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A monitoring procedure for detecting structural breaks in factor copula models

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Abstract: We propose a new monitoring procedure based on moving sums (MOSUM) for detecting single or multiple structural breaks in factor copula models. The test compares parameter estimates from a rolling window to those from a historical data set and analyzes the behavior under the null hypothesis of no parameter change. The case of multiple breaks is also treated. In the model, the joint copula is given by the copula of random variables which arise from a factor model. This is particularly useful for analyzing high dimensional data. Parameters are estimated with the simulated method of moments (SMM). We analyze the behavior of the monitoring procedure in Monte Carlo simulations and a real data application. We consider an online procedure for predicting the day-ahead Value-at-risk based on the suggested monitoring procedure.

Keywords: factor copula model; monitoring procedure; simulated method of moments; value at risk.

JEL classification: C12; C32; C58.

1 Introduction

Analyzing time-variant parameters in models for financial data is a research topic of wide importance. In this paper, we consider factor copula models which have been recently proposed by Oh and Patton (2017) and Krupskii and Joe (2013), and we focus on the first approach. In such models, the joint copula between random variables is given by the copula of random variables which arise from a factor model. The time-varying parameters are factor loadings and the parameters describing the distributions of the common and idiosynchratic factors.

The advantage of these models is that they can be used in relatively high dimensional applications and nevertheless capture the dependence structure by a fairly low number of parameters. Alternative copula models suitable for high-dimensional data are hierarchical Archimedean copulas (see Savu and Trede 2010) and vine copulas (see Bedford and Cooke 2002). We focus on factor copula models to have both considerable model flexibility and parsimonious parametrizations that allow for reliable statistical inference.

For the estimation of the model parameters, we use the simulated method of moments (SMM) as suggested by Oh and Patton (2013), which is different to standard method of moments applications, since the theoretical moment-counterparts are simulated and not as usual analytically derived. This makes asymptotic theory such as deriving consistency and asymptotic distribution results of the estimators more difficult. The reason is that the objective function is not continuous and furthermore not differentiable in the parameters and standard asymptotic approaches cannot be used here.

There are many papers which deal with monitoring procedures for detecting structural changes; some go back to the seminal paper by Chu, Stinchcombe, and White (1996) on monitoring the regression parameters in a linear regression model. The basic idea is that an initial training sample with constant parameters is available and the goal is to monitor for changes in the correlation as new data become available. A more recent paper

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concerned with this problem is Kurozumi (2017) who considers endogenous regressors. In the context of univariate financial time series Garthoff (2014) provides a sequential analysis of financial time series, where mean and variance of time series are simultaneously monitored. Hoga and Wied (2017) construct a sequential monitoring procedure for changes in the tail index and extreme quantiles of beta-mixing random variables, which can be based on a large class of tail index estimators. Furthermore, Pape, Wied, and Galeano (2017) propose a model-independent multivariate sequential procedure to monitor changes in the vector of component wise unconditional variances in a sequence of p-variate random vectors. In the context of monitoring dependence measures, Wied and Galeano (2013) develop a monitoring procedure to test for the constancy of the correlation coefficient of a sequence of random variables. Na and Lee (2014) propose a monitoring test for stability of copula parameter in time series. Finally, Dette and Goesmann (2019) propose a new approach for sequential monitoring a parameter of a d-dimensional time series, where a closed-end-method motivated by the likelihood ratio testing principle is considered.

The aim of this paper is to construct a new parametric monitoring procedure, based on moving sums (MOSUM), for the parameters in factor copula models. Rolling window parameter estimates are compared to the parameter estimates of an initial training sample for which we can assume constant parameter values. Concerning the assumption of constant parameters for the initial training period, we suggest applying the retrospective changepoint test in Manner, Stark, and Wied (2019) to pre-test this crucial assumption. These two tests complement each other in the sense that the monitoring procedure proposed here is meant for real-time monitoring of change-points, whereas the test in Manner, Stark, and Wied (2019) detects structural change in factor copulas in a retrospective way.

We study the asymptotic properties of the test and suggest a bootstrap procedure to approximate its resulting asymptotic distribution. We then analyze size and power properties of our procedure in single and multi break situations in Monte Carlo simulations. Finally, we use the monitoring procedure in a real-data application for a data set covering the last financial crisis. We also propose an online procedure for predicting the 1-day ahead Value-at-risk using simulations from the considered factor model accounting for the detected change-points.

The rest of the paper is structured as follows: Section 2 presents the model and the monitoring procedure, whereas in Section 3 we study its asymptotic distribution under the setting of simulated method of moments estimation. Results from the Monte Carlo simulations can be found in Section 4. Section 5 presents our empirical application and Section 6 concludes the paper. The main proof can be found in the appendix.

2 Model, null hypothesis, detectors and monitoring

In this section we present the factor copula model (Section 2.1), followed by our testing problem and the monitoring procedure (Section 2.2).

2.1 Factor copula model

We consider the same class of data-generating process as in Manner, Stark, and Wied (2019), i.e. the factor copula model proposed by Oh and Patton (2017). In this class the dynamics of the marginal distributions are determined by a parameter vector $\phi_0 \in \mathbb{R}^r$. We have d cross sectional dimensions and each variable can have time varying conditional mean $\mu_t(\phi_0) := [\mu_{1t}(\phi_0), \ldots, \mu_{dt}(\phi_0)]'$ and variance $\sigma_t(\phi_0) := \text{diag}\{\sigma_{1t}(\phi_0), \ldots, \sigma_{dt}(\phi_0)\}$. The dependence function of the joint distribution of the innovations η_t, namely the copula $C(\cdot, \theta_t)$, depends on the unknown parameter vector θ_t for $t = 1, \ldots, T$, which we allow to be time-varying in general. The data-generating process is given by

$$[Y_{1t}, \ldots, Y_{dt}]' =: Y_t = \mu_t(\phi_0) + \sigma_t(\phi_0)\eta_t,$$

where $[\eta_{1t}, \ldots, \eta_{dt}] =: \eta_t$ with distribution function $F_\eta = C(F_1(\eta_1), \ldots, F_d(\eta_d); \theta_t)$ by Sklar's theorem. This means that the joint distribution of the innovations is given by the copula C, capturing the contemporaneous dependence, evaluated at the marginal distributions F_i, $i = 1, \dots, d$. Moreover μ_t and σ_t are $\mathcal{F}_{t-1}$-measurable and independent of η_t. $\mathcal{F}_{t-1}$ is the sigma field containing information from the past $\{Y_{t-1}, Y_{t-2}, \dots\}$. Note that ϕ_0 is assumed to be $\sqrt{T}$ consistently estimable, which is fulfilled by most commonly used time series models, e.g. ARMA and GARCH models (see, e.g. Francq and Zakoian 2004), and the corresponding estimator is denoted as $\hat{\phi}$. For the contemporaneous dependence of the vector η_t, estimated using standardized residuals $\hat{\eta}_t$, we assume the factor copula model $C(\cdot, \theta_t)$, which is implied by the following linear factor structure

$$[X_{1t}, \dots, X_{dt}]' = :X_t = \beta_t Z_t + q_t, \tag{2.1}$$

i.e., $X_{it} = \sum_{k=1}^{K} \beta_{ik}^{t} Z_{kt} + q_{it}$ with idiosyncratic factors $q_{it} \overset{iid}{\sim} F_q(\alpha_t)$ and common factors $Z_{kt} \overset{iid}{\sim} F_{z_k}(\gamma_{kt})$, for $i = 1, \dots, d$, $t = 1, \dots, T$ and $k = 1, \dots, K$. Here K denotes the number of factors. Note that Z_{kt} and q_{it} are independent $\forall i, k, t$. The distribution function of X_t, F_x implies the factor copula $C(\cdot, \theta_t)$, i.e.,

$$F_x(x_{1t}, \dots, x_{dt}; \theta_t) = C(G_1(x_{1t}; \theta_t), \dots, G_d(x_{dt}; \theta_t); \theta_t) \tag{2.2}$$

with continuous marginal distributions $G_i(\cdot, \theta_t)$ and $\theta_t = [vec(\beta_t)', \alpha_t', \gamma_{1t}', \dots, \gamma_{Kt}']'$. Note that in this model we are only interested in the implied factor copula $C(\cdot, \theta_t)$ from the (latent) factor structure (2.1). We completely ignore the marginal distributions $G_i(\cdot, \theta_t)$ of the factor model, which are in general different from $F_i(\cdot)$, the marginal distributions of η_t. The advantage of these models is that they can be applied in high dimensions and nevertheless capture the dependence structure by a relatively low numbers of parameters. Through the choice of the distributions of the common factor F_{z_k} and the idiosyncratic error distribution F_q one can adapt asymmetry and tail dependence properties to the copula, which is useful when dealing with financial data. A (block-) equidependence structure can be accommodated by placing appropriate restrictions on θ_t. See Oh and Patton (2017) for more details on the properties of factor copulas. The estimation of the $p \times 1$ vectors $\theta_t \in \Theta$ of the copula is based on the simulated method of moments described in Section 3.1 below.

As the notation suggests, we consider a constant model structure and allow the parameters to be time-varying, having a piece-wise constant model in mind, i.e. $F_q(\alpha_t)$ and $F_{z_k}(\gamma_{kt})$ are only time-varying through their parameter vectors α_t and γ_{kt}. We make this more precise in the next subsection.

2.2 Null hypothesis and detectors

In this paper we want to test the null hypothesis of no parameter change of the factor copula model that is assumed to describe the residual dependence. The main idea is to compare parameter estimates from a training sample of size $\lfloor mT \rfloor$ (that we call "initial sample" for the remainder of the paper), for which constant dependence is assumed, to sequentially estimated parameters from a rolling data window of the same size. Thus, we are considering a MOSUM type procedure; see Chu, Hornik, and Kuan (1995). Here T is the length of the monitored time series and m a value in $(0, 1]$. Since we are interested in sequentially monitoring whether or not the parameter θ_t changes in $t = mT + 1, \dots, T$, we assume that the parameters remain constant over the initial sample $t = 1, \dots, mT$, meaning that:

Assumption 1.

$$\theta_1 = \dots = \theta_{mT}. \tag{2.3}$$

In practice, if a sufficient amount of initial data is available, this assumption can be tested by using the test for parmeter constancy in factor copulas proposed in Manner, Stark, and Wied (2019).

