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Functional approach to $\mathcal{N} = 4$ super-Yang-Mills theory

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Abstract

Symmetries are of crucial importance to theoretical physics. The idea is to keep the theory invariant under continuous or discrete transformations. Supersymmetry (SUSY) is such a transformation between bosons and fermions. Due to that symmetry various new concepts have been developed, both in mathematics as well as in physics. Nevertheless until now no particles associated to SUSY have been detected. In this thesis $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) will be studied in Landau gauge. SYM is as a supersymmetric extension of Yang-Mills theory, where $\mathcal{N} = 4$ refers to the number of supersymmetric charges Q. To solve the propagator equations a functional method called Dyson-Schwinger equations (DSEs) is used. A powerlaw like behaviour of the solutions is observed. To support that this powerlaw or scaling solutions are valid the already analytically known beta function $\beta(g) = 0$ is derived.

In contrast to Yang-Mills Theory $\mathcal{N} = 4$ SYM is highly constrained and therefore a valuable tool to test the understanding of the first. We use $\mathcal{N} = 4$ SYM to take a look at non-perturbative Landau gauge as well as to understand how good the quality of the employed truncation is. We also want to understand if there is a Gribov-Singer ambiguity in $\mathcal{N} = 4$ SYM, since this ambiguity has a vast impact on Yang-Mills theory.

Kurzfassung

Symmetrien spielen eine zentrale Rolle in der theoretischen Physik. Das Konzept einer Symmetrie ist es, die Theorie invariant zu lassen unter einer bestimmten Transformation, sei diese kontinuierlich oder diskret. Im Falle von Supersymmetrie wird eine Symmetrie zwischen den bosonischen und fermionischen Freiheitsgraden einer Theorie erzeugt. Diese Idee lieferte über Jahre mathematische und physikalische Erkenntnisse, es zeigt sich jedoch, dass es bis heute keinen experimentellen Beweis für ihre tatsächliche Existenz gibt. Diese Arbeit beschäftigt sich mit Super-Yang-Mills Theorie (SYM), einer supersymmetrischen Erweiterung der Yang-Mills Theorie mit $\mathcal{N} = 4$ supersymmetrischen Ladungen Q. Hierfür werden Propagatoren über die funktionale Methode der Dyson-Schwinger Gleichungen im Rahmen der Landau Eichung untersucht und deren Lösungen auf Potenzgesetze untersucht. Um zu zeigen, dass die Potenzgesetze eine legitime Lösung darstellen wird auch die $\beta(g)$ berechnet und mit der analytisch Bekannten verglichen.

Im Gegensatz zu Yang-Mills Theorie ist $\mathcal{N} = 4$ SYM eine sehr eingeschränkte Theorie, dadurch wird sie zu einem wertvollen Werkzeug um Probleme der Yang-Mills Theorie zu vereinfachen. In dieser Arbeit verwenden wir $\mathcal{N} = 4$ SYM um einen Blick in die nicht pertubative Landau Eichung zu gewinnnen und um herauszufinden, wie hoch die Qualität der verwendeten Trunkierung ist. Zuletzt wollen wir uns mit der Frage beschäftigen, ob und wie die Gribov-Singer Ambiguität in dieser Theorie aussieht. Denn bis heute konnte noch nicht geklärt werden wie wichtig deren Effekt auf die Yang-Mills Theorie ist.

Contents

Chapter 1

Introduction

Physical theories are the best tool to describe nature. When one is doing theoretical particle physics, one is mainly interested in very small scales, for example a few femto meters. But there is also the astronomical scale which is essentially different from the quantum scale. These two scales are associated with two different theories, the quantum scale with qunatum field theory (QFT) [1–3] and the astronomical scale with general relativity (GR) [4–6]. Every process in phyiscs, wheter it is connected to gravity or particle physics, is governed by energy, or more accurate by minimizing the energy. But as Feynman said, we still do not know what energy is [7].

Leaving the conceptional questions aside, Johann Wolfgang Goethe already asked in his famous book Faust: "*...whatever holds the world together in its inmost folds*?"

The right answer seems to be the strong force. The strong force is described by quantum chromodynamics (QCD) [1, 2, 8] a non-Abelian gauge theory. QCD has been under study for decades now, but is still not fully understood [9]. The force particle of QCD is the gluon, a spin 1 boson which interacts with itself. The pure bosonic part of QCD can be described by Yang-Mills theory, gluo-dynamics plays a central role in the theory. It is thought, that the phenomena called confinement [10] can be described by Yang-Mills theory. Also the high temperature phase of QCD seems to be governed by gluo-dynamics [11, 12].

Yet no one solved QCD exactly. This of course is no problem since many approximations are needed for hard problems, but it is strange, that it is not possible to calculate the full 2-point correlation function for the gluon, as it would be the simplest non vanishing correlation function. Nevertheless, one way to overcome this result, is to simply use less dimensions.

The effect of decreasing the dimensions from 4 to only 2 was studied in [13–17]. The problem which arises is, that the 2-dimensional case is different and therefore not relatable to the case of 4 dimensions. Therefore a new ansatz to the problem of Yang-Mills theory has to be used. In this thesis a new artificial symmetry will be used, that relates bosons to fermions. This symmetry is called supersymmetry (SUSY) and it is the only symmetry that can be used with Poincare group [18, 19]. SUSY also offers a hope for solving one of the hardest problems in theoretical particle physics namely the hierarchy problem [18, 20].

To investigate quantum field theories or in this case the Yang-Mills sector of QCD one need methods, in recent years the ones listed below have been widely used:

(i) **Dyson-Schwinger equations**

The Dyson-Schwinger equations (DSEs) allow to solve QFT's non-perturbativly [21]. This would be a great deal, but due to their intrinsic nature, it is impossible to solve them exactly. This intrinsic nature yields an infinite tower of coupled integral equations. Since it is impossible to solve an infinite system of coupled equations, only a subset of these equations is taken into account, this is called truncation. Finding a suitable truncation is the major task. A lot of progress has been made understanding how this truncations affect the solutions [22–25]. Many solutions from DSEs, for example propagators, are in agreement with other methods [25–27] like lattice field theory, which generates credibility.

(ii) **Lattice field theory**
Lattice simulations are another tool to handle QFT's. Within this particular field a lot of manpower was put during the last thirty years [8, 28–30]. The idea of Lattice field theory is to discretize space-time. Such a discretization yields from the mathematical point of view a well defined pathintegral [8]. One issue that arises using Lattice field theory is the continuum limit. Since space-time is mapped on a grid with finite lattice spacing a and length the limit a going to zero is mathematically not well defined [8], in contrast to that it seems that the continuum limit is supported by experimental data, at least for QCD [31]. Another problem is the computing power needed for such calculations, which is a limiting factor for the whole method. Nevertheless this method is in best agreement with experimental data when it comes to e.g. hadron spectroscopy.

(iii) **Functional renormalization group**
The functional renormalization group is another non-perturbative method [32, 33]. Like the Dyson-Schwinger equations the functional renomalization group is a functional method. Starting with the functional renormalization group equation [1] this method makes it possible to calculate so-called Flows. The flow describes the change of the coupling with respect to the change of the scale. The whole method is built up using a one loop diagram [32]. The computation again is involved and as in the Dyson-Schwinger case not exact when it comes to representative calculations. Recently it has been used as a tool to tackle quantum gravity [34].

(iv) **Effective field theories**
Within the framework of effective field theories one is only interested in effective degrees of freedom. Since the dynamics of a given theory to arbitrary high energy scales are unachievable one makes a cut of the energy at which the only certain degrees of freedom are important [35]. Meaning, that there are various energy scales [1], i.e. Λ_{QCD}, Λ_{GUT}, Λ_{Planck}, etc.. Knowing the theory to arbitrary high energy scales would solve the problem because the low energy theory is described by the former. Unfortunately no one knows how a theory has to look like that includes all forces. Therefore it is an natural ansatz to describe what phenomenology can be observed. As an example of this approach one can think of the Fermi theory of β-decay, where the 4-fermion interaction is approximated by a point like interaction [36]. Also χ-perturbation theory should be mentioned here [37].

To understand Yang-Mills theory better, a different approach is employed. We use a more constrained theory that resemble many features of Yang-Mills theory, hence $\mathcal{N} = 4$ super-Yang-Mills (SYM) [18] will be the object of study.

The same tools will be employed to $\mathcal{N} = 4$ SYM as to Yang-Mills theory [26, 27]. This is not only interesting for the sake of $\mathcal{N} = 4$ SYM but will also show how good the quality of the used truncation is. Therefore new propagator equations will be worked out and related to those of pure Yang-Mills. As $\mathcal{N} = 4$ SYM is highly constrained we want to check how good the quality of the employed one-loop truncation is. This can be checked since the propagator solutions need to be powerlaws, if one-loop truncation would fail one can not observe these solutions. Hopefully the results will shed new light on some unknown features of Yang-Mills theory concerning the Gribov problem [38, 39] as well as the conformal solution [25, 27].

This thesis is organized as follows: In Chapter 2 SUSY will be introduced. A short discussion of SUSY and why it is a good tool for quantum field theories will be given. In Chapter 3 an introduction to DSEs follows, this will be based on pure Yang-Mills theory. Chapter 4 will be for the DSEs of the full theory, namely for $\mathcal{N} = 4$ SYM. The derivation of the scalar- as well the Majorana fermion propagators is of central interest in this chapter. Due to the conformality of the theory an ansatz for the propagators is made in Chapter 5. With this ansatz it is possible to show that the truncated DSEs can be self-consistently solved. This Chapter ends with some remarks on the Gribov-Singer ambiguity. Chapter 6 will contain a conclusion and also some remarks on the Gribov problem as well as a short outlook.

The Appendix will hold all conventions as well as a Feynman rules and some Γ-function calculus.

Chapter 2

Supersymmetry

The idea of supersymmetry (SUSY) is to build up a symmetry between the building blocks of theoretical particle physics, namely the bosons and fermions [2]. The difference between a boson and a fermion is their statistic. The wavefunction Ψ of a bosonic state is symmetric under change of a particle

$$\Phi(x_1, x_2, ..., x_i, x_j, ...) = \Phi(x_1, x_2, ..., x_j, x_i, ...). \tag{2.1}$$

For a given fermionic state the wavefunction Ψ has to be antisymmetric

$$\Psi(x_1, x_2, ..., x_i, x_j, ...) = -\Psi(x_1, x_2, ..., x_j, x_i, ...). \tag{2.2}$$

Even though this two equations only differ in a sign, fermions and bosons show completely different behaviour. For an introduction to their statistics one can take a look at [40].

The approach now is to build a symmetry between those two particles. This is what SUSY does [18]. To generate such a transformation, a supersymmetry charge Q is introduced which, when acting on a bosonic state, creates a fermionic state and vice versa [19], therefore changing the statistics of a given state.

With this symmetry one, so to say, enlarges space-time including further degrees of freedom, fermionic degrees of freedom. These additional degrees of freedom widen space-time to a superspace [19, 41].

There is a lot of literature about SUSY, for any details one can look at [18]. In the case of Standardmodel physic and the MSSM a nice introduction is given at [20]. Also some papers which are used for this thesis [19, 41, 42]

2.1 Boson = Fermion

Boson = Fermion, is as it stands a wrong statement. As already stated, these particles obey a different statistic. What this acquisition means is that a given representation space H can be divided into subspaces. SUSY is generated by a graded algebra [18]. Therefore it demands a grading of the vectorspace on which a representation acts [19]. Since SUSY relates fermions and bosons, H can be divided into a bosonic and a fermionic one. Following the line of argumentation from [19] (, also a rather short and nice line of arguments can be found at [41]):

$$R(H) \to R(bosonic) + R(fermionic). \tag{2.3}$$

The even elements of the algebra, the bosonic ones, namely P^μ the translation operator, $M^{\mu\nu}$ the rotation operator and A_I arbitrary internal operators, map the subspace into themselves [19]:

$$R(bosonic, fermionic) \xrightarrow{P^\mu} R(bosonic, fermionic), \tag{2.4}$$

whereas the odd elements or fermionic ones Q, the so-called supercharges, map the bosonic subspace into the fermionic and vice versa [19]:

$$R(bosonic, fermionic) \xrightarrow{Q} R(fermionic, bosonic) \tag{2.5}$$

These supercharges Q are fermionic objects, therefore one can define an anticommutator consisting of them. Such an anticommutator can be seen as two consecutive mappings, because

$$\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = 2\sigma^\mu P_\mu, \tag{2.6}$$

with σ^μ are the Pauli matrices. Therefore the double use of a fermionic element will yield a bosonic transformation [19]

$$R(bosonic, fermionic) \xrightarrow{Q} R(fermionic, bosonic) \xrightarrow{Q} R(bosonic, fermionic) \tag{2.7}$$

This means, that there is a certain pattern of how to move from one subspace to another. Important to mention is, that from the action of P^μ on the representation space no degrees of freedom are lost. Which implies, that the fermionic subspace must be of the same order as the bosonic subspace [19]. Two arguments for that are

- Every picture has at most the same dimension as the original
- A double mapping like the anticommutator $\{Q, Q^\dagger\}$ preserves the dimensions.

From this one can deduce that the fermionic and bosonic subspace of a representation space of a graded algebra must have the same dimension.

Note that this holds for the case of "off-shell representations" and several cases are known where this *rule* does not hold, for example when $R(P_\mu) = 0$ [19]. Coming to the question: "What is an graded algebra and how to generate them?"

The second part is easily answered by supersymmetric charge.

2.2 Supersymmetry generator

Supersymmetry is a symmetry that connects bosonic degrees of freedom with fermionic ones. Think of a charge Q that does

$$Q|fermion\rangle = |boson\rangle \quad \text{and} \quad Q|boson\rangle = |fermion\rangle. \tag{2.8}$$

Clearly this is no ordinary symmetry operator, since Q acting on any given quantum state $|s\rangle$ will yield

$$Q\Big|s\Big\rangle = \Big|s \pm \frac{1}{2}\Big\rangle. \tag{2.9}$$

For all known bosonic symmetries this feature will not hold. The symmetry operator alters the spin of any given state by 1/2 [19]. No scalar transformation is known that can do this. Therefore the charge Q which generates SUSY has to be spinor like. In fact Q has to be a spinor. This can be proven by rotating Q by 2π - the charge will then pick up a sign analogue to a fermionic state [19].

To be a symmetry of the theory the commutator of the charge Q and the Hamiltion operator has to vanish:

$$[Q, H] = 0. \tag{2.10}$$

This can be shown using Eq.(2.6), which yields a general statement for supersymmetric theories [18].

SUSY states, that there is a fermionic state for every bosonic state. Therefore if Q acts on a known bosonic state, like a photon, a fermion with the **same** mass [42], a so-called photiono must be created:

$$Q|photon\rangle = |photino\rangle, \tag{2.11}$$

Due to this it follows, that every supermultiplet contains a boson and a fermion who's spins differs by $\frac{1}{2}$ [18, 19, 42, 43]. As in general particle physics degenerate particle states produce multiplets [1], the super- of a supermultiplet refers to the boson/fermion created from a fermion/boson. This is the intuitive picture of $\mathcal{N} = 1$ SUSY. In principle, there could be more that just one supercharge Q, this is then called extended SUSY. For extended SUSY there is still only one boson and only one fermion per supermultiplet.

For any SUSY theory a new symmetry arise, the so-called R-symmetry. The R-symmetry is global symmetry that transforms the supercharge Q, or for extended SUSY, the supercharges Q^i into each other [18, 41]. The R-symmetry of $\mathcal{N} = 1$ SUSY is just a $U(1)$ symmetry [41]

$$[Q_\alpha, R] = Q_\alpha, \tag{2.12}$$

$$\left[Q^\dagger_\alpha, R\right] = -Q^\dagger_\alpha, \tag{2.13}$$

where α is a spinor index. Note that the generators, the supercharges Q, Q^* have R-charge +1 and -1 [41]. For higher $\mathcal{N}$ the R-symmetry group becomes non-abelian [18].

To generate such behaviour the group algebra has to be special. Since neither only commuting nor only anticommuting generators could produce such a symmetry, a valid solution is to mix them. This can be archived by a *graded* algebra [18]. A graded algebra is almost a bosonic algebra, defined by the commutator relations of the group elements. The special feature a graded algebra posses is, that two fermionic elements are related by an anticommutator. Then one can define a graded algebra [18, 19] using

$$t_a t_b - (-1)^{\eta_a \eta_b} t_b t_a = i \sum_c C^c_{ab} t_c, \tag{2.14}$$

the η_a and η_b are the so-called gradings of the generators t_a and t_b, they can either be 0 or 1. If η_a is for example 1 then the generator t_a is fermionic, if η_a is 0 then t_a is bosonic. The C^c_{ab} are numerical structure constants, which are determined by the group structure. These structure constants must obey the relations [18]

$$C^c_{ba} = -(-1)^{\eta_a \eta_b} C^c_{ab} \quad \text{and} \quad C^{c*}_{ab} = -C^c_{ba}. \tag{2.15}$$

One can also show that a these structure constants satisfy a non-linear constrain, which follows from a super-Jacobian identity

$$(-1)^{\eta_c\eta_a}[[t_a,t_b\},t_c\} + (-1)^{\eta_a\eta_b}[[t_b,t_c\},t_a\} + (-1)^{\eta_b\eta_c}[[t_c,t_a\},t_b\} = 0. \tag{2.16}$$

Again this symmetry is special in the sense that is compatible with the Poincare group [44], therefore in the next section the famous Coleman-Mandula theorem [45] will be stated. It is this powerful theorem that makes SUSY such an interesting candidate even though there is no experimental legitimation yet.

2.2.1 Coleman-Mandula no-go theorem

The line of arguments is closely related to those of [19], for further reading and deeper insight one can take a look at [18]. To represent very formally the findings of Coleman and Mandula one thinks of the symmetries QFTs can posses and still not over restrict the scattering processes. They found that any bosonic symmetry of the S-matrix of a given field theory is the direct product of the Poincare group with an internal symmetry group [45]. This internal symmetry group has to be the direct product of a semi-simple Lie group with $U(1)$ factors. This was generalized by [46].

The bosonic generators are thus the four-momenta P^μ and the angular momentum tensor $M^{\mu\nu}$ plus a certain number of Hermitian internal symmetry generators. Those internal symmetries must be translational invariant Lorentz scalars.

Given the Casimir operators of the Poincare group

$$P^\mu P_\mu = P^2, \quad W^\mu W_\mu = W^2, \tag{2.17}$$

where W^μ is the Pauli-Lubanski pseudovector [3] defined in the following way

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \tag{2.18}$$

Every internal operator must commute with them. Introducing such an internal symmetry operator A_I we require

$$[A_I, P^2] = 0, \quad [A_I, W^2] = 0. \tag{2.19}$$

Even though those two commutators are simple, their implications are far from trivial. The first commutator tells us that all members of the irreducible multiplet of A_I have the same mass. This is also known as O'Raifeartaugh's theorem [47]. The second commutator says that they also must have the same spin.

For massless states with discrete helicities one can calculate

$$W^\mu = sP^\mu, \tag{2.20}$$

where s has to be a half integer number [19]. Concluding that if $P^2 = W^2 = 0$, no such A_I can change either spin or mass of a given state. Every symmetry that change mass and spin would therefore over restrict the scattering processes.

Restating Coleman-Mandula no-go theorem in a positive way:

Supersymmetry is generated by fermionic operators, i.e. they must change the statistics of the state and the spin by a half odd integer [19].

Therefore only SUSY can be added to the Poincare group and still ensure reasonable scattering processes. This means that a larger symmetry group can be used which allows to put together QFT with gravity. Since the momentum operator and the supersymmetric charge Q are directly connected, the theory becomes a theory of gravity as soon as the momentum operator is local [18].

Stated that SUSY is compatible with the Poincare group one can go even further and find out that for a theory with extended SUSY namely $\mathcal{N} = 4$ extended SUSY the symmetry group of theory becomes the **conformal group** [48, 49].

2.2.2 Confromal transformation

The conformal symmetry is very restraining one on the one hand and a very rich one the other. The important message is that the $\beta(g)$ function [50] vanishes. The $\beta(g)$ can be computed using the Callan-Symanzik equation [51, 52] which is given by [1]

$$\beta(g) = M \frac{\partial}{\partial M} g \Big|_{g_0,\Lambda}. \tag{2.21}$$

$\beta(g)$ is the rate of change of the renormalized coupling at some arbitrary scale M corresponding to a fixed coupling g_0 [1]. Λ is the cutoff scale, hence one requires $M < \Lambda$.

Therefore a vanishing β function states that the coupling is scale invariant and such the theory becomes scale invariant [50]. Nevertheless this makes such theories also a very interesting testing tool for functional methods. $\mathcal{N} = 4$ SYM possesses (super-)conformal symmetry. Due to conformal symmetry only massless particles can appear. This conformal group is spanned by the generators given in Tab.(2.1). Note that the dilation symmetry D, as well as special

Table 2.1: Table of the transformation contained by the conformal group [53].

Generators	Transformations
K^μ	special conformal
D	dilation
P^μ	translation
$M^{\mu\nu}$	rotation

conformal symmetry K^μ is broken in pure Yang-Mills theory at quantum level [54]. Two of those generators span the Poincare group, namely P^μ and $M^{\mu\nu}$ [3]. The commutation relations are [18]:

$$\begin{aligned}
&[P^\mu, D] = iP^\mu, \quad [K^\mu, D] = -iK^\mu, \\
&[P^\mu, K^\nu] = 2i\eta^{\mu\nu} D + 2iM^{\mu\nu}, \quad [K^\mu, K^\nu] = 0, \\
&[M^{\rho\sigma}, K^\mu] = i\eta^{\mu\rho} K^\sigma - i\eta^{\mu\sigma} K^\rho, \quad [M^{\rho\sigma}, D] = 0.
\end{aligned} \tag{2.22}$$

What is still missing are the commutation relations of the Poincare group:

$$\begin{aligned}
&i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu} \\
&i[P^\mu, M^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \\
&[P^\mu, P^\rho] = 0
\end{aligned} \tag{2.23}$$

An infinitesimal group element is of the form [18]

$$U(1+\omega,\epsilon,\lambda,\beta) = 1 + \frac{i}{2}M^{\mu\nu}\omega_{\mu\nu} + iP^{\mu}\epsilon_{\mu} + i\lambda D + iK^{\mu}\beta_{\mu} \tag{2.24}$$

and induces an infinitesimal space-time transformation [18]

$$x^{\mu} \to x^{\mu} + \omega^{\mu\nu}x_{\nu} + \epsilon^{\mu} + \lambda x^{\mu} + \beta^{\mu}x^{\nu}x_{\nu} - 2x^{\mu}\beta^{\nu}x_{\nu} \tag{2.25}$$

The 4 simple cases which can occur using these transformations are:

- Only $\epsilon \neq 0$: $x^{\mu} \to x^{\mu} + \epsilon^{\mu}$, translational transformation
- Only $\omega \neq 0$: $x^{\mu} \to x^{\mu} + \omega^{\mu\nu}x_{\nu}$, rotational transformation
- Only $\lambda \neq 0$: $x^{\mu} \to x^{\mu}(1+\lambda)$, dilation
- Only $\beta \neq 0$: $x^{\mu} \to x^{\mu} + \beta^{\mu}x^{\nu}x_{\nu} - 2x^{\mu}\beta^{\nu}x_{\nu}$, special conformal transformation

The dilation is a rescaling of the whole system, meaning that if one changes the scale the theory would still look the same. Fractal structures like the *Koch snowflake* or the *Mandelbrot set* for example can be said to be scale invariant, although only up to a discrete rescaling a [55].

For the special conformal transformation there is no pictorial example. The whole point of the special spacial mapping is to preserve angles. Therefore everything except angles got changed by this mapping.

Having all ingredients of the conformal symmetry, one is already able to understand, that the observables of such a conformal theory themselves are very restrained. Therefore they need to be massless and the solutions of the correlation functions need to be described by pure powerlaws [48].

Note that conformal mapping and transformations are not only used for $\mathcal{N}=4$ SYM, but for various theories [53, 54]. One example concerning electrodynamics can be found at [56].

2.3 Maximal supersymmetry

The question one can ask is, if there is a limit on how much supersymmetry generators a theory can bear. Indeed there is a limit of how much SUSY a theory can bear which is determined by the spin [19].

Non-extended SUSY, or $\mathcal{N}=1$ gets generated by just one supersymmetric charge Q. This one generator *creates* a state which has a different statistics, fermion for a boson and vice versa. The Coleman Mandula no-go theorem [45] holds already for one SUSY charge Q. What happens if one considers not only one but N of those charges Q?

