Supersymmetry

Lecture in WS 2022/23 at the KFU Graz

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Contents

Chapter 1

Introduction

1.1 Preliminaries

The prerequisites for this courses are quantum mechanics, and knowledge of quantum field theory, including some basic knowledge about non-Abelian gauge theories and the Brout-Englert-Higgs effect. The necessary basics can also be obtained in parallel in the lectures on relativistic quantum mechanics and quantum field theory. Some knowledge of the phenomenology of particle physics is also helpful, as can be obtained in various lectures.

There are quite a number of good books on supersymmetry. This lecture is largely based on

- Supersymmetrie by H. Kalka and G. Soff (Teubner)
- Supersymmetry in particle physics by I. Aitchison (Cambridge)
- The quantum theory of fields III (as well as I and II) by S. Weinberg (Cambridge)
- Fields by W. Siegel (available for free from arxiv.org/abs/hep-th/9912205)
- Supersymmetry and supergravity by J. Wess and J. Bagger (Princeton)
- Advanced topics in quantum field theory by M. Shifman (Cambridge)

This is only a subjective choice, and many other excellent texts exists. As always, one should test these, to find the most suitable book (and presentation) for oneself.

There are also a number of reviews on the topic available on the arXiv and in journals. Particular recommendable are

- A Supersymmetry primer by S. Martin (arxiv.org/abs/hep-ph/9709356)

- TASI 2002 lectures: Nonperturbative supersymmetry by J. Terning (arxiv.org/abs/hep-th/0306119)

- The Search for Supersymmetry: Probing Physics Beyond the Standard Model by H. Haber and G. Kane (Phys.Rept.117:75-263,1985)

- Supersymmetry, Supergravity and Particle Physics by H. Niles (Phys.Rept.110:1-162,1984)

which, however, go significantly beyond the scope of this lecture.

1.2 Why Supersymmetry?

Supersymmetry, as will become clear during this lecture, is much more than a particular theory. It is a conceptual idea, on which a multitude of theories rest. Supersymmetric quantum field theories have furthermore unique features, not shared by any other type of quantum field theories. They are therefore particular interesting candidates for our understanding of nature. Also, theories more complex than quantum field theories, like string theories, very often induce as the low-energy effective quantum field theories supersymmetric theories. Furthermore, even in theories which are not supersymmetric, like nuclear physics, tools based on supersymmetry can be used to simplify complicated problems significantly. Finally, supersymmetric theories are from a purely mathematical (physics) point of view very interesting entities, and therefore a subject worth of study in themselves.

Though in this lecture all of these aspects will be touched upon, it will be mainly concentrated on the application of supersymmetry to particle physics. The interest in supersymmetry in this particular arena stems not from experimental evidence. In fact, at the current time the results from particle physics experiments are rather discouragingly indicating that we will likely not discover anything supersymmetric in nature any time soon. But many conceptual issues in particle physics let supersymmetry appear an attractive, albeit not necessary, solution.

These conceptual problems have mostly to do with questions of ultraviolet completeness, i. e. what happens at very large energies. The standard model appears to be failing in these regions. This is mainly signaled by the need of renormalization, i. e. the appearance of infinities, and challenges like triviality and arbitrarily rising interaction strengths in the Higgs sector. Though the latter may still be within the realm of the standard model to solve, and they are just technically complicated, the former are clearly requiring a different physics. There are also many features of the standard model, which just are, like the values of masses and coupling constants, for which an explanation is desirable.

But not theoretical considerations induce the desire for more. One thing which is clearly missing in the standard model is gravity. Supersymmetry, in the form of supergravity, provides a rather natural way of including it, though again not a uniquely necessary one. This will be a feature which can only be touched upon in this lecture. However, it can be shown, known as the Coleman-Mandula theorem, that supersymmetry is the only way to combine gravity and the standard model in a non-trivial way in a quantum field theory, where here non-trivial means not in the form of just a direct product structure.

Furthermore, during the last couple of years a number of experimental observations have been made, which cannot be explained in the framework of the standard model and classical general relativity. Among these are the indications for dark matter, as has been deduced from several observables, which leads to the critical density of the universe, when including the (classical) dark energy. Also the period of inflation at very early times cannot be explained, though there is mounting evidence for it.