We are interested in testing the null hypothesis

$$H_0: \quad \theta_1 = \dots = \theta_{mT} = \theta_{mT+1} = \dots$$

versus the alternative

$$H_1 : \theta_1 = \cdots = \theta_{mT} = \cdots = \theta_{mT+k^*-1} \neq \theta_{mT+k^*} = \theta_{mT+k^*+1} = \cdots,$$

by using the detector

$$D_T(s) := m^2 T\left(\widehat{\theta}_{1+(s-m)T:sT} - \widehat{\theta}_{1:mT}\right)'\left(\widehat{\theta}_{1+(s-m)T:sT} - \widehat{\theta}_{1:mT}\right), \tag{2.4}$$

where $s \in [m, 1]$, $k^\star \geq 1$ and $mT + k^\star$ is the unknown change point and $\widehat{\theta}_{t_1:t_2}$ a consistent estimator for θ that is based on the subsample ranging from t_1 to t_2. Note that for the sake of thrift, we use the same parameter m for the initial period and further rolling window periods. Furthermore, we do not need a certain weighted deviation factor, due to the fact we consider a MOSUM-type test statistic in contrast to for example Pape, Wied, and Galeano (2017), where an expanding window is used. The monitoring procedure is stopped if the MOSUM-type detector defined in (2.4) exceeds the appropriately chosen constant critical value c for the first time k. This yields the stopping rule

$$\tau_T := \inf_k \left\{ k \leq T : D_T\left(\frac{k}{T}\right) > c \right\},$$

where τ_T is the stopping time of the monitoring procedure. Here c is chosen in a way that under H_0 the monitoring procedure holds the size level $\lim_{T,S\to\infty} P(\tau_T < \infty | H_0) = \alpha$, with $\alpha \in (0, 1)$. The quantity S refers to the number of simulations used to approximate the moments, see Section 3.1.

We write $\tau_T < \infty$ to indicate that the monitoring has been terminated during the testing period, meaning that the detector crossed the boundary value c at a time point $k \leq T$. On the other hand, we write $\tau_T = \infty$, if D_T does not cross the boundary value during the testing period. Note that the detected stopping time τ_T is not meant to be an estimator of change point, as the actual change point is likely to be earlier. This is due to the fact the monitoring procedure needs a sufficient number of observations after a change point before it can be detected. In the next chapter we present a procedure for estimating the change point conditional on H_0 having been rejected.

Similar to the detector defined in (2.4) we consider an alternative detector that is based directly on the moment conditions used to estimate the model.

$$M_T(s) := m^2 T\left(\widehat{m}_{1+(s-m)T:sT} - \widehat{m}_{1:mT}\right)'\left(\widehat{m}_{1+(s-m)T:sT} - \widehat{m}_{1:mT}\right) \tag{2.5}$$

This allows for monitoring the corresponding dependence measures in a model-free way. Under the assumed factor copula model it can be used to monitor the stability of the model parameters. Furthermore, it has the added advantage of being computationally much less demanding since no model parameters have to be estimated and it does not depend on any simulated quantities.

3 Estimation and asymptotics

In this section we describe our theoretical results. The estimation of the factor copula model by the SMM is reviewed in Section 3.1, whereas the asymptotic behavior of our monitoring procedures is studied in Section 3.2. A bootstrap algorithm to approximate the asymptotic distribution is presented in Section 3.3 and a procedure for detecting multiple breaks is described in Section 3.4.

3.1 SMM estimation

We are interested in estimating the parameter vector $\theta_{uT:vT}$ for the subsample ranging from $\lfloor uT \rfloor$ to $\lfloor vT \rfloor$, where $u < v$ and $u, v \in [\varepsilon, 1]$, with $\varepsilon > 0$. The value ε is chosen by the applicant; typical values are 0.1 or 0.2. This is achieved by using the SMM, where the estimator is defined as

$$\widehat{\theta}_{uT:vT,S} := \arg\min_{\theta\in\Theta} Q_{uT:vT,S}(\theta),$$

where $Q_{uT:vT,S}(\theta) := g_{uT:vT,S}(\theta)'\widehat{W}_{(uT:vT)} g_{uT:vT,S}(\theta)$ is the objective function,

$$g_{a:b,S}(\theta) := \widehat{m}_{a:b} - \tilde{m}_S(\theta),\ a < b, \tag{3.1}$$

and $\widehat{W}_{(uT:vT)}$ a positive definite weight matrix which convergence in probability to W. For simplicity one can chose the $k \times k$ identity matrix. The moment conditions $\widehat{m}_{uT:vT}$ are $k \times 1$ vectors of appropriately chosen pairwise dependence measures $\widehat{m}^{ij}_{uT:vT}$ (possibly averaged over equidependent pairs), computed from the residuals $\{\widehat{\eta}_t\}_{t=uT}^{vT}$, whereas $\tilde{m}_S(\theta)$ is an approximation for the corresponding vector of true dependence measures. Note that the dependence measures implied by the factor copula model are typically not available in closed form and they have to be obtained by simulation. Therefore, the classical method of moments (MM) or generalized method of moments (GMM) cannot be used here. The true dependence measures are approximated using S simulations $\{\tilde{\eta}_t\}_{t=1}^{S}$ from F_x from equation (2.2), and hence the objective function, the estimator, and consequently our detector defined in equation (2.4) depend on the number of simulations S. Following the simulation studies in Oh and Patton (2013), we chose $S = 25 \cdot (vT - uT)$ and we need to ensure that the sub-sample ranging from uT to vT is large enough to receive reasonable SMM estimates. In our simulation studies we find that our procedure still results in reasonable size and power properties by choosing $uT - vT = mT = 250$ data points. For the dependence measures of the pair (η_i, η_j), we use Spearman's rank correlation ρ^{ij} and the quantile dependence λ^{ij}_q. These are defined as

$$\rho^{ij} := 12\int_0^1\int_0^1 C_{ij}(u_i, v_j)du_i dv_j - 3$$

$$\lambda^{ij}_q := \begin{cases} P\left[F_i(\eta_i) \le q \middle| F_j(\eta_j) \le q\right] = \dfrac{C_{ij}(q,q)}{q}, & q \in (0, 0.5] \\ P\left[F_i(\eta_i) > q \middle| F_j(\eta_j) > q\right] = \dfrac{1-2q+C_{ij}(q,q)}{1-q}, & q \in (0.5, 1) \end{cases}.$$

The sample counterparts for the observations between uT and vT are defined as

$$\widehat{\rho}^{ij} := \frac{12}{\lfloor vT - uT \rfloor}\sum_{t=\lfloor uT\rfloor}^{\lfloor vT\rfloor} \widehat{F}_i^{uT:vT}(\widehat{\eta}_{it})\widehat{F}_j^{uT:vT}(\widehat{\eta}_{jt}) - 3$$

$$\widehat{\lambda}^{ij}_q := \begin{cases} \dfrac{\widehat{C}_{ij}^{uT:vT}(q,q)}{q}, & q \in (0, 0.5] \\ \dfrac{1-2q+\widehat{C}_{ij}^{uT:vT}(q,q)}{1-q}, & q \in (0.5, 1) \end{cases},$$

where $\widehat{F}_i^{uT:vT}(y) := \frac{1}{vT-uT}\sum_{t=uT}^{vT} 1\{\widehat{\eta}_{it} \le y\}$ and $\widehat{C}_{ij}^{uT:vT}(u,v) := \frac{1}{vT-uT}\sum_{t=uT}^{vT} 1\{\widehat{F}_i^{uT:vT}(\widehat{\eta}_{it}) \le u, \widehat{F}_j^{uT:vT}(\widehat{\eta}_{jt}) \le v\}$. The simulated counterparts of these dependence measures based on the simulations $\{\tilde{\eta}_t\}_{t=1}^{S}$ are defined analogically and are denoted by $\tilde{\rho}^{ij}$ and $\tilde{\lambda}^{ij}_q$.

In summary, the SMM estimator minimizes the weighted difference between suitable sample dependence measures and their model counterparts obtained by simulation. Depending on the precise model specification, the pairwise dependence measures are averaged for groups, which have the same factor loadings. For more information on SMM estimation and a suitable way to average the pairwise dependence measures for equidependence or block equidependence models see Oh and Patton (2013, 2017).

3.2 Asymptotics

To derive the asymptotic distribution of our detector (2.4), we consider Assumption 1 and Assumptions 3–6 (given in the appendix), which are fullfilled by the considered ARMA-GARCH factor copula model, see Oh and Patton (2013) and references therein. We follow similar steps as in Manner, Stark, and Wied (2019) where the difference is that we replace the scale factor $s\sqrt{T}$ by $m\sqrt{T}$ and that we derive the following distributional limit for the process $s \mapsto m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta)$ with $g_{\cdot:\cdot,S}(\theta)$ from equation (3.1), $\frac{S}{T} \to k \in (0, \infty]$ and $T, S \to \infty$:

$$\begin{aligned} m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta) &= m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - \tilde{m}_S(\theta)\right) \\ &= m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - m\sqrt{T}\left(\tilde{m}_S(\theta) - m_0(\theta)\right) \\ &= m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - \sqrt{\frac{T}{S}}m\sqrt{S}\left(\tilde{m}_S(\theta) - m_0(\theta)\right) \\ &\overset{d}{\Rightarrow} A(s) - \frac{m}{\sqrt{k}}B. \end{aligned}$$

Here and in the following, $\Rightarrow$ denotes convergence of stochastic processes in certain metric spaces. In this particular case, the convergence takes place in the Cádlág space $D[m, 1]$ for $m \geq \varepsilon > 0$. Moreover, $A(s)$ is a Gaussian process defined in the proof of Theorem 1 in the Appendix and $B: = N(0, \Sigma_0)$ a centered Gaussian distribution with covariance matrix Σ_0, for details see Oh and Patton (2013). The limit result follows by using the independence of the moment process calculated from the data and the moment process corresponding to the simulated data. Note that the term $\frac{m}{\sqrt{k}}B$ cancels out in later considerations, e.g. to determine the critical value c using the bootstrap procedure proposed in Section 3.3.

Theorem 1. *Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and under Assumption 1 in Section 2.2 and Assumptions 2–5 in the Appendix, we obtain for $m \geq \varepsilon > 0$*

$$m\sqrt{T}\left(\widehat{\theta}_{1+(s-m)T:sT,S} - \theta_0\right) \overset{d}{\Rightarrow} A^*(s)$$

as $T, S \to \infty$ in the space of Càdlàg functions on the interval $[m, 1]$ and $\frac{S}{T} \to k \in (0, \infty]$. Here, $A^*(s) = (G'WG)^{-1}G'W(A(s) - \frac{m}{\sqrt{k}}B)$ and θ_0 is the (constant) value of θ_t under the null. Note that G is the derivative matrix of $g_0(\theta)$ with $g_{1:mT,S}(\theta) \to_p g_0(\theta)$ for $T, S \to \infty$.

With Theorem 1 we obtain for $T, S \to \infty$

$$\begin{aligned} &m\sqrt{T}\left(\widehat{\theta}_{1+(s-m)T:sT,S} - \widehat{\theta}_{1:mT,S}\right) \\ &= m\sqrt{T}\left(\widehat{\theta}_{1+(s-m)T:sT,S} - \theta_0\right) - m\sqrt{T}\left(\widehat{\theta}_{1:mT,S} - \theta_0\right) \\ &\overset{d}{\Rightarrow} A^*(s) - A^*(m). \end{aligned}$$

From this we can conclude the asymptotic behavior of our parameter detector (2.4) under H_0, which we state in Corollary 1.