With 2 charges Q^i, Q^j and use of Eq.(2.6) one can already find [43]

$$\{Q^i_{\alpha}, Q^{\dagger j}_{\beta}\} = 2\sigma^m_{\alpha\beta}P_m\delta_{ij}, \tag{2.26}$$

where $\sigma^m = (1,\vec{\sigma})^m$ and $\overline{\sigma}^m = (1,-\vec{\sigma})^m$ are the Pauli matrices [43], α, β are spinor indices. This equation states that there is a new superparnter, due to the Kronecker delta, it is impossible to get back to the starting state [19]. So introducing just two Q's already yields that there must be at least 3 particles. Since the spin/helicity has to differ, taking the same example as in Eq.(2.11)

$$|photon\rangle \xrightarrow{Q_i} |photino_1\rangle \xrightarrow{Q_j} |particle, \lambda = 1\rangle. \tag{2.27}$$

Using Q further will then also yield a second $|photino_2\rangle$. So for $\mathcal{N} = 2$ the massless particle content is already at four, one photon, two photions and one spin/helicity 1 particle. Adding more and more SUSY to the theory will therefore result in a higher particle content. That said one important observation can be made:

Any multiplet of $\mathcal{N}$-extended SUSY will contain particles with spin/helicity at least large as $\frac{1}{4}\mathcal{N}$ [19].

The title of this work is $\mathcal{N} = 4$ SYM which means it is an $\mathcal{N} = 4$ extended supersymmetric theory. Now following the argument that the spin or helicity, depends on whether one is interested in massive or massless states, is at least as large as $\frac{1}{4}\mathcal{N}$ which means that for a $\mathcal{N} = 4$ SUSY theory the spin/helicity is at least 1. Luckily the force particle of the strong interaction, the gluon, also has spin 1, so one can think of a gauge multiplet with 1 gluon or gauge particle [19].

In 4 dimensional flat space time, due to renormalizabilty the spin/helicity of a given supermultiplet should not surpass 1 [57]. The game is sustainably different when allowing higher $\mathcal{N}$, one is then able to merge gravity with quantum field theory. For example $\mathcal{N} = 8$ super gravity (SUGRA) would be a quantum theory of gravity which is also the low energy theory of string theory [58], see section Maldacenas conjecture: the AdS/CFT correspondence. Also the maximal spin/helicity for $\mathcal{N} = 8$ is 2, which would then be carried by the graviton [18]. Note that usually the dimensionality of SUGRA calculations surpass 4 dimensions.

Briefly summarize:

- $\mathcal{N} = 4$ (SYM) for flat euclidian 4-dimensional space-time
- $\mathcal{N} = 8$ (SUGRA) for curved (higher-dimensional) space-times

Note that gravity is pretty easy to implement in the framework of SUSY, since there is already a commutation relation which connects the Q's with P^μ, therefore if $P^\mu(x)$ is a local generator gravity is automatically introduced [18].

As a subsection here the famous Wess-Zumino multiplet which in the first place introduced supersymmetry should be mentioned [59], as well as some aspects on the gauge multiplet which is the multiplet of $\mathcal{N} = 4$ SYM.

2.3.1 Wess-Zumino and gauge multiplets

The simplest multiplet that has SUSY, with only 1 supercharge Q, is the Wess-Zumino multiplet [59]. This multiplet contains 2 helicity 0 scalars and 1 fermion for helicity $\pm\frac{1}{2}$.

In fact there is no way to create a smaller supermultiplet since CPT should be conserved, for further details on CPT symmetry I refer to [60]. Note that this CPT symmetry is very general feature of every well behaved QFT. Due to that introducing a helicity $\lambda = 1/2$ and $\lambda = 0$ state, a fermion and a boson, will require the spectrum of a Lorentz-covariant field theory to contain their CPT-conjugate multipet [19]. As a general condition for massless supermultiplets one findes

While λ_0 is defined to be the maximal negative helicity [42]

$$\lambda_0 = -(\text{max. helicity}). \tag{2.28}$$

Helicity:	λ_0	$\lambda_0 + \frac{1}{2}$	$\lambda_0 + 1$	...	$\lambda_0 + \frac{n}{2}$
Number of states:	$\binom{N}{0} = 1$	$\binom{N}{1} = N$	$\binom{N}{2}$	...	$\binom{N}{N} = 1$

For the Wess-Zumino case this table reads as follows

Helicity:	$-\frac{1}{2}$	0	$\frac{1}{2}$
Number of states:	1	2	1

As for the gauge multiplet [18] of 4 times extended SUSY, or $\mathcal{N} = 4$, due to the extended SUSY the particle content needs to be higher. Starting form the gauge state with helicity 1 will produce 4 fermions with helicity $\frac{1}{2}$, using Q again will then yield 6 scalars with helicity 0, which can be visualized by the table

helicity:	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
number of states:	1	4	6	4	1

Since CPT needs to be preserved [1] as well as SUSY being extended to its maximum in 4-dimensional flat space with spin ≤ 1, the particle content is significantly higher than in Yang-Mills where only the gluons are considered. Nevertheless this gauge multiplet will be the starting point of the analysis of the Dyson-Schwinger equations for $\mathcal{N} = 4$ SYM. Note that because of this high particle content it is rather cumbersome to calculate the DSEs and the $SU(4)$ R-symmetry [18] will play a central role in their derivation.

This chapter already introduced conformal symmetry and it was mentioned that this symmetry is connected with $\mathcal{N} = 4$ SYM. Therefore a short calculation on how the theory becomes finite will be presented, this finiteness then requires the theory to be conformal.

2.3.2 Finiteness of $\mathcal{N} = 4$ SYM

One outstanding feature of $\mathcal{N} = 4$ SYM is that the theory is finite (at least in Feynman-Wess-Zumino gauge [19]), due to the fact that all *quantum anomalies* cancel each other precisely. This behaviour is closely connected to the fact that the dilation operator do not get any quantum corrections [48]. Therefore a natural starting point is to calculate the β function Eq.(2.21). Following the line of arguments from [18], the β function can be rewritten in terms of Casimir operators [50]

$$\beta(g) = -\frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right) + O(g^5). \tag{2.29}$$

This would hold for gluons and gluinos, the gluons superpartners. Due to the presence of scalars in $\mathcal{N} = 4$ SYM, see table of the gauge multiplet or Chapter 4 the $\mathcal{N} = 4$ SYM action, this equation needs to be modified

$$\beta(g) = \frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{6}C_2^f - \frac{1}{12}C_2^s\right), \tag{2.30}$$

to all orders in perturbation theory [18], the Casimirs C_2^f and C_2^s are defined by

$$C_{abc}C_{abd} = g^2 C_1 \delta_{cd} \tag{2.31}$$

$$Tr[t_c t_d]_{Majoranas} = g^2 C_2^f \delta_{cd} \tag{2.32}$$

$$Tr[t_c t_d]_{complex\ scalars} = g^2 C_2^s \delta_{cd}. \tag{2.33}$$

For theories with $\mathcal{N} = 2$ SUSY there are generally two Majorana fermions λ_a, ψ_a in the adjoint representation and H pairs of Majorana fermions ψ'_n, ψ''_n. The left- and right-handed parts of ψ'_n and ψ''_n are in a representation generated by t'_a or $-t_a^{T\prime}$. Therefore the fermionic Casimir is given by

$$C_2^f = 2C_1 + 2HC'_2, \tag{2.34}$$

with

$$Tr[t'_c, t'_d] = g^2 C'_2 \delta_{cd}. \tag{2.35}$$

The same argument which has been applied to fermions also holds for scalars. We get

$$C_2^s = C_1 + 2HC'_2. \tag{2.36}$$

Now it is possible to plug in the terms for C_2^f and C_2^s, respectively Eq.(2.32) and Eq(2.33), which finally yields

$$\beta(g) = -\frac{g^2}{8\pi^2}(C_1 - HC'_2). \tag{2.37}$$

What makes $\mathcal{N} = 4$ SYM special is that there is just one pair of each fermion and complex scalar of $\mathcal{N} = 2$ supermultiples in the adjoint representation, meaning $H = 1$. Therefore, $C'_2 = C_1$ and thus

$$\beta(g) = -\frac{g^2}{8\pi^2}(C_1 - C_1) = 0, \tag{2.38}$$

therefore the coupling g is energy independent [18, 46]. Within a specific gauge, the Feynman-Wess-Zumino gauge, one can show that this is related to the cancellations of all loop diagrams [18]. Without loop diagrams a theory is finite, because there is nothing to renormalize. This can even be pushed beyond perturbation theory which was shown in [61].

2.4 Maldacena's conjecture: the AdS/CFT correspondence

As a closure of this chapter the famous AdS/CFT correspondence should be mentioned. Again the conformal symmetry makes it possible to connect $\mathcal{N} = 4$ SYM in the large N limit with gravity. Stating that observables could be calculated in both the conformal QFT or general relativity.

Almost twenty years ago Juan Maldacena conjectured that a conformal field theory could *live* on the boundary, a so-called brane, of a curved space time, not any space-time but an anti-de Sitter space-time [49]:

$$\partial(AdS_5 \times S^5) = CFT, \tag{2.39}$$

where ∂ is the boundary operator [62].

AdS is a space-time with a negative curvature everywhere, i.e. a hyperparabolid.

This is realized by the holographic principle [63]: every information the space-time is carrying is somehow encoded in its boundary. As a rather intuitive picture one can think of a photo, usually a 2-dimensional picture of a given 3-dimensional space, or a hologram after which the principle is named.

In this pictorial presentation the field theory framework can be seen as the photo, while the string theory framework would be the world in which the photo was taken. The informations are observables of the theory. This *duality* is not as well defined as one wants it to be: it should be possible to calculate, in a given limit [49], observables in *superstring* theory [58] as well as in a conformal field theory, referred to as *gauge-gravity* duality.

Many introductions on this field are available [64–68], due to the fact that this work is mainly related to $\mathcal{N} = 4$ SYM only a very short representation will be given of the main ingredients of AdS/CFT, namely: string theory, branes, AdS and gravity and as a conclusion so to say an example for this duality.

2.4.1 A short overview of string theory

To understand why the conjecture can be seen as a powerful tool it is necessary to at least have a little insight on both sides of duality. Since the aim of this work is to find valid solutions of $\mathcal{N} = 4$ SYM within DSE's, the CFT part should be satisfied. Therefore a very short review of what string theory is, will be given here. For further reading one can use e.g. [58, 62, 69].

We start with the most basic physical string theory, the bosonic string in flat d-dimensional spacetime give by the Polaykov action [58]

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X^\nu\eta_{\mu\nu}, \tag{2.40}$$

This is rather different from the Lagrangians which will be presented in Chapters 3 and 4, therefore some explanations:

- $X^\mu(\tau,\sigma)$ are fields which arise due to embedding of the 2-dimensional string worldsheet into spacetime, the σ's are the coordinates.
- $h^{\gamma\delta}$ is the worldsheet metric,
- α' is some constant and called slope parameter. It is related to the string tension T by $T = \frac{1}{2\pi\alpha'}$.

Since Eq(2.40) exhibits conformal invariance [58], it is possible to rewrite it using conformal gauge, i.e. $h_{\gamma\delta} = \eta_{\gamma\delta}$, to

$$S_C = -\tfrac{1}{4\pi\alpha'}\textstyle\int d^2\sigma(\dot{X}^2 - X'^2), \quad \text{with} \tag{2.41}$$
$$\dot{X}^2 = \partial_\tau X^\mu\partial_\tau X^\nu \quad \text{and} \quad X'^2 = \partial_\sigma X^\mu\partial_\sigma X^\nu.$$

From this, one is able to derive the equations of motion for X^μ, which are 2-dimensional wave functions [58]. There are still constraints associated with the gauge symmetry the so-called *Virasoro constraints* [70]. Taking these into account the equations of motion for the worldsheet metric are

$$\frac{\delta S_P}{\delta h^{\gamma\delta}} \propto T_{\gamma\delta} = 0, \tag{2.42}$$

i.e. the energy-momentum tensor of a given worldsheet vanishes. This leaves a set of equations given by

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \tag{2.43}$$
$$(X' \pm \dot{X})^2 = 0. \tag{2.44}$$

In addition one has to deal with the variation of Eq(2.41) at the boundary:

$$\delta S_{C|_{\text{boundary}}} \propto \int_{-\infty}^{\infty} d\tau (X'\delta X|_{\sigma=\pi} - X'\delta X|_{\sigma=0}), \tag{2.45}$$

which in the end has to vanish. To ensure these two scenarios one can take into account

- **Open string**: Distinguished endpoints of the string ($\sigma = 0, \pi$), can be realized by two possibilities:
$$X'^{\mu}(\tau, \sigma') = 0, \quad X^{\mu}(\tau, \sigma') = x^{\mu}_{\sigma'}, \tag{2.46}$$
for some constant or fixed $x^{\mu}_{\sigma'}$ at an string endpoint σ'. Leaving the boundary of a string fixed at x^{μ} leads naturally the concept of D-branes [58], which are objects on which open strings end. The D refers to the Dirichlet-boundary condition, while in case $X'^{\mu}(\tau, \sigma') = 0$ the Neumann-boundary condition is fulfilled.

- **Closed string**: For a closed string both endpoints can be identified with each other. Without specifying whether they are periodic or antiperiodic we have
$$X^{\mu}(\tau, 0) = \pm X^{\mu}(\tau, \pi), \tag{2.47}$$
where the positive sign is for the periodic case.

Now one can expand X^{μ} in modes, see [70] and define commutation relations (at least for $x^{\mu}_{\sigma'} = 0$), which lead to a quantization of the theory:

$$[X^{\mu}(\tau, \sigma), X^{\nu}(\tau, \sigma')] = 0, \tag{2.48}$$
$$\left[X^{\mu}(\tau, \sigma), P^{\nu}(\tau, \sigma')\right] = \eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{2.49}$$

These equations would require 26 dimensions to be Lorentz invariant, one can overcome this by for example *Kaluza-Klein reduction* see [68]. Another problem is given by tachionic states [58], which cannot be interpreted as physical states. Also this is a bosonic string theory stating that there are no fermions.

Since a desirable theory consists of bosons and fermions, supersymmetry will so to say complete the bosonic string theory [69]. Therefore worldsheet fermions Ψ^{μ} are introduced which modifies the action given by Eq.(2.40):

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_{\gamma} X_{\mu} \partial^{\gamma} X^{\mu} + i\overline{\Psi}^{\mu} \rho^{\gamma} \partial_{\gamma} \Psi_{\mu}), \tag{2.50}$$

with

$$\{\rho^{\gamma}\rho^{\delta}\} = 2\eta^{\gamma\delta}, \quad \overline{\Psi} = i\Psi^{\dagger}\rho^{0}. \tag{2.51}$$

Using this complete description will then lead again for the fermionic sector to wave equations, but in a different way since they can be split into two Weyl spinors [58]. Some steps are omitted here, but it can be shown that the tachionic states can be projected out and the theory consists of a physical chiral massless spectrum [58]. As a remarkable feature of this string theory it turns out that the low energy effective space time theory is supergravity (SUGRA) [49, 70].

But what does string theory have to do with gravity? The concept of branes got introduced as a hyperplane on which an open string ends. These objects play a central role for the conjecture since they are both connected to strings as well as to gravity.

2.4.2 D-branes

One fact already derived is, that D-branes naturally arise from open strings. They are hyperplanes upon which the endpoints of open strings are fixed. D refers to the fact that they obey Dirichlet boundary conditions, meaning the boundary of a string ends at a fixed value x^μ, setting this fixed value to 0 is also legit and would yield Neumann boundary conditions. Both boundary conditions are given by Eq.(2.46).

They also can be seen as a solution to SUGRA, since the AdS/CFT correspondence connects gauge theories with string theories (gravity). The advantage of this is that gauge theories *live* on the worldvolume of a brane [49]. Such a worldvolume is analogous to the worldsheet of a string, in the sense of a (d+1) *volume*. Starting from the gravitational point of view following [68].

Not going too far into details a gauge field A_μ can be seen as a p-form on a given manifold [62]. Then one is able to construct an action by

$$S \propto \int_M A_p, \tag{2.52}$$

where M refers to the dimensionality (rank) of the manifold which has to be equal to the rank of the gauge field, otherwise there would be no diffeomorphism invariance [68]. The field-strength tensor is then defined to be the exterior derivative of the gauge field

$$F_{p+1} = dA_p, \tag{2.53}$$

changing the Lorentz index μ to some arbitrary form-rank p. Note that the field-strength tensor is gauge invariant due to the nilpotency of the exterior derivative [62]. As a solution of the SUGRA field equations one can define a p-brane with a charge associated with the gauge field A_p. Depending on the p-forms the brane solutions are limited, for type IIB SUGRA [70] there are four. The interesting ones are the Dp-branes, which refer to a special sector of IIB SUGRA [58]. Of importance for AdS/CFT are only $D3$-branes [49].

Due to symmetry arguments each brane solution has to reduce the SUSY of a SUGRA by $\frac{1}{2}$, or break half of the SUSY generators [68]. An ansatz that satisfies the SUGRA field equations is given by [71]:

$$ds^2 = \frac{1}{\sqrt{H(\vec{y})}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(\vec{y})}d\vec{y}^2, \tag{2.54}$$

where x^μ are coordinates parallel to the brane, while y^μ are coordinates which are normal to the brane. Also $H(\vec{y})$ has to be a harmonic function of $\vec{y}$. In the limit $y \to \infty$, with $y = \sqrt{\vec{y}\vec{y}}$, the metric should be flat, which fixes $H(\vec{y})$ such that

$$H(\vec{y}) = 1 + \left(\frac{L}{y}\right)^{d-p-3}, \tag{2.55}$$

with some scale factor L. The interesting case is a stack of N coincident Dp-branes, for which one can derive this scale factor [71]

$$L^{d-p-3} = N g_S (4\pi)^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right) \alpha'^{\frac{d-p-3}{2}}. \tag{2.56}$$

The N is given by N units of 5-form flux generated by N branes. The interesting case is the case of $d = 10$, and as stated before $D3$-branes will be of special interest, therefore $p = 3$. In the limit $\alpha' \to 0$ one ends up with SUGRA starting from superstring theory. In the weak coupling limit $g_S \to 0$, which is a perturbative regime, Dp-brane solutions become localized defects ($y = 0$) in spacetime [68].

Why is this important at all?

The brane dynamics for a stack of N $D3$-branes is described by 4-dimensional $\mathcal{N} = 4$ SYM, with a $SU(N)$ gauge group [49]. So far the string picture has be introduced, now the general relativity picture will employed to introduce the AdS space.

2.4.3 Anti-de Sitter spacetime

The AdS is a solution of the Einstein equation [6] and therefore of the Einstein-Hilbert action [64], i.e.,

$$S_{EH} = \frac{1}{16\pi G_d} \int d^d x \sqrt{|g|}(R - \Lambda), \tag{2.57}$$

with euclidean signature in d-dimensions. The quantities are named as follow

- G_d is the gravitational constant in d-dimensions
- $g_{\mu\nu}$ is the metric
- $R_{\mu\nu}$ is the Ricci tensor [72]
- R is the Ricci scalar which can be obtained by contraction of the Ricci tensor [72]
- Λ is the so-called cosmological constant

Varying the Eq.(2.57) yields the equations of motion, i.e.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2}\Lambda g_{\mu\nu} \quad \text{with} \quad R = \frac{d}{2-d}\Lambda \quad \text{and} \quad R_{\mu\nu} = \frac{\Lambda}{2-d} g_{\mu\nu}. \tag{2.58}$$

The (anti) de Sitter spaces are Einstein spaces, meaning that the Ricci tensor is proportional to the metric tensor. Considering those spaces with maximal symmetry [64]

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}), \tag{2.59}$$

which have to be spheres, S^d, de Sitter dS_d, or anti-de Sitter AdS_d. The difference which arises from the word 'anti' is just the sign of the cosmological constant Λ. For AdS one ends up with $\Lambda < 0$. These maximally symmetric spaces have the feature of constant curvature scalars as Eq.(2.58) states. In the AdS case the curvature is negative everywhere. After embedding AdS into a higher dimensional space one can express the invariant line element as [71]

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2, \tag{2.60}$$

and define AdS_d as the set of solutions of

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = L^2, \tag{2.61}$$

with a radius L. The space we are interested in this work is AdS_5 in which one can find that the isometry group is given by O(2,4) [6], which can be referred as the conformal group in 4-dimensional Minkowski space. This allows to rewrite the invariant line element as

$$ds^2 = \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho}{X^2 + L^2}\right) dX^\mu dX^\nu. \tag{2.62}$$

Another ingredient for AdS/CFT is the fact that the boundary of AdS_5 is a conformal one [73]. Not going into details one can show [71]

$$\partial(AdS_d) = \mathbb{R}^{1,D-2}, \tag{2.63}$$

which also refers to the holographic nature. The feature that the boundary is just the flat space makes calculation feasible.

2.4.4 Gauge theory and gravity

Up to now in this section, many aspects of the AdS/CFT correspondence have been mentioned, but how can one relate a gauge theory to gravity? It is clear that the boundary of an AdS_5 is the flat Euclidean space on which the gauge theory is defined. On the other hand branes can be related to classical gravity and gauge theories live on the worldsheet of those branes. Let us make a short statement:

A 4-dimensional $\mathcal{N} = 4$ SYM gauge theory with gauge group $SU(N)$ and a (single) coupling constant g_{YM} is dual/equivalent to a type IIB string theory on $AdS_5 \times S^5$, both AdS_5, S^5 having radius L, with a 5-form flux N and g_S as string coupling [49].

As mentioned before these theories can be related by

$$g_S = g^2_{YM}, \quad L^4 = 4g_S N\alpha'^2. \tag{2.64}$$

This is the *strong* statement and has yet not been proven in general. It is extremely difficult to show that this holds due to the fact that quantization of strings on curved manifolds is not yet well understood. Therefore, a *weak* form of this conjecture exists, which in fact can be related to numerical results [74, 75]. The *weak* or 't Hooft form can be obtained by [49]

$$g_S = \frac{\lambda}{N}, \quad \lambda = g^2_{YM}N = fixed, \tag{2.65}$$

but holds only for $N \to \infty$ and $g_{YM} \to 0$.

As a first check it can be shown that the global symmetries of each side of the correspondence match. Meaning that the global symmetry for $\mathcal{N} = 4$ SYM is given by the conformal group $SU(2, 2|4)$ Eqs.(2.22-25), while the isometry group of $AdS_5 \times S^5$ is also given by $SU(2, 2|4)$.

Up to now there are many examples stating that the *weak* form is valid [74, 75]. Still the mathematical prof remains, and of course for the *weak* form one has to accept the large N limit as well as the weak coupling limit.

After introducing the concept of SUSY as well as the restrains conformal symmetry generates we want to start with the Dyson-Schwinger equations from the already known Yang-Mills case [27], followed by $\mathcal{N} = 4$ SYM [76]. The solutions of the derived DSEs will then be connected to the conformal symmetry and therefore restricted to be pure powerlaws [48].

Chapter 3

Green functions and Dyson-Schwinger equations

Historically Green's functions were derived to solve differential equations [77]. In fact they are a far more powerful tool than Green would have expected. Today Green's functions play a central role in QFT. This is because if one is able to calculate all Green functions one knows everything about a theory either classical or quantum. By calculating the n-point Green's function of a QFT the results represent the vacuum expectation values (VEV) of its n-point correlation function [1]. This is important, because knowing how fields are correlated in the vacuum is knowing everything about the theory [21]. Therefore, we desire a way to calculate these Green's functions. For this, the path-integral method will be employed [3]. The path-integral allows to introduce a generating functional $\mathcal{Z}$. With this generating functional various Green functions can be classified. Derivatives of $\mathcal{Z}$ can be connected with n-point correlation functions of arbitrary fields [21]. As a starting point we introduce what the generating functional is.

Note that it is also possible to derive DSEs from canonical quantization [78], obviously this is less efficient than the path-integral formalism.

3.1 Generating functional

An introduction to the path-integral formalism can be found in [1, 3, 21, 79], where the derivation of n-piont Green's functions from a generating functional is given.

The action of a QFT with fermionic and bosonic degrees of freedom can be represented by its Lagrangian $\mathcal{L}$:

$$\mathcal{S}[\phi^i, \psi^i, \overline{\psi}^i] = \int dx \mathcal{L}[\phi^i(x), ...], \tag{3.1}$$

with ϕ^i are the bosonic fields and the ψ^i and $\overline{\psi}^i$ are the fermionic and antifermionic fields, see Chapter 2 for bosons or fermions or [3]. The index i is some multi-index of several quantum numbers. Since quantum physics is build on statistics a natural step is to compare it to thermodynamics [1]. Therefore, analogous to the thermodynamic partition sum a generating functional $\mathcal{Z}$ can be introduced:

$$\mathcal{Z}[j,\xi,\overline{\xi}] = \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{\int[-\mathcal{L}+\phi^i(x)j^i(x)+\overline{\xi}^i(x)\psi^i(x)+\overline{\psi}^i(x)\xi^i(x)]d^dx} \quad , \tag{3.2}$$

from which all full n-point functions can be obtained. This functional $\mathcal{Z}$ only depends on external sources j^i, ξ^i and $\overline{\xi}^i$ [1, 3].

Implicitly it is assumed throughout this thesis that the measures $\mathcal{D}\{\phi\psi\overline{\psi}\}$ are well-defined. In general this assumption only holds for lattice field theory [8].