Supersymmetry offers not always a compelling solution to these problems, but often an attractive one. Although, there are many technical details unsolved of how these solutions should be implemented. It is therefore worthwhile to understand the basics of it. With the upcoming next runs of the LHC, more will be said on whether nature has chosen supersymmetry at the comparatively low energy scale of a few TeV.

However, there are also several reasons which make supersymmetry rather suspect. The most important one is that supersymmetry is not realized in nature. Otherwise the unambiguous prediction of supersymmetry would be that for every bosonic particle (e. g. the photon) an object with the same mass, but different spin-statistics (for the photon the spin-1/2 photino), should exist, which is not observed. The common explanation for this is that supersymmetry in nature has to be broken either explicitly or spontaneously. However, how such a breaking could proceed such that the known standard model emerges is not known. It is only possible to parametrize this breaking, yielding an enormous amount of free constants and coupling constants, for the standard model more than a hundred, while the original standard model has only about thirty.

1.3 A first encounter with supersymmetry

Supersymmetry is a symmetry which links the two basic types of particles one encounters in quantum physics[1]: bosons and fermions. In particular, it is possible to introduce

[1]Note that in two dimensions there are more options, so-called anyons, with arbitrary real spin.

symmetry transformations in the form of operators which change a boson into a fermion and vice versa. Supersymmetric theories are then theories which are invariant under these transformations. This entails, of course, that such theories necessarily include both, bosons and fermions. In fact, it turns out that the same number of bosonic and fermionic degrees are necessary to obtain a supersymmetric theory. Hence to each particle there should exist a superpartner. Superpartners of bosons are usually denoted by the ending ino, so for a bosonic gluon the fermionic superpartner is called gluino. For the superpartners of fermions an s is put in front, so the superpartner of the fermionic top quark is the bosonic stop (and for quarks in general squarks).

Besides the conceptual interest in supersymmetric theories there is a number of phenomenological consequences which make these theories highly attractive. The most interesting one of these is that quite a number of divergences, which have to be removed in ordinary quantum field theories by a complicated process called renormalization, drop out automatically.

A simple example for this concept is given by an infinite chain of harmonic oscillators. The Hamilton function for a harmonic oscillator with[2] $\omega = 1$ is given by

$$H_B = \frac{p^2}{2m} + \frac{m}{2}x^2. \tag{1.1}$$

It is useful to replace the operators p and x with the bosonic creation and annihilation operators a and $a^\dagger$

$$\begin{aligned} a^\dagger &= \sqrt{\tfrac{m}{2}}\left(x - i\tfrac{p}{m}\right) \\ a &= \sqrt{\tfrac{m}{2}}\left(x + i\tfrac{p}{m}\right) \end{aligned}$$

These operators fulfill the commutation relations

$$\begin{aligned} \left[a, a^\dagger\right] &= aa^\dagger - a^\dagger a = 1 \\ [a, a] &= [a^\dagger, a^\dagger] = 0. \end{aligned} \tag{1.2}$$

which follow directly from the canonic commutation relations

$$[x, p] = i.$$

Replacing p and x in (1.1) with these new operators yields

$$H_B = \frac{1}{2}\left(a^\dagger a + aa^\dagger\right) \tag{1.3}$$

[2]Everything will be in natural units, i. e., $\hbar = c = k_B = 1$.

Using the commutation relation (1.2) yields the Hamilton operator of the bosonic harmonic oscillator

$$H_B = a^\dagger a + \frac{1}{2}$$

A noninteracting chain of these operators is therefore described by the sum

$$H_B = \sum_{n=1}^{N} \left(a^\dagger a + \frac{1}{2} \right).$$

If the number of chain elements N is sent to infinity, the contribution from the vacuum term diverges. Of course, in this simple example it would be possible to measure everything with respect to the vacuum energy, and just drop the diverging constant.