Corollary 1. *Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and if the assumptions from Theorem* 1 *hold, we obtain for our detector*

$$\begin{aligned} D_{T,S}(s) &= m^2T\left(\widehat{\theta}_{1+(s-m)T:sT,S} - \widehat{\theta}_{1:mT,S}\right)'\left(\widehat{\theta}_{1+(s-m)T:sT,S} - \widehat{\theta}_{1:mT,S}\right) \\ &\overset{d}{\Rightarrow} \left(A^*(s) - A^*(m)\right)'\left(A^*(s) - A^*(m)\right) = :Q(s) \end{aligned}$$

as $T, S \to \infty$ and $\frac{S}{T} \to k \in (0, \infty]$.

The asymptotic behavior of our moment detector (2.5) can be found in Corollary 2.

Corollary 2. *Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and if the assumptions from Theorem 1 hold, we obtain*

$$M_T(s) = m^2T\left(\widehat{m}_{1+(s-m)T:sT} - \widehat{m}_{1:mT}\right)'\left(\widehat{m}_{1+(s-m)T:sT} - \widehat{m}_{1:mT}\right)$$

$$\overset{d}{\Rightarrow} (A(s) - A(m))' (A(s) - A(m)) = :R(s)$$

as $T \to \infty$.

With the limit distribution of our detector $Q(s)$, we define the boundary value c in our monitoring procedure as the upper α-quantile of

$$\sup_{s\in[m,1]} Q(s) = \sup_{s\in[m,1]} (A^*(s) - A^*(m))' (A^*(s) - A^*(m)), \quad m \geq \varepsilon > 0. \tag{3.2}$$

Thus, $\lim_{T,S\to\infty} P(\tau_T < \infty | H_0) = \lim_{T,S\to\infty} P(\inf_k \{k \leq T : D_{T,S}(k) > c\} < \infty | H_0) = \alpha.$

In the same way the critical value of the moment monitoring procedure is determined as the upper α-quantile of $\sup_{s\in[m,1]} R(s)$. For the estimation of the break point $mT + k^\star$, once H_0 is rejected, we propose $mT + \hat{k}$, with

$$\hat{k} := \underset{\lfloor\gamma(\tau_T - mT)\rfloor \leq i \leq \tau_T - mT}{\operatorname{argmax}} \frac{i^2}{\tau_T - mT} \left(\hat{\theta}_{1+mT:mT+i-1,S} - \hat{\theta}_{1+mT:\tau_T-1,S}\right)' \left(\hat{\theta}_{1+mT:mT+i-1,S} - \hat{\theta}_{1+mT:\tau_T-1,S}\right), \tag{3.3}$$

where we only consider the information from $mT + 1$ to $\tau_m - 1$. Note that we need to trim a sufficient fraction $\gamma(\tau_T - mT)$ of the beginning, where $\gamma > 0$ to receive reasonable SMM estimates. In a similar way, the size of the rolling window mT should not be chosen to small. Note that the stopping time and the break point estimator for the moment monitoring procedure are defined analogically to the parameter monitoring procedure. As mentioned above, the moment based monitoring procedure is easy to implement and can be calculated fast, but in general it has lower power than the parametric procedure. Furthermore, as outlined in Manner, Stark, and Wied (2019), another disadvantage is that it does not allow testing the constancy of a subset of the parameters, but only can detect breaks in the whole copula. It may, however, be used to test for breaks in the dependence in selected regions of the support such as the lower tail. We leave this possibility for future research.

The limit distributions of $D_{T,S}$ and M_T are not known in closed form. To overcome this issue we have to simulate the critical values using an i.i.d. bootstrap procedure, which is described in the next section.

3.3 Bootstrap distribution

First note that the limit result mainly consists of the limit distribution of the moment vectors, which can be computed relatively fast, compared to the detector that requires solving a minimization problem. This fact is used for the construction of the bootstrap. In order to approximate the limiting distribution under the null we use a parametric i.i.d. bootstrap consisting of the following steps:

i. Sample with replacement from $\{\tilde{\eta}_i\}_{i=1}^T$ to obtain B bootstrap samples $\{\tilde{\eta}_i^{(b)}\}_{i=1}^T$, for $b = 1, \ldots, B$, where $\{\tilde{\eta}_i\}_{i=1}^T$ stacks the initial residual data $\{\hat{\eta}_i\}_{i=1}^{mT}$ and simulated data $\{\tilde{\eta}_i^\star\}_{i=mT+1}^T$ from the assumed model, using the parameter estimate $\hat{\theta}_{1:mT,S}$ from the initial sample period.
ii. Use $\{\tilde{\eta}_i^{(b)}\}_{i=1+t-mT}^t$ to compute $\hat{m}_{1+t-mT:t}^{(b)}$ for $t = mT, \ldots, T$ and use $\{\tilde{\eta}_i^{(b)}\}_{i=1}^T$ to obtain $\hat{m}_{1:T}^{(b)}$, for $b = 1, \ldots, B$.
iii. For obtaining the critical values of $Q(s)$, calculate the bootstrap version of the limiting distribution of our detector

$$K^{(b)} := \max_{t\in\{mT,\ldots,T\}} \left(A^{\star(b)}\left(\frac{t}{T}\right) - A^{\star(b)}(m)\right)' \left(A^{\star(b)}\left(\frac{t}{T}\right) - A^{\star(b)}(m)\right),$$

with $A^{\star(b)}\left(\frac{t}{T}\right) := (\hat{G}'\hat{W}_T\hat{G})^{-1}\hat{G}'\hat{W}_T A^{(b)}\left(\frac{t}{T}\right)$ and $A^{(b)}\left(\frac{t}{T}\right) = m\sqrt{T}(\hat{m}_{1+t-mT:t}^{(b)} - \hat{m}_{1:T}^{(b)})$, where $\hat{G}$ is the two sided numerical derivative estimator of G, evaluated at point $\theta_{1:mT,S}$, computed with the historical sample $\{\hat{\eta}_i\}_{i=1}^{mT}$. We can compute the k-th column of $\hat{G}$ by

$$\widehat{G}^k = \frac{g_{T,S}\left(\widehat{\theta}_{1:mT,S} + e_k\varepsilon_{T,S}\right) - g_{T,S}\left(\widehat{\theta}_{1:mT,S} - e_k\varepsilon_{T,S}\right)}{2\varepsilon_{T,S}}, \quad k \in \{1, \ldots, p\},$$

where e_k is the k-th unit vector, whose dimension is $p \times 1$ and $\varepsilon_{T,S}$ has to be chosen in a way that it fulfills $\varepsilon_{T,S} \to 0$ and $\min\{\sqrt{T}, \sqrt{S}\}\varepsilon_{T,S} \to \infty$.

For obtaining the critical values of $R(s)$, replace $A^{\star(b)}$ with $A(b)$.

iv. Compute B versions of $K^{(b)}$ and determine the boundary value c such that

$$\frac{1}{B}\sum_{b=1}^{B} 1\left\{K^{(b)} > c\right\} \stackrel{!}{=} 0.05.$$

This bootstrap method is similar to the bootstrap used in Manner, Stark, and Wied (2019), where iii) is adapted to the monitoring situation. Under the following assumption we obtain that both detectors are valid under the null hypothesis and a suitable alternative.

Assumption 2. *Both the parametric factor copula model and the rank-based estimators fulfill the regularity conditions in* Genest and Rémillard (2008) *(Definition 1 and 4).*

Theorem 2. *Let c_Q^B be the bootstrapped critical value for $Q(s)$ and c_R^B the bootstrapped critical value for $R(s)$, based on B bootstrap replications, respectively. Consider the hypotheses $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and $H_1 : \theta_1 = \cdots = \theta_{mT} = \cdots = \theta_{rT} \neq \theta_{rT+1} = \ldots$ for some $m < r < 1$. Moreover, let Assumption 1 in Section* 2.2, *Assumption* 2 *and Assumptions* 3–6 *in the Appendix be true. Then,*

$$\lim_{T,S,B\to\infty} P\left(\inf_k \left\{k \le T : D_{T,S}(k) > c_Q^B\right\} < \infty \middle| H_0\right) = \lim_{T,B\to\infty} P\left(\inf_k \left\{k \le T : M_T(k) > c_R^B\right\} < \infty \middle| H_0\right) = \alpha$$

and

$$\lim_{T,S,B\to\infty} P\left(\inf_k \left\{k \le T : D_{T,S}(k) > c_Q^B\right\} < \infty \middle| H_1\right) = \lim_{T,B\to\infty} P\left(\inf_k \left\{k \le T : M_T(k) > c_R^B\right\} < \infty \middle| H_1\right) = 1,$$

whereas, for the last equation, we impose the additional assumption that $m^{mT+1} = \cdots = m^{rT} \neq m^{rT+1} = \ldots$, where m^t is the vector of true dependence measures at time t.

Clearly, Assumption 2 is high-level, but Genest and Rémillard (2008) and subsequent papers such as Rémillard (2017) showed that this holds for a wide range of models and estimators. Our Monte Carlo simulations below confirm that the bootstrap indeed results in reasonably sized tests and we leave it as a task for further research to show that the assumption also holds under lower-level assumptions.

3.4 Multiple break testing

In practice if one is interested in detecting multiple structural breaks in factor copula models in real time, we propose the following procedure that consists of steps applying the monitoring procedure proposed in this paper and the retrospective change point test for factor copulas from Manner, Stark, and Wied (2019). In particular, the retrospective test is used to test for the constant parameter assumption (2.3) in the initial sample period and to detect the break point location once the monitoring procedure stops.

(1) Compute the retrospective change point statistic $\sup_{s\in[\varepsilon,m]} P_{sT,S}$ from Manner, Stark, and Wied (2019) for the initial mT observation. If a changepoint is detected go to step 2a). If no changepoint is detected go to step 2b).

(2a) Estimate the breakpoint location and remove all pre-change observations. Restock the subsample to mT observations and return to step 1). If there are not enough observations left to restock the subsample to mT observations go to step 4).

(2b) Take the sample as initial sample period. Apply the monitoring procedure to the residuals, i.e. compute $D_{T,S}(s)$ for $s \in (m, 1]$. Compute the bootstrap critical value c as described in Section 3.3. If a changepoint is detected go to step 3). If no changepoint is detected go to step 4).

(3) Estimate the location of the changepoint. Then, remove the pre-change observations, use the first mT observations of the resulting dataset as the new initial sample and return to step 1). If there are not enough observations left to restock the subsample to mT observations go to step 4).

(4) Terminate the procedure.