To derive a n-point Green's function one has to take the n-th derivative of $\mathcal{Z}$. For example is the three gluon vertex given by the derivative of $\mathcal{Z}$ with respect to j^a_μ, j^b_ν and j^b_ρ, which is a 3-point correlation function. But the full Green's function includes all Feynman diagrams [3] one can think of. Not only 1PI- but also disconnected- as well reducible one. The 1PIs are the one particle irreducible Green's function which correspond in a sense to the connected diagrams [1] and these 1PIs are the quantities of interest.

In principle these correlation functions depend on the non-zero sources. Since these sources are simply a tool to introduce the path-integral we have to set them to zero. This is done after differentiating and usage of the normalized generating functional, i.e.

$$\mathcal{Z}[0] = 1. \tag{3.3}$$

The first three full Green's functions are then given by a product of connected diagrams denoted with a c and vacuum bubbles $\mathcal{Z}[j_i j_j j_k]$, e.g.

$$\langle\phi_i\rangle = \langle\phi_i\rangle_c\mathcal{Z}[j_ij_jj_k] \ , \tag{3.4}$$

$$\langle\phi_i\phi_j\rangle = \langle\phi_i\phi_j\rangle_c\mathcal{Z}[j_ij_jj_k] + \langle\phi_i\rangle_c\langle\phi_j\rangle_c\mathcal{Z}[j_ij_jj_k], \tag{3.5}$$

$$\begin{aligned}\langle\phi_i\phi_j\phi_k\rangle = \langle\phi_i\phi_j\phi_k\rangle_c\mathcal{Z}[j_ij_jj_k] + \langle\phi_i\phi_j\rangle_c\langle\phi_k\rangle_c\mathcal{Z}[j_ij_jj_k] + \text{cyclic permutations}\\ +\langle\phi_i\rangle_c\langle\phi_j\rangle_c\langle\phi_k\rangle_c\mathcal{Z}[j_ij_jj_k].\end{aligned} \tag{3.6}$$

Where j_i, j_j and j_k are the sources for the fields ϕ_i, ϕ_j and ϕ_k respectively. Note the ϕ's and j's in this example can either be fermionic or bosonic. These connected diagrams can be defined using the generating functional

$$W[j_ij_jj_k] = ln\ \mathcal{Z}[j_ij_jj_k], \tag{3.7}$$

with W also referred as the free energy [3], analogues to thermodynamics.

The generating functional of 1PI Green's functions is defined as the Legendre transform of W with respect to all fields [38]. It is useful to introduce all fields which are used in Chapter 3,4 and 5 hence we get

$$\Gamma[A,\phi,\phi^*,\psi,\overline{\psi},c,\overline{c}] = -W[j,k,k^*,\overline{\xi},\xi,\overline{\eta},\eta] + \int d^dx\Big(A_\mu j_\mu + \phi^i k^i + \phi^{i*}k^{i*} + \overline{\psi}^i\xi^i + \overline{\xi}^i\psi^i + \overline{c}\eta + \overline{\eta}c\Big), \tag{3.8}$$

with a gluon field A and its source j, a complex scalar ϕ and its source k, a fermion field ψ with its corresponding antifermion $\overline{\psi}$ and their source $\overline{\xi}$ and ξ and the ghost $\overline{c}, c$ [80] with sources $\eta, \overline{\eta}$. All these fields arise, except the ghost, due to $\mathcal{N} = 4$ SUSY's R-symmetry [18], see Chapter 2 gauge multiplet.

While the gluon as well as the scalar field are bosonic fields, the ghost as well as the fermion are fermionic fields. The ghost field [1] arises due to gauge fixing this is discussed in the subsection: Gauge fixing and Gribov-Singer ambiguity. Γ is the so-called quantum effective action [1, 3] and the desired quantity for yielding the 1PI Green functions.

At this point one is able to use functional methods in the form of Dyson-Schwinger equations.

3.2 Dyson-Schwinger equations

The Dyson-Schwinger equations [78, 81] are the equations of motion of the Green's functions and therefore of the QFT [21]. By knowing them one in principle knows everything about the theory. In this section a formal presentation of how to derive these equations from the action of a theory will be given. The emerging need for truncation will be encountered [22]. Within this chapter we derive the DSEs for Yang-Mills theory. In Chapter 5 we show how one solves the propagator DSEs for $\mathcal{N}=4$ SYM self-consistently.

The derivation of DSE's can be found in the textbook [21] or in the report [22]. For the sake of simplicity we use the functional integral approach. The Euclidian version of the DSEs can be derived from

$$\left(\left(-\frac{\delta S}{\delta\phi^a(x)}\,|_{\phi=\frac{\delta}{\delta j^a(x)}}+j^a(x)\right)\mathcal{Z}[j^a(x)]\right)\Bigg|_{j=0}=0, \tag{3.9}$$

with one bosonic field ϕ and its source j. However, ϕ is a bosonic field this equation also holds for fermionic fields. $\mathcal{Z}$ is the generating functional see Eq.(3.2) and S is some action. From here on all calculations will be within Euclidian space-time, see App. Conventions.

We defined Γ in Eq.(3.8), this quantity generates the 1PI Green's functions. Instead of $\mathcal{Z}$ which would produce all diagrams we are only interested in the 1PIs. Therefore, one can rewrite the field and the source in Eq.(3.9) using W,Γ [1, 82], i.e.

$$\phi^a=\frac{\delta W}{\delta j^a}, \tag{3.10}$$

$$j^a=\frac{\delta\Gamma}{\delta\phi^a}. \tag{3.11}$$

This works just for bosonic fields. If one is interested in fermionic fields, two independent Grassmann valued sources are necessary, hence we get

$$\sigma^a = \frac{\delta W}{\delta\overline{c}^a} \quad \overline{\sigma}^a=\frac{\delta W}{\delta c^a}, \tag{3.12}$$

$$c^a = \frac{\delta\Gamma}{\delta\overline{\eta}} \quad \overline{c}^a=\frac{\delta\Gamma}{\delta\eta}, \tag{3.13}$$

where all derivatives with respect to Grassmann variables act to the right. This way sources fields are linked by Eqs.(3.10-13). Also their derivatives can be identified with the propagators, i.e.

$$\frac{\delta}{\delta\phi^a}=\frac{\delta^2\Gamma}{\delta\phi^a\phi^b}\frac{\delta}{\delta j^b}, \quad \frac{\delta}{\delta j^a}=\frac{\delta^2 W}{\delta j^a j^b}\frac{\delta}{\delta\phi^b}, \tag{3.14}$$

where the second derivative of Γ yields the inverse propagator, while the second derivative of W yields the propagator. How to derive the actual propagator DSE is discussed in great detail in the next section. We also comment on the fact, that a non-perturbative gauge-fixing, here Landau gauge is the choice, generates the Gribov-Singer ambiguity [38].

3.3 DESs of Yang-Mills theory

With Eq.(3.9) and an action one is able to derive arbitrary DSEs. Since $\mathcal{N} = 4$ is related to Yang-Mills theory we want to start with the seemingly simpler one, the Yang-Mills theory. As we show in Chapter 4 the DSEs only related to pure Yang-Mills theory, i.e. $F_{\mu\nu}F_{\mu\nu}$, are exactly the same in $\mathcal{N} = 4$ SYM. Due to the fact that this field is studied for centuries the DSEs for pure Yang Mills theory can be found, for example at [17, 22, 27, 83, 84]. The theory describes the gluo-dynamics generated by the gluon field A^a_μ, which is a part of QCD [22]. The derivation will closely follow [84]. The action is given by

$$\mathcal{S} = \int d^dx \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \tag{3.15}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu, \tag{3.16}$$

with f^{abc} is the structure constant of the gauge group and g is the coupling constant.

Due to the arguments of the following subsection the action needs to be gauge fixed. This will require to introduce a ghost term for Eq.(3.16). Landau gauge is the gauge we use which is defined by

$$\partial_\mu A_\mu = 0. \tag{3.17}$$

Landau gauge is used throughout this thesis, to compare the results we derive to Yang-Mills results of the same truncation [22, 38, 85–88], since it is not possible to use a supersymmetric preserving gauge condition such as Feynman-Wess-Zumino gauge for Yang-Mills theory.

The gauge fixed action is then

$$\mathcal{S} = \int d^dx \Big(\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{c}^a \partial_\mu D^{ab}_\mu c^b \Big) \tag{3.18}$$

$$D^{ab}_\mu = \delta^{ab} \partial_\mu + g f^{abc} A^c_\mu, \tag{3.19}$$

where D^{ab}_μ is the so-called covariant derivative.

To derive the corresponding DSEs for the propagators, see Eq.(3.9), one needs to differentiate the action with respect to its fields. For example: The ghost propagator is defined to be the second functional derivative of Eq.(3.18) with respect to $\bar{c}$ and c.

For simplicity reasons it is useful to split the action into the interaction parts of the desired field, i.e.

$$\mathcal{L}_{YM} = \mathcal{L}_{gluon} + \mathcal{L}_{ghost}. \tag{3.20}$$

First we want to show why a ghost term gets introduced. This is needed because one should only take different field configurations into account not those which are connected by gauge transformations [25, 38]. After gauge fixing we encounter another problem the so-called Gribov-Singer ambiguity [89, 90]. The impact of this ambiguity on the DSEs is a long standing problem [22, 25, 38].

3.3.1 Gauge-fixing and the Gribov-Singer ambiguity

To derive the DSE we want to use the path-integral. A closer look at Eq.(3.2) reveals that this yields two problems

- The integral yields an infinite constant, the volume of the gauge group [3] and,

- Taking Eq.(3.15) one can show that the gluon 2-point function has zero eigenvalues [25].

The first point arises due to the fact that the integral is over all gauge configurations. Therefore, also those which are connected by gauge transformations, thus which describe the same physics [1]. The integral overcount so to say the gauge part. The second point states that the gluon 2-point function is singular and cannot be inverted, which makes it impossible to define a propagator [25].

Gauge-fixing

Fixing the gauge is a tool to handle both problems. Therefore all gauge fields which describe the same phyiscs, or in other words are related by gauge transformations are grouped into an equivalence classes $[A_\mu^U]$. A^U are the transformations of the gauge field under which Eq.(3.15) is invariant and given by [22, 25]

$$A_\mu^U(x) = U(x)A_\mu U(x)^{-1} + \frac{i}{g}\big(\partial_\mu U(x)\big)U(x)^{-1}, \tag{3.21}$$

with the transformation matrix $U(x)$ is

$$U(x) = e^{ig\omega(x)}, \tag{3.22}$$

where $\omega(x)$ is a Lie algebra valued gauge parameter [25, 38].

In the following we want to refer to this equivalence classes as gauge orbit defined by

$$O[A] = \{A'_\mu | A'_\mu = A_\mu^U\}. \tag{3.23}$$

For the path integral we want that only one representative per gauge orbit is taken into account. Faddeev and Popov were the first who thought of a way to solve this issue. Their idea was to restrict the integration to a hyperplane [80]. Therefore, they introduced a unity given by

$$1 = \Delta[A]\int DU\delta(f[A^U]), \tag{3.24}$$

into Eq.(3.2). Here the integral is performed over a group, hence the measure DU is the so-called Haar measure [8], while the delta functional defines the hyperplane $f(A^U) = 0$ [25]. The $\Delta[A]$ term is the Jacobian which is given by

$$\Delta[A] = det\left(\frac{\delta f[A(x)]}{\delta\omega(y)}\right), \tag{3.25}$$

which is also defined to be the Faddeev-Popov determinant such that

$$det\left(\frac{\delta f[A(x)]}{\delta\omega(y)}\right) = detM(x,y). \tag{3.26}$$

Note that the integration over U in Eq.(3.24) can be absorbed into $\mathcal{Z}[0]$ [1, 25]. Therefore, one is able to rewrite the measure of Eq.(3.2) such that $\mathcal{Z}$ is given by

$$\mathcal{Z}[j,\xi,\bar{\xi}] = \int D\{A\overline{\psi}\psi\}\Delta[A]\delta(f[A^U])e^{-\int d^4x\mathcal{L}+\int d^4x\left(A_\mu^a j_\mu^a+\overline{\eta}\psi+\overline{\psi}\eta\right)}. \tag{3.27}$$

For linear covariant gauges, like Landau gauge the $\Delta[A]$ term can be written in exponential form, i.e.

$$\Delta[A] = det[-\partial_\mu D^{ab}_\mu] = \int D\{c\bar{c}\} e^{-\int d^4x \bar{c}^a \partial_\mu D^{ab}_\mu c^b}, \tag{3.28}$$

where D^{ab}_μ is the derivative given by Eq.(3.19). These ghosts fields are spin zero particles, which means they have to be scalars [1]. In contrast to that they obey Fermi statistics. Therefore, these ghost fields are auxiliary and not physical fields. Nevertheless, the ghosts ensure the unitarity of the S-Matrix by canceling unphysical longitudinal polarized gluon states [3].

The right approach instead of the restriction to a hyperplane is to relax the condition to a Gaussian distribution over $O[A^U]$. The mean value of the Gaussian is defined to be the original gauge fixing condition, the Landau gauge fixing condition $\partial_\mu A_\mu = 0$. With the Gaussian we introduce a gauge fixing parameter α which corresponds to the width of the Gaussian around the Landau gauge condition. In the limit $\alpha \to 0$ the original Landau gauge condition is recovered. Since the Gaussian is normalized the integral is well defined, in difference to the unnormalized integration [25]. This gauge fixing conditions have to be added to the Lagrangian, therefore we get

$$\mathcal{L}_{ghost} = \frac{\left(\partial_\mu A^a_\mu\right)^2}{2\alpha} + \bar{c}^a \partial_\mu D^{ab}_\mu c^b. \tag{3.29}$$

In Eq.(3.20) it is already stated that the full Lagrangian is given by the gluonic part as well as the gauge fixing, or ghost part. As one can see due to gauge fixing the ghost part in the action gets generated. Note only after the gluon 2-point function has been inverted one can set $\alpha = 0$ [25]. This is not a problem because

$$e^{-\frac{\left(\partial_\mu A^a_\mu\right)^2}{2\alpha}} \to \delta \quad \text{in the limit} \quad \alpha \to 0. \tag{3.30}$$

this factor of the path-integral simply reduces to a delta functional.

The Gribov-Singer ambiguity

Due to gauge fixing one thinks that the path-integral is well defined. However Gribov and Singer showed already in the 70's that this is not the case [89, 90]. They found out that the Faddeev-Popov determinate is not sufficient to single out only one representative per gauge orbit [22, 25, 38]. This can be shown by performing a gauge transformation analogues to Eq.(3.21)

$$\partial_\mu A_\mu = 0 \to \partial_\mu A_\mu - \partial_\mu D^{ab}_\mu \omega = 0, \tag{3.31}$$

with the Faddeev-Popov operator $M = -\partial_\mu D^{ab}_\mu$. To fulfill this equation one require

$$M^{ab}_\mu \omega = 0. \tag{3.32}$$

However, if the Faddeev-Popov operator has zero modes then there are still gauge equivalent configurations left [38]. In fact, one can show that the operator has zero modes on the Gribov horizon [38].

To fix this, Gribov suggested that one can further restrict the path-integral to a region where the Faddeev-Popov operator is strictly positive [89]:

$$GR = \{A | \partial_\mu A_\mu = 0, M(A) > 0\}. \tag{3.33}$$

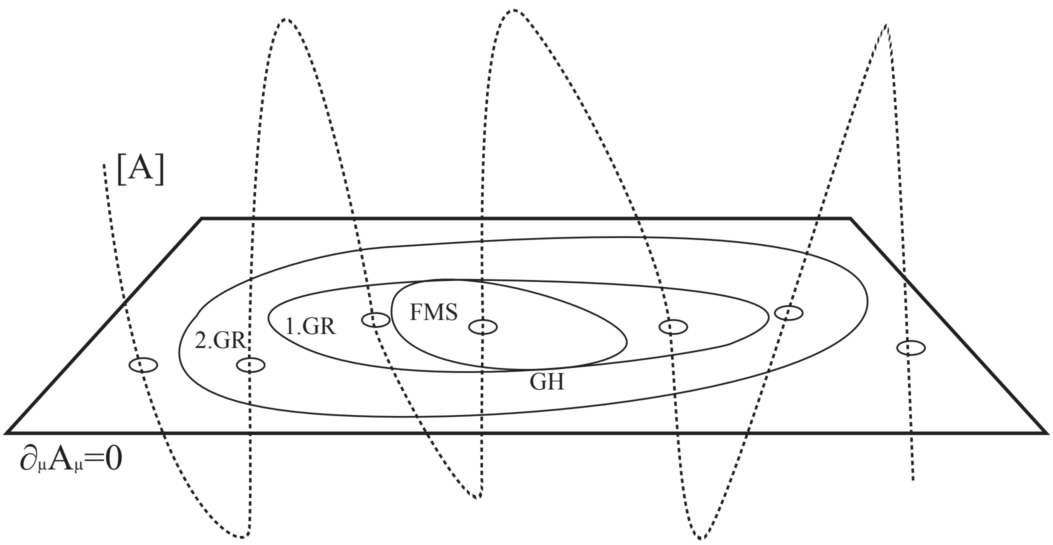


Figure 3.1: A sketch of the geometry of a gauge configuration space. $[A]$ is a gauge orbit defined in Eq.(3.33). The gauge conditions is $\partial_\mu A_\mu = 0$ and defines the hyperplane. With GH as the Gribov horizon and FMS as the fundamental modular region (FMR). The gauge orbit $[A]$ intersects the total hyperplane several times, however the FRM is intersected only once. Note that due to the intersection inside the first Gribov region Gribov copies arise.

Due to its positivity the operator remains invertible. Hence one can define the ghost propagator as the inverse of the Faddeev-Popov operator. This region is called Gribov region and denoted by GR in Fig.(3.1). It has the following properties (only Landau gauge is considered):

- The vacuum $A_\mu = 0$ lies within the first GR, this can be shown due to the fact that the Faddeev-Popov operator reduces to the Laplacian [91, 92].
- All gauge orbits pass at least once through the first Gribov region [93] therefore no physical information is lost.
- The first Gribov region is bounded [91].
- One can construct a new gauge configuration by combining two known and can show that this new gauge configuration lies within the Gribov region, therefore the region is convex [25, 91].
- However there are still gauge copies, the so-called Gribov copies within the first Gribov region [38, 94].

As one can see in Fig.(3.1) there are more than just the first Gribov region. This corresponds to the fact that the eigenvalues of the Faddeev-Popov operator are strictly positive inside the region but vanishing at the boundary, the so-called Gribov horizon [38]. The further Gribov regions, see for example Fig.(3.1) 2.GR correspond to the second Gribov region, each having an increasing number of negative eigenvalues of the Faddeev-Popov operator [38].

The ultimate goal is to restrict the path-integral to the fundamental modular region (FMR) where every gauge orbit passes once and only once through [25, 38]. This of course corresponds to the global minimum of a functional of the form

$$R[A] = \frac{1}{2} \int dx A_\mu(x) A_\mu(x), \tag{3.34}$$

which till today is still under research, as it is cumbersome to find the global minimum of such a functional.

The Gribov-Singer ambiguity has been discussed but what are the implications for the DSEs? Without going too far into details there are three implications:

- DESs for the whole field space as well as those restricted to a Gribov region look exactly the same [95, 96].
- The infrared behaviour of the ghost and gluon propagator (actually the dressing function, see Chapter 5) is influenced [97, 98] and,
- Due to the IR propagator behaviour confinement scenarios are closely connected, e.g. Kugo-Ojima [25, 38, 99]

For further reading on this topic one should take a look at ref.[38], which is state of the art concerning this ambiguity.

We now proceed calculating the propagator DSEs of Yang-Mills theory. In Chapter 5 we comment on the results we found and try to put them in context of this interesting ambiguity.

3.3.2 The ghost propagator DSE

The ghost sector is given by the action

$$\mathcal{S}_{gh} = \int d^d z \bar{c}^e(z) \partial_\mu D_\mu^{ef} c^f(z). \tag{3.35}$$

To derive the ghost propagator DSE we put the ghost action into Eq.(3.9) and differentiate it with respect to $\bar{c}^a(x)$, which yields

$$\left(-\partial^2 c^a(x) - g f^{afe} \partial_\mu A_\mu^e(x) c^f(x) + \frac{\delta \Gamma}{\delta \bar{c}^a(x)} \right) e^W = 0, \tag{3.36}$$

where the $\frac{\delta\Gamma}{\delta\bar{c}^a(x)} = \eta^a(x)$ and η is the source of $\bar{c}$ from the generating functional Eq.(3.2). After performing the derivative, we divide by e^W and get

$$-\partial^2 c^a(x) - g f^{afe} \partial_\mu A_\mu^e(x) c^f(x) + \frac{\delta \Gamma}{\delta \bar{c}^a(x)} = 0, \tag{3.37}$$

and replace the fields with derivatives of their sources, such that

$$\left(-\partial^2 c^a(x) - g f^{afe} \partial_\mu \Big(\frac{\delta W}{\delta j_\mu^e(x)} \frac{\delta W}{\delta \bar{\eta}^f(x)} + \frac{\delta^2 W}{\delta j_\mu^e(x) \delta \bar{\eta}^f(x)} \Big) \right) + \frac{\delta \Gamma}{\delta \bar{c}^a(x)} = 0. \tag{3.38}$$

At this point there is still a product of only a single derivative of W. Differentiating this product again with respect to a source yields

$$\frac{\delta}{\delta j(y)}\frac{\delta W}{\delta j^e_\mu(x)}\frac{\delta W}{\delta \overline{\eta}^f(x)} = \frac{\delta^2 W}{\delta j^e_\mu(x)\delta j(y)}\frac{\delta W}{\delta \overline{\eta}^f(x)} + \frac{\delta W}{\delta j^e_\mu(x)}\frac{\delta^2 W}{\delta \overline{\eta}^f(x)\delta j(y)}. \tag{3.39}$$

Always at least one term remains with a single derivative, which can be replaced by a classical field.

When setting the classical source to 0, one also sets the classical fields to 0 and due to that these terms will not contribute [84]. Therefore, these terms can already be neglected and will not appear in any further calculation [22]. This simplifies Eq.(3.38) further to

$$-\partial^2 c^a(x) - g f^{afe}\partial_\mu \frac{\delta^2 W}{\delta j^e_\mu(x)\delta\overline{\eta}^f(x)} + \frac{\delta \Gamma}{\delta \overline{c}^a(x)} = 0. \tag{3.40}$$

The next step is to differentiate with respect to $c^b(y)$, which yields

$$-\partial^2 \delta^{ab}\delta(x-y) - g f^{afe}\partial_\mu \frac{\delta}{\delta c^b(y)}\Big(\frac{\delta^2 W}{\delta j^e_\mu(x)\delta\overline{\eta}^f(x)}\Big) + \frac{\delta^2 \Gamma}{\delta c^b(y)\delta \overline{c}^a(x)} = 0. \tag{3.41}$$

The last term defines the inverse ghost propagator:

$$\frac{\delta^2 \Gamma}{\delta c^b(y)\delta \overline{c}^a(x)} = \big(D^{ab}_G(x-y)\big)^{-1}. \tag{3.42}$$

While W refers to the ghost propagator:

$$\frac{\delta^2 W}{\delta \eta^b(y)\delta \overline{\eta}^a(x)} = D^{ab}_G(x-y). \tag{3.43}$$

The fact that Γ is the inverse of W can be shown by

$$\int d^d z \frac{\delta^2 W}{\delta \eta^c(z)\delta \overline{\eta}^a(x)}\frac{\delta^2 \Gamma}{\delta c^b(y)\delta \overline{c}^c(z)} = \int d^d z \frac{\delta c^a(x)}{\delta \eta^c(z)}\frac{\delta \eta^c(z)}{\delta c^b(y)} = \frac{\delta c^a(x)}{\delta c^b(y)} = \delta^{ab}\delta(x-y). \tag{3.44}$$

The interaction part of the ghost propagator DSE is given by the mixed derivative of W, using

$$\frac{\delta^2 W}{\delta j^a_\mu(x)\delta\overline{\eta}^b(x)} = -\int d^d y d^d z \frac{\delta^2 W}{\delta j^h_\nu(y)\delta j^a_\mu(x)}\frac{\delta^2 \Gamma}{\delta \overline{c}^f(z)\delta A^h_\nu(y)}\frac{\delta^2 W}{\delta \eta^f(z)\delta\overline{\eta}^b(x)}, \tag{3.45}$$

where the anti-commuting derivatives produce a sign. Also note that $\delta^3 W$ terms vanish due the fact the that [84]

$$\frac{\delta^2 \Gamma}{\delta c^a(x)\delta A^b_\mu(y)}\ |_{j=\eta=\overline{\eta}=0} = 0, \tag{3.46}$$

which is true for all 2-point functions [22]. Applying Eq.(3.45) to Eq.(3.41) yields the result in position space, i.e.

$$\big(D^{ab}_G(x-y)\big)^{-1} = \partial^2\delta(x-y) + g f^{ade}\partial_\mu \int d^d z_1 d^d z_2 D^{ef}_{\mu\nu}(x-z_1) D^{dh}_G(x-z_2)\Gamma^{c\overline{c}A;bfh}_\nu(y,z_1,z_2). \tag{3.47}$$

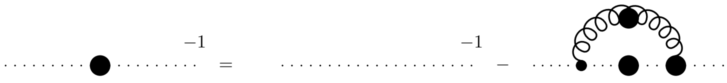


Figure 3.2: The diagrammatically interpretation of Eq.(3.47), showing how the ghost can interact by propagating form one point to another. For clarification the dotted lines are ghost propagators while the curled line denotes a gluon propagator. Full propagators are lines with a dot, while bare propagators lack a dot. The intersection points are the vertices, one bare which is denoted by smaller dot and one full vertex with a full dot.