However, imagine one adds a harmonic oscillator, which has fermionic quanta instead of bosonic ones. Such an oscillator is most easily obtained by replacing (1.3) with

$$H_F = \frac{1}{2}(b^\dagger b - b b^\dagger) \tag{1.4}$$

with the bosonic creation and annihilation operators replaced by their fermionic counterparts $b^\dagger$ and b. These satisfy anticommutation relations[3]

$$\begin{aligned} \{b, b^\dagger\} &= b b^\dagger + b^\dagger b = 1 && (1.5) \\ \{b, b\} &= \{b^\dagger, b^\dagger\} = 0. && (1.6) \end{aligned}$$

The form (1.4) can be motivated by introducing fermionic coordinates, but this will not be done here. In the end, it is a postulate, as fermions are inherently quantum objects, and thus the fermionic oscillator cannot be obtained by the canonical quantization of a classical system.

Using these relations, the Hamiltonian can be recast as

$$H_F = \sum_{n=1}^{M} \left(b^\dagger b - \frac{1}{2} \right)$$

In this case the vacuum energy is negative.

Now, adding both systems with the same number of degrees of freedom, $N = M$, the resulting total Hamiltonian

$$H = H_B + H_F = \sum_{n=1}^{N} \left(a^\dagger a + b^\dagger b \right) \tag{1.7}$$

[3]This type of algebra is a Clifford algebra, not incidentally similar to the one encountered when dealing with fermions in quantum field theory.

has no longer a diverging vacuum energy. Actually, the Hamilton operator (1.7) is one of the simplest supersymmetric systems. Supersymmetry here means that if one exchanges bosons for fermions the Hamilton operator is left invariant. This will be discussed in detail below. The mechanism to cancel the vacuum energy is a consequence of supersymmetry, and this mechanism is one of the reasons which make supersymmetric theories very interesting.

More generally, in a quantum-field setting this type of cancellation will yield a 'natural' explanation why scalar particles should be of about the same mass as the other particles in a theory, as will be discussed in detail much later.

The rest of this lecture will proceed in the following way: In the next chapter 2, the first encounter with supersymmetry in quantum mechanics will be extended, to introduce many of the central concepts of supersymmetry in a technically less involved setting. After this, a short preparation in Grassmann mathematics will be made in chapter 3, which will be necessary in many places. Then, things will turn to the first and simplest supersymmetric theory, a non-interacting one in chapter 4. For this, a brief repetition of fermions in quantum field theories will be necessary. Interactions in the simplest case, the so-called Wess-Zumino model, will be introduced in chapter 5. After this first encounter with supersymmetric field theories the concept of superspace will be introduced in chapter 6, which is very useful in constructing supersymmetric theories. To be able to discuss supersymmetric extensions of the standard model, the introduction of supersymmetric gauge theories is necessary, which will be done in chapter 7. Also necessary to extend to the standard model is supersymmetry breaking, which will be discussed in chapter 8. This is then enough to introduce the minimal supersymmetric standard model in chapter 9.

1.4 The conceptual importance of supersymmetry

After having seen how a supersymmetry theory can look like, it is now time to contemplate what supersymmetry implies. For this, it is worthwhile to have a look at a, somewhat hand waving, version of the Coleman-Mandula theorem.

All previous symmetries in particle physics are of either of two kinds. One are the external symmetries, like translational and rotational ones. These are created by the momentum and angular momentum operators. The other one are internal ones, like electric charges. The difference between both is that external charge operators carry a Lorentz index, while internal ones are Lorentz scalars. The natural question from a systematic point of view is, whether there are other conserved quantities besides momentum and angular momentum, which have a Lorentz index[4].

The Coleman-Mandula theorem essentially states that this is impossible in a quantum field theory. Since the most general vector and anti-symmetric tensors are already assigned to the momentum and angular momentum operator, the simplest one would be a symmetric tensor operator $Q_{\mu\nu}$. Acting with it on a single particle state would yield

$$Q_{\mu\nu}|p\rangle = (\alpha p_\mu p_\nu + \beta\eta_{\mu\nu})|p\rangle,$$

where the eigenvalue is the most general one compatible with Poincare symmetry, with eigenvalues α and β. Since a symmetry is looked for, Q must be diagonalizable simultaneously with the Hamiltonian, and therefore these must be momentum-independent. One could ask what if the eigenvalues themselves would have a direction, and the single-particle state would thus be characterized by two vectors. In this case, the scalar product of these two vectors would single out a direction, and therefore break the isotropy of space-time, and thus the Poincare group. Lacking any experimental evidence for this so far, this possibility is excluded, and would anyhow alter the complete setting.