In the same way this procedure can be adapted for the moment monitoring procedure. Simulation results for single and multiple break testing, using the moment or the parameter monitoring procedure can be found in the next section. An obvious issue with this procedure is its multiple testing nature, in particular given that a pre-test has to be applied to the initial sample period to ensure that Assumption 1 holds. One should adapt the significance levels accordingly and be aware of this when interpreting testing results. In our simulation study and the empirical analysis below we adapt the significance levels to $\alpha_k = 1 - (1-\alpha_0)^{\frac{1}{k}}$ for the $k^t h$ hypothesis test, where α_0 is some initially chosen significance level.

4 Simulations

We now want to investigate the size and power and the estimation of the break point location of our monitoring procedure. We consider the simple one factor copula model, i.e. the copula implied by

$$[X_{1t}, \ldots, X_{dt}]' =: X_t = \beta_t Z_t + q_t, \tag{4.1}$$

where $\beta_t = (\beta_t, \ \ldots \beta_t)'$ and $q_t = (q_t, \ \ldots q_t)'$ are $d \times 1$ vectors, $Z_t \sim \text{Skew}\, t(\sigma^2, \nu^{-1}, \lambda)$ and $q_t \overset{iid}{\sim} \text{t}(\nu^{-1})$ for $t = 1, \ldots, T$. We fix $\sigma^2 = 1$, $\nu^{-1} = 0.25$ and $\lambda = -0.5$, so that our model is parametrized by the factor loading parameter β_t.

The sequential parameter estimates $\widehat{\beta}_t = \widehat{\beta}_{1-mT+t:t}$ for $t = mT, \ldots, T$ in the detector are computed using the SMM approach with $S = 25 \cdot mT$ simulations. For this we use five dependence measures, namely Spearman's rank correlation and the 0.05, 0.10, 0.90, 0.95 quantile dependence measures, averaged across all pairs. Critical values for the monitoring procedure are computed using $B = 500$ bootstrap replications.

The nominal size of the tests is chosen to be 5%. We use 700 Monte Carlo replications to compute the size of the test and 301 Monte Carlo replications for all other settings.[1]

Before reporting the simulation results, we report the computation times (in hours) of the procedure in Table 1. It shows the time it takes to perform the monitoring procedure for a single break, including the computation of the bootstrap distribution on a standard PC using parallel computation on four cores. It can be seen that the computations are feasible for all reported cases and that the parameter based detector runs approximately two to four times longer than the moment detector.

4.1 Size and single break case

We begin with the case of testing against a single break. The rejection rates under the null are presented in Table 2 for $\beta_t = 1$ for $t = 1, \ldots, T$, for various combinations of the length of the initial sample mT and dimension d, where the critical values are calculated using one of the following two possibilities:

---

1 The computational complexity of the simulations was extremely high due to the fact that for every monitoring procedure the parameter values need to be estimated a large number of times using the computationally heavy SMM estimator and because critical values have to be bootstrapped. This explains why we had to restrict ourselves to a limited number of situations for a fairly simple model. Furthermore, numerical instabilities were present in more complex models when repeatedly estimating the model parameters. Such problems can be dealt with in empirical applications, but further restrict the potential model complexity in simulations. The computations were implemented in Matlab, parallelized and performed using CHEOPS, a scientific High Performance Computer at the Regional Computing Center of the University of Cologne (RRZK) funded by the DFG.

Table 1: Computation times in hours for monitoring the breakpoint based on the parameter (β) and the vector of dependence measure (m_T) for different combinations of T and d with $\beta_0 = 1.0$ and $m = 0.25$. Procedures implemented and performed in MATLAB. Calculations parallelized on four kernels with Intel(R) Core(TM) i7-6700 CPU 3.40 GHz.

		$d = 5$	$d = 10$	$d = 20$	$d = 40$
β	$T = 1000$	0.11	0.22	0.53	1.43
	$T = 1500$	0.23	0.47	1.10	2.91
	$T = 2000$	0.40	0.82	1.86	4.88
	$T = 4000$	1.51	3.27	8.23	19.86
m_T	$T = 1000$	0.03	0.07	0.21	0.70
	$T = 1500$	0.05	0.12	0.37	1.26
	$T = 2000$	0.08	0.20	0.59	1.95
	$T = 4000$	0.23	0.64	1.92	6.13

i. Calculate the critical value c using the whole, in general not known, data up to time T. This mimics the situation that the test is used in a retrospective fashion, i.e. once all T observations are available.
ii. Calculate the critical value c using the initial data set together with the data from $mT + 1$ up to T, based on the estimated parameter $\widehat{\beta}_{1:mT,S}$.

The test shows acceptable size for both settings. The empirical size is slightly higher than the nominal level for the second procedure ii), most likely due to the fluctuation in the parameter estimation in the SMM procedure. The size of the testing period is always fixed to be $T = 1500$.

To study the power of the procedure, we generate data with a break point at $\frac{T}{2}$, where the data is simulated with $\beta_t = 1$ for $t \in \left\{1, \ldots, \frac{T}{2}\right\}$, denoted as β_0 and with $\beta_t = \{1.2, 1.4, 1.6, 1.8, 3.0\}$ for $t \in \left\{\frac{T}{2} + 1, \ldots, T\right\}$, denoted as β_1. The dimension d is set equal to 10 in this case. With power we mean the probability that our monitoring procedure stops with in the monitored testing period ($\tau_T < \infty$). The upper panel of Table 3 reveals that the power of the procedure increases with the size of the initial sample for the two possibilities i) and ii). The moment monitoring procedure based on M_T has similar size characteristics but lower power compared to the parameter-based procedure. This result is in line with the results for the retrospective test in Manner, Stark, and Wied (2019).

The second and third panels of the table present the (average) relative stopping times and break point estimates using (3.3). The table reveals that the averaged stopping time, given that a break has been detected, occurs with a significant delay after the true break point. It is closer to the true location $\frac{1}{2}$ for a smaller monitoring window, due to the greater impact of new data and, of course, for an increase of the step size between β_0 and β_1. If the step size is large enough ($\beta_1 = 3.0$) the monitoring procedure consistently stops shortly after the true break point.

The averaged estimated break point locations based on equation (3.3) are closer to the true break point. It always detects the break before the stopping time. For small shifts in θ it estimates the break too late, whereas

Table 2: Empirical size for $\beta_0 = 1.0$, $T = 1500$ and 700 simulations, using i) the whole sample up to time point T and using ii) the initial data set and simulated data from $mT + 1$ up to T.

		$d = 10$	$d = 20$	$d = 30$
i)	$mT = 250$	0.051	0.052	0.051
	$mT = 400$	0.055	0.054	0.050
	$mT = 500$	0.057	0.047	0.054
	$mT = 250$	0.062	0.064	0.061
ii)	$mT = 400$	0.062	0.065	0.065
	$mT = 500$	0.061	0.054	0.055

Table 3: Rejection frequency (rej), average stopping time $\frac{\tau_T}{T}$ and average breakpoint estimate $\frac{\hat{k}}{T}$ for $\beta_0 = 1$, $T = 1500$ $d = 10$ and 301 simulations for the parameter monitoring procedure, where critical values c computed with the two possibilities i) and ii) and for the moment monitoring procedure. Data was generated with a break at $\frac{T}{2}$ and post-break parameter β_1.

			$\beta_0 = 1.0$	$\beta_1 = 1.2$	$\beta_1 = 1.4$	$\beta_1 = 1.6$	$\beta_1 = 1.8$	$\beta_1 = 3.0$
Rej	i)	$mT = 250$	0.047	0.375	0.787	0.973	1.000	1.000
		$mT = 400$	0.059	0.435	0.877	0.993	1.000	1.000
		$mT = 500$	0.053	0.465	0.910	1.000	1.000	1.000
	ii)	$mT = 250$	0.059	0.409	0.787	0.967	1.000	1.000
		$mT = 400$	0.053	0.468	0.860	0.990	1.000	1.000
		$mT = 500$	0.053	0.485	0.894	1.000	1.000	1.000
	m_T	$mT = 250$	0.057	0.193	0.482	0.780	0.944	1.000
		$mT = 400$	0.049	0.223	0.671	0.944	0.993	1.000
		$mT = 500$	0.051	0.306	0.738	0.960	1.000	1.000
$\frac{\tau_T}{T}$	i)	$mT = 250$		0.715	0.667	0.625	0.579	0.513
		$mT = 400$		0.751	0.689	0.629	0.588	0.523
		$mT = 500$		0.767	0.677	0.639	0.596	0.525
	ii)	$mT = 250$		0.698	0.660	0.619	0.581	0.518
		$mT = 400$		0.733	0.675	0.626	0.587	0.525
		$mT = 500$		0.759	0.699	0.638	0.595	0.527
	m_T	$mT = 250$		0.718	0.695	0.662	0.627	0.525
		$mT = 400$		0.738	0.725	0.672	0.625	0.528
		$mT = 500$		0.765	0.741	0.679	0.626	0.530
$\frac{\hat{k}}{T}$	i)	$mT = 250$		0.516	0.487	0.483	0.473	0.457
		$mT = 400$		0.544	0.508	0.493	0.487	0.479
		$mT = 500$		0.562	0.522	0.497	0.492	0.487
	ii)	$mT = 250$		0.511	0.484	0.479	0.471	0.464
		$mT = 400$		0.538	0.502	0.491	0.485	0.483
		$mT = 500$		0.562	0.518	0.497	0.491	0.489
	m_T	$mT = 250$		0.517	0.495	0.487	0.486	0.485
		$mT = 400$		0.541	0.518	0.500	0.495	0.492
		$mT = 500$		0.561	0.534	0.507	0.499	0.496

for large shifts in θ break are estimated a little too early. It seems that a larger initial sample always results in slightly later stopping times, which can be explained due to the greater impact on the detector by new observed data in small rolling window sample sizes. However the usage of smaller window sizes imply lower power of the procedure. Note that the moment monitoring tends to result in later stopping times and break point estimates in all cases.

Next, we consider a setting similar to that in Table 3, but where we now consider a break in the skewness parameter to study whether the test is able to detection breaks in the shape of the copula. We fix $\beta_t = 2$ and vary the skewness parameter λ under the alternative. Similarly to the previous case, we denote its value before the break as $\lambda_0 = -0.5$ and its value after the break as $\lambda_1 = \{-0.4, -0.3, -0.2, -0.1, 0\}$. The results in Table 4 show that the test is close in size to its nominal value and the power increases with the size of the break in λ. Furthermore, the parameter based test again has higher power than the one based on the moments.

