitive-ergonomic principles into the current model. A systematic account of the first three principles would require a procedure to estimate the similarity of any two pitch-time patterns, such that the principles would only be invoked when sufficiently similar patterns (i.e., patterns that are similar beyond some preset threshold) could realistically be played with the same fingering. Two keyboard patterns may be regarded as similar from the point of view of fingering (1) if interval sizes are maintained (as in exact transposition through any interval), (2) if the pattern of black/white keys is maintained (as sometimes happens in transpositions through the interval of a fifth), or (3) if both these constraints are almost met (where "almost" implies the existence of adjustable thresholds). The repertoire of fingering patterns known to a given pianist or to a group of expert pianists could be represented either serially (as a simple list) or by a connectionist system (neural net); in any case, the relative familiarity of individual fingering patterns would need to be accounted for. The fourth principle could be modeled by including procedures for low-level structural analysis at the input to the model.

Clarke et al. (1997) discuss several other important influences on fingering that are neglected by the present model:

5. Rhythm. Other things being equal, pianists tend to use stronger fingers at metrically stronger positions and at dynamic accents (Bach, 1753/1957).
6. Tempo. The tendency to avoid ergonomic difficulties of the kinds encapsulated by this model increases as tempo increases: the higher the speed of performance, the greater the need to avoid unnecessary movements of the hand (Czerny, 1839). So, at high speeds, relatively greater weight may need to be placed on rules that address changes of hand position and the placing of short fingers (*1, 5*) on short (black) keys.
7. Articulation. A fingering that is optimal for legato playing is not necessarily so in staccato (see, e.g., Werkenthin, 1888); changes in hand position may occur more frequently in staccato.
8. Register. A given fingering may be easier to play in one register than another because of the angle the hand makes with the keyboard. So, for example, individual white keys in the highest register of the piano may easily and reliably be played with the right-hand thumb; in lower registers, the right thumb becomes a less comfortable choice for isolated white keys.

9. Style. It may be argued that the principles encapsulated by the model are limited with regard to musical style (Neuhaus, 1958/1973), for the experimental data against which we have tested the model consisted only of fingerings for Czerny studies. However, if—as we propose—the rules indeed are motor-anatomic, then they should apply equally well to all single-line melodic fragments, regardless of style.

Future improvements to the model, apart from accounting for some of the above factors, may also make existing parts of the model more concise. The rules as currently formulated are not necessarily orthogonal (i.e., independent of one another): two or more rules often have a similar effect on the calculated difficulty of a fingering. Sloboda et al. (in press) did a factor analysis on difficulty estimates according to the separate rules for 7 of the 10 Czerny studies referred to in this paper. Five factors emerged, which together accounted for 78% of the variance in fingerings performed by pianists as they sight-read the studies. Rules 1, 3, 10, and 11 combined to form a general "stretch" factor; rules 2, 4, and 5 formed a general "position change" factor; and rules 6, 7, and 8 represented "agility." The other two factors corresponded to the individual rules 9 and 12.

As mentioned earlier, many of the rules in the model have historical precedents. But not all such fingering rules play an important role in our model, and conversely, not all rules that are important in our model were important historically. Because this is the first study of fingering that includes a computational model, we have been forced to specify our constraints quite precisely and to include concepts that might previously have been ignored or taken for granted. Conversely, the rules laid down by early writers deal primarily with issues that pianists need to think about in order to improve their technique, such as the use of more or less strong or agile fingers for different purposes. For example, the "stretch" and "span" concepts developed here have not been central issues in historical writings. Pianists tend to learn about these concepts intuitively, and so little guidance or discussion is needed in books that aim to give practical assistance to pianists. Moreover, the piano repertoire has changed: many 20th century piano works, for example, require fingering procedures that were unknown in earlier music. Finally, the piano itself has changed, so rules for avoiding weak fingerings may be more relevant to the modern piano than to the fortepiano, because the modern piano has a heavier action.

In closing, we wish to emphasize that the model is not intended to mimic the cognitive processes that occur when a pianist chooses fingerings in performance situations (sight-reading, rehearsed reading, memorized performance). We suppose this process to be rather different from that implied by the model. All pianists have at their disposal a vast repertoire of fingerings associated with specific musical patterns that they have played in pieces of music, technical exercises, and improvisations. Presumably, it is from this repertoire that fingerings are primarily drawn during performance and practice.