So far, there is no contradiction. Acting with $Q_{\mu\nu}$ on a two-particle state of two identical particles of the type would yield

$$Q_{\mu\nu}|p, q\rangle = (\alpha(p_\mu p_\nu + q_\mu q_\nu) + 2\beta\eta_{\mu\nu})|p, q\rangle.$$

In this case, it was assumed that Q is a one-particle operator, i. e. its charge is localized on a particle, and the total charge is obtained by the sum of the individual charges. Though operators of other types can be considered, even in lack of physical evidence, this would only complicate the argument in the following, without changing the outcome. This will therefore be ignored.

Now consider elastic scattering of these two particles. Since Q should describe a symmetry, the total charge before and afterwards must be the same. Furthermore, 4-momentum

[4]Note that certain theories, so-called conformal theories, exist, in which the Poincare group is enlarged by scale/dilatation transformations and so-called special conformal transformations, which essentially remap coordinates in a non-linear way. This enlarged Poincare symmetry is called the conformal group. This extension is not an internal symmetry, but the Poincare symmetry itself is larger. Therefore, this case is not in contradiction to the Coleman-Mandula theorem. Such theories require that the dynamics are scaleless, and especially that all particles are massless. Since any non-trivial scattering process breaks conformal symmetry, such theories do not have any observables. They are therefore certainly not a theory of nature, and will therefore not be discussed here in detail. However, both from a conceptual point of view, as well as in case the conformal symmetry is slightly broken, they are of a certain interest to particle physics, which also not be dwelt upon here to a large extent.

conservation must hold. This implies that

$$\begin{aligned} p_\mu p_\nu + q_\mu q_\nu &= p'_\mu p'_\nu + q'_\mu q'_\nu \\ p_\mu + q_\mu &= p'_\mu + q'_\mu \end{aligned}$$

The only solution to these equations is $p = p'$ and $q = q'$ or $p = q'$ and $q = p'$. Hence no interaction occurs, since the second possibility is indistinguishable for two identical particles. Hence, any theory with such a conserved symmetric tensor charge would be non-interacting, and therefore not interesting. The generalized version of this statement is the Coleman-Mandula theorem, which includes all the subtleties and possible extensions glossed over here.

How does supersymmetry change the situation? For this simple example of elastic scattering, this is rather trivial. Since supersymmetry, by definition, changes the spin of the particles, it plays no role in this process, as in an elastic scattering the particle identities are not changed. In the more general case, it can be be shown that for the more general Coleman-Mandula theorem it is actually an assumption that this never happens. Hence, introducing such a symmetry violates the assumption, and therefore invalidates the argument. Again, the complete proof is rather subtle. This leaves the question open, whether any interesting, consistent, non-trivial, let alone experimentally relevant, theories actually harbor supersymmetry. The second question is still open, and no experimental evidence in strong favor of supersymmetry has so far been found.

The first question is, what this lecture is about. In the following, in very much detail, it will be shown that there are interesting, consistent, and non-trivial supersymmetric theories. Furthermore, it will be shown that supersymmetry is, from a conceptual point, a fundamental change. This can already be inferred from the following simple argumentation.