Here we introduced the full gluon propagator $D^{ef}_{\mu\nu}$ and the full ghost-gluon vertex $\Gamma^{c\bar{c}A;bfh}_{\nu}(y, z_1, z_2)$, the later can be identified with

$$\frac{\delta^3\Gamma}{\delta c^b \delta\bar{c}^f \delta A^h_\nu} = \Gamma^{c\bar{c}A;bfh}_{\nu}, \tag{3.48}$$

this way of rewriting is more compact and holds the same information, spacial arguments are suppressed.

The word full in both, propagators and vertices refer to the fact that for every full quantity there is a DSE. As nobody is able to solve infinitely many DSEs it is useful to model full quantities [21]. The simplest approach is to model them with their respective bare quantities and some unknown functions, which are the so-called dressing functions. The bare, or tree-level quantities are known from the derivation of the propagator DSE. For example the bare inverse ghost propagator is just the first term in Eq.(3.47), namely

$$\left(D^{bare;ab}_G(x-y)\right)^{-1} = -\partial^2\delta(x-y). \tag{3.49}$$

Given Eq.(3.47) in momentum space yields [100]

$$\left(D^{ab}_G(p)\right)^{-1} = -\delta^{ab}p^2 - igf^{ade}\int\frac{d^dq}{(2\pi)^d}D^{ef}_{\mu\nu}(p-q)D^{dh}_G(q)\Gamma^{c\bar{c}A;bfh}_{\nu}(-p,q,p-q), \tag{3.50}$$

with the tree-level or bare vertex given by

$$\Gamma^{tl;c\bar{c}A;abc}_{\mu}(p,q,k) = igf^{abc}q_\mu\delta(p+q+k). \tag{3.51}$$

Note that momentum conservation is explicitly used at the vertices. This equation can also be represented diagrammatically, which is done in Fig.(3.1).

Different line styles correspond to different propagators. In this diagrammatic representation the intersection point of arbitrarily many lines correspond to a vertex. Fig.(3.1) has already two such intersection points, meaning that there are two vertices. One for the tree-level vertex which is given by the derivation of the DSE similar to the bare propagator, corresponding to the pre-factors of $\frac{\delta^2 W}{\delta j^a_\mu \delta c^b}$ in Eq.(3.41) and one full vertex, which is unknown as it requires in principle its own DSE. Since we are only interested in rainbow ladder truncation [22] we model the full ghost gluon vertex in Chapter 5. Therefore we come to the second propagator DSE of Yang-Mills theory.

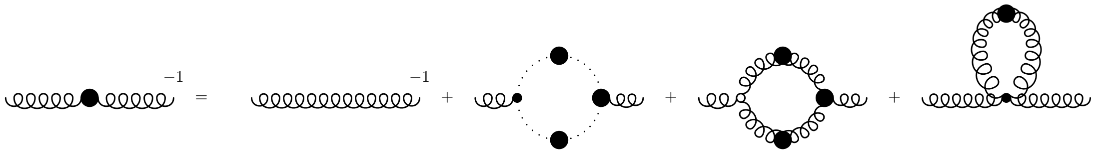


Figure 3.3: The propagator DSE for the Yang-Mills gluon with all interactions. This equation consists of a bare propagator, a ghost-loop, a gluon-loop and a gluontadpole. The dotted lines are ghost propagators, the curled lines are the gluon-propagators. All internal lines are full propagators.

3.3.3 The gluon propagator DSE

Like in the ghost case the same procedure is employed to generate a 2-point correlation function which corresponds to the full inverse propagator. Because of the bose nature of the gluon field the derivation will be simpler in the sense that no anti-commuting derivatives will be performed. The gluonic sector of the action is given by

$$\mathcal{S}_{Gl} = \frac{1}{4}\int d^dz\Big(F^c_{\rho\sigma}(z)F^c_{\rho\sigma}(z) + gf^{gfc}\,\bar{c}^g(z)\partial_\rho A^e_\rho c^f(z)\Big). \tag{3.52}$$

For the derivation of the DSE it is desirable to rewrite this equation into one where only A^a_μ's appear, i.e.

$$\begin{aligned}\mathcal{S}_{Gl} = \int d^dz\Big[&\frac{1}{2}\big(-A^c_\rho(z)\partial^2 A^c_\rho(z) + A^c_\sigma(z)\partial_\sigma\partial_\rho A^c_\rho(z)\big)\\ &- gf^{cfe}A^c_\rho(z)A^f_\sigma(z)\partial_\rho A^e_\sigma(z) + \frac{g^2}{4}f^{cgf}f^{ced}A^g_\rho(z)A^f_\sigma(z)A^e_\rho(z)A^d_\sigma(z)\\ &+ gf^{cfe}\,\bar{c}^c(z)\partial_\rho A^e_\rho(z)c^f(z)\Big],\end{aligned} \tag{3.53}$$

since we need to differentiate this equation with respect to the gluon-field A^a_μ. With this equation one is already able to draw all diagrams, represented in Fig.(3.3). In Eq.(3.53) there are two terms with a product of two gluons A, which correspond to the inverse bare propagator. Then there are also two 3-point couplings one consisting of three gluons A and one consisting of one gluon A and a ghost c and an antighost $\bar{c}$, see Fig.(3.4). Therefore, there is only a single term left which corresponds to the 4-point coupling of the gluon, the so-called gluon tadpole [1], see Fig(3.5).

To derive the DSEs we presume as in the ghost case. Therefore, the fist derivative is with respect to $A^a_\mu(x)$ and the second with respect to $A^b_\nu(y)$. The gluon propagator and its inverse are defined by

$$\big(D^{ab}_{\mu\nu}(x-y)\big)^{-1} = \frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta A^a_\mu(x)}, \tag{3.54}$$

$$D^{ab}_{\mu\nu}(x-y) = \frac{\delta^2 W}{\delta j^b_\nu(y)\delta j^a_\mu(x)}. \tag{3.55}$$

The first 2 terms of Eq.(3.53) will then generate the bare propagator

$$\delta^{ab}\big(\delta_{\mu\nu}\,\partial^2\,\delta(x-y) - \partial_\mu\,\partial_\nu\,\delta(x-y)\big), \tag{3.56}$$

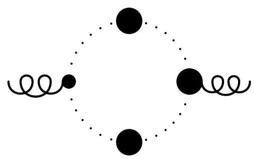


Figure 3.4: The diagrammatic expression of the gluon-ghost coupling in the gluon equation, which can be referred as the ghost-loop. The dotted lines represent ghost propagators while the curled lines represent gluon propagators. The two dots at the intersection points denote the vertices, the small dot denotes the bare vertex, while the other one denotes the full ghost-gluon vertex.

and after performing a Fourier transformation, we get

$$(D_{\mu\nu}^{bare;ab}(p) = \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu). \tag{3.57}$$

The next terms are the 3-point couplings, called the gluon-loop and the ghost-loop.

Starting with the ghost-loop, which is given by the last term of Eq.(3.53). After taking the derivative with respect to A_μ^a, we get

$$gf^{cfa}\int d^dz \frac{\delta}{\delta\sigma^c(x)}\partial_\mu \frac{\delta W}{\delta\overline{\sigma}^f(x)}, \tag{3.58}$$

this can be rewritten using [84]

$$\partial_\mu \frac{\delta}{\delta\overline{\sigma}^d(x)} = \int d^dz \left(\partial_\mu \frac{\delta^2\Gamma}{\delta\overline{\sigma}^d(x)\delta c^c(z)}\right)\frac{\delta}{\delta\overline{\sigma}^c(z)}, \tag{3.59}$$

into two propagators and a vertex term Γ

$$-gf^{cfa}\int d^dz d^dw \frac{\delta^2 W}{\delta\overline{\sigma}^e(y)\delta\sigma^f(x)}\frac{\delta^2\Gamma}{\delta c^f(w)\overline{c}^e(y)}\partial_\mu \frac{\delta^2 W}{\delta c^g(y)\delta\overline{c}^c(x)}. \tag{3.60}$$

Performing the second derivative with respect to $A_\nu^b(y)$, yields

$$-gf^{cfa}\int d^dz d^dw \frac{\delta^2 W}{\delta\overline{\sigma}^e(y)\delta\sigma^f(x)}\frac{\delta^2\Gamma}{\delta c^f(w)\overline{c}^e(y)A_\nu^b(y)}\partial_\mu \frac{\delta^2 W}{\delta c^g(y)\delta\overline{c}^c(x)}. \tag{3.61}$$

Again we use the abbreviation

$$\Gamma_\mu^{c\overline{c}A;abc} = \frac{\delta^3\Gamma}{\delta c^a\delta\overline{c}^b\delta A_\mu^c}. \tag{3.62}$$

Now we Fourier transform Eq.(3.61) and get [38]

$$\int \frac{d^dq}{(2\pi)^d} igf^{adc}q_\mu D_G^{de}(q)D_G^{cf}(p+q)\Gamma_\nu^{c\overline{c}A;feb}(-q,p+q,-p). \tag{3.63}$$

the corresponding diagram to this integral is given by Fig.(3.4). Clearly this shows why the diagram is called a loop. In Chapter 4 due to SUSY Yang-Mills theory gets extended, therefore there are not only two loop diagrams but four. These additional loops correspond to the scalar and the fermionic interaction. The bare vertex from the ghost loop has to be similar to the bare vertex from ghost DSE, Eq.(3.51). As a consistency check for the derivation of the diagrams one should take a look at the indices. There are exactly four external indices namely a, μ and b, ν. Those should be the only indices which are not contracted. Every internal index has to appear twice [22]. Seemingly this is the case for the ghost loop.

The next loop that is generated is the gluon loop. This term corresponds to the term in Eq.(3.53) with the three gluons A. The diagram looks like the one in Fig(3.2), only the ghost lines are replaced by gluon lines (curled ones).

The first derivative yields

$$-g\big(f^{afe}A^f_\sigma(x)\partial_\mu A^e_\sigma(x) + f^{cae}A^c_\rho(x)\partial_\rho A^e_\mu(x) + f^{cfa}A^c_\rho(x)A^f_\mu(x)\partial_\rho\big). \tag{3.64}$$

Using Eq.(3.59) yields the result in position space

$$\begin{aligned}-gf^{acd}\int d^dz_0 d^dz_1 (D^{ce}_{\sigma\omega}(x-z_0)\partial_\mu D^{df}_{\sigma\omega}(x-z_1) - D^{ce}_{\sigma\lambda}(x-z_0)\partial_\sigma D^{df}_{\mu\omega}(x-z_1)-\\ - \partial_\sigma(D^{ce}_{\sigma\lambda}(x-z_0)D^{df}_{\mu\omega}(x-z_1)))\Gamma^{bfe}_{\nu\omega\lambda}(y,z_0,z_1).\end{aligned} \tag{3.65}$$

Here one has to be careful since

$$\partial_\mu \frac{\delta^2 W}{\delta j^c_\mu(x)\delta j^d_\nu(x)} \neq 0, \tag{3.66}$$

not even at zero momentum [84].

To obtain the result in momentum space Fourier transformation is employed twice, such that the momentum of both propagators is assigned differently. Abbreviate the 3-gluon vertex with

$$\Gamma^{A^3;abc}_{\mu\nu\rho} = \frac{\delta^3\Gamma}{\delta A^a_\mu \delta A^b_\nu \delta A^c_\rho}, \tag{3.67}$$

using the bose symmetry of the vertex and adding both Fourier transformed yields

$$\begin{aligned}\frac{igf^{acd}}{2}\int \frac{d^dq}{(2\pi)^d}\Big(\big((p-2q)_\mu\delta_{\sigma\chi} - (2p-q)_\chi\delta_{\mu\sigma} + (p+q)_\sigma\delta_{\chi\omega}\big)\times\\ \times D^{cf}_{\sigma\omega}(q)D^{de}_{\chi\lambda}(p-q)\Big)\Gamma^{A^3;bfe}_{\nu\omega\lambda}(-p,q,p-q).\end{aligned} \tag{3.68}$$

Where the tree-level vertex can be identified with

$$\Gamma^{tl;A^3;abc}_{\mu\nu\rho}(p,q,k) = -igf^{abc}((q-k)_\mu\delta_{\nu\rho} + (k-p)_\nu\delta_{\rho\mu} + (p-q)_\rho\delta_{\mu\nu})\delta(p+q+k), \tag{3.69}$$

with $k = p - q$. This concludes the 3-point couplings of Yang-Mills theory [22].

The next diagram is easier to calculate since it lacks momentum dependency [84]. In principle there would be three 4-point couplings: The tadpole, the sunset and the squint diagram [38]. To be consistent with the following chapter only the derivation of the tadpole diagram is shown. As the other two are already tow-loop order, which is neglected in the following calculations.

Employing the first derivative on the 4-point coupling term in Eq.(3.53) yields

$$\begin{aligned}g^2 f^{abc}f^{ade}\Big(\frac{\delta^2 W}{\delta j^d_\mu(x)\delta j^e_\sigma(x)}\frac{\delta W}{\delta j^c_\sigma(x)} + \frac{\delta^2 W}{\delta j^e_\mu(x)\delta j^c_\sigma(x)}\frac{\delta W}{\delta j^d_\sigma(x)} + \frac{\delta^2 W}{\delta j^c_\sigma(x)\delta j^d_\sigma(x)}\frac{\delta W}{\delta j^e_\mu(x)}\Big)\\ + \text{two-loop....}\end{aligned} \tag{3.70}$$

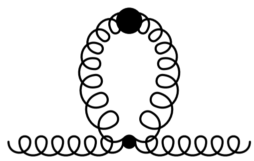


Figure 3.5: The diagram that corresponds to the gluon tadpole. The curled line denotes the gluon propagator. Tadpole diagrams are special in the sense that they only have a bare vertex and a single full propagator.

Now the second derivative has to be applied to all of these three terms, the result in position space is given by

$$\begin{gathered} g^2 f^{abc} f^{ade} \int d^d z d^d w \delta(x-w)\delta(x-z)\delta(x-y) \\ \big(\delta^{ef} D^{dc}_{\mu\nu}(w-z) + \delta^{cb} D^{de}_{\mu\nu}(w-z) + \delta_{\mu\nu}\delta^{db} D^{ce}_{\sigma\sigma}(w-z)\big). \end{gathered} \tag{3.71}$$

Same expression after Fourier transf.

$$g^2 f^{abc} f^{ade} \int \frac{d^d q}{(2\pi)^d} \big(\delta^{eb} D^{dc}_{\mu\nu}(q) + \delta^{cb} D^{de}_{\mu\nu}(q) + \delta_{\mu\nu}\delta^{db} D^{ce}_{\sigma\sigma}(q)\big). \tag{3.72}$$

It is possible to make the tree-level vertex explicit [84]. With those permutations we get

$$\begin{aligned} &\frac{g^2}{2} \int \frac{d^d q}{(2\pi)^d} \big(f^{eab} f^{ecd} (\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \\ &\quad + f^{gac} f^{gbd} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + f^{gad} f^{gbc} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\nu\rho})\big) D^{cd}_{\sigma\rho}(q), \end{aligned} \tag{3.73}$$

with the tree-level vertex contained in the parenthesis $\big(...\big)$, e.g.

$$\begin{aligned} \Gamma^{tl;A^4;abcd}_{\alpha\beta\gamma\delta}(p,q,k,l) \;=\;& g^2 \big(f^{eab} f^{ecd} (\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \\ & + f^{gac} f^{gbd} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + f^{gad} f^{gbc} (\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\nu\rho})\big) \times \\ & \times \delta(p+q+k+l). \end{aligned} \tag{3.74}$$

Diagrammatically this tadpole is represented in Fig.(3.5). For Yang-Mills theory this diagram introduces a scale since it is a primitively divergent diagram. Therefore, it introduces a mass dependent counter term [1]. For $\mathcal{N}=4$ SYM this is not the case, which is described in Chapter 4 and 5.

All the diagrams are known therefore we can put together the gluon propagator DSE

$$\begin{aligned}\left(D^{ab}_{\mu\nu}(p)\right)^{-1} &= \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\\ &+\int\frac{d^dq}{(2\pi)^d}igf^{adc}q_\mu D^{de}_G(q)D^{cf}_G(p+q)\Gamma^{c\bar{c}A;feb}_\nu(-q,p+q,-p)\\ &+\frac{igf^{acd}}{2}\int\frac{d^dq}{(2\pi)^d}((p-2q)_\mu\delta_{\sigma\chi}-(2p-q)_\chi\delta_{\mu\sigma}+(p+q)_\sigma\delta_{\chi\omega}\times\\ &\times D^{cf}_{\sigma\omega}(q)D^{de}_{\chi\lambda}(p-q))\Gamma^{A^3;bfe}_{\nu\omega\lambda}(-p,q,p-q)\\ &+\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}g^2(f^{eab}f^{ecd}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})\\ &+f^{gac}f^{gbd}(\delta_{\mu\nu}\delta_{\sigma\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})+f^{gad}f^{gbc}(\delta_{\mu\nu}\delta_{\sigma\rho}-\delta_{\mu\sigma}\delta_{\nu\rho}))D^{cd}_{\sigma\rho}(q).\end{aligned} \tag{3.75}$$

This can be rewritten using the tree-level terms of each vertex

$$\begin{aligned}\left(D^{ab}_{\mu\nu}(p)\right)^{-1} &= \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\\ &-\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;c\bar{c}A;acd}_\mu(-p-q,q,p)D^{de}_G(q)D^{cf}_G(p+q)\Gamma^{c\bar{c}A;feb}_\nu(-q,p+q,-p)\\ &+\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;A^3;acd}_{\mu\sigma\chi}(p,q-p,-q)D^{cf}_{\sigma\omega}(q)D^{de}_{\chi\lambda}(p-q)\Gamma^{A^3;bfe}_{\nu\omega\lambda}(-p,q,p-q)\\ &+\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;A^4;abcd}_{\mu\nu\sigma\rho}(-p,p,-q,q)D^{cd}_{\sigma\rho}(q).\end{aligned} \tag{3.76}$$

The propagator DSE to one-loop order contains a bare propagator, a ghost loop and a gluon loop and a tadpole [17, 22, 38]. Diagrammatically this equation can be found at Fig.(3.3)

This concludes the derivation of the DSEs for Young-Mills theory. Here a very simple truncation scheme is used: Only one-loop diagrams were taken into account and no separate DSEs for the 3-point couplings are derived, which require to model the full vertices. Further literature, for rainbow ladder truncation and beyond, can be found at [17, 22, 27, 38].

In the next chapter Yang-Mills theory is getting extended by SUSY. This theory resembles many features of Yang-Mills theory such as spin 1 gauge boson [18]. While the conformal window of Yang-Mills theory and therefore the scaling solutions are thought to be only valid in the IR [27], $\mathcal{N}=4$ SYM is sustainably different in this perspective, see Chapter 2.

Chapter 4

Propagators of $\mathcal{N}$=4 super Yang-Mills theory

Every new result which has been derived and was unknown before this thesis can be accessed at the arXiv:1512.0664. The paper which is also published in European Physics Journal C follows the structure of Chapter 4 to 6.

In the previous chapter the DSEs have been derived for the Yang-Mills case within a one-loop truncation, see Eqs.(3.50,76). The derivations for $\mathcal{N} = 4$ SYM are similar. In fact the actions of both theories have the $F_{\mu\nu}F_{\mu\nu}$ term as well as the ghost term in common. Due to Landau gauge Eq.(3.17) SUSY is explicitly broken. Due to gauge fixing, a ghost term arises. This will be of interest in the next chapter.

$\mathcal{N} = 4$ SYM is a superconformal theory, see Eqs.(2.22-25) and since Landau gauge preserves the conformal symmetry [101], results should be in principle conformal invariant, as mentioned in Chapter 2.

For the derivation of the DSEs an action or Lagrangian is needed. The $\mathcal{N} = 4$ SYM Lagrangian will contain the Yang-Mills Lagrangian but also several other terms. The theory requires the Lagrangian to be invariant under the superconformal symmetry group $SU(2, 2|4)$ [49], this restrains the theory. In Feynman-Wess-Zumino gauge [19] it was shown, that the theory not only has a vanishing β function Eqs.(2.29-38), but is finite. Nevertheless within Landau gauge the theory is interacting and comparable to Yang-Mills theory.

4.1 The action

It is possible to construct all $\mathcal{N} > 1$ from $\mathcal{N} = 1$ SUSY [18]. Due to $\mathcal{N} = 4$ SUSY one has to deal with the gauge multiplet, see Chapter 2. This multiplet consists of several particles. The action can be written in terms of an explicit particle content [18]

$$\begin{aligned}
\mathcal{S} \;=\; & \int d^dx\Big[(D_\mu\phi')^*_a(D^\mu\phi')_a-(D_\mu\phi'')^*_a(D^\mu\phi'')_a-(D_\mu\phi)^*_a(D^\mu\phi)_a \\
& -\frac{1}{2}\big(\overline{\psi'_a}(\not{D}\psi')_a\big)-\frac{1}{2}\big(\overline{\psi''_a}(\not{D}\psi'')_a\big) \\
& -\frac{1}{2}\big(\overline{\psi_a}(\not{D}\psi)_a\big)-\frac{1}{2}\big(\overline{\lambda_a}(\not{D}\lambda)_a\big) \\
& -2\sqrt{2}Re\ f^{abc}\phi_a(\psi'^T_{bL}\epsilon\psi''_{cL})-2\sqrt{2}Re\ f^{abc}(\lambda^T_{aL}\epsilon\psi_{cL}\phi^*_b) \\
& -2\sqrt{2}Re\ f^{abc}\phi'_b(\psi''^T_{cL}\epsilon\psi_{aL})-2\sqrt{2}Re\ f^{abc}\phi''_c(\psi'^T_{bL}\epsilon\psi_{aL}) \\
& +2\sqrt{2}Re\ f^{abc}(\psi'^T_{bL}\epsilon\lambda_{aL})\phi'^*_c+2\sqrt{2}Re\ f^{abc}(\psi''^T_{bL}\epsilon\lambda_{aL})\phi''^*_c \\
& -\frac{1}{4}F_{a\mu\nu}F_{a\mu\nu}+\frac{g^2\theta}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{a\mu\nu}F_{a\rho\sigma}-V\Big].
\end{aligned} \tag{4.1}$$

With the covariant derivative D_μ defined at Eq.(3.19), the ϕ,ϕ',ϕ'' are complex scalar fields, the $\psi,\psi',\psi'',\lambda$ are gauginos the fermionic degrees of freedom, $F^{\mu\nu}$ is the field-strength tensor defined in Eq.(3.16). V again is the potential:

$$\begin{aligned}
V \;=\; & f^{ade}f^{bce}\big(\phi_a\phi^*_b+\phi_b\phi^*_a\big)\big(\phi'_c\phi'^*_d+\phi''^*_c\phi''_d\big) \\
& +\frac{1}{2}|f^{abc}\big(\phi'^*_b\phi'_c-\phi''_b\phi''^*_c\big)|^2 \\
& -\frac{1}{2}f^{abc}f^{ade}\phi^*_b\phi_c\phi^*_d\phi_e+2|f^{abc}\phi'_b\phi''_c|^2.
\end{aligned} \tag{4.2}$$

ϵ is some matrix defined in the Appendix. The sub- and superscripts of the fields like $a,b,c,d,...$ correspond to the gauge symmetry, e.g. a $SU(N)$. The L and R subscript on the fermions correspond to the handiness of the fermion. The antisymmetric tensor f^{abc} is the real structure constant of the gauge group and it contains a factor of gauge coupling:

$$f^{abc}=g\tilde{f}^{abc} \tag{4.3}$$

This Lagrangian consists of three Wess-Zumino multiplets given in Chapter 2 and a multiplet containing the gluon and gluino, which verifies the statement that $\mathcal{N}=4$ is a special case of $\mathcal{N}=1$. Nevertheless this is a rather complicate expression to derive the DSEs.