The purpose of the present model is not, then, to explain how fingerings are determined in real time, but to explain why certain fingering patterns predominate over others in the fingering repertoires of individual pianists and in the collective fingering knowledge of all pianists. The process by which a certain fingering becomes established in this repertoire may best be explained by a kind of evolutionary theory. First, a pianist chances upon a new fingering for a given passage by trial and error. If the fingering competes well with other possible fingerings, then the pianist is likely to use it in performance and to tell other pianists (such as students) about it. If others agree that the fingering is a good one, then it will find an established place in the repertoire. In this sense, fingerings are determined by the principle of survival of the fittest. The model described in the present paper is an attempt to predict, from a purely ergonomic viewpoint, which fingerings are most likely to survive—provided that they have, at some time, been discovered and experimented with.[38]

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Appendix
Mathematical Specification of the Stretch, Large-Span, and Small-Span Rules

Table 1 presents an overview of values of maximum and minimum practical, comfortable, and relaxed spans. The table does not, however, specify how the rules may be programmed in a way that elegantly accounts both for rising and falling intervals and for thumb passing.

Let *i* be the interval between two notes played by right-hand fingers *f* and *g*, in that order. Assume for the moment that *g* > *f*. If *i* is positive, the interval rises in pitch, that is, the note played by *g* is higher in pitch than the note played by *f*; if *i* is negative, the interval falls in pitch.

Let MaxPrac (*f*, *g*) be the maximum value of *i* that may comfortably be played by *f* followed by *g*; MaxComf and MaxRel may be defined similarly. Here, the maximum of a set of negative numbers is defined as the *least negative*; for example, a falling interval of 8 semitones, denoted *i* = –8, is here regarded as "greater" than a falling interval of 12 semitones (*i* = –12).

Referring to Table 1, we may now define three matrices of "maximum spans". The rows represent the first finger *f* in a pair; the columns, the second finger *g*:

MaxPrac				
0	10	12	14	15
5	0	5	7	10
4	–1	0	4	7
3	–1	–1	0	5
1	–2	–1	–1	0

MaxComf				
0	8	10	12	13
3	0	3	5	8
2	–1	0	2	5
1	–1	–1	0	3
–1	–2	–1	–1	0

MaxRel				
0	5	7	9	10
–1	0	2	4	6
–3	–1	0	2	4
–5	–3	–1	0	2
–7	–5	–3	–1	0

Each cell in each matrix may be referred to by its (*f*, *g*) coordinates; for example, MaxComf (*1*,*2*) = 8. Values under the leading diagonal correspond to finger pairs (*f*, *g*) played in descending rather than ascending order (*g* < *f*). Positive numbers under the leading diagonal denote crossed fingers; for example, MaxPrac (*2*,*1*) = *5* means that the finger *2* may comfortably be passed up to *5* semitones over finger *1*. Negative numbers under the leading diagonal are smallest distances between finger pairs when next to each other but not crossing.

For a further example of the "maximum" of a set of negative numbers, consider the entry MaxRel (*5*,1) = –7. This means that the most positive (or smallest absolute) relaxed span between *5* followed by *1* is a descending interval of 7 semitones. MinRel (*5*,*1*), the "minimum" relaxed span between *5* followed by *1* is actually bigger than MaxRel (*5*,*1*) in absolute terms; referring to Table 1, it is –10 semitones.

The three corresponding "minimum" matrices may be obtained by reflecting the corresponding "maximum" matrices about their leading diagonals and changing the sign of all entries. Expressed mathematically,

MinPrac (*f*, *g*) = –1 x MaxPrac (*g*, *f*)
MinComf (*f*, *g*) = –1 x MaxComf (*g*, *f*)
MinRel (*f*, *g*) = –1 x MaxRel (*g*, *f*)

Using these matrices, the Stretch Rule may be formulated in a general way that applies to both positive and negative intervals *i*, and to both ascending and descending orders of fingers *f* and *g*:

if *i* > MaxComf, assign 2 x (*i* – MaxComf); and
if *i* < MinComf, assign 2 x (MinComf – *i*).

Similarly, the Small-Span Rule may be written:

if *i* < MinRel and the thumb is involved, assign (MinRel – *i*); and
if *i* < MinRel and the thumb is not involved, assign 2 x (MinRel – *i*).

Finally, the Large-Span Rule may be written:

if *i* > MaxRel and the thumb is involved, assign (*i* – MaxRel); and
if *i* > MaxRel and the thumb is not involved, assign 2 x (*i* – MaxRel).

The preceding definitions of the Small-Span and Large-Span Rules are based on the assumption (maintained throughout the above argument) that "large" means "relatively positive" and "small" means "relatively negative." To make the definitions apply instead to the absolute size of intervals and ignore their direction (as in the everyday meaning of the words "large" and "small"), the preceding two definitions must be interchanged in cases where $f > g$ (e.g., when passing the thumb). In other words, whenever $f > g$, the Small-Span Rule is given by the above definition of the Large-Span Rule, and vice versa. (This final modification was implemented before calculating Table 4. It does not affect the total calculated difficulty of each fingering or the rank order of fingerings.)