Since Poincare symmetry remains conserved in a supersymmetric theory, any operator which changes a boson into a fermion must carry itself a half-integer spin, and thus be a spinor Q_a. Otherwise, the spin on the left-hand side and the right-hand side would not be conserved. Furthermore, if it is a symmetry, it must commute with the Hamiltonian $[Q_a, H] = 0$. In addition, so must its anti-commutator

$$[\{Q_a, Q_b\}, H] = 0$$

In the simplest case, a spinor has two independent components, and therefore the anti-commutator can have up to four independent components. Since two spinors form together an object of integer spin, it must therefore be (at least) a vector. The only-vector-valued operator, however, is the momentum operator P_μ. Therefore, one would expect that

$$\{Q_a, Q_b\} \sim P_\mu. \tag{1.8}$$

It will be shown that this is indeed the correct structure. However, this means that supersymmetry enlarges indeed the Poincare symmetry non-trivially, since otherwise all (anti)commutator relation would be closed within the supersymmetry operators Q_a, therefore fulfilling the original goal. Moreover, the operators Q_a therefore behave, in a certain sense to be made precise later, like a square-root of the momentum operator. Similar like the introduction of i as the square-root of -1, this concept will require to enlarge the concept of space-time by adding additional, fermionic dimensions, giving birth to the concept of superspace. This already shows how conceptually interesting supersymmetry is, and that it is therefore worthwhile to pursue it for its own sake, even if no experimental evidence in favor of it exists.

In the following, all of these concepts will be made more precise, and all the gaps will be filled in.

Chapter 2

Supersymmetric quantum mechanics

Supersymmetry is not a concept which requires either a field theory nor does it require special relativity. Its most simple form it takes in non-relativistic quantum mechanics. However, as fermions are involved, quantum effects are necessary. Such things as spin 1/2-objects, and thus fermions, cannot be described in the context of a classical theory.

2.1 Generators of supersymmetry

To be able to discuss a supersymmetric theory, it is of course necessary to have a Hilbert space which includes both bosons and fermions. One example would be the Hilbert space of the harmonic oscillator with only one element, the simplification of (1.7),

$$H = a^\dagger a + b^\dagger b. \tag{2.1}$$

The Hilbert space of this Hamilton operator is given by

$$|n_B n_F\rangle \tag{2.2}$$

where n_B is the number of bosons and n_F is the number of fermions. Note that no interaction between bosons and fermions occur. Such an interaction is not necessary to obtain supersymmetry. However, supersymmetry imposes very rigid constraints on any interactions appearing in a supersymmetric theory, as will be seen later. Furthermore, the number of degrees of freedom is the same: one bosonic and one fermionic oscillator.

The conventional bosonic creation and annihilation operators $a^\dagger$ and a act on the Hilbert space in the usual way as

$$\begin{aligned}
a^\dagger|n_B n_F\rangle &= \sqrt{n_B+1}|n_B+1 n_F\rangle \\
a|n_B n_F\rangle &= \sqrt{n_B}|n_B-1 n_F\rangle \\
a|0 n_F\rangle &= 0.
\end{aligned}$$

The number of bosonic quanta in the system therefore ranges from 0 to infinity. Due to the Pauli principle the action of the fermionic creation and annihilation operators $b^\dagger$ and b is much more limited. The only possibilities are

$$\begin{aligned} b^\dagger|n_B 0\rangle &= |n_B 1\rangle \\ b|n_B 1\rangle &= |n_B 0\rangle \\ b^\dagger|n_B 1\rangle &= b|n_B 0\rangle = 0. \end{aligned}$$

This can be shown to be a direct consequence of the (postulated) anti-commutation relations (1.5-1.6) of b and $b^\dagger$. Thus, there can be only either zero or one fermionic quanta in this system.

This delivers the complete spectrum of the theory, which is given by

$$H|n_B n_F\rangle = n_B + n_F.$$

Now, supersymmetry is based on a relation between fermions and bosons. Therefore, it will be necessary to introduce operators which can change a boson into a fermion. Such operators can be defined by their action on the states as

$$\begin{aligned} Q_+|n_B n_F\rangle &\sim |n_B - 1 n_F + 1\rangle \\ Q_-|n_B n_F\rangle &\sim |n_B + 1 n_F - 1\rangle. \end{aligned} \tag{2.3}$$

Note that since the maximum number of fermionic quanta is 1, Q_+ annihilates the state with one fermionic quanta, just as the fermionic creation operator. Furthermore, the operator Q_- annihilates the fermionic vacuum, and Q_+ the bosonic vacuum, where vacuum is the state with zero quantas.