Additionally, we consider a heterogeneous two-factor model with several parameters, i.e.

$$\left[X_{1t}, \ldots, X_{\frac{d}{2}t}\right]' = \beta_{11}^t Z_{1t} + \beta_{12}^t Z_{2t} + q_t \tag{4.2}$$

$$\left[X_{(\frac{d}{2}+1)t}, \ldots, X_{dt}\right]' = \beta_{21}^t Z_{1t} + \beta_{22}^t Z_{2t} + q_t,$$

where $\beta_{11}^t = \beta_{21}^t = (\beta_1^t, \ldots, \beta_1^t)$, $\beta_{12}^t = (\beta_{12}^t, \ldots, \beta_{12}^t)$, $\beta_{22}^t = (\beta_{22}^t, \ldots, \beta_{22}^t)$ and the factors again follow a Skew t distribution with fixed parameters as above. Thus there are two factors and two groups of variables that have the same loading for the first factor, but a different factor loading for the second factor. The value β_1^t is fixed to 0.5

Table 4: Rejection frequency (rej), average stopping time $\frac{\tau_T}{T}$ and average breakpoint estimate $\frac{\hat{k}}{T}$ for $\lambda_0 = -0.5$, $T = 1500$ $d = 10$ and 301 simulations for the parameter monitoring procedure, where critical values c computed with the two possibilities i) and ii) and for the moment monitoring procedure. Data was generated with a break at $\frac{T}{2}$ and post-break parameter λ_1. We fixed $\beta = 2$ and $v = 4$.

			$\lambda_0 = -0.5$	$\lambda_1 = -0.4$	$\lambda_1 = -0.3$	$\lambda_1 = -0.2$	$\lambda_1 = -0.1$	$\lambda_1 = 0$
rej	i)	$mT = 500$	0.053	0.130	0.468	0.837	0.967	1.000
	ii)	$mT = 500$	0.059	0.249	0.591	0.870	0.973	1.000
	m_T	$mT = 500$	0.064	0.116	0.362	0.661	0.924	0.977
$\frac{\tau_T}{T}$	i)	$mT = 500$		0.822	0.797	0.767	0.718	0.682
	ii)	$mT = 500$		0.782	0.761	0.717	0.677	0.645
	m_T	$mT = 500$		0.787	0.811	0.773	0.744	0.707
$\frac{\hat{k}}{T}$	i)	$mT = 500$		0.624	0.570	0.550	0.530	0.522
	ii)	$mT = 500$		0.585	0.563	0.538	0.520	0.506
	m_T	$mT = 500$		0.590	0.577	0.549	0.534	0.519

and only the parameters β_{12}^t and β_{22}^t are estimated. We consider breaks only in β_{12}^t, i.e., the loading of the second factor for the first group. It changes from $\beta_{12}^0 = 1.5$ to $\beta_{12}^1 = \{1.7, 1.9, 2.1, 2.3, 2.5\}$. The results in Table 5 show some small size distortions, likely due to the increased estimation error as a consequence of the increased model complexity. The power increases with the break size as expected. As before, the moment-based test performs worse compared to the parameter-based test.

Next, we consider the problem of detecting the correct number of breaks using the procedure proposed in Section 3.4 in the case a single break occurs at time $2T/3$ with the parameter changing from $\beta_0 = 1$ to $\beta_1 = 1.5$. For every conducted test $k = 1, 2, \ldots$ we adapted the significance levels to $\alpha_k = 1 - (1 - \alpha_0)^{\frac{1}{k}}$ with $\alpha_0 = 0.05$ for correcting the multiple testing setup of our procedure based on Galeano and Wied (2014). The results in Table 6 reveal that in most cases the correct number of breaks is identified. The parameter based test detector performs much better here, whereas the test based on m_T suffers from the general weakness of low power and therefor often does not detect a single break. The results improve slightly going from $d = 10$ to $d = 20$, whereas a larger size of moving window has a strong effect on the results.

4.2 Two breaks

For the analysis of two breaks we allow for breaks at $\frac{T}{3}$ and $\frac{2T}{3}$ with sample size $T = 1500$, and dimensions $d = 10$ and $d = 20$. The parameter varies from $\beta_0 = 1.0$ for $t \in \left\{1, \ldots, \frac{T}{3}\right\}$ to $\beta_1 = 1.5$ for $t \in \left\{\frac{T}{3} + 1, \ldots, \frac{2T}{3}\right\}$ and

Table 5: Rejection frequency (rej), average stopping time $\frac{\tau_T}{T}$ and average breakpoint estimate $\frac{\hat{k}}{T}$ for the null parameter $\beta_{12}^0 = 1.5$, $T = 1500$ $d = 10$ and 301 simulations for the parameter monitoring procedure, where critical values c computed with the two possibilities i) and ii) and for the moment monitoring procedure. Data was generated with a break at $\frac{T}{2}$ and post-break parameter β_{12}^1.

			$\beta_{12}^0 = 1.5$	$\beta_{12}^1 = 1.7$	$\beta_{12}^1 = 1.9$	$\beta_{12}^1 = 2.1$	$\beta_{12}^1 = 2.3$	$\beta_{12}^1 = 2.5$
rej	i)	$mT = 500$	0.078	0.123	0.216	0.455	0.691	0.864
	ii)	$mT = 500$	0.073	0.120	0.196	0.429	0.665	0.891
	m_T	$mT = 500$	0.062	0.093	0.123	0.209	0.392	0.518
$\frac{\tau_T}{T}$	i)	$mT = 500$		0.646	0.690	0.697	0.688	0.673
	ii)	$mT = 500$		0.653	0.682	0.698	0.691	0.681
	m_T	$mT = 500$		0.852	0.799	0.797	0.814	0.801
$\frac{\hat{k}}{T}$	i)	$mT = 500$		0.515	0.545	0.531	0.525	0.514
	ii)	$mT = 500$		0.514	0.534	0.526	0.522	0.514
	m_T	$mT = 500$		0.601	0.593	0.569	0.574	0.564

Table 6: Fraction of no, exact one or more found breaks in a single break setting. Constructed break at $\frac{2T}{3}$ with $\beta_0 = 1.0$ and $\beta_1 = 1.5$, $T = 1500$ and 301 simulations, using ii) the initial data set and simulated data from $mT + 1$ up to T. Results are based on the parameter based detector $D_{T,S}$ (top panel) and the moment based detector (bottom panel).

		no breaks		one break		more breaks	
		$d = 10$	$d = 20$	$d = 10$	$d = 20$	$d = 10$	$d = 20$
β	$mT = 250$	0.226	0.193	0.641	0.651	0.133	0.156
	$mT = 400$	0.120	0.093	0.794	0.811	0.086	0.096
m_T	$mT = 250$	0.535	0.492	0.439	0.475	0.027	0.033
	$mT = 400$	0.326	0.309	0.645	0.641	0.030	0.050

$\beta_2 = 0.8$ for $t \in \left\{\frac{2T}{3} + 1, \ldots, T\right\}$. As in the previous section, we adapted the significance level of the test to $\alpha_k = 1 - (1 - \alpha_0)^{\frac{1}{k}}$ for the k^{th} test using $\alpha_0 = 0.05$. The results using the procedure proposed in Section 3.4 can be found in Table 7. The tables report the averaged stopping times, averaged break point estimates and rejection rates for the first, second, and the joint first and second break events.

The rejection rates increase with the size of the initial sample period mT. Power increases in the dimension d, although this effect is only moderate for both tests. As before, the tests based on $D_{T,S}$ has larger power than the one based on M_T. We also note that the second break point is detected more frequently than the first one, which can be explained by the higher magnitude of the second break compared to the first break. Furthermore, if the monitoring procedure detects the first break point it is very likely that the second break point is detected as well, which can be seen by the almost identical rejection rates of rej_1 and rej_{all}. Again, the average stopping time is much later than the true break, but the estimated break point $\widehat{k}$ is able to detect the breaks reasonably well. Thus, we can conclude that the procedure works fairly well for the case of two breaks and that both the power of detecting changes and estimating the break locations can be achieved in a reasonably reliable manner.

5 Empirical application

In this section we apply our test to a real data set. We use daily log returns of stock prices over a time span ranging from 29.01.2002 to 01.07.2013 of 10 large firms, namely Citigroup, HSBC Holdings ($), UBS-R, Barclays, BNP Paribas, HSBC Holdings (ORD), Mitsubishi, Royal Bank, Credit Agricole and Bank of America. This implies

Table 7: Average detected break point location $\frac{\widehat{k}^i}{T}$, stopping time $\frac{\tau_T^i}{T}$ and rejection frequency using 301 simulations for the parameter monitoring procedure. Data was generated with breaks at $\frac{T}{3}$ and $\frac{2T}{3}$, with $T = 1500$, $d = 10, 20$, $\beta_0 = 1.0$, $\beta_1 = 1.5$, $\beta_2 = 0.8$. Results are based on the parameter based detector $D_{T,S}$ (top panel) and the moment based detector (bottom panel).

		$\frac{\tau_T^1}{T}$	$\frac{\widehat{k}^1}{T}$	rej_1	$\frac{\tau_T^2}{T}$	$\frac{\widehat{k}^2}{T}$	rej_2	$(\frac{\tau_T^1}{T}\ \frac{\tau_T^2}{T})$	$(\frac{\widehat{k}^1}{T}\ \frac{\widehat{k}^2}{T})$	rej_{all}
Parameter based										
$d = 10$	$mT = 250$	0.476	0.351	0.833	0.822	0.671	0.724	(0.457 0.822)	(0.341 0.671)	0.724
	$mT = 400$	0.502	0.383	0.923	0.854	0.720	0.854	(0.479 0.856)	(0.365 0.721)	0.841
$d = 20$	$mT = 250$	0.479	0.355	0.874	0.818	0.666	0.767	(0.457 0.818)	(0.341 0.665)	0.764
	$mT = 400$	0.494	0.378	0.927	0.849	0.718	0.900	(0.475 0.848)	(0.364 0.720)	0.870
Moment based										
$d = 10$	$mT = 250$	0.520	0.365	0.588	0.800	0.668	0.542	(0.497 0.797)	(0.348 0.669)	0.528
	$mT = 400$	0.551	0.395	0.777	0.858	0.733	0.744	(0.520 0.857)	(0.373 0.736)	0.714
$d = 20$	$mT = 250$	0.522	0.362	0.648	0.809	0.665	0.588	(0.494 0.809)	(0.342 0.667)	0.575
	$mT = 400$	0.549	0.395	0.821	0.853	0.727	0.774	(0.513 0.852)	(0.370 0.730)	0.741

a monitored period of size $T = 2980$ and $d = 10$. Figure 1 is a plot of the stock prices in US-$ of the 10 assets over the whole monitored period.

We use the same factor copula model as in (4.1) and we fix the parameters $\nu = 2.855$ and $\lambda = -0.0057$ for the monitoring procedure, i.e. we only monitor the factor loading parameter. These fixed values correspond the parameter estimates from the initial sample period of size $mT = 400$. For the conditional mean and variance we specify the following AR(1)-GARCH (1,1).

$$r_{i,t} = \alpha + \beta r_{i,t-1} + \sigma_{i,t}\eta_{it},$$

$$\sigma_{it}^2 = \gamma_0 + \gamma_1\sigma_{i,t-1}^2 + \gamma_2\eta_{i,t-1}^2,$$

for $t = 2, \ldots, 2980$, and $i = 1, \ldots, 10$. Note that for the monitoring procedure the parameters of the conditional mean and variance models are always reestimated on the same rolling window sample of size mT.