To simplify the derivation one can rearrange the fermionic fields in terms of a $SU(4)$ vector, which is due to the R-symmetry of $\mathcal{N}=4$ SYM. Under R-symmetry the scalars need to transform like a matrix, because they have to relate to the Yukawa interaction. Line four to six in Eq.(4.1) correspond to the Yukawa interaction. Writing down the $SU(4)$ R-symmetry invariant action yields [18]:

$$\begin{aligned}
\mathcal{S} \;=\; & \int d^dx\Big[-\frac{1}{2}\big(D_\rho\phi^{ij}\big)_a\big(D^\rho\phi^{ij}\big)^*_a \\
& -\frac{1}{2}\psi^T_{iaL}\epsilon\big(\not{D}\psi^i_R\big)_a+\frac{1}{2}\psi^{iT}_{aR}\epsilon\big(\not{D}\psi_{iL}\big)_a \\
& -\sqrt{2}Ref^{abc}\phi^{ij}_a\left(\psi^T_{ibL}\epsilon\psi_{jcL}\right)-V \\
& -\frac{1}{4}F_{a\rho\sigma}F_{a\rho\sigma}+\frac{g^2\theta}{64\pi^2}\varepsilon_{\sigma\epsilon\delta}F_{a\rho\sigma}F_{a\epsilon\delta}\Big].
\end{aligned} \tag{4.4}$$

The point of this rewriting is that the theories R-symmetry is now explicit. Therefore, the fermions ψ and their antifermions $\overline{\psi}$ get an R-index i, $i\in\{1,2,3,4\}$. The complex scalars get

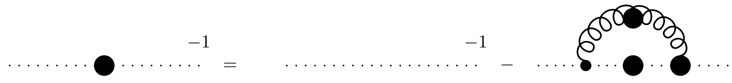


Figure 4.1: Diagrammatic representation of the ghost Dyson-Schwinger equation. As in Chapter 3 the ghost only couples to the gluon. The dotted lines are ghost propagators and the curled lines are gluon propagators. A dot within a line defines a full propagator, while a small dot correspond to a bare vertex.

two R-indices $i, j \in \{1, 2, 3, 4\}$. The gluon is a singlet under R-symmetry therefore no R-index. The potential V can be written in more compact form than in Eq.(4.2):

$$V = \frac{1}{8}|f^{abc}\phi_b^{ij}\phi_c^{kl}|^2. \tag{4.5}$$

This is an elegant way of presenting the action and it is a much simpler starting point to derive the DSEs. Note that a table of how to connect the implicit- and the explicit R-symmetric action is given in [18].

To simplify this action even further for the DSE derivation we want to get rid of the subscript which denotes right- and left handed fermions. Following the convention of [18]:

$$\psi_{iaL} = \frac{1}{2}(1+\gamma_5)\psi_{ia} \ , \quad \psi_{iaR} = \frac{1}{2}(1-\gamma_5)\psi_{ia}. \tag{4.6}$$

This will make the DSEs independent of the handiness of the spinor. Of course it is also useful to drop the handiness of the spinors to relate them to Dirac spinors.

Same argument can be employed to the R-index. Since the action Eq.(4.4) permits interaction like $\psi^i\psi_i$ we choose to rewrite this such that both R-indices are subscripts. This R-index shift can be derived from the $SU(4)$ invariant Majorana condition [18]

$$(\psi_{iaL})^* = -\beta\epsilon\psi^i_{aR}, \tag{4.7}$$

the β and ϵ are some matrices which can be found in the Appendix. After two to three lines of algebra one can observe:

$$\psi^i_a = -\psi_{ia}. \tag{4.8}$$

Lowering the R-index produces a change of sign.

4.2 Parallels with Yang-Mills theory

In the previous chapter the derivation DSE of Yang-Mills theory was done, see Eq.(3.50,76). This is a theory without SUSY and without conformal symmetry. Note that the conformal symmetry is thought to be restored in the conformal window of Yang-Mills theory [102]. Still, as already mentioned, the Yang-Mills theory is part of $\mathcal{N} = 4$ SYM. Therefore, everything that has been previously calculated can be used. In last line of the action Eq.(4.4) the term

$$-\frac{1}{4}\int d^dx F_{a\mu\nu}F_a^{\mu\nu}, \tag{4.9}$$

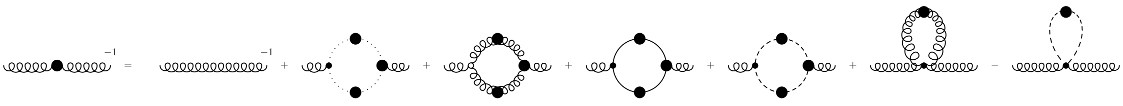


Figure 4.2: Diagrammatic representation of the Dyson-Schwinger equation for the $\mathcal{N} = 4$ SYM gluon propagator. The Full propagator is then given by its bare, four loop diagrams and two tadpoles. The nomenclature is as follows: dotted lines are ghost propagators, curled lines are gluon propagators, full lines are Majorana fermion propagators and dashed lines are scalar propagators. All internal lines are full propagators. The small dot corresponds to the bare vertex.

can be identified with action Eq.(3.15). Here the same steps as in Chapter 3 can be applied. The gauge sector of both theories is equal.

One remark has to be made: Using Landau gauge will fix the gauge and therefore "hide" SUSY. To compare Yang-Mills with $\mathcal{N} = 4$ SYM we introduce a ghost field by hand, such that the action Eq.(4.4) gets an extra term $\overline{c^a}\partial_\mu D^{ab}_\mu c^b$. Taking the results already been derived produces the first parts of the full $\mathcal{N} = 4$ SYM propagator equations:

$$\left(D^{ab}_G(p)\right)^{-1} = -\delta^{ab}p^2 - if^{ade}\int \frac{d^dq}{(2\pi)^d} D^{ef}_{\mu\nu}(p-q) D^{dh}_G(q) \Gamma^{c\bar{c}A;bfh}_\nu(-p,q,p-q) \tag{4.10}$$

Again the ghost only couples to the gluon which makes the $\mathcal{N} = 4$ SYM ghost equation equal with Eq.(3.35). Diagrammatically this corresponds to Fig.(4.1).

$$\begin{aligned}
\left(D^{ab}_{\mu\nu}(p)\right)^{-1} \quad = \quad & \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) \\
& + \int \frac{d^dq}{(2\pi)^d} if^{adc} q_\mu D^{de}_G(q) D^{cf}_G(p+q) \Gamma^{Ac\bar{c};feb}_\nu(-q,p+q,-p) \\
& + \frac{if^{acd}}{2}\int \frac{d^dq}{(2\pi)^d}\big((p-2q)_\mu\delta_{\sigma\chi} - (2p-q)_\chi\delta_{\mu\sigma} + (p+q)_\sigma\delta_{\chi\omega} \times \\
& \times D^{cf}_{\sigma\omega}(q) D^{de}_{\chi\lambda}(p-q)\big)\Gamma^{A^3;bfe}_{\nu\omega\lambda}(-p,q,p-q) \\
& + \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\big(f^{eab}f^{ecd}(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \\
& + f^{gac}f^{gbd}(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + f^{gad}f^{gbc}(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\nu\rho})\big) D^{cd}_{\sigma\rho}(q).
\end{aligned} \tag{4.11}$$

The gluon equation is much less cumbersome in the Yang-Mills case. As already mentioned due to the covariant derivative, see Eq.(3.19), loop diagrams are produced. Within $\mathcal{N} = 4$ SYM new loop diagrams will arise due to fermions and scalars. This leads to a more lenghty equation which is only diagrammatically represented, see Fig.(4.2) and is worked out later.

Diagrammatically one can already write down the DSE for the full $\mathcal{N} = 4$ gluon propagator, see Fig.(4.2). However, the equation is generated by

$$\begin{aligned}
\mathcal{S}_{Gl} \quad = \quad & \int d^dx\Big[-\frac{1}{2}(D_\rho\phi^{ij})_a(D^\rho\phi^{ij})^*_a \\
& -\frac{1}{2}\psi^T_{iaL}\epsilon(\not{D}\psi^i_R)_a + \frac{1}{2}\psi^{iT}_{aR}\epsilon(\not{D}\psi_{iL})_a \\
& -\frac{1}{4}F_{a\rho\sigma}F^{\rho\sigma}_a + \frac{g^2\theta}{64\pi^2}\varepsilon_{\rho\sigma\epsilon\delta}F^{\rho\sigma}_a F^{\epsilon\delta}_a\Big],
\end{aligned} \tag{4.12}$$

the θ represents the vacuum angle, or instanton angle and is the only free parameter of the theory. In this thesis we set $\theta = 0$. This does of course no harm to the DSE, because the θ term corresponds to a boundary condition, see [103]. With this choice of θ the gluonic action simplifies. Left are the Yang-Mills term and the covariant derivatives for every field, which are the loops or 3-point couplings. Additional to the Yang-Mills diagrams there are a fermi-loop, a scalar-loop and a scalar tadpole. Starting with the fermi-loop, the corresponding term in the action is:

$$-\int d^dx \left(\frac{1}{2}\psi^T_{iaL}\epsilon(\not{D}\psi^i_R)_a + \frac{1}{2}\psi^{iT}_{aR}\epsilon(\not{D}\psi_{iL})_a\right), \tag{4.13}$$

Performing the transformations Eq.(4.6) and Eq.(4.8) and replacing $\not{D}$ by its right hand side yields

$$\frac{1}{8}\int d^dx \big[\psi^{iT}_a(x)(1+\gamma_5)\epsilon\gamma_\rho f^{abc}A^b_\rho(x)(1-\gamma_5)\psi^i_c - \psi^{iT}_a(x)(1-\gamma_5)\epsilon\gamma_\rho f^{abc}A^b_\rho(x)(1+\gamma_5)\psi^i_c(x)\big]. \tag{4.14}$$

The first derivative with respect to $A^d_\mu(y)$ produces:

$$\frac{1}{8}\int d^dx \big[\psi^{iT}_a(x)(1+\gamma_5)\epsilon\gamma_\mu f^{adc}(1-\gamma_5)\psi^i_c(x) - \psi^{iT}_a(x)(1-\gamma_5)\epsilon\gamma_\mu f^{adc}(1+\gamma_5)\psi^i_c(x)\big]\delta(x-y). \tag{4.15}$$

After some algebra the corresponding term in position space reads

$$\frac{1}{2}f^{adc}\gamma_\mu\int d^dz_0 d^dz_1 D^{ag}_{ij}(y-z_0)D^{ce}_{ik}(y-z_1)\Gamma^{A\psi\overline{\psi};geh}_{\nu;jk}(x,z_0,z_1). \tag{4.16}$$

With the fermion-propagator:

$$\left(D^{ab}_{ij}(x-y)\right)^{-1} = \frac{\delta^2\Gamma}{\delta\psi^b_j(y)\delta\overline{\psi}^a_i(x)}. \tag{4.17}$$

Defining the 2-fermion-gluon vertex as

$$\frac{\delta^3\Gamma}{\delta A^d_\mu(x)\delta\overline{\psi}^b_i(y)\delta\psi^e_j(z)} = \Gamma^{A\overline{\psi}\psi;dbe}_{\mu;ij}(x,y,z), \tag{4.18}$$

and after Fourier transforming to momentum space

$$\frac{1}{2}f^{adc}\gamma^\mu\int\frac{d^dq}{(2\pi)^d}D^{ag}_{ij}(q)D^{ce}_{ik}(p-q)\Gamma^{A\psi\overline{\psi};geh}_{\nu;jk}(-p,q,p-q), \tag{4.19}$$

the tree level term can be identified with

$$\Gamma^{tl;A\overline{\psi}\psi;abc}_{\mu;ij}(p,q,k) = -f^{abc}\gamma_\mu\delta_{ij}\delta(p+q+k). \tag{4.20}$$

The next loop contribution is generated by the coupling to the scalar. This expression is structural similar to the gluon-Higgs term in Yang-Mills theory with an adjoint Higgs field [104]. However, without SUSY there is no R-symmetry. Using Eq.(3.45), produces

$$f^{abc}\int d^dz_0 d^dz_1 D^{bd}_{ijkl}(y-z_0)\partial^x_\mu D^{cf}_{mnpq}(y-z_1)\Gamma^{A\phi^*\phi;gdf}_{\nu;klpq}(x,z_0,z_1), \tag{4.21}$$

with two yet unknown functions $D^{ab}_{ijkl}(x-x_0)$ and $\Gamma^{A\phi^*\phi;gdf}_{\nu;klpq}(x,z_0,z_1)$. The first one is the full scalar-propagator defined as

$$D^{ab}_{ijkl}(x-x_0) = \frac{\delta^2\Gamma}{\delta\phi^{a*}_{ij}(x)\delta\phi^{b}_{kl}(x_0)}, \tag{4.22}$$

Our convention here is that a scalar propagator consists of a derivative with respect to a scalar and one with respect to a complex-conjugated scalar, there are also different conventions [105]. The second one denotes the 2-scalar-gauge vertex

$$\frac{\delta^3\Gamma}{\delta A^d_\mu(x)\delta\phi^{b*}_{ij}(y)\delta\phi^{e}_{kl}(z)} = \Gamma^{A\phi^*\phi;dbe}_{\mu;ijkl}(x,y,z). \tag{4.23}$$

Note that if one wants to change ϕ^* to ϕ one has to take into account the reality condition which connects them, i.e.,

$$(\phi^{ij}_a)^* = \frac{1}{2}\varepsilon_{ijkl}\phi^{kl}_a. \tag{4.24}$$

Interchanging the R-indices will yield a sign change. Fourier transform Eq.(4.21) yields

$$\frac{1}{2}\int\frac{d^dq}{(2\pi)^d} if^{abc}(2q-p)_\mu D^{de}_{ijmn}(q) D^{cf}_{klpq}(p-q)\Gamma^{A\phi^*\phi;bef}_{\nu;ijkl}(-p,q,p-q). \tag{4.25}$$

The corresponding tree-level vertex is then defined:

$$\Gamma^{tl;A\phi^*\phi;abc}_{\mu;ijkl}(p,q,k) = if^{abc}\delta_{ijkl}(q-k)_\mu, \tag{4.26}$$

where δ_{ijkl} is an abbreviation for an antisymmetric combination of δ_{ik} and δ_{jl} such as

$$\delta_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{4.27}$$

All loop diagrams from Fig.(4.2) are known now. As in Chapter 3 we only consider one-loop order. Only one more contribution of the full propagator is unknown, namely the scalar-tadpole. The derivation of the scalar tadpole is analogues to the gluon tadpole:

$$\begin{aligned} f^{abc}f^{ade}\Bigg(&\frac{\delta^2 W}{\delta j^d_\mu(x)\delta k^c_{ij}(x)}\frac{\delta W}{\delta k^{e*}_{kl}(x)} + \frac{\delta^2 W}{\delta j^d_\mu(x)\delta k^{e*}_{kl}(x)}\frac{\delta W}{\delta k^c_{ij}(x)} \\ &+ \frac{\delta^2 W}{\delta k^c_{ij}(x)\delta k^{e*}_{kl}(x)}\frac{\delta W}{\delta j^d_\mu(x)}\Bigg) + \text{2-loop order....} \end{aligned} \tag{4.28}$$

Only the last term survives due to the fact that mixed functional derivatives have to vanish, e.g. $\frac{\delta\phi^a(x)}{\delta A^b_\nu(x)} = 0$. Therefore, Eq.(4.28) reduces to

$$f^{abc}f^{ade}\left(\frac{\delta^2 W}{\delta k^c_{ij}(x)\delta k^{e*}_{kl}(x)}\frac{\delta W}{\delta j^d_\mu(x)}\right), \tag{4.29}$$

and after Fourier transformation

$$-\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\delta_{\mu\nu}\delta_{ijkl}\big(f^{abc}f^{adf} + f^{abf}f^{adc}\big)D^{df}_{mnmn}(q), \tag{4.30}$$

with the tree-level term written explicitly is

$$\Gamma^{tl;A^2\phi^*\phi;abdf}_{\mu\nu;ijkl}(p,q,k,l) = \delta_{\mu\nu}\delta_{ijkl}\big(f^{abc}f^{adf} + f^{abf}f^{adc}\big). \tag{4.31}$$

Note that the contributions form the tadpoles come with opposite signs.

If conformal symmetry stays valid, then this primitively divergent diagrams have to cancel each other exactly. Otherwise this would imply massive counter-terms in the Lagrangian as well as a cutoff Λ and therefore the scale invariance would be broken.

The full one-loop gluon-propagator equation, which is diagrammatically shown in Fig.(4.2), is given by

$$\begin{aligned}
\big(D^{ab}_{\mu\nu}(p)\big)^{-1} &= \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) \\
&- \int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;Ac\bar{c};adc}_{\mu}(-p-q,q,p)D^{de}_G(q)D^{cf}_G(p+q)\Gamma^{Ac\bar{c};feb}_{\nu}(-q,p+q,-p) \\
&+\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;A^3;dca}_{\mu\sigma\chi}(p,q-p,-q)D^{cf}_{\sigma\omega}(q)D^{de}_{\chi\lambda}(p-q))\Gamma^{A^3;bfe}_{\nu\omega\lambda}(-p,q,p-q) \\
&-\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;A\overline{\psi}\psi;acd}_{\mu;ij}(p,q-p,-q)D^{de}_{ik}(q)D^{cf}_{jl}(p-q)\Gamma^{A\overline{\psi}\psi;bef}_{\nu;kl}(-p,q,p-q) \\
&+\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;A\phi^*\phi;acd}_{\mu;ijkl}(p,q-p,-q)D^{de}_{ijmn}(q)D^{cf}_{klpq}(p-q)\Gamma^{tl;A\phi^*\phi;bef}_{\nu;mnpq}(-p,q,p-q) \\
&+\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;A^4;abcd}_{\mu\nu\sigma\rho}(p,-p,q,-q)D^{cd}_{\sigma\rho}(q) \\
&-\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;A^2\phi^*\phi;abcd}_{\mu;ijkl}(p,-p,q,-q)D^{cd}_{mnmn}(q).
\end{aligned} \tag{4.32}$$

This finishes the $\mathcal{N}=4$ SYM gluon DSE.

4.3 Majorana Fermions

Instead of the ghost in Yang-Mills theory $\mathcal{N}=4$ SYM has fermions, either Weyl or Majorana fermions [18]. In the case of 4 dimensions their representation coincides [106]. One major advantage of DSEs is, that for fermion correlation functions one does not have to compute a fermion determinant. Still one needs to be careful of mixed derivatives. Since the Majorana fermion is particle as well as its own antiparticle at the same time, the derivative of ψ with respect to $\overline{\psi}$ does not vanish. In fact

$$\frac{\delta\overline{\psi}}{\delta\psi} = \gamma_5\epsilon, \tag{4.33}$$

which is different from Dirac fermions, where the right side of the equation would just vanish. Due to Eq.(4.33) the use of an algorithmic program to derive the DSEs was not feasible [107]. Spinor indices will be suppressed during the whole derivation.

The fermionic part of the action is given by

$$\begin{aligned}
\mathcal{S}_F &= \int d^dx\Big[-\frac{1}{2}\psi^T_{iaL}\epsilon(\not{D}\psi^i_R)_a + \frac{1}{2}\psi^{iT}_{aR}\epsilon(\not{D}\psi_{iL})_a \\
&\quad -\sqrt{2}Ref^{abc}\phi^{ij}_a(\psi^T_{ibL}\epsilon\psi_{jcL})\Big],
\end{aligned} \tag{4.34}$$

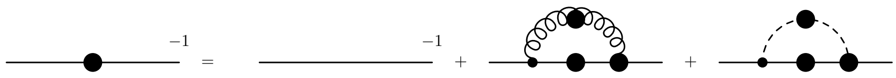


Figure 4.3: Diagrammatic representation of the $\mathcal{N} = 4$ SYM Majorana fermion DSE with all its interactions. The full propagator is given by the bare propagator a self-energy term and a Yukawa interaction with a scalar. All internal propagators are full propagators, the full line corresponds to a fermion-propagator, the curled line to a gluon-propagator and the dashed line to a scalar-propagator.

containing the bare propagator, a selfenergy contribution and a Yukawa contribution. To derive the DSE it is necessary to perform two functional derivatives. The first one is taken with respect to $\psi^i_a(x)$ while the second one with respect to $\overline{\psi}^j_b$. The first part of Eq.(4.34) will yield the propagator

$$\left(D^{ab}_{ij}(x-y)\right)^{-1} = \frac{\delta^2\Gamma}{\delta\psi^i_a(x)\delta\overline{\psi}^j_b(y)}, \tag{4.35}$$

$$D^{ab}_{ij}(x-y) = \frac{\delta^2 W}{\delta\xi^i_a(x)\delta\overline{\xi}^j_b(y)}, \tag{4.36}$$

with some Grassmann valued sources ξ and $\overline{\xi}$ which generate the fermions.

The bare or tree-level fermion-propagator already in position space is then given by the first term in the fermionic action Eq.(4.34):

$$\left(D^{tl;ab}_{ij}(p)\right)^{-1} = -i\delta^{ab}\delta_{ij}\not{p}. \tag{4.37}$$

Diagrammatically the DSE is shown in Fig.(4.3) and consists only of three diagrams. Therefore after calculating the bare propagator, e.g. Eq.(4.37) only 2 diagrams are left. One due to the covariant derivative and one due to the *Re* contribution in Eq.(4.34). Starting with the gluon coupling, again already in momentum space

$$\int \frac{d^dq}{(2\pi)^d} f^{abc}\gamma_\mu\delta_{ik}D^{ef}_{\mu\nu}(p-q)D^{dg}_{kl}(q)\Gamma^{\overline{\psi}\psi A;bgf}_{jl;\nu}(-p,q,p-q), \tag{4.38}$$

where it has been used that

$$\Gamma^{\overline{\psi}\psi A;abc}_{jl;\mu} = \frac{\delta^3\Gamma}{\delta\overline{\psi}^a_j\delta\psi^b_l\delta A^c_\mu}. \tag{4.39}$$

The tree-level vertex is already known and given by Eq.(4.20), inserting the tree-level yields

$$-\int \frac{d^dq}{(2\pi)^d}\Gamma^{tl;\overline{\psi},\psi A;dae}_{ik;\mu}(-q,p,q-p)D^{ef}_{\mu\nu}(p-q)D^{dg}_{kl}(q)\Gamma^{\overline{\psi}\psi A;bgf}_{jl;\nu}(-p,q,p-q). \tag{4.40}$$

The last diagram missing is the one from the 2-fermion-scalar interaction. This is a rather new coupling and is almost never found in any quark DSE calculations [22, 85]. Writing it down in position space yields

$$\frac{\sqrt{2}}{2}f^{dae}\int d^dz_0d^dz_1\left(\delta_{ijkl}(1+\gamma_5)-\epsilon_{ijkl}(1-\gamma_5)\right)D^{ef}_{mnrs}(x-z_0)D^{dg}_{il}(x-z_1)\Gamma^{\overline{\psi}\psi\phi^*;bgf}_{jl;rs}(y,z_0,z_1), \tag{4.41}$$

where it has been used that

$$\Gamma^{\overline{\psi}\psi\phi^*;bgf}_{jl;rs}(x,y,z) = \frac{\delta^3\Gamma}{\delta\overline{\psi}^b_j(x)\delta\psi^g_l(y)\delta\phi^f_{rs}(z)}. \tag{4.42}$$

It is also possible to directly read of the tree-level term from Eq.(4.41) given in momentum space

$$\Gamma^{tl;\overline{\psi}\psi\phi^*;bgf}_{ij;kl}(p,q,k) = \big(\delta_{ijkl}(1+\gamma_5) - \epsilon_{ijkl}(1-\gamma_5)\big)\delta(p-q-k). \tag{4.43}$$

The last diagram of the Majorana fermion can be identified with

$$-\int d^dq\Gamma^{tl;\overline{\psi}\psi\phi^*;dae}_{ik;mn}(-q,p,q-p)D^{ef}_{mnrs}(p-q)D^{dg}_{il}(q)\Gamma^{\overline{\psi}\psi\phi^*;bgf}_{jl;rs}(-p,q,p-q) \tag{4.44}$$

Which finishes the derivation for the Majorana fermion-propagator DSE.

The diagrammatic representation is given in Fig.(4.3). The mathematical representation is

$$\begin{aligned}\left(D^{ab}_{ij}(p)\right)^{-1} &= -i\delta^{ab}\delta_{ij}\not{p} \\ &-\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\overline{\psi}\psi A;dae}_{ik;\mu}(-q,p,q-p)D^{ef}_{\mu\nu}(p-q)D^{dg}_{kl}(q)\Gamma^{\overline{\psi}\psi A;bgf}_{jl;\nu}(-p,q,p-q) \\ &+\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\overline{\psi}\psi\phi^*;dae}_{ik;mn}(-q,p,q-p)D^{ef}_{mnrs}(p-q)D^{dq}_{il}(q)\Gamma^{\overline{\psi}\psi\phi^*;bgf}_{jl;rs}(-p,q,p-q).\end{aligned} \tag{4.45}$$

Leaving only one propagator DSE open, namely the one for the scalar.