To obtain a symmetry, Q_+ and Q_- have to be symmetry transformation operators. However, for a theory to be supersymmetric, its Hamilton operator must be invariant under supersymmetry transformations, i. e.,

$$[H, Q_\pm] = 0. \tag{2.4}$$

To be able to check this, it is necessary to deduce the commutation relations of the bosonic and fermionic annihilation operators with $Q_\pm$. For this an explicit form of $Q_\pm$ in terms of a and b is necessary. Based on the action on the base state (2.3) it can be directly inferred that

$$\begin{aligned} Q_+ &= ab^\dagger \\ Q_- &= a^\dagger b. \end{aligned} \tag{2.5}$$

is adequate. From this also follows that $Q_\pm^\dagger = Q_\mp$, and both operators are adjoint to each other. Testing now the condition (2.4) yields for Q_+

$$[a^\dagger a + b^\dagger b, ab^\dagger] = a^\dagger a a b^\dagger - ab^\dagger a^\dagger a + b^\dagger b a b^\dagger - ab^\dagger b^\dagger b. \tag{2.6}$$

Since two consecutive fermionic creation or annihilation operators always annihilate any state, it follows that

$$b^\dagger b^\dagger = bb = 0.$$

This property of operators is called nilpotency, the operators are called nilpotent. This is a very important concept, and will appear throughout this lecture. However, currently this only implies that the last term in (2.6) is zero. Using the brackets (1.2) and (1.5-1.6), it then follows

$$b^\dagger a^\dagger a a - b^\dagger a - b^\dagger a^\dagger a a + ab^\dagger - ab^\dagger b^\dagger b = b^\dagger a - ab^\dagger = 0. \tag{2.7}$$

Here, once more the nilpotency of b and $b^\dagger$ was used, and the fact that bosonic and fermionic creation and annihilation operators commute. The same can be repeated for Q_-. Hence the combined oscillator (2.1) is supersymmetric, and is called the supersymmetric oscillator. Note that this required that the oscillator frequencies ω were identical. This is a portent of a more general requirement of supersymmetry: In a supersymmetric particle physics theory, for each fermion there must be a boson of the same mass, and vice versa.

That this particular theory is supersymmetric can be seen easier, as it is possible to rewrite (2.1) in terms of $Q_\pm$ as

$$\begin{aligned} H &= a^\dagger a + b^\dagger b + a^\dagger a b^\dagger b - b^\dagger b a^\dagger a \\ &= a a^\dagger b^\dagger b + a^\dagger a b b^\dagger \\ &= \{ab^\dagger, a^\dagger b\} = \{Q_+, Q_-\}. \end{aligned}$$

From this it follows directly that

$$\begin{aligned} [H, Q_+] &= [Q_+Q_-, Q_+] + [Q_-Q_+, Q_+] \\ &= Q_+Q_-Q_+ - Q_+Q_+Q_- + Q_-Q_+Q_+ - Q_+Q_-Q_+ = 0 \end{aligned}$$

This vanishes, as the operators $Q_\pm$ inherit the property of nilpotency directly from the fermionic operators. This can be repeated for Q_-, yielding the same result. As a consequence, any system which is described by the Hamilton operator $\{Q_+, Q_-\}$, where the $Q_\pm$ can now be more complex nilpotent operators, is automatically supersymmetric.

The only now not manifest property is the hermiticity of the Hamilton operator. Defining the hermitian operators

$$\begin{aligned} Q_1 &= Q_+ + Q_- \\ Q_2 &= i(Q_- - Q_+) \end{aligned}$$

this can be remedied. By explicit calculation it follows that

$$H = Q_1^2 = Q_2^2. \tag{2.8}$$

Note that the hermitian operators Q_1 and Q_2 are no longer nilpotent. Also, since energy eigenstates are not changed by the Hamiltonian this implies that performing twice the supersymmetry transformation Q_1 or Q_2 will return the system to its original state. Finally, this implies that the supercharge(s) of a system, as eigenvalues of the Q_i, are observable quantities.