5.1 Monitoring procedure

Figure 2 shows the factor loading parameter estimated over a rolling window of size 400. From this one can see some notable parameter changes between 2006 and 2009. The results of the monitoring procedure of the whole considered period can be seen in Table 8, where again we used a significance level of $\alpha_k = 1 - (1 - \alpha_0)^{\frac{1}{k}}$ for the k^th test with $\alpha_0 = 0.05$. We choose the initial sample as $mT = 400$ from 29.01.2002 to 11.08.2003, where we first estimate the marginal AR(1)-GARCH(1,1) model to obtain the residuals. We use the retrospective test from Manner, Stark, and Wied (2019) to test the hypothesis of no parameter change in the initial sample and the null

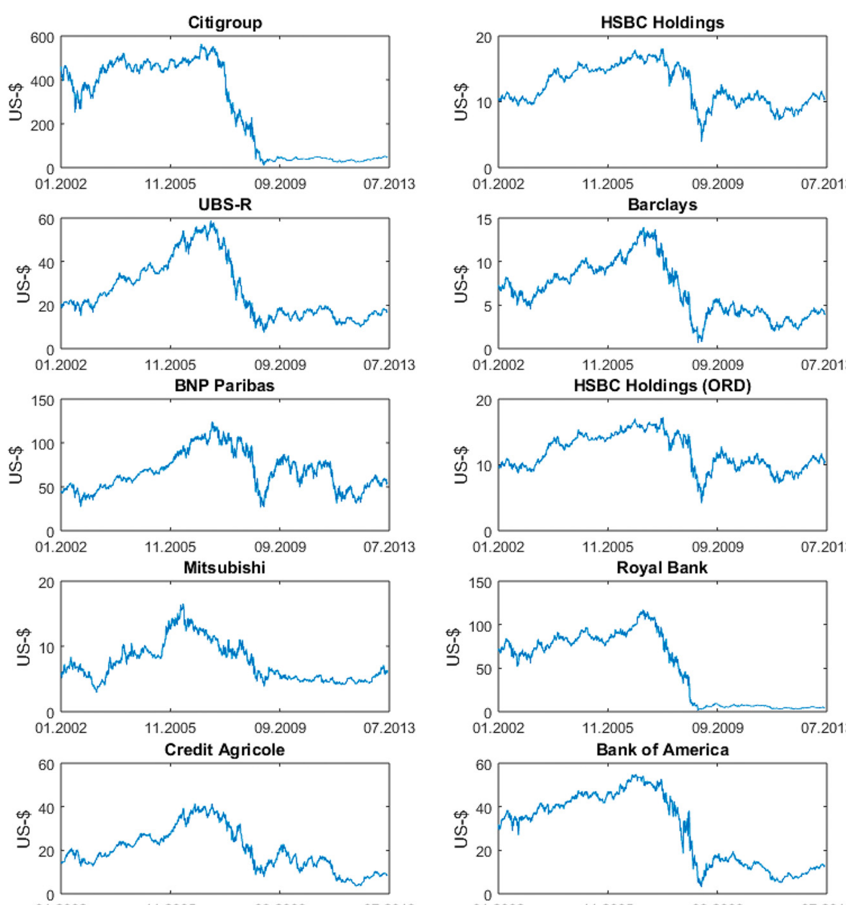

Figure 1: Asset values S_t^i in US-$ in our considered portfolio for data between 29.01.2002 and 01.07.2013, $T = 2980$ and $d = 10$.

Table 8: Stopping time τ_T, estimated break point location $\hat{k}$ and associated sample size T for monitored or tested periods using the monitoring procedure or the retrospective parameter test.

Monitored/testing Period	τ_T	$\hat{k}$	T
29.01.2002–11.08.2003			400
12.08.2003–01.07.2013	18.09.2008	19.07.2007	2580
20.07.2007–29.01.2009		08.08.2008	400
11.08.2008–22.02.2010			400
23.02.2010–01.07.2013			875

hypothesis cannot be rejected. Note that for the retrospective parameter test a burn in period of 20 % of the behold data is used. We then apply our constructed monitoring procedure. The monitoring procedure stops at the 18.09.2008 and the estimated break point location is found at the 19.07.2007, where we used the retrospective parameter break point estimate with data from the end of the historical data set 12.08.2003 to the stopping time 18.09.2008.

Figure 3 is a plot of $D_{T,S}$ for every time point between $mT + 1$ (12.08.2003) and the stopping point, where $D_{T,S}$ exceeds the critical value of (3.2) equal to 3.4566.

We then cut of all the data in front of the estimated break point location (19.07.2007) and test for the null hypothesis of no parameter change in the period from 20.07.2007 to 29.01.2009 of size $mT = 400$, using again the retrospective parameter test and the null is rejected. The estimated break point is found at the 08.08.2008.

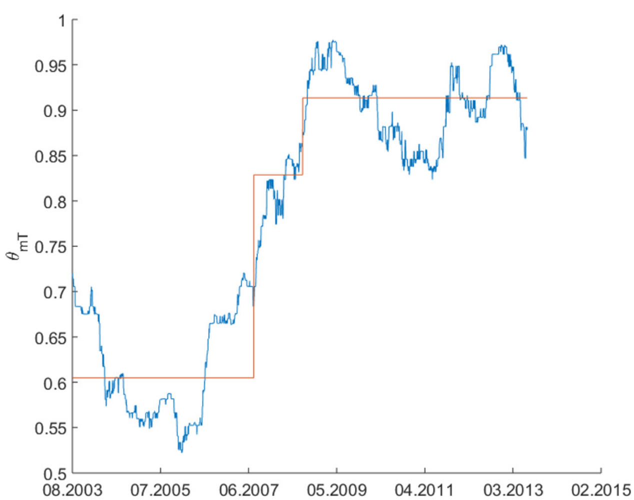

Figure 2: Rolling window estimate of θ_{mT} for $mT = 400$ and $d = 10$ between 11.08.2003 and 01.07.2013, with parameter values estimated from break to break. Each parameter value is associated to the end time point of the rolling window.

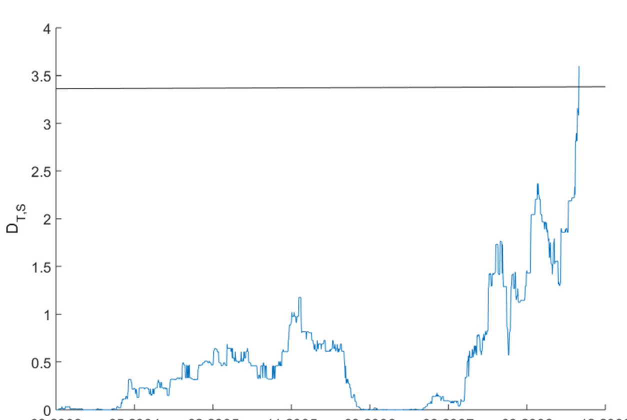

Figure 3: $D_{T,S}(s)$ for $T = 2980$, $mT = 400$ and $d = 10$. Stopping date at 18.09.2008 and $c = 3.4566$.

For the next subsample we try the period from 11.08.2008 to 22.02.2010 and get a retrospective test statistic value $S_{T,S}$ of 2.0269 with a critical value of 4.1138. Hence, the null hypothesis cannot be rejected and we choose this period as our new historical period and restart our monitoring procedure from 23.02.2010 to 01.07.2013. The detector $D_{T,S}$ does not cross the boundary value $c = 15.5073$ and the procedure stops at the end of the monitored period, without rejecting the null. The piecewise constant factor loadings can be seen in Figure 2 and we observe that they track the evolution of the rolling window estimates fairly well.

5.2 Value-at-risk predictions

Given the growing need for managing financial risk, risk prediction plays an increasing role in banking and finance. The value-at-risk (VaR) is one of the most prominent measure of financial risk. Despite it having been criticized as being theoretically not efficient and numerically problematic (see Dowd and Blake 2006), it is still the most widely used risk measure in practice. The number of methods for its computation continues to increase. The theoretical and computational complexity of VaR models for calculating capital requirements is also increasing. Some examples include the use of extreme value theory (McNeil and Frey 2000), quantile regression methods (Manganelli and Engle 2004), and Markov switching techniques (Gray 1996 and Klaassen 2002).

First, we want to define the Value at Risk (VaR). We define the log return of a single asset i at time t as $r_t^i = ln(S_t^i) - ln(S_{t-1}^i)$, where S_t^i is the time t stock price of asset i. The change in the portfolio value over the time interval $[t-1, t]$ is then

$$\Delta V_t = \sum_{i=1}^{d} w_i r_t^i,$$

where w_i are portfolio weights. The (negative) α-quantile of the distribution of $\Delta V := \{\Delta V_t\}_{t=1}^{T}$ is the day t Value-at-risk at level α.

Here we want to show that our monitoring procedure can help improve the day-ahead predictions of the VaR based on a factor copula model. The VaR predictions based on the monitoring procedure for the factor copula model are computed as follows. In general, based on $\mathcal{F}_t$, the information available at time t, we want to predict the VaR for period $t+1$. The prediction of the VaR is always based on the following four steps.

(1) Simulate M draws from the copula model $\tilde{u}_{t+1} \sim C(\cdot, \widehat{\theta}_t)$, where $\tilde{u}_{t+1} = [\tilde{u}_{1,t+1}, \ldots, \tilde{u}_{d,t+1}]$ is an $M \times d$ matrix of simulated observation and $\widehat{\theta}_t$ is an appropriate parameter estimate based on information up to time t.
(2) Use the inverse marginal distribution function of the standardized residuals η to transform every component of $\tilde{u}_{t+1}$ to $\tilde{\eta}_{t+1} = [F_1^{-1}(\tilde{u}_{1,t+1}), \ldots, F_d^{-1}(\tilde{u}_{d,t+1})]$, where $F_i^{-1}(\cdot)$ is estimated by the inverse integrated kernel density estimator of the residuals $\widehat{\eta}$ with a sufficiently large number of evaluation points.
(3) Compute the simulated returns $\tilde{r}_{t+1} := [\tilde{r}_{t+1}^1, \ldots, \tilde{r}_{t+1}^d]' = \mu(\widehat{\phi}_t) + \sigma(\widehat{\phi}_t)\tilde{\eta}_{t+1}$, where $\widehat{\phi}_t$ are the estimated parameters from models for the conditional mean and variance using information up to time t.
(4) Form the portfolio of interest from the simulated returns and compute the appropriate quantile from the distribution of the portfolio to obtain the VaR prediction for time $t+1$.