4.4 Scalars

The reason to choose the scalar DSE as last one is that the derivation due to the scalars R-structure is involved. Therefore it is easier to first derive the tree-level vertices in the other DSEs.

Here one is again confronted with the problem that the derivative of ϕ with respect to ϕ^* is not vanishing. In fact one finds the relation

$$\frac{\delta\phi^{a*}_{ij}}{\delta\phi^a_{ij}} = \frac{1}{2}\varepsilon_{ijij}. \tag{4.46}$$

with ε defined to be the total antisymmetric Levi-Civita tensor.

The scalar section of the action Eq.(4.4) is then given by

$$\begin{aligned}\mathcal{S}_{Sc} &= \int d^dx\Big[-\frac{1}{2}(D\phi^{ij})_a(D^\rho\phi^{ij})^*_a \\ &-\sqrt{2}Ref^{abc}\phi^{ij}_a(\psi^T_{ibL}\epsilon\psi_{jcL}) \\ &-\frac{1}{8}|f^{abc}\phi^b_{ij}\phi^c_{kl}|^2\Big].\end{aligned} \tag{4.47}$$

Already at the level of the action one is able to draw the diagrammatic representation of the scalar propagator DSE, see Fig.(4.4).

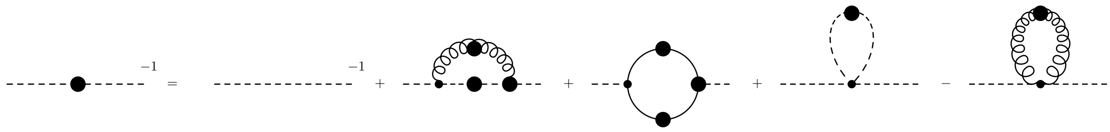


Figure 4.4: Diagrammatic representation of the $\mathcal{N} = 4$ scalar-propagator. As in the fermionic equation there is a self-energy term. Also one loop diagram gets generated due to fermion-scalar interaction. There are also two tadpole diagrams, one for the gluon and one for the scalar.

To derive the scalar-propagator we need two derivatives. One with respect to ϕ^a_{rs} and one with respect to ϕ^{b*}_{uv}. This is to keep the scalar-propagator diagonal in R-space. Note that this is a convention we also could define the scalar propagator to be the derivative of Γ with respect to $\phi^a_i j$ and $\phi^b_k l$.

We choose our scalar-propagator such that

$$\left(D^{ab}_{rsuv}(x-y)\right)^{-1} = \frac{\delta^2\Gamma}{\delta\phi^{b*}_{uv}(y)\delta\phi^a_{rs}(x)}, \tag{4.48}$$

$$D^{ab}_{rsuv}(x-y) = \frac{\delta^2 W}{\delta k^{b*}_{uv}(y)\delta k^a_{rs}(x)}, \tag{4.49}$$

with k and k^* are the sources which generate the scalar fields.

Starting the derivation with the fist term of Eq.(4.47) which yields the bare propagator

$$\left(D^{tl;ab}_{rsuv}(x-y)\right)^{-1} = \delta_{rsuv}\delta^{ab}\partial^2\delta(x-y), \tag{4.50}$$

and after Fourier transformation

$$\left(D^{tl;ab}_{rsuv}(p)\right)^{-1} = \delta_{rsuv}\delta^{ab}p^2, \tag{4.51}$$

which is beside the R-structure similar to the massless Higgs-Yang-Mills-propagator [104].

As first 3-point coupling we start with the 2-scalar-gluon interaction, or scalar selfenergy, generated by the covariant derivative

$$-f^{ace}\int d^dz_0d^dz_1\Big(\partial_\mu\big(D^{eg}_{\mu\nu}(x-z_0)D^{fc}_{mnkl}(x-z_1)\big)+D^{eg}_{\mu\nu}(x-z_0)\partial_\nu D^{fc}_{mnkl}(x-z_1)\Big)\Gamma^{\phi^*\phi A;gbf}_{kluv;\mu}(y,z_0,z_1), \tag{4.52}$$

and after Fourier-transformation we get

$$if^{ace}\int\frac{d^dq}{(2\pi)^d}(p-q)_\rho D^{cg}_{\rho\sigma}(p+q)D^{fc}_{mnkl}(q)\Gamma^{\phi^*\phi A;gbf}_{kluv;\sigma}(p+q,-p,-q), \tag{4.53}$$

where the tree-level vertex has already been worked out, see Eq.(4.26). Inserting the tree-level then yields

$$\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\phi A;ace}_{ijmn;\mu}(-p-q,p,q)D^{cg}_{\mu\nu}(q+p)D^{fc}_{mnuv}(q)\Gamma^{\phi^*\phi A;gbf}_{kluv;\nu}(p+q,-p,-q). \tag{4.54}$$

The next contribution is the fermi-loop generated by the Yukawa-interaction, which is given by

$$\frac{\sqrt{2}}{2}f^{abc}\int d^dz_0d^dz_1\big((\delta_{rsuv}(1+\gamma_5)-\epsilon(1-\gamma_5)\big)D^{cf}_{mu}(x-z_0)D^{de}_{nv}(x-z_1)\Gamma^{\phi^*\phi\psi;feb}_{kl;uv}(y,z_0,z_1), \tag{4.55}$$

and after Fourier-transformation we get

$$\frac{\sqrt{2}}{2}f^{dca}\int\frac{d^dq}{(2\pi)^d}\big((\delta_{rsuv}(1+\gamma_5)-\epsilon(1-\gamma_5)\big)D^{cf}_{mu}(p+q)D^{de}_{nv}(q)\Gamma^{\phi^*\phi\psi;feb}_{kl;uv}(-q,p+q,-p),\quad(4.56)$$

the tree-level structure is already known from the Majorana fermion-propagator, see Eq.(4.43), inserting yields

$$-\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\psi\overline{\psi};dca}_{ij;mn}(-p-q,q,p)D^{cf}_{mu}(p+q)D^{de}_{nv}(q)\Gamma^{\phi^*\psi\overline{\psi};feb}_{kl;uv}(-q,p+q,-p).\quad(4.57)$$

The remaining terms describe 4-point couplings, which produce tadpole diagrams: One due to the 2-scalar 2-gluon coupling and one due to the 4-scalar coupling. Starting with the more simple one, the 2-gluon 2-scalar tadpole, because the same arguments as in the gluon DSE Eqs.(4.28-29) reduce the derivation to

$$f^{fac}f^{fed}\left(\frac{\delta^2W}{\delta j^c_\mu(x)\delta j^d_\nu(x)}\frac{\delta W}{\delta k^e_{rs}(x)}\right)+\text{2-loop contributions ....}\quad(4.58)$$

The tadpole in position space is then given by

$$f^{fac}f^{fed}\delta_{rsuv}\int d^dz_0d^dz_1D^{cd}_{\mu\mu}(z-z_0)\delta(x-y)\delta(z_0-x)\delta(z_1-z_0),\quad(4.59)$$

and after Fourier-transformation we get

$$\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\delta_{rsuv}\big(f^{fca}f^{fdb}+f^{fcd}f^{fda}\big)\delta_{\mu\nu}D^{cd}_{\mu\nu}(q).\quad(4.60)$$

with the corresponding tree-level term

$$\Gamma^{tl;\phi^*\phi A^2;abc}_{ijkl;\mu\nu}(p,q,k)=\delta_{\mu\nu}\delta_{ijkl}(f^{fca}f^{fdb}+f^{fcd}f^{fda}).\quad(4.61)$$

The last term to this truncation is the scalar tadpole from the scalar equation, see Fig.(4.4). Since the tadpole is generated by the potential V Eq.(4.5), we write down all possible combinations

$$\begin{aligned}f^{fac}f^{fed}\Bigg(&\frac{\delta^2W}{\delta k^c_{mn}(x)\delta k^{d*}_{ij}(x)}\frac{\delta W}{\delta k^{e*}_{rs}(x)}+\frac{\delta^2W}{\delta k^{e*}_{rs}(x)\delta k^c_{mn}(x)}\frac{\delta W}{\delta k^{d*}_{ij}(x)}\\&+\frac{\delta^2W}{\delta k^{d*}_{ij}(x)\delta k^{e*}_{rs}(x)}\frac{\delta W}{\delta k^c_{mn}(x)}\Bigg)+\text{2-loop ....}\end{aligned}\quad(4.62)$$

In position space this gives

$$-f^{fac}f^{fed}\int d^dz_0d^dz_1D^{cd}_{ijij}(z-z_0)\delta(x-y)\delta(z_0-x)\delta(z_1-z_0),\quad(4.63)$$

and after performing a Fourier-transformation we get

$$\begin{aligned}-\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Big(&(f^{afi}f^{agh}+f^{afh}f^{agi})\tfrac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnop}\\&+(f^{afg}f^{aih}+f^{afh}f^{aig})\delta_{ijmn}\delta_{klop}\\&+(f^{afg}f^{ahi}+f^{afi}f^{ahg})\delta_{klmn}\delta_{ijrs}\Big)D^{cd}_{mnop}(q).\end{aligned}\quad(4.64)$$

Writing down the 4-scalar tree-level vertex explicitly

$$\begin{aligned}
\Gamma^{tl;\phi\phi\phi^*\phi^*;figh}_{ijklmnop} &= \frac{1}{2}\Big((f^{afi}f^{agh}+f^{afh}f^{agi})\frac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnop}\\
&+(f^{afg}f^{aih}+f^{afh}f^{aig})\delta_{ijmn}\delta_{klop}\\
&+(f^{afg}f^{ahi}+f^{afi}f^{ahg})\delta_{klmn}\delta_{ijop}\Big). \qquad (4.65)
\end{aligned}$$

Which is analogous to the gluon-tadpole in the gluon DSE in the scenes that this vertex accumulates eight R-indices, while the gluon-tadpole accumulates four Lorentz-indices.

Since all contributions to this truncation are worked out, it is now possible to write down the full scalar-propagator DSE:

$$\begin{aligned}
D^{ab-1}_{ijkl}(p) &= \delta^{ab}\delta_{ijkl}p^2\\
&+\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\phi A;eac}_{\mu;ijmn}(-p-q,p,q)D^{cg}_{\mu\nu}(q+p)D^{fc}_{mnuv}(q)\Gamma^{\phi^*\phi A;gbf}_{\nu;kluv}(p+q,-p,-q)\\
&-\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\phi\psi;dca}_{ij;mn}(-p-q,q,p)D^{cf}_{mu}(p+q)D^{de}_{nv}(q)\Gamma^{\phi^*\phi\psi;feb}_{kl;uv}(-q,p+q,-p)\\
&+\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\phi A^2;bcef}_{\mu\nu;ijkl}(-p,p,-q,q)D^{ef}_{\mu\nu}(q)\\
&-\frac{1}{4}\int\frac{d^dq}{(2\pi)^d}\Gamma^{tl;\phi^*\phi^*\phi\phi;bcef}_{ijklmnop}(-p,p,-q,q)D^{ef}_{mnop}(q). \qquad (4.66)
\end{aligned}$$

4.5 Propagators of $\mathcal{N}=4$ SYM

A short resume of the chapter is given here. Starting with the action Eq.(4.4) employing the derivation proceeder developed in Chapter 3 yields four DSEs. Those four equations have to be solved selfconsistently, which is due to the fact, that full propagators appear on the right hand side as well as on the left hand side. The full propagators are then given by their bare ones multiplied by an unknown dressing function $G(p^2), Z(p^2), F(p^2)$ and $S(p^2)$, which contains all the interesting physics

Table 4.1: Table of all $\mathcal{N}=4$ SYM propagators, namely the ghost, the gluon, the fermion and the scalar, with their respective dressing functions as well as their bare propagators. Note that all of them denoted as D, but those four propagators differ in the number of indices.

Propagatos	Particle	Function	Dressing function
$D^{ab}_G(p)$	Ghost	$-\frac{\delta^{ab}}{p^2}G(p^2)$	$G(p^2)$
$D^{ab}_{\mu\nu}(p)$	Gluon	$\delta^{ab}(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2})\frac{Z(p^2)}{p^2}$	$Z(p^2)$
$D^{ab}_{ij}(p)$	Fermion	$-i\delta_{ij}\delta^{ab}\not{p}_{\alpha\beta}\frac{F(p^2)}{p^2}$	$F(p^2)$
$D^{ab}_{ijkl}(p)$	Scalar	$\delta_{ijkl}\delta^{ab}\frac{S(p^2)}{p^2}$	$S(p^2)$

Summarizing all four DSEs with their mathematical representation, first the ghost, second the gluon, third the fermion and last the scalar:

$$
\begin{aligned}
D_G^{ab-1}(p) &= -\delta^{ab}p^2 \\
&\quad -igf^{ade}\int\frac{d^dq}{(2\pi)^d}D_{\mu\nu}^{ef}(p-q)D_G^{dh}(q)\Gamma_\nu^{c\bar{c}A;bfh}(-p,q,p-q),
\end{aligned}
$$

$$
\begin{aligned}
D_{\mu\nu}^{ab-1}(p) &= \delta^{ab}(\delta_{\mu\nu}p^2-p_\mu p_\nu) \\
&\quad -\int\frac{d^dq}{(2\pi)^d}\Gamma_\mu^{tl;A\bar{c}c;adc}(-p-q,q,p)D_G^{de}(q)D_G^{cf}(p+q)\Gamma_\nu^{c\bar{c}A;feb}(-q,p+q,-p) \\
&\quad +\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma_{\mu\sigma\chi}^{tl;A^3;dca}(p,q-p,-q)D_{\sigma\omega}^{cf}(q)D_{\chi\lambda}^{de}(p-q))\Gamma_{\nu\omega\lambda}^{A^3;bfe}(-p,q,p-q) \\
&\quad +\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma_{ij;\mu}^{tl;A\overline{\psi}\psi;acd}(p,q-p,-q)D_{ik}^{de}(q)D_{jl}^{cf}(p-q)\Gamma_{kl;\nu}^{A\overline{\psi}\psi;bef}(-p,q,p-q) \\
&\quad +\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma_{ijkl;\mu}^{tl;A\phi^*\phi;acd}(p,q-p,-q)D_{ijmn}^{de}(q)D_{klpq}^{cf}(p-q)\Gamma_{mnpq;\nu}^{tl;A\phi^*\phi;bef}(-p,q,p-q) \\
&\quad +\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma_{\mu\nu\sigma\rho}^{tl;A^4;abcd}(p,-p,q,-q)D_{\sigma\rho}^{cd}(q) \\
&\quad -\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\Gamma_{ijkl;\mu}^{tl;A^2\overline{\phi}\phi;abcd}(p,-p,q,-q)D_{mnmn}^{cd}(q),
\end{aligned}
$$

$$
\begin{aligned}
D_{ij}^{ab-1}(p) &= -i\delta^{ab}\delta_{ij}\not{p} \\
&\quad -\int\frac{d^dq}{(2\pi)^d}\Gamma_{\mu;ik}^{tl;\overline{\psi}\psi A;dae}(-q,p,q-p)D_{\mu\nu}^{ef}(p-q)D_{kl}^{dg}(q)\Gamma_{\nu;jl}^{\overline{\psi}\psi A;bgf}(-p,q,p-q) \\
&\quad +\int\frac{d^dq}{(2\pi)^d}\Gamma_{mn;ik}^{tl;\overline{\psi}\psi\phi^*;dae}(-q,p,q-p)D_{mnrs}^{ef}(p-q)D_{il}^{dq}(q)\Gamma_{rs;jl}^{\overline{\psi}\psi\phi^*;bgf}(-p,q,p-q),
\end{aligned}
$$

$$
\begin{aligned}
D_{ijkl}^{ab-1}(p) &= \delta^{ab}\delta_{ijkl}p^2 \\
&\quad +\int\frac{d^dq}{(2\pi)^d}\Gamma_{\mu;ijmn}^{tl;\phi^*\phi A;eac}(-p-q,p,q)D_{\mu\nu}^{cg}(q+p)D_{mnuv}^{fc}(q)\Gamma_{\nu;kluv}^{\phi^*\phi A;gbf}(p+q,-p,-q) \\
&\quad -\int\frac{d^dq}{(2\pi)^d}\Gamma_{ij;mn}^{tl;\phi^*\phi\psi;dca}(-p-q,q,p)D_{mu}^{cf}(p+q)D_{nv}^{de}(q)\Gamma_{kl;uv}^{\phi^*\phi\psi;feb}(-q,p+q,-p) \\
&\quad +\int\frac{d^dq}{(2\pi)^d}\Gamma_{\mu\nu;ijkl}^{tl;\phi^*\phi A^2;bcef}(-p,p,-q,q)D_{\mu\nu}^{ef}(q) \\
&\quad -\frac{1}{4}\int\frac{d^dq}{(2\pi)^d}\Gamma_{mnmn}^{tl;\phi^*\phi^*\phi\phi;bcef}(-p,p,-q,q)D_{ijkl}^{ef}(q).
\end{aligned}
$$

This concludes Chapter 4. In the next chapter an ansatz will be employed to solve this coupled equations simultaneously. This is in fact possible and easier than in the Yang-Mills case due to the absence of an overall scale Λ_{YM}. So far we still got tadpole diagrams. Chapter 5 will show that they cancel each other.

Chapter 5

Self-consistent solutions

In Chapter 4 the DSEs for all four propagators were derived, see Eqs.(4.10,32,45,66). Now one is interested in solving them. This is a rather complex task. If one wants to solve a DSE self-consistently one will end up with an infinite tower of coupled integral equations [21]. In the previous chapter it was shown that n-point functions depend on higher n-point functions. A simple example for this behaviour are the propagator DESs since all of them already depend on 3-point functions, for 3-point DSEs see [108–110].

To derive a mathematically correct result would mean that one is confronted with solving the 3- and 4-point functions in a self consistent way too. Stating the arguments in other words: Solve a DSE for every coupling. This is impossible [79]. Even for the simple ghost case, trying to solve the ghost-gluon vertex self consistently will yield so called triangle and swordfish diagrams which then have to be solved too [111].

Despite the fact that DSEs in principle would yield an exact result, their structure makes it impossible to observe that exact result. But it is possible to cut this infinite tower of equations and take into account only a subset of couplings [22]. This is a well established method and various analysis of different orders have been performed in the Yang-Mills case [22, 38, 85–87].

$\mathcal{N} = 4$ SYM is a superconformal theory [19]. The solutions should be superconformal as well. Introducing a cutoff scale Λ would not be in agreement with Eqs.(2.22-25). Therefore the solutions have to be scale invariant, see Chapter 2 dilation. Since we use Landau gauge SUSY gets "hidden". If one is interested in restoring SUSY, one could of course use Feynman-Wess-Zumino gauge [59]. This on the other hand would make it impossible to compare results with ordinary Yang-Mills theory.

5.1 The problem

The Problem is to find a self-consistent solution for the integral equations given in Chapter 4. All of them are of the kind (, excluding the tadpoles)

$$\left(D^{\cdots}_{\cdots}(p)\right)^{-1} = \int \frac{d^d q}{(2\pi)^d} \Gamma^{tl;\ldots;\ldots}_{\cdots}(-q, q+p, p) D^{\cdots}_{\cdots}(p-q) D^{\cdots}_{\cdots}(q) \Gamma^{\ldots;\ldots}_{\cdots}(-p, q, p-q), \tag{5.1}$$

the dots denote

- Gauge symmetry,
- R-symmetry,
- And within vertices the particles.

The only thing known are the tree-level propagators and vertices, so all full propagators and all full vertices have to be modelled.

One should notice that the tadpole integrals can be computed, since only the tree-level vertices are involved. From perturbation theory it is already known that the tadpoles vanish [61]. In the gluon- as well as scalar propagator DSE the tadpoles come with different signs.

Ansätze

First of all Eqs.(4.10,32,45,66) are rather hard to handle. Meaning that these equations are matrix like in the R-space, the gauge-group-space (e.g. $SU(N)$) and the Lorentz-space. What we want is to work with scalar equations. This can be done by simply multiplying Eqs.(4.10,32,45,66) with their respective bare propagators, given in Tab.(4.1), and contracting the remaining indices. The program **FORM** [112] was used to carry out this contractions and doing the Dirac-traces.

While the results of Chapters 3 and 4 are within d dimensions, the interesting case considered in this work are 4 dimensions. The integrals change to

$$\int \frac{d^d q}{(2\pi)^d} \quad \rightarrow \quad \int \frac{d^4 q}{(2\pi)^4}. \tag{5.2}$$

Still there are unknown functions. To make progress we need to make ansätze for the dressing functions and the full vertices. These dressing function ansätze are dictated by conformal symmetry. Those need to be powerlaws, see Chapter 2 and [48]. For the vertices we model them with their bare ones, this is called Rainbow ladder approximation [22, 38, 85–87, 97]. The unknown functions then take the form

$$D^{ab}_{\dots}(p) = cp^{\kappa} \tag{5.3}$$
$$\Gamma^{\dots;abc}_{\dots}(p,q,k) = \Gamma^{tl;\dots;abc}_{\dots}(p,q,k). \tag{5.4}$$

For simplicity and comparison reasons we decide not to go beyond Rainbow ladder.

There are more sophisticated truncations, see [108, 110, 113–115], nevertheless qualitative features seem to be captured correctly

What is left now to do is to use the ansätze to self-consistently solve the propagator DSEs. Therefore we look for a scaling solution [102] of Eqs.(4.10,32,45,66).

5.2 Scaling solution

Since it is expected that the conformal symmetry is not broken, conformal solutions should exist, i.e. scaling solutions [76]. Such scaling solutions can be represented using Eq.(5.3)

$$G(x) = ax^{\kappa_1}, \tag{5.5}$$
$$Z(x) = bx^{\kappa_2}, \tag{5.6}$$
$$F(x) = fx^{\kappa_3}, \tag{5.7}$$
$$S(x) = sx^{\kappa_4}. \tag{5.8}$$

where $x = p^2$. These scaling solutions can also be found in the IR region of Yang-Mills theory [27, 102, 116]. This is then called conformal window of Yang-Mills theory. For $\mathcal{N} = 4$ SYM these scaling solutions have to be valid for the whole momentum spectrum.

The four propagator dressing functions introduce due to the scaling ansatz new "free" parameter. These are some constants a, b, f, s and $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ some yet undetermined exponents. One major step towards a self-consistent solution is the requirement of tadpole cancellation. These primitively divergent diagrams can in fact be computed and a ratio of

$$\frac{b}{s} = 3, \tag{5.9}$$

exactly cancels them in the propagator DSEs. If they do not cancel there will be quadratic divergences, which will require a gauge-noninvariant mass counter-term for the gluon. Therefore, only due to counter-term reasons we can either eliminate b or s.

5.2.1 The ghost equation

Starting with the simplest of these four DSEs, we insert the scaling ansatz for the propagator dressing and the tree-level model for the full vertex and get

$$\frac{1}{G(x)} = 1 - C_A g^2 \int \frac{d^4q}{(2\pi)^4} K(x,y,z) G(y) Z(z). \tag{5.10}$$

This equation almost looks like Eq.(4.10) but is in fact the result of plugging in the discussed ansätze. C_A is the adjoint Cassimir operator of the gauge group generated by $f^{abc} f^{abc}$. For a $SU(N)$ gauge group this would be N_C, which corresponds to the number of colors. The momenta are

$$x = p^2, \tag{5.11}$$
$$y = q^2, \tag{5.12}$$
$$z = (p-q)^2. \tag{5.13}$$

What is left is of Eq.(5.10) is the integral kernel $K(x,y,z)$ which is given by

$$K(x,y,z) = -\frac{x^2 + (y-z)^2 - 2x(y+z)}{4xyz^2}. \tag{5.14}$$

This is already everything which can be computed for the ghost equation for now. What remains is to integrate this equation. This will be done when all integral kernels are known.

5.2.2 The gluon equation

The gluon equation has significantly more terms, but less terms than Eq.(4.32) since the tadpole contributions already dropped out. Inserting the ansätze yields

$$\begin{aligned} \frac{1}{Z(x)} = {} & 1 + C_A g^2 \int \frac{d^4q}{(2\pi)^4} M(x,y,z) G(y) G(z) + C_A g^2 \int \frac{d^4q}{(2\pi)^4} N(x,y,z) Z(y) Z(z) \\ & + C_A g^2 \int \frac{d^4q}{(2\pi)^4} Q(x,y,z) S(y) S(z) + C_A g^2 \int \frac{d^4q}{(2\pi)^4} R(x,y,z) F(y) F(z), \end{aligned} \tag{5.15}$$

where the only unknown functions are the integral kernels M, N, Q, R. They are given by

$$\begin{aligned}
M(x,y,z) &= -\frac{x^2+(y-z)^2-2x(y+z)}{12x^2yz}, && (5.16)\\
N(x,y,z) &= -\frac{1}{24x^2y^2z^2}\Big(x^4+8x^3(y+z)+(y-z)^2(y^2+10yz+z^2)\\
&\quad -2x^2(9y^2+16yz+9z^2)+8x(y^3-4y^2z-4yz^2+z^3)\Big), && (5.17)\\
Q(x,y,z) &= -\frac{16}{x^2yz}\Big(x^2+(y-z)^2-2x(y+z)\Big), && (5.18)\\
R(x,y,z) &= -\frac{4}{xyz}(-x+y+z). && (5.19)
\end{aligned}$$

Those without fermions, respectively M, N and Q are in full agreement with the Yang-Mills theory with a Higgs [84], since the R-structure does not alter the momentum dependence.