This procedure for predicting the VaR is generic. The monitoring procedure for the copula parameter θ_t is used to determine the appropriate information set on which the parameter estimate in Step 1 is based. The basic idea is to use as much information as possible as long as no changepoint is detected. In case a changepoint is found only the most recent observations should be used to estimate θ_t. Recall that mT observations for which the dependence is assumed to be constant are available at the beginning of the sample. Further, denote $\widehat{\theta}_{s:t}$ the estimator of the copula parameter based on the observations from time s to t. At each point in time t, compute $D_{T,S}(t)$.

i. Before a changepoint is detected, i.e. as long as $D_{T,S}(t) < c$ the draws from the copula in Step 1 above are based on $\widehat{\theta}_{1:t}$

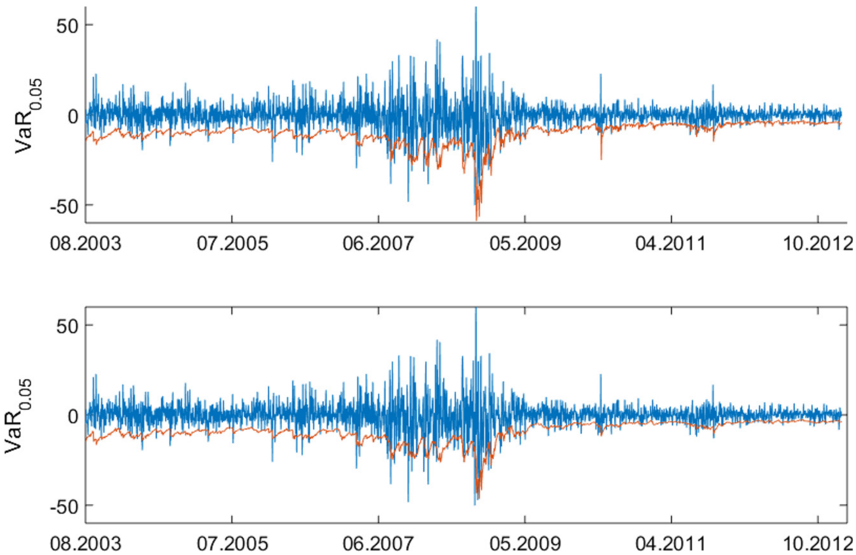

Figure 4: Portfolio returns ΔV_t and the $\alpha = 0.05$ predicted Value-at-Risk based on the monitoring procedure, allowing for structural breaks (upper panel) and without (lower panel) for the period between 29.01.2002 and 01.07.2013.

ii. Assume the monitoring procedure stops at time $t = \hat{\tau}$, i.e. when $D_{T,S}(t) > c$. Compute the breakpoint estimate $\hat{k}$ using (3.3). Use the estimate $\hat{\theta}_{\hat{k}:t}$ in Step 1 above. If $\hat{k} - t < 400$, i.e. if less than 400 observations are available use $\hat{\theta}_{t-400:t}$. In other words, after a breakpoint is identified use either all observations after the breakpoint estimate or the most recent 400 observations to estimate the copula parameter.[2]

iii. If $\hat{k} - t \leq mT$ proceed as in Step ii. Otherwise use the window $[\hat{k}, \hat{k} + mT]$ as the new initial sample and apply the monitoring procedure. As long as no further breakpoint is detected the parameter estimate $\hat{\theta}_{\hat{k}:t}$ is used. When the monitoring procedure stops again return to Step ii.

The results for the online VaR evaluation based on $M = 1500$ simulations for each period and for $\alpha = 0.05$ can be seen in Figure 4. As an alternative, we consider the same model without the monitoring procedure. In that case the copula parameter is estimated using the full sample available at time t using an expanding window. The model for the margins is an AR(1)-GARCH(1,1) in both cases. Visually, the online procedure tracks the 5 % VaR well. The empirical VaR exceedance rate is, in fact, 5.39% (139 exceedances in 2580 days) and therefore reasonably close to 5 %. In the model without structural breaks, where the parameters are estimated from the beginning of the sample on, the exceedance rate is higher with 6.78% (175 exceedances). With a binomial test (compare Berens et al. 2014), we test the null hypothesis of unconditional coverage, i.e.,

$$\mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T} I_t(0.05)\right) = \alpha = 0.05,$$

where α is the VaR coverage probability and

$$I_t(0.05) = \begin{cases} 0, & \text{if } \Delta V_t \geq -VaR_{0.05} \\ 1, & \text{if } \Delta V_t < -VaR_{0.05}. \end{cases}$$

One expects 129 exceedances under H_0 and at the 1% significance level the critical value of the test is 158 exceedances. This implies that the null of unconditional coverage is rejected in the model without structural breaks, but not in the model with structural breaks.

2 The minimum number of observations required for model estimation depends on the complexity of the chosen model. However, for the type of model we are considering here we found that one needs at least 400 observations to obtain reliable and numerically stable parameter estimates.

6 Conclusion

We propose a new monitoring procedure for detecting structural breaks in factor copula models and analyze the behavior under the null hypothesis of no change. Due to the discontinuity of the SMM objective function this requires additional effort to derive a functional limit theorem for the model parameters. The presence of nuisance parameters in the asymptotic distribution of the two proposed detectors requires a bootstrap approximation for parts of the asymptotic distribution. The case of detecting two breaks is also treated. In simulations, the proposed procedures show good size and power properties in single and multiple break settings in finite samples. An empirical application to a set of 10 stock returns of large financial firms indicates the presence of break points around July 2007 and August 2008, time points of the heights of the last financial crisis. The proposed online Value-at-Risk procedure shows the usefulness of the monitoring procedure in portfolio management.

7 Assumptions and Proof

7.1 Assumption

Assumption 3 and Assumption 4 ensure that the estimated rank correlation and quantile dependencies converge to their respective population counterparts.

Assumption 3. i. *The distribution function of the innovations F_η and the joint distribution function of the factors $F_X(\theta)$ are continuous.*

ii. *Every bivariate marginal copula $C_{ij}(u_i, u_j\,; \theta)$ of $C(u\,;\theta)$ has continuous partial derivatives with respect to $u_i \in (0,1)$ and $u_j \in (0,1)$.*

The assumption is similar to Assumption 1 in (Oh and Patton 2013), but the assumption on the copula is relaxed in the sense that the restriction of u_i and v_i is relaxed to the open interval $(0,1)$.

Assumption 4. *The first order derivatives of the functions $\phi \mapsto \mu_t(\phi)$ and $\phi \mapsto \sigma_t(\phi)$ exist and are given by $\dot{\mu}_t(\phi) := \frac{\partial \mu_t(\phi)}{\partial \phi'}$ and $\dot{\sigma}_{kt}(\phi) := \frac{\partial [\sigma_t(\phi)]_{k-th\, column}}{\partial \phi'}$ for $k = 1, \ldots, d$. Moreover, define $\gamma_{0t} := \sigma_t^{-1}(\hat{\phi})\dot{\mu}_t(\hat{\phi})$ and $\gamma_{1kt} := \sigma_t^{-1}(\hat{\phi})\dot{\sigma}_{kt}(\hat{\phi})$ such as*

$$d_t := \eta_t - \hat{\eta}_t - \left(\gamma_{0t} + \sum_{k=1}^{d} \eta_{kt}\gamma_{1kt}\right)(\hat{\phi} - \phi_0),$$

with η_{kt} is the k-th row of η_t and γ_{0t} such as γ_{1kt} are *$\mathcal{E}_{t-1}$-measurable, where $\mathcal{E}_{t-1}$ contains information from the past as well as possible information from exogenous variables.*

i. $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}\gamma_{0t} \overset{p}{\Rightarrow} s\Gamma_0$ *and* $\frac{1}{T}\sum_{t=1}^{sT}\gamma_{1kt} \overset{p}{\Rightarrow} s\Gamma_{1k}$, *uniformly in $s \in [\varepsilon, 1]$, $\varepsilon > 0$, where Γ_0 and Γ_{1k} are deterministic for $k = 1, \ldots, d$.*

ii. $\frac{1}{T}\sum_{t=1}^{T}E(\|\gamma_{0t}\|)$, $\frac{1}{T}\sum_{t=1}^{T}E(\|\gamma_{0t}\|^2)$, $\frac{1}{T}\sum_{t=1}^{T}E(\|\gamma_{1kt}\|)$ *and* $\frac{1}{T}\sum_{t=1}^{T}E(\gamma_{1kt}^2)$ *are bounded for $k = 1, \ldots, d$.*

iii. *There exists a sequence of positive numbers $r_t > 0$ with $\sum_{i=1}^{\infty} r_t < \infty$, such that the sequence $\max_{1\le t\le T}\frac{\|d_t\|}{r_t}$ is tight.*

iv. $\max_{1\le t\le T}\frac{\|\gamma_{0t}\|}{\sqrt{T}} = o_p(1)$ *and* $\max_{1\le t\le T}\frac{|\eta_{kt}|\|\gamma_{1kt}\|}{\sqrt{T}} = o_p(1)$ *for $k = 1, \ldots, d$.*

v. $(\alpha_T(s, u), \sqrt{T}(\hat{\phi} - \phi_0))$ *weakly converges to a continuous Gaussian process in $D((0, 1] \times [0, 1]^d) \times \mathbb{R}^r$, where $\mathscr{D}((0, 1] \times [0, 1]^d)$ is the space of all Càdlàg-functions on $(0,1] \times [0,1]^d$, with*

$$\alpha_T(s,u) := \frac{1}{\sqrt{T}} \sum_{t=1}^{sT} \left\{ \prod_{k=1}^{d} 1\{U_{kt} \le u_k\} - C(u\,;\theta) \right\}.$$

vi. $\frac{\partial F_\eta}{\partial \eta_k}$ *and* $\eta_k \frac{\partial F_\eta}{\partial \eta_k}$ *are bounded and continuous on* $\overline{\mathbb{R}}^d = [-\infty, \infty]^d$ *for* $k = 1, \dots, d$.

vii. *For* $u \in [0,1]^d$, $s \in [m, 1]$ *and* $\widehat{F}^{1+(s-m)T:st}(\widehat{\eta}_t) = (\widehat{F}_1^{1+(s-m)T:st}(\widehat{\eta}_{1t}), \dots, \widehat{F}_d^{1+(s-m)T:st}(\widehat{\eta}_{dt}))$, *the sequential empirical copula process*

$$\frac{1}{\sqrt{T}} \left[\sum_{t=1+(s-m)T}^{\lfloor sT \rfloor} 1\left\{\widehat{F}^{1+(s-m)T:st}(\widehat{\eta}_t) \le u\right\} - C(u) \right]$$

converges in distribution to some limit process $A^*(s, u)$ on $[0,1]^d \times [m, 1]$

Parts i) to vi) of this assumption are similar to Assumption 2 in (Oh and Patton 2013), only part i) and v) are more restrictive. We need this because we consider successively estimated parameters. Part vii) ensures that the empirical copula process of the residuals has some well-defined limit. Note that Assumption vii) is plausible and follows from a combination of the results in Bücher et al. (2014) and Rémillard (2017).