5.2.3 The fermion equation

The fermion equation consists only of one additional term compared to the ghost, e.g. the Yukawa term, employing Eq(5.3-4) we get

$$\frac{1}{F(x)} = 1 - C_A g^2 \int \frac{d^4q}{(2\pi)^4} L(x,y,z)F(y)Z(z) + C_A g^2 \int \frac{d^4q}{(2\pi)^4} H(x,y,z)F(y)S(z), \quad (5.20)$$

with the integral kernels

$$\begin{aligned}
L(x,y,z) &= \frac{24}{xyz}(x+y-z), && (5.21)\\
H(x,y,z) &= \frac{192}{xyz}(x+y-z). && (5.22)
\end{aligned}$$

Note that the signs for both kernels are positive.

5.2.4 The scalar equation

The last of the four propagator DSEs is the scalar DSE. After cancelling the tadpoles only two interaction diagrams are present. The DSE reduces to a selfenergy term and a Yukawa term. Using the ansatz for the dressing functions Eq.(5.5-8) as well as the tree-level vertices produces

$$\frac{1}{S(x)} = 1 - C_A g^2 \int \frac{d^4q}{(2\pi)^4} D(x,y,z)S(y)Z(z) + C_A g^2 \int \frac{d^4q}{(2\pi)^4} B(x,y,z)S(y)F(z), \quad (5.23)$$

with the kernels

$$\begin{aligned}
D(x,y,z) &= \frac{96}{xyz^2}\Big(x^2+(y-z)^2-2x(y-z)\Big), && (5.24)\\
B(x,y,z) &= -\frac{96}{xyz}(-x+y+z). && (5.25)
\end{aligned}$$

As a side remark it is *interesting* that the scalar kernels are the only ones with different signs.

This finishes the work done by **FORM** [112]. Since there are no more unknown functions one can integrate with respect to the loop momentum q. Due to the fact that only conformal solutions should appear, i.e. powerlaws, the bare propagator contributions needed to be cancelled. These bare propagators correspond to the 1 on the right hand side of Eqs.(5.10,15,20,23). This can be done by a global shifting of the renormalization constants, which leave the propagator equations with terms which only depend, on powers of the momenta.

5.2.5 Renormalization

What remains, is to integrate these equations with respect to the loop momentum. The integrals can be calculated using [1]

$$\int \frac{d^dq}{(2\pi)^d} q^{2\alpha}(q-p)^{2\beta} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(-\alpha-\beta-\frac{d}{2})\Gamma(\frac{d}{2}+\alpha)\Gamma(\frac{d}{2}+\beta)}{\Gamma(d+\alpha+\beta)\Gamma(-\alpha)\Gamma(-\beta)} y^{2(\frac{d}{2}+\alpha+\beta)}. \tag{5.26}$$

Again we can relate this to Yang-Mills theory in the IR-region [17, 117, 118] where the same integral formula is used. This is only correct if the integrals are finite. For infinite integrals we have to deal with regulated versions. For arbitrary exponents κ_i these integrals can have divergences. Since we choose Landau gauge this is actually expected. We employ here a renormalization scheme, in which all counter-terms δZ_i are chosen [76]

$$\delta Z_i = -1 + f(\Lambda), \tag{5.27}$$

where all divergences are put into the function $f(\Lambda)$ such that the integrals evaluate to the form Eq.(5.26). The term -1 is a possible finite shift of the renormalization constants. This is needed to find an exponent κ that is not 0. Otherwise the tree-level terms would spoil the calculation. Also no mass counter-terms are necessary, since there are no divergences require them to appear.

5.3 Analysis of the critical exponent

As stated above, it is expected that one exponent will solve the whole system of coupled propagator equations. Therefore, all the κ_i need to be in some relation to κ,

$$\kappa_i \approx a_i \kappa. \tag{5.28}$$

Finding these relations is now the main task. For now it is not even clear if such an exponent exists, because a truncation was employed which can in principle alter this scaling solution property. In fact with this simple vertex modelling it is not possible to find a single κ that solves all four equations, Eqs.(5.10,15,20,23). Without the freedom to renomarlize the only valid κ would have to be 0. But if $\kappa = 0$ only the ghost DSE can be solved, all other DSEs contain inconsistencies.

For all full propagators the tree-level structure was taken and multiplied with a scalar dressing function. Generally this can be done with the full vertices too. This will be done for the scalar gluon vertex, but is not generally needed. Another option is to include the full tensor structures for a vertex. This is by no means comparable to the simple scalar dressing functions of the propagators. For the 3-gluon vertex with transverse projection this will be something like

$$\Gamma^{A^3} = \sum_{i=1}^{4} a_i(x,y,z)\gamma_i^{A^3}, \tag{5.29}$$

where all indices are omitted. Why the 3-gluon vertex is omitted will be discussed in a later subsection, see the irrelevance of the gluon-loop.

To keep the conformal symmetry powerlaw-like dressing will be employed. Meaning that the functions a_i are some kind of powerlaw of the involved momenta. In fact they have to look like

$$a_i(x,y,z) = c_i x^{\alpha} y^{\beta} z^{\gamma}. \tag{5.30}$$

Note that the exponents α, β and γ are arbitrary but still have to represent the nature of the coupling. Meaning that for bose-symmetric vertices like the 3-gluon vertex one requires $\alpha = \beta = \gamma$. We only employ a vertex dressing to those vertices which do not have a matching exponent. As an example one can think of

$$\frac{1}{c}x^p = D_1 x^{p+m}, \tag{5.31}$$

and dressing such that

$$\frac{1}{c}x^p = D_1 D_d x^{p+m} x^{-m}, \tag{5.32}$$

with a vertex dressing which ensures that the momentum dependence drops out.

5.3.1 Pure Yang-Mills solution

Up to now all integrals have been performed and four propagator equations can be formulated. Since the physical spectrum of $\mathcal{N} = 4$ SYM is at least at low energy irrelevant, lets briefly discuss this analysis for pure Yang-Mills theory. DSE calculation also finds a scaling solution for Yang-Mills theory [27, 102, 116]. In Chapter 3 already all equations are known. Therefore, the coupled equations read

$$\frac{1}{a}x^{-\kappa_1} = g_1 x^{\kappa_1+\kappa_2}, \tag{5.33}$$

$$\frac{1}{b}x^{-\kappa_2} = z_1 x^{2\kappa_1} + z_2 x^{2\kappa_2} + \text{tadpol}. \tag{5.34}$$

Actually this system of coupled equations is simpler than the one of $\mathcal{N} = 4$ SYM. Also diagrams can be evaluated in order of couplings in pure Yang-Mills and only the IR-leading terms are taken into account [26]. Due to this behaviour it is possible to identify the leading contribution, therefore

$$\frac{1}{a}x^{-\kappa_1} = g_1 x^{\kappa_1+\kappa_2} \tag{5.35}$$

$$\frac{1}{b}x^{-\kappa_2} = z_1 x^{2\kappa_1} \tag{5.36}$$

Note that only the ghostloop survives while both, the tadpole as well as the gluonloop, are subleading and therefore of no interest. The natural choice here is $\kappa_2 = -2\kappa_1$. After that one require the ghost to be IR-divergent, therefore $\kappa = -\kappa_1$. Doing so yields

$$\frac{1}{a} = g_1 \tag{5.37}$$

$$\frac{1}{b} = z_1 \tag{5.38}$$

Since g_1 and z_1 are known functions of the form $C \times \prod_i \frac{\Gamma_i(f(\kappa))}{\Gamma_i(f(\kappa))}$ with arbitrary functions $f(\kappa)$, we are able to solve Eqs.(5.10,5.15). Deforming the Γ-functions, see Appendix, such that the right hand side of both equations are equal yields

$$\frac{(2+\kappa)(1+\kappa)}{(3-2\kappa)} = \frac{\Gamma^2(2-\kappa)\Gamma(2\kappa)}{\Gamma(4-2\kappa)\Gamma^2(1+\kappa)}\frac{g^2 C_A}{48\pi^2}a^2 b, \tag{5.39}$$

$$\frac{4\kappa-2}{1} = \frac{\Gamma^2(2-\kappa)\Gamma(2\kappa)}{\Gamma(4-2\kappa)\Gamma^2(1+\kappa)}\frac{g^2 C_A}{48\pi^2}a^2 b. \tag{5.40}$$

Cancelling the right hand side of both equations and equating them, yields as solution for κ:

$$\kappa = \frac{93-\sqrt{1201}}{98} \approx 0.5953. \tag{5.41}$$

Therefore this value of κ defines the IR-scaling solution for Yang-Mills theory. It will be useful to come back to this later for comparison with $\mathcal{N}=4$ SYM. As a remark Yang-Mills theory also allows a different solution for the DSEs, the so-called decoupling solution [26, 119]

5.3.2 The κ of $\mathcal{N}=4$ SYM

The DSEs define a set of coupled equations for the exponent κ. As shown earlier in this chapter it is possible to renormalize and cancel the bare propagators as well as integrate the loop integrals according to Eq.(5.26). This has been done with the help of **Mathematica** and yields a general structure of a product of Γ-functions and $x^{\sum_i \kappa_i}$ such that the propagator equations, see Eqs.(5.10,15,20,23) reduce to

$$\frac{1}{a}x^{-\kappa_1} = g_1 x^{\kappa_1+\kappa_2}, \tag{5.42}$$

$$\frac{1}{3s}x^{-\kappa_2} = z_1 x^{2\kappa_1} + z_2 x^{2\kappa_2} + z_3 x^{2\kappa_3} + z_4 x^{2\kappa_4}, \tag{5.43}$$

$$\frac{1}{f}x^{-\kappa_3} = f_1 x^{\kappa_3+\kappa_2} + f_2 x^{\kappa_3+\kappa_4}, \tag{5.44}$$

$$\frac{1}{s}x^{-\kappa_4} = s_1 x^{\kappa_4+\kappa_2} + s_2 x^{2\kappa_3}. \tag{5.45}$$

The functions g_i, z_i, f_i and s_i are known functions of Γ-functions, coupling constant g and the adjoint Casimir of the gauge group C_A.

The task now is to find the relations between the four κ's introduced in Eqs.(5.5-8) such that these four reduce to just one κ, which then simultaneously solves Eqs.(5.10,15,20,23). To do so we match the exponents of the momenta x on the left- with the ones on the right hand side. Starting with the ghost equation yields

$$\kappa_2 = -2\kappa_1, \tag{5.46}$$

which is actually the same relation that is found in the Yang-Mills case [116–118]. This is due to the fact that the ghost equation Eq.(5.10) consists in both theories of only one term and is therefore very stringent. Modifying the equations with this relation

$$\frac{1}{a}x^{-\kappa_1} = g_1 x^{-\kappa_1}, \tag{5.47}$$

$$\frac{1}{3s}x^{2\kappa_1} = z_1 x^{2\kappa_1} + z_2 x^{-4\kappa_1} + z_3 x^{2\kappa_3} + z_4 x^{2\kappa_4}, \tag{5.48}$$

$$\frac{1}{f}x^{-\kappa_3} = f_1 x^{\kappa_3 - 2\kappa_1} + f_2 x^{\kappa_3 + \kappa_4}, \tag{5.49}$$

$$\frac{1}{s}x^{-\kappa_4} = s_1 x^{\kappa_4 - 2\kappa_1} + s_2 x^{2\kappa_3}. \tag{5.50}$$

The next equation is the fermion equation. It is required, that

$$\kappa_1 = \kappa_3, \quad \kappa_4 = -2\kappa_3, \tag{5.51}$$

which means that the fermions as well as the bosons have to have the same exponent. Therefore the fermion and scalar mirroring the behaviour of the Yang-Mills sector.

Since it is observed that the ghost and fermion have the same exponent it is now possible to write the equations with just a single κ, i.e.

$$\kappa_3 = \kappa_1 = -\kappa, \tag{5.52}$$

and thus

$$\frac{1}{a}x^{-\kappa} = g_1 x^{-\kappa}, \tag{5.53}$$

$$\frac{1}{3s}x^{2\kappa} = z_1 x^{2\kappa} + z_2 x^{-4\kappa} + z_3 x^{2\kappa} + z_4 x^{-4\kappa}, \tag{5.54}$$

$$\frac{1}{f}x^{-\kappa} = f_1 x^{-\kappa} + f_2 x^{-\kappa}, \tag{5.55}$$

$$\frac{1}{s}x^{2\kappa} = s_1 x^{-4\kappa} + s_2 x^{2\kappa}. \tag{5.56}$$

This choice of κ solves both the ghost and the fermion equation. Still the bosonic equations, the scalar and gluon, are not momentum independent. To resolve this issue a dressing for the vertices is needed. Since almost all momentum dependency already cancels, it is only necessary to dress 2 vertices, the three gluon vertex and the gluon-scalar vertex. The later one needs to be dressed both in the scalar as well as the gluon equation, Eqs(5.54,56). Since we already expect that the four equations Eqs(5.42-45) can not be solved within this simple truncation, a conformal vertex dressing, see Eq.(5.30) was introduced. There also more sophisticated dressings see [120]. The conformal dressing for the scalar gluon vertex is of the form

$$\Gamma^{\phi^*\phi A}(p^2, q^2, k^2) = S p^{2\alpha_1} q^{2\alpha_1} k^{2\alpha_2}, \tag{5.57}$$

where S is simply a multiplicative constant. α_i with $i = 1, 2$ refers to the propagator, either gluon or scalar. For the three gluon vertex we used the dressing

$$\Gamma^{A^3}(p^2, q^2, k^2) = S_G p^{2\alpha} q^{2\alpha} k^{2\alpha}. \tag{5.58}$$

Note S and S_G are different constants. On the left hand side of the bosonic propagator equations a momentum behaviour of $x^{2\kappa}$ is given. It is now possible to match the scalar-gluon vertices with the left hand side, taking into account only the exponents of Eqs.(5.54,56)

$$2\kappa = -4\kappa + 2\alpha_2 + \alpha_1 \quad \text{for the gluon eq.,} \tag{5.59}$$

$$2\kappa = -4\kappa + 2\alpha_2 + \alpha_1 \quad \text{for the scalar eq..} \tag{5.60}$$

These equations have several solutions. The choice for this thesis was to set the gluonic α to 0 and vary the scalar α. This yields

$$2\kappa = -4\kappa + 2\alpha_2 \quad \text{for the gluon eq.,} \tag{5.61}$$

$$2\kappa = -4\kappa + 2\alpha_2 \quad \text{for the scalar eq..} \tag{5.62}$$

and therefore

$$\alpha = 6\kappa. \tag{5.63}$$

This leaves just one more issue, namely the gluon-loop. Again there is a matching condition for the power of the momentum which, after using the dressed vertex, is given by

$$2\kappa = -4\kappa + 6\alpha_G \tag{5.64}$$

This equation has one solution, since there is only a single exponent to vary. Choosing $\alpha = -\kappa$ leads to a $\Gamma(-1)$ after integration. Which is not a well defined value. At this point the employed truncation comes to its limit. As it is shown later it is justifiable to simply drop the gluon-loop, see subsection: The irrelevance of the gluon-loop.

The remaining equations are

$$\frac{1}{a}x^{-\kappa} = g_1 x^{-\kappa}, \tag{5.65}$$

$$\frac{1}{3s}x^{2\kappa} = z_1 x^{2\kappa} + z_3 x^{2\kappa} + z_{4d} x^{2\kappa}, \tag{5.66}$$

$$\frac{1}{f}x^{-\kappa} = f_1 x^{-\kappa} + f_2 x^{-\kappa}, \tag{5.67}$$

$$\frac{1}{s}x^{2\kappa} = s_{1d} x^{2\kappa} + s_2 x^{2\kappa}. \tag{5.68}$$

The subscript indicates that $s_1 \neq s_{1d}$ and $z_4 \neq z_{4d}$. As already said the z_i, f_i, s_i are a product of the Γ-functions and constants. Since all momentum dependencies dropped out it is now possible to extract a common factor from all equations, given by

$$\omega = \frac{C_a g^2 \Gamma^2(2-\kappa)\Gamma(2\kappa) 3a^2 s}{48\pi^2 \Gamma(4-2\kappa)\Gamma^2(1+\kappa)}. \tag{5.69}$$

One ends up with the equations

$$\omega = -\frac{(1+\kappa)(2+\kappa)}{18(-3+2\kappa)} \quad \text{for the ghost,} \tag{5.70}$$

$$\omega = \frac{4\kappa - 2}{3(1 + 864B + 16F^2(2\kappa - 3))} \quad \text{for the gluon,} \tag{5.71}$$

$$\omega = -\frac{(1+\kappa)(2+\kappa)}{14400F^2(3 - 8\kappa + 4\kappa^2)} \quad \text{for the fermion,} \tag{5.72}$$

$$\omega = \frac{(2\kappa - 1)(2 + 3\kappa + \kappa^2)}{864(2\kappa - 2)(2F^2 + 72B + 3F^2\kappa - 144B\kappa + F^2\kappa)} \quad \text{and the scalar,} \tag{5.73}$$

where F, B are relations between the constants of Eqs.(5.5-8), given by

$$F = \frac{a}{f}, \tag{5.74}$$

$$B = S\left(\frac{b}{a}\right)^2 = 9S\left(\frac{s}{a}\right)^2, \tag{5.75}$$

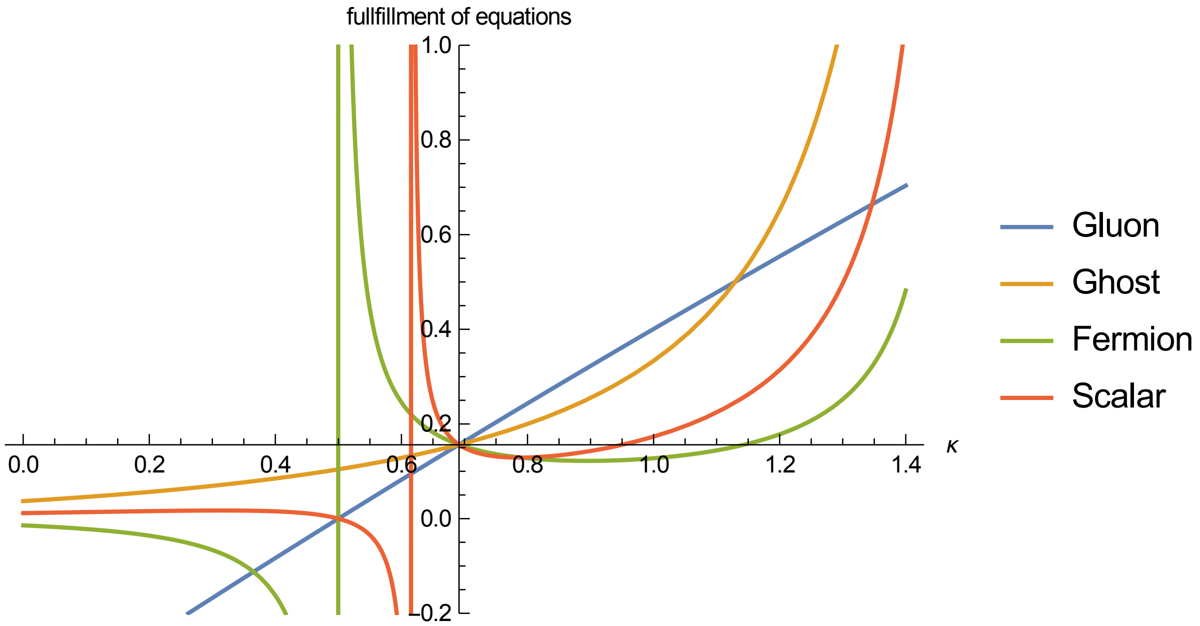


Figure 5.1: A plot of all ω's which describe the propagators. All of those four ω's intersect at a single point κ which corresponds to the exponent for which all propagator equations are fulfilled. The colouring denotes the particles: the yellow line corresponds to the ghost, the blue to the gluon, the green to the fermion and the red one to the scalar.

where S is the constant which arises due to the vertex dressing of the scalar-gluon vertex. These are four equations for the four unknown quantities a^2s, F, B and κ. Plotting all ω's against each other, already gives a hint that there is one κ smaller than 1, see Fig(5.1).

After a graphical presentation of the solution for κ in Fig.(5.1) we want to calculate the exact result. Eliminating the three constants yields a conditional equation for κ,

$$-1 = \frac{50(1-2\kappa)^2(146-381\kappa+193\kappa^2)}{(1+\kappa)(2+\kappa)(218-823\kappa+819\kappa^2)}, \tag{5.76}$$

which is a fourth-order polynomial for κ. Two solutions are complex one $\kappa > 1$ and only one $\kappa \in (0,1]$ [76]. This is important because otherwise the propagators cannot be interpreted as a well tempered distribution. The desired κ is then given approximately by

$$\kappa \approx 0.691354,$$

which is quite close to the value of the corresponding exponent in the so-called scaling solution of Yang-Mills theory where $\kappa \approx 0.59$. With the known value of κ it is possible to calculate the

Table 5.1: Table of particles and their related dressing functions, as well as their functional form. All of the dressing functions given with the scaling found in Chapter 5. The exponent κ found to be $\kappa \approx 0.69$

Particle	Dressing	Function
Ghost	$G(x)$	$ax^{-\kappa}$
Gluon	$Z(x)$	$3s(a)x^{2\kappa}$
Fermion	F(x)	$f(a)x^{-\kappa}$
Scalar	$S(x)$	$s(a)x^{2\kappa}$

other quantities, quoting only real positive ones, i.e.

$$F = \frac{a}{f} = \frac{\sqrt{2+3\kappa+\kappa^2}}{20\sqrt{2}\sqrt{2\kappa^3+5\kappa^2+\kappa-2}} \approx 0.0571506, \tag{5.77}$$

$$B = 9S\frac{s^2}{a^2} = \frac{1906-8445\kappa+11747\kappa^2-4902\kappa^3}{43200(-2+\kappa+5\kappa^3-2\kappa^3)} \approx 0.00828921, \tag{5.78}$$

$$3a^2sg^2C_A = \frac{2^{5-2\kappa}\pi^{\frac{3}{2}}(1+\kappa)(2+\kappa)\Gamma(\frac{3}{2}-\kappa)\Gamma(1+\kappa)^2}{3\Gamma(2\kappa)} \approx 110.876, \tag{5.79}$$

which shows that the result is valid for any gauge coupling and gauge group. Still the total value of a, f, s and S are undetermined but their ratios are fixed. This leaves an overall scale open. Most importantly all of these can have positive values creating suitable propagators. One interesting fact is that the pre-factor of the gluon and the scalar are of the same size while the pre-factor of the fermion is three orders smaller than the one of the ghost. With the truncation used in this thesis there is no relation between the scalar and ghost pre-factor. Those two can substantially differ, which also again possibly yields a very different order of magnitude for the pre-factor of the scalar-gluon vertex.

Stating the propagator dressing functions with the scaling we observed in Tab.(5.1).

Note that the gluon-loop was dropped out of the calculation and the gluon-scalar vertex got a dressing given by

$$\Gamma^{\phi^*\phi A}(x,y,z) = S(a)x^{\alpha_1}y^{\alpha_1}z^{6\kappa-2\alpha_1}. \tag{5.80}$$

The ghost per-factor a is the only independent constant and α_1 is some arbitrary exponent. All other vertices remain bare. The value of κ is about 0.69 and the ratios of all remaining pre-factors are fixed.

This solution is highly non-trivial and at the same time realizes the conformal properties of the theory [76]. As in the scaling solution of Yang-Mills theory the ghost diverges, while the gluon vanishes. The same is true for the relation of the fermion and scalar, meaning the fermion diverges and the scalar vanishes, mirroring the Yang-Mills sector [76].

5.3.3 Running coupling $\alpha(\mu)$

It is desirable to test this results against some known quantities. A well established result from $\mathcal{N} = 4$ is that $\beta(g)$ has to vanish, see Eqs.(2.29-38). This is a general feature of a CFT.

Instead of g we use α, which is connected to g by

$$\alpha(x) = \frac{g^2(x)}{4\pi}, \tag{5.81}$$

with Eq.(2.21) is rewritten into

$$\beta(\alpha) = \frac{\partial \alpha(\mu)}{\partial \mu}. \tag{5.82}$$

This is a well established quantity [121–124]. In the case of Landau gauge this α is an easy accessible quantity due to the relations of the renormalization constants [27]. Following the section: "The running coupling of the strong interaction" in [27] one ends up with the condition

$$\alpha(x) = \alpha(\mu) G^2(p) Z(p). \tag{5.83}$$

The running coupling in the scheme we employed is defined analogous to the miniMOM scheme given by Eq.(5.27). This ensures as in Yang-Mills theory that the $\alpha(x)$ is some constant [123]. As Eq.(5.82) states $\beta(\alpha)$ is the derivative of α with respect to some scale μ. Within our truncation we can observe $\alpha = const.$, therefore $\beta(\alpha) = 0$, which is in agreement with conformal symmetry Eq.(2.22-25).

The same procedure can be used to calculate the remaining running coupling constants. For the fermion scalar [124] one finds

$$\alpha_1 F^2(x) S(x) = \alpha_1 f^2 s x^{-2\kappa+2\kappa}, \tag{5.84}$$

and for the fermion gluon coupling

$$\alpha_2 F^2(x) Z(x) = \alpha_2 3 f^2 s x^{-2\kappa+2\kappa}. \tag{5.85}$$

Both of them are just numerical constants and in fact independent of the momentum x. The remaining one is the scalar gluon coupling [124], which is given at the symmetric point by

$$\alpha_3 x^{6\kappa} S^2(x) Z(x) = \alpha_3 3 s^3 x^{6\kappa-6\kappa}. \tag{5.86}$$

The only one left is the three gluon coupling. In the next subsection it is shown that we can neglect that coupling.

This shows that the conformal behaviour survives the DSEs, which was not expected due to the simple expressions and just two vertex dressings.

5.3.4 The irrelevance of the gluon-loop

The three gluon vertex is the only diagram we had to drop during calculation. Come to the question whether this is legit or not. In Eq.(5.54) we are forced to dress the three gluon vertex. When doing so no valid solution for the system can be found.

Dropping this diagram yields a solution κ for the coupled system. We can argue that this result is not harmed by the fact that we neglect the gluon-loop. The argument for this is as follows: Due to vertex dressing the gluon-loop vanishes for a yet undetermined dressing exponent α , see Fig.(5.2).

One possible way to introduce the gluon-loop and get rid of the $\Gamma(-1)$ term was to introduce a full transverse base. For this one can use many bases, for example Ball, Chiu [125] or the one used here [108]. Therefore, it is necessary to introduce new functions, i.e.

$$\mathcal{T} = T^{\mu\alpha}_{p_j} T^{\alpha\nu}_{p_k}, \quad t^{\mu}_{i} = T^{\mu\nu}_{p_i} q_i = T^{\mu\nu}_{p_i} (p_j - p_k)^{\nu}, \tag{5.87}$$

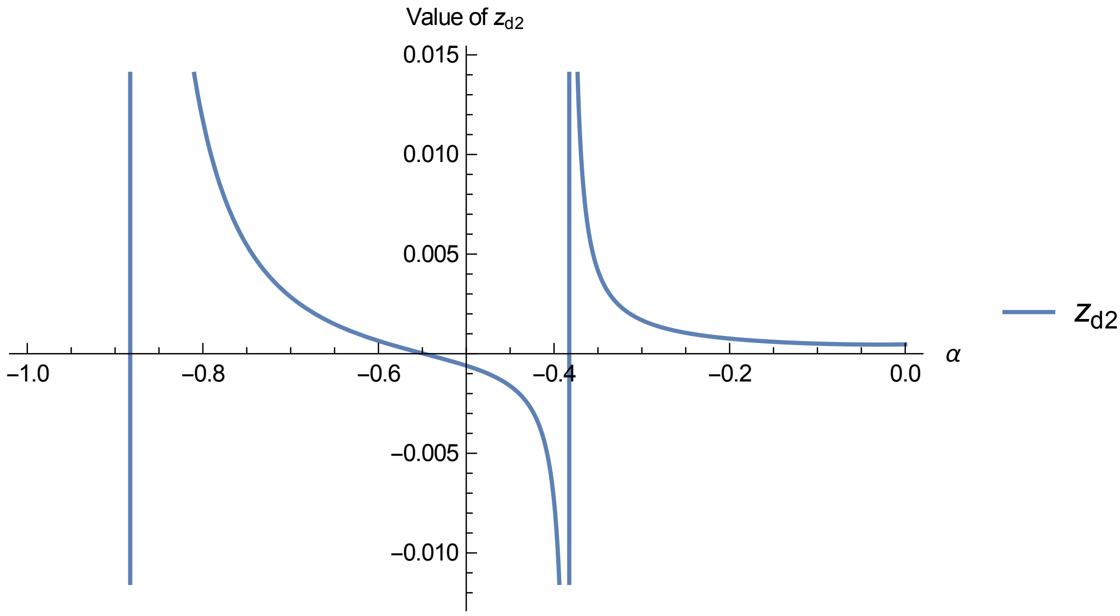


Figure 5.2: The dressed three gluon coupling $z_d 2$ of Eq.(5.43). Plotted for the dressing exponent α and the known κ. The gluon-loop vanishes for $\alpha = -0.55$.

where $T_k^{\mu\nu}$ denotes the usual transverse projector [108]. Then the basis can be expanded as

$$\tau_{1\perp}^{\mu\nu\rho} = t_1^\mu \mathcal{T}^{\nu\rho} + t_2^\nu \mathcal{T}^{\mu} + t_3^\rho \mathcal{T}^{\mu\nu}, \tag{5.88}$$

$$a\tau_{2\perp}^{\mu\nu\rho} = t_1^\mu t_2^\nu t_3^\rho, \tag{5.89}$$

$$a\tau_{3\perp}^{\mu\nu\rho} = p_1^2 t_1^\mu \mathcal{T}^{\nu\rho} + p_2^2 t_2^\nu \mathcal{T}^{\rho\mu} + p_3^2 t_3^\rho \mathcal{T}^{\mu\nu}, \tag{5.90}$$

$$b\tau_{4\perp}^{\mu\nu\rho} = \omega_1 t_1^\mu \mathcal{T}^{\nu\rho} + \omega_2 t_2^\nu \mathcal{T}^{\rho\mu} + \omega_3 t_3^\rho \mathcal{T}^{\mu\nu}, \tag{5.91}$$

The three gluon vertex is then given by

$$\Gamma_{\mu\nu\rho}^{A^3}(p,q,k) = \sum_{i=1}^{4} F_i(a,b,c)\tau_{i\perp}^{\mu\nu\rho}(p,q,k). \tag{5.92}$$

For further information on the unknown a,b,c,p and ω we refer to [108]. It is important that the four base functions for the transverse vertex have an antisymmetric behaviour. This is needed because the color structure is also antisymmetric.

A new **FORM** [112] code was written which has implemented the new basis. With the results produced by **FORM** the integral kernel N Eq.(5.17) got recalculated by **Mathematica**.

Even with the inclusion of the full tensor-structures of the three gluon vertex the $\Gamma(-1)$, which arise due to exponent matching, remains because the tree-level term is a part of the basis [108]. The functions F_i of the the sum are the conformal dressing functions for the vertex, so there is one constant for every structure.

Since the tree-level term yields the problem it is possible to only set the tree-levels dressing constant to 0. One can claim that all structures have to cancel or none of them have to cancel, but setting only the tree-level structure to 0 by choosing its constant equal to 0 is not valid. Nevertheless, it is possible to switch off the tree-level term and let the others survive. Doing

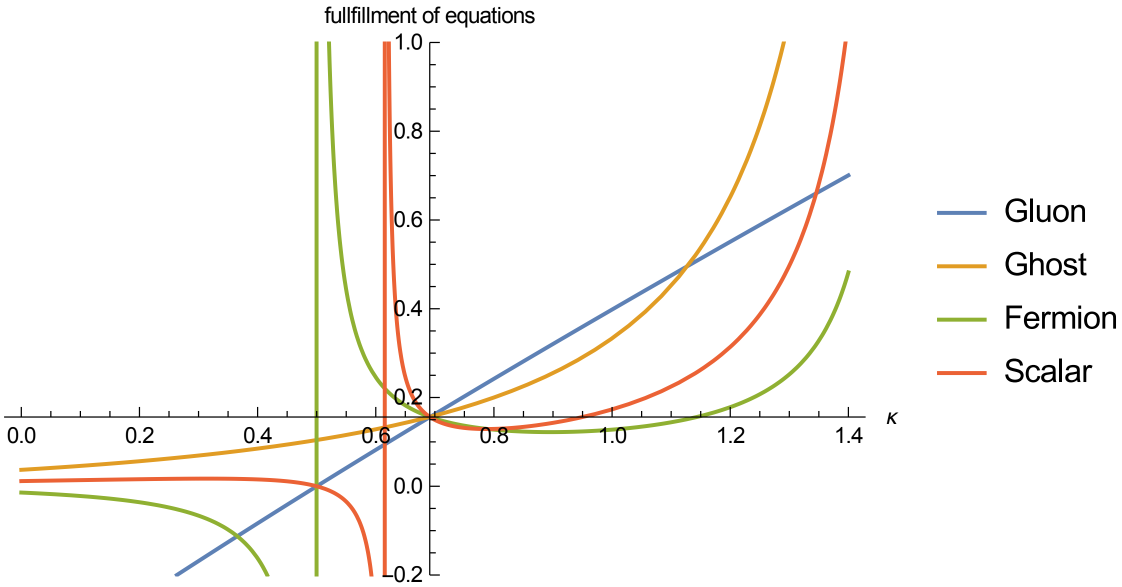


Figure 5.3: Plotting the ω's Eq.(5.70-73) which correspond to the four propagators. The difference from Fig.(5.1) is the inclusion of the gluon-loop. Due to that the system is overdefined. In principle the gluon line could be bent to agree with almost any thinkable behaviour. The yellow line corresponds to the ghost, the blue to the gluon, the green to the fermion and the red to the scalar.

so yields that another tensor structure is 0 and only two out of these four tensor structures are relevant. A real drawback is, that it is possible again to solve the propagator equations but now the solutions need to be parametrized with these F's yielding multiple solutions, since the system is now overdefined. The additional term z_2 of Eq.(5.48) is then

$$z_2^{\text{full tensors.}} = \frac{25\pi^2 b(84F_2 - 71F_4)\Gamma(4-2\kappa)\Gamma(\kappa+1)^2}{48\sqrt[3]{2}\sqrt{3}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{8}{3}\right)^2\Gamma(2-\kappa)^2\Gamma(2\kappa)} \tag{5.93}$$

One choice, $F_2 = -5, F_4 = -4$ is plotted in Fig.(5.3).

5.4 The Gribov problem

In Chapter 3 the Gribov-Singer ambiguity [89, 90] was briefly adressed. As it is mentioned the IR behaviour of the dressing functions $G(x)$ and $Z(p)$ is influenced by this ambiguity [38]. What we have found is, that the gluon dressing function is vanishing at zero momentum while the ghost dressing function diverges at zero momentum, see Tab.(5.1). If the inverse ghost dressing function $ax^\kappa \neq 0$ then the gluon propagator becomes infrared finite [76].

The problem that arises is that a IR finite gluon propagator is not in agreement with conformality as it would define a mass scale. In this thesis it has been shown that conformality is unbroken. Therefore the case could be as well that $\mathcal{N} = 4$ SYM has no Gribov-Singer ambiguity [126]. But there are even more options. It can be argued that these statements only relate to gauge-depended quantities and therefore break conformal symmetry analogues to Landau gauge breaking SUSY. Note that gauge invariant quantities still remain conformal and supersymmetric.

Another option is to relate $\mathcal{N} = 4$ SYM to 2-dimensional Yang-Mills theory. In 2-dimensional Yang-Mills theory the ghost dressing function is necessarily infrared divergent [127, 128]. One want to compare this with results from lattice field theory. There it has been observed that the approach $G(x) \to \infty$ may be modified [38, 98, 129]. Within this scenario there is the possibility of a non trivial Gribov-Singer ambiguity and conformality. This is of course interesting as 2-dimensional Yang-Mills theory is in the same sense different from higher dimensional Yang-Mills theory [130], as $\mathcal{N} = 4$ [126]. Therefore $\mathcal{N} = 4$ SYM is not only related to 2-dimensional Yang-Mills but the Gribov-Singer ambiguity is the same in both cases, which can be understood by the absence of dynamics in both theories [76].

To check whether there is a Gribov-Singer ambiguity for $\mathcal{N} = 4$ one has to find full non-conformal solutions to the system. This is already for Yang-Mills theory a rather cumbersome problem [38, 97]. What is also possible is to include two-loop diagrams and/or modification of the vertices. With the truncation we used there is only one κ in between 0 and 1. Due to the amount of work related to self-consistent solutions of the 3-point vertex equations we had to stop at the propagator level. For Yang-Mills theory an all order construction can be performed [116–118]. $\mathcal{N} = 4$ SYM is in an different spot, as it is a priori not clear if dominate diagrams can be identified to simplify the solution.

In lattice field theory it is in principle possible to count the Gribov copies. If the arguments of [126] are legit then lattice methods should not find any Gribov copies. Recently configurations for $\mathcal{N} = 4$ SYM have been generated by [131, 132], therefore this study is under way.

This concludes Chapter 5 and the search of a self-consistent solution for the propagator DSEs of $\mathcal{N} = 4$ SYM. We found a single exponent which can solve all propagator equations $\kappa \approx 0.69$.

Chapter 6

Conclusion and outlook

Let us summarize what has been done in this thesis. First of all an introduction on the present scope of usable tools for QFT calculations was given. In the Chapter 2 SUSY was introduced, which is in agreement with the symmetries of QFT as stated by Coleman-Mandula [45]. After that a functional method, the Dyson-Schwinger equations, was introduced and used to derive propagator equations for Yang-Mills theory. In this context the problem of Gribov copies was briefly discussed. In Chapter 4 we used an extended SUSY version of Yang-Mills theory, the so-called $\mathcal{N} = 4$ SYM theory. Again deriving the DSEs for propagators, but in contrast to Yang-Mills theory, fermions and scalars were added by SUSY. Therefore, instead of two propagator equations there were four propagator DSEs. All of them got derived to 1-loop level. In Chapter 5 the main result of this thesis is the exponent of the scaling solution which solved all propagator equations simultaneously. The exponent is given by

$$\kappa \approx 0.69.$$

This result is not trivial, since a scaling or powerlaw like solution must be valid for all momenta. In fact this can only happen for CFTs. Due to gauge fixing a conformal gauge was used, i.e. Landau gauge. Even from a diagrammatic point of view the DSEs are far more tedious to work out than the pure Yang-Mills IR solutions. In Yang-Mills theory only the leading order diagrams were taken into account, i.e. the ghost-loop. For $\mathcal{N} = 4$ there is no leading order in the coupling: all diagrams are equally important, or at least this is the current understanding. The calculations here ended at 1-loop level, including 2-loop level can already change the derived results.

To be in agreement with the theory it was necessary that the tadpoles vanish. This cancellation of the primitively divergent diagrams led to a relation between the dressing function of the gluon and the dressing function of the scalar, demanding that they are of same order, or to be precise $3sp^{2\kappa}$ and $sp^{2\kappa}$. Looking at the final results such a relation between the ghost and the fermion was also found. It is interesting to note that the fermion dressing function is around three magnitudes smaller than the ghost one: $fp^{-\kappa}$ and $ap^{-\kappa}$ with their ratio $\frac{a}{f} \approx 0.057$.

The main result of this thesis is that a scaling solution can in fact be found. Even within this simple truncation it was already possible to find one exponent κ which can solve all equations. Therefore, we have shown that the quality of the truncation is sufficient to capture the conformal behaviour. Note that for this result it was not necessary to dress all vertices, but only the scalar-gluon vertices. The 3-gluon vertex was dropped out of the equations due to its tree-level behaviour. The derived $\kappa \approx 0.69$ is in fact close to the $\kappa \approx 0.59$ derived from pure Yang-Mills theory [27]. This κ allows to construct suitable propagators in form of well tempered distributions, which will be not possible if $\kappa > 1$. Also in context to Yang-Mills theory, it is remarkable that the fermion and scalar of $\mathcal{N} = 4$ SYM mimic the behaviour of the ghost and the gluon [26]. Such that all fermions and bosons behave alike.

As a short outlook we want to motivate, that the Gribov-Singer ambiguity in $\mathcal{N} = 4$ is still not fully understood and therefore it would be helpful to introduce a new gauges, one example would be Landau b gauge [38]. The solution found in this work corresponds to a b-value of 0. It could be possible to find a valid gauge with $b \neq 0$, which would shed new light on the Gribov-Singer ambiguity in $\mathcal{N} = 4$ SYM.

Also within the lattice community a first set of configurations for $\mathcal{N} = 4$ SYM has been generated [131, 132]. These data makes it possible to look for Gribov copies directly, which unfortunately is not the case with DSEs.

Another possible scenario is that $\mathcal{N} = 4$ SYM has no Gribov problem, which is motivated in [126].

As a last statement it should be said that an all order expansion of $\mathcal{N} = 4$ SYM is also in the interest of the AdS/CFT correspondence. The κ we found $\kappa \approx 0.69$ is close to the Yang-Mills one $\kappa \approx 0.59$. Stating that both exponents are not that far apart. More interesting is the fact that the gauge sector of $\mathcal{N} = 4$ SYM coincided with the gauge sector Yang-Mills theory.

Appendix A

Conventions

- The metric $g_{\mu\nu}$ defined by $p.x = g_{\mu\nu}p^\mu x^\nu$ is Euclidian throughout this thesis

$$g_{\mu\nu} = \delta_{\mu\nu}, \tag{A.1}$$

  therefore a distinction between co- and contravariant tensorial structures is not needed.

- For every calculation we use natural units

$$\hbar = c = 1 \tag{A.2}$$

  this defines the system such

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}.$$

- For all indices Einstein's sum convention is applied.
- The squared momentum always refers to

$$p^2 = p_0^2 + \vec{p}^2. \tag{A.3}$$

- C_A the Casimir of a compact, semi-simple Lie group G in the adjoint representation with structure constants f^{abc}. C_A is defined to be

$$f^{abc}f^{abc} = C_A. \tag{A.4}$$

  If that compact, semi-simple Lie group is a $SU(N)$ then, $C_A = N_c$.

- Fourier transformations are defined by

$$f(p) = \int d^dx \tilde{f}(x)e^{i(p.x)}, \tag{A.5}$$

$$f(x) = \int \frac{d^dp}{(2\pi)^d}\tilde{f}(p)e^{i(p.x)}. \tag{A.6}$$

- The matrices which we refer to follow the notation of [18]:

$$\epsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ . \end{pmatrix} \tag{A.7}$$

- A spinor can be transform to it's antispinor by

$$\overline{\psi} = \psi^\dagger \beta = \psi^T \epsilon \gamma_5, \tag{A.8}$$

where $\psi^\dagger$ is simply $(\psi^*)^T$

$$\psi^* = -\beta \gamma_5 \epsilon \psi, \tag{A.9}$$

and γ_5 is given by

$$\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ . \end{pmatrix} \tag{A.10}$$

Note that this relation would not hold for the spinors given in Eq.(4.4), due the subscript L and R. Due to its wide use in the DSEs of $\mathcal{N} = 4$ SYM:

$$\delta_{ijkl} = \epsilon_{ijmn} \epsilon_{mnkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \tag{A.11}$$

which can also be found in Chapter 4 Eq.(4.27). Note that despite the fact the this tensor is called δ it is an antisymmtric tensor.

Appendix B

Γ-functions

In Chapter 5 we introduced a integral formula Eq.(5.26). This formula produces several Γ-functions. Since a common factor can be extracted from all propagator equations, Eqs(5.10,15,20,23)

$$\omega = \frac{C_a g^2 \Gamma^2(2-\kappa)\Gamma(2\kappa) 3a^2 s}{48\pi^2 \Gamma(4-2\kappa)\Gamma^2(1+\kappa)}, \tag{B.1}$$

we need several Γ-function identities.

- Γ is defined by

  $$\Gamma(n) = (n-1)!, \tag{B.2}$$

  for integer numbers and

  $$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{B.3}$$

  for complex numbers with $Re(z) > 0$.
- The $\Gamma(z)$ is analytical everywhere except at $z = 0, -1, -2, ...$, therefore we had to exclude the gluon-loop.
- One can shift the argument of a Γ-function according to

  $$a\Gamma(a) = \Gamma(a+1), \quad -a\Gamma(-a) = \Gamma(1-a). \tag{B.4}$$

- Products of Γ-functions can be combined to a trigonometric function by

  $$\Gamma(a)\Gamma(-a) = -\frac{\pi}{a\ sin(\pi x)}, \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{sin(\pi x}. \tag{B.5}$$

- The argument of a Γ-function can be multiplied by an integer

  $$\Gamma(2a) = (2\pi)^{-1/2} 2^{2a-1/2} \Gamma(a)\Gamma(a+\frac{1}{2}). \tag{B.6}$$

- Γ-functions without a κ can be computed, e.g.

  $$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi} \tag{B.7}$$

Appendix C

Feynman rlues

For completeness we want to present the Feynman rules for $\mathcal{N} = 4$ SYM. The bare vertices correspond to these rules, therefore the DSEs produce them. These were checkt against [105], but note that here different conventions were used.

C.1 Propagators

The bare ghost propagator:

$$c^b \cdots\cdots\cdots \overline{c^a} \quad = \quad -\delta^{ab}\frac{1}{p^2} \tag{C.1}$$

The bare gluon propagator:

$$A^b_\nu \; \text{(gluon line)} \; A^a_\mu \quad = \quad \delta^{ab}\frac{1}{p^2}\big(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\big) \tag{C.2}$$

The bare fermion propagator:

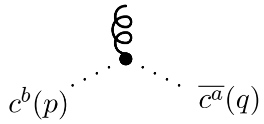
$$\psi^b_j \; \text{———} \; \overline{\psi^a_i} \quad = \quad -\delta^{ab}\delta_{ij}\frac{i}{\not{p}} \tag{C.3}$$

The bare scalar propagator:

$$\phi^b_{kl} \; \text{- - - - -} \; \phi^{a*}_{ij} \quad = \quad \delta^{ab}\delta_{ijkl}\frac{1}{p^2} \tag{C.4}$$

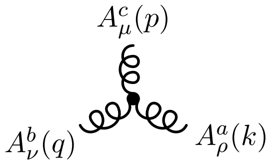
C.2 3-point couplings

The ghost-gluon coupling:



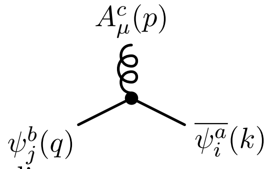
$$= \quad igf^{abc}q_\mu \tag{C.5}$$

The 3-gluon coupling:



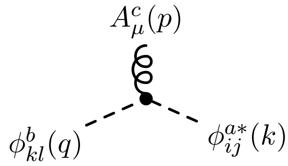
$$= \quad -igf^{abc}\Big((q-k)_\mu\delta_{\nu\rho} + (k-p)_\nu\delta_{\mu\rho} + (q-p)_\rho\delta_{\mu\nu}\Big) \tag{C.6}$$

The fermion gluon coupling:



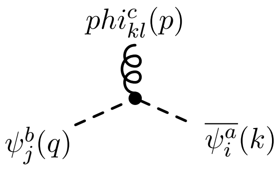
$$= \quad -gf^{abc}\delta^{ij}\gamma_\mu \tag{C.7}$$

The scalar gluon coupling:



$$= \quad igf^{abc}\delta_{ijkl}(k-q)_\mu \tag{C.8}$$

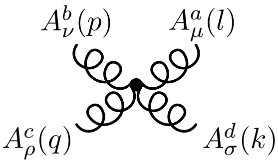
The Yukawa coupling:



$$= \quad g\frac{\sqrt{2}}{2}f^{abc}\Big(\delta_{ijkl}(1+\gamma_5) - \varepsilon_{ijkl}(1-\gamma_5)\Big) \tag{C.9}$$

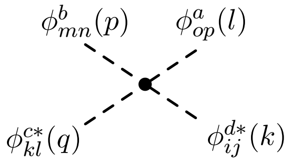
C.3 4-point couplings

The 4-gluon coupling:



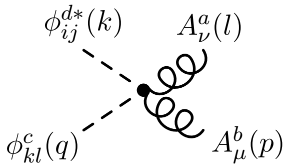
$$\begin{aligned} = \quad & g^2\Big(f^{eab}f^{ecd}(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \\ & + f^{gac}f^{gbd}(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + f^{gad}f^{gbc}(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\nu\rho})\Big) \end{aligned} \tag{C.10}$$

The 4-scalar coupling:



$$\begin{aligned} = \quad & \frac{g^2}{2}\Big((f^{ead}f^{abc} + f^{eac}f^{ebd})\frac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnop} \\ & + (f^{eab}f^{edc} + f^{eac}f^{edb})\delta_{ijmn}\delta_{klop} \\ & + (f^{eab}f^{ecd} + f^{ead}f^{ecb})\delta_{klmn}\delta_{ijop}\Big) \end{aligned} \tag{C.11}$$

The 2-scalar 2-gluon coupling:



$$= \quad g^2\delta_{\mu\nu}\delta_{ijkl}\left(f^{abc}f^{adf} + f^{abf}f^{adc}\right) \tag{C.12}$$

Which completes the set of n-point functions used for this thesis.

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