The next assumption is needed for consistency of the successively estimated parameters. It is the same as Assumption 3 in (Oh and Patton 2013) with the difference that part (iv) is adapted to our situation and that a regularity condition on the moment simulating function (which is missing both in (Oh and Patton 2013) and (Manner, Stark, and Wied 2019) is added in part (v). Note that part i) ensures the identifiability of the factor model.

Assumption 5. i. *For* $g_0(\theta)$, *defined by the limit* $g_{1:mT,S}(\theta) \to_p g_0(\theta)$ *for* $T, S \to \infty$, *it holds that* $g_0(\theta) = 0$ *only for* $\theta = \theta_0$ *(the value of all* θ_t *under the null).*

ii. *The space* Θ *of all* θ *is compact.*

iii. *Every bivariate marginal copula* $C_{ij}(u_i, u_j; \theta)$ *of* $C(u\,;\theta)$ *is Lipschitz-continuous for* $(u_i, u_j) \in (0,1) \times (0,1)$ *on* Θ.

iv. *The sequential weighting matrix* $\widehat{W}_{(s-m)T:sT}$ *is* $O_p(1)$ *and* $\sup_{s \in [m,1]} \| \widehat{W}_{(s-m)T:sT} - W \| \xrightarrow{p} 0$ *for* $m \ge \varepsilon > 0$.

v. *It holds for the moment simulating function* $\tilde{m}_S(\theta)$ *that, for* $\theta_1, \theta_2 \in \Theta$,

$$|\tilde{m}_S(\theta_1) - \tilde{m}_S(\theta_2)| \le C_S |\theta_1 - \theta_2|$$

with a random variable C_S *that is independent of* $\theta_1 - \theta_2$ *and that fulfills* $E(C_S^{2+\delta}) < \infty$ *for some* $\delta > 0$.

The compactness of Θ is not too restrictive and the parameter space can be determined from outside information such as constraints from economic arguments. Further, we checked Assumption 5 v) for the case of $\widehat{m}_{ij} = \widehat{\rho}_{ij}$ and $\widehat{m}^{ij} = \widehat{\lambda}_{0.1}^{ij}$ using Model 4.1. We considered $\theta_1 = \theta_2 + h$ where $h = \frac{1}{i}$ for $i = 1, \dots, 1000$, $\theta_2 = 1.0$ and $d = 10$. We varied $S = \{250, 500, 1000, 2000, 4000\}$ and the results can be seen in Figure 5.

Figure 5 reveals that the quotient $q(h) := \frac{|\tilde{m}_S(\theta_1) - \tilde{m}_S(\theta_2)|}{|\theta_1 - \theta_2|}$ seems to be bounded for increasing S independently of the parameter difference $\frac{1}{i}$.

Finally, we need an assumption for distributional results, which is the same as Assumption 4 in (Oh and Patton 2013) with a difference in part iii).

Assumption 6. i. θ_0 *is an interior point of* Θ.

ii. $g_0(\theta)$ *is differentiable at* θ_0 *with derivative* G *such that* $G'WG$ *is non-singular.*

iii. $\forall s \in [m, 1], \varepsilon > 0:\ g_{.,S}(\theta_{(s-m)T:sT,S})' \widehat{W} g_{.,S}(\theta_{(s-m)T:sT,S}) \le \inf_{\theta \in \Theta} g_{.,S}(\theta)' \widehat{W} g_{.,S}(\theta) + d_T$, *where* $d_T = o_p^*((m^2 T)^{-1})$ *and* $d_T \ge 0$.

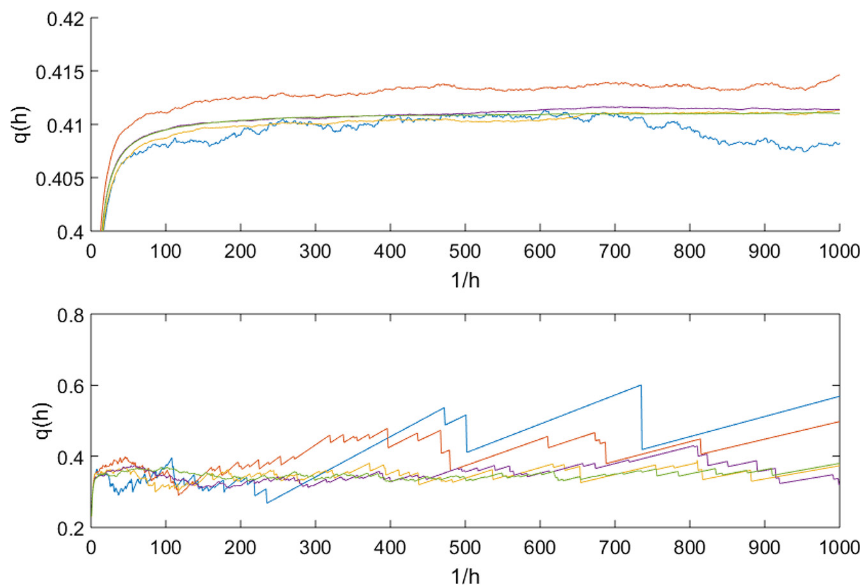

Figure 5: Quotient $q(h)$ for $h = \frac{1}{i}$ for $i = 1, \ldots, 1000$, $\theta_2 = 1.0$ and $d = 10$ such as S={250 (blue), 500 (orange), 1000 (yellow), 2000 (purple), 4000 (green)}. Results for $\hat{m}_{ij}=\hat{\rho}_{ij}$ (upper panel) and $\hat{m}^{ij}=\hat{\lambda}^{ij}_{0.1}$ (lower panel) using Model 4.1.

7.2 Proofs

Proof of Theorem 1. We consider the dependence measures Spearman's rho and quantile dependence measures, which are functions only depending on bivariate copulas. Under the null and all mentioned Assumptions, we first want to show

$$m\sqrt{T}\left(\hat{m}_{(s-m)T:sT} - m_0(\theta_0)\right) \overset{d}{\Rightarrow} A(s), \quad T \to \infty, \quad \forall\, s \in [m, 1],\ m \geq \varepsilon > 0$$

where $A(s)$ is a Gaussian process and θ_0 the value of all θ_t under the null.

By Assumption iiv) (1) the sequential empirical copula of the d-dimensional random vectors fulfills

$$\begin{aligned}
\mathbb{C}_T := {} & m\sqrt{T}\left[\hat{C}_{1+(s-m)T:sT}(u) - C(u)\right] \\
= {} & \frac{1}{\sqrt{T}}\left[\sum_{t=1+(s-m)T}^{\lfloor sT \rfloor} 1\left\{\hat{F}^{1+(s-m)T:sT}(\hat{\eta}_t) \leq u\right\} - C(u)\right] \\
\underset{(1)}{\overset{d}{\Rightarrow}} {} & = A^*(s, u), \quad T \to \infty, \quad \forall\, s \in [m, 1],\ m \geq \varepsilon > 0,
\end{aligned}$$

where $u \in [0, 1]^d$ and $\hat{F}^{1+(s-m)T:sT}(\hat{\eta}_t) := (\hat{F}_1^{1+(s-m)T:sT}(\hat{\eta}_{1t}), \ldots, \hat{F}_d^{1+(s-m)T:sT}(\hat{\eta}_{dt}))$. Here, $\hat{F}_j^{1+(s-m)T:sT}$ denotes the marginal empirical distribution function of the j-th component and $\hat{C} := \hat{C}_{1+(s-m)T:sT}(u)$ the empirical copula both calculated from the data between the time point $1 + \lfloor (s-m)T \rfloor$ and time point $\lfloor sT \rfloor$. Note that Spearman's rho between the i-th and j-th component is given by

$$12 \int_0^1 \int_0^1 C(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1) du_i du_j - 3$$

and that the quantile dependencies are projections of the d-dimensional copula onto one specific point divided by some prespecified constant. Define the function $m^{ij}(C)$ as the function which generates a vector of all considered dependence measures (Spearman's rho and/or quantile dependencies for different levels) between the i-th and j-th component out of the copula C. Without loss of generality consider the equidependent case (averaging over all possible pairs, for details see Oh and Patton (2017)), then the function

$$m(C) : D[0,1]^d \to \mathbb{R}^k$$
$$C \to m(C) = \frac{2}{d(d-1)} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} m^{ij*}(C)$$

is continuous and we directly obtain

$$m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) = m\sqrt{T}\left[m\left(\widehat{C}\right) - m(C)\right]$$
$$\overset{d}{\Rightarrow} \frac{2}{d(d-1)}\left(\sum_{i,j} m^{ij}\left(A^*(s,u)\right)\right) =: A(s)$$

as $T \to \infty$ with $s \in [m, 1]$, $m \geq \varepsilon > 0$. Here, $m^{ij}(\cdot)$ is the same function as $m^{ij*}(\cdot)$ with the only difference that the formula for Spearman's rho between the i-th and j-th component is replaced by

$$12\int_0^1 \int_0^1 C(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)du_i du_j.$$

Then we receive for $\frac{S}{T} \to k \in (0, \infty]$ and $T, S \to \infty$

$$m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta) = m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - \tilde{m}_S(\theta)\right)$$
$$= m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - m\sqrt{T}\left(\tilde{m}_S - m_0(\theta)\right)$$
$$= m\sqrt{T}\left(\widehat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - \sqrt{\frac{T}{S}} m\sqrt{S}\left(\tilde{m}_S - m_0(\theta)\right)$$
$$\overset{d}{\Rightarrow} A(s) - \frac{m}{\sqrt{k}}B,$$

where $B = N(0, \Sigma_0)$ is a centered Gaussian distribution with covariance matrix Σ_0, for details see Oh and Patton (2013). The limit result then follows with the same proof steps as in Manner, Stark, and Wied (2019), using the given limit result for $m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta)$ and replacing the scale factor $s\sqrt{T}$ by $m\sqrt{T}$.

This completes the proof.

Proof of Theorem 2. Due to Assumption 2, conditionally on the original data, as $T \to \infty$, the process

$$A^{(b)}(s) = m\sqrt{T}\left(\widehat{m}^{(b)}_{1+(s-m)T:sT} - \widehat{m}^{(b)}_{1:T}\right)$$

converges in distribution to the process $A(s)$ defined in Theorem 1 in Manner, Stark, and Wied (2019), see Proposition 4 in Genest and Rémillard (2008). Then, the results for the null hypothesis follow, as all transformations of the process of the empirical moments from the proof of Theorem 1 are applicable for the bootstrap sample as well. Under the alternatives, it holds that, for some $s \in (0, 1)$ the quantities $\widehat{\theta}_{1+(s-m)T:sT,S}$ and $\widehat{\theta}_{1:mT,S}$ resp. $\widehat{m}_{1+(s-m)T:sT}$ and $\widehat{m}_{1:mT}$ have different limits so that the detectors tend to ∞. On the other hand, the bootstrapped critical values remain stochastically bounded, as they are generated under the assumption that the model does not change over time. □

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