Furthermore, it can be directly inferred that

$$\{Q_1, Q_2\} = 0.$$

It is then possible to formulate the supersymmetric algebra of the theory as

$$\begin{aligned} [H, Q_i] &= 0 \\ \{Q_i, Q_j\} &= 2H\delta_{ij}. \end{aligned} \tag{2.9}$$

It is the first manifestation of the generic result (1.8) derived heuristically in the introduction. This algebra is always available when the Hamilton operator takes the form (2.8), which will be done in the following. These algebras are often labeled with the number $\mathcal{N}$ of linearly independent generators of supersymmetric transformations. In the present case this number is $\mathcal{N} = 2$. If there would be only a single supercharge Q with $Q^2 = H$, the theory would hence be a $\mathcal{N} = 1$ theory. Many of the following results apply, partly slightly modified, to system with arbitrary $\mathcal{N}$. However, for simplicity, in this chapter only the case $\mathcal{N} = 2$ will be explicitly considered.

2.2 Spectrum of supersymmetric Hamilton operators

There are quite general consequences which can be deduced just from the algebra (2.9), which implies the form (2.8) of the Hamilton operator. Note that the only properties which have been used in the calculations of the supersymmetric operators were the nilpotency of the operators $Q_\pm$, and so the results are not restricted to simple harmonic oscillators.

The first general consequence is that the spectrum is non-negative. This follows trivially from the fact that (2.8) can be written as the square of a hermitian operator, which has real eigenvalues. Given, e. g., that $|q\rangle$ is an eigenstate with eigenvalue q to the operator Q_1, it follows immediately that

$$H|q\rangle = Q_1^2|q\rangle = Q_1 q|q\rangle = q^2|q\rangle$$

and thus $|q\rangle$ is an energy eigenstate with energy q^2, which is positive or zero by the hermiticity of Q_1. Furthermore, let Q_2 act on $|q\rangle$, yielding

$$Q_2|q\rangle = |p\rangle.$$

This state is in general not an eigenstate of Q_1, although it is necessarily by virtue of (2.8) an eigenstate of Q_1^2. However, acting with Q_1 on $|p\rangle$ yields

$$Q_1|p\rangle = Q_1 Q_2|q\rangle = -Q_2 Q_1|q\rangle = -qQ_2|q\rangle = -q|p\rangle. \tag{2.10}$$

Here, it was assumed that q is not zero. Therefore, it is in fact an eigenstate, but to the eigenvalue $-q$, $|p\rangle = |-q\rangle$, so it is different from the original state $|q\rangle$. Therefore, this state is also an eigenstate of the Hamilton operator to the same eigenvalue. Hence, for all non-zero energies the spectrum of the Hamilton operator is doubly degenerate. Furthermore, the states q and p differ by the application of a supersymmetry transformation. Hence, they differ by their bosonic and fermionic content, one state has an even number of fermionic quantas, and the other one an odd number of fermionic quanta.

2.3 Breaking supersymmetry and the Witten index

If a symmetry is exact, its generator G commutes with the Hamilton operator

$$[H, G] = 0. \tag{2.11}$$

The operators Q_i are the generators of supersymmetry. If there is a zero-energy state, this is automatically fulfilled, because then $(Q_+ \pm Q_-)|0\rangle = Q_i|0\rangle = H|0\rangle = 0$. On the other hand, if there is no zero-energy state then for the ground-state $|g\rangle$ follows

$$[H, Q_i]|g\rangle = HQ_i|g\rangle + Q_i E_g|g\rangle = H|g'\rangle + Q_i E_g|g\rangle = E_g Q_i|g\rangle + Q_i E_g|g\rangle = 2E_g Q_i|g\rangle \neq 0.$$

Thus, the ground-state is not supersymmetric. Supersymmetry is then necessarily broken. Hence, the energy of the ground state in in one-to-one correspondence with whether the system exhibits manifest supersymmetry or not. Of course, there exists one superposition of the two degenerate ground states, for which $[H, Q_i]$ vanishes for one of the Q_i, but not for the other. But then the degeneracy is lifted, as now one of the Q_i can be used to distinguish both states.

This furthermore implies that if supersymmetry is unbroken the ground state has in fact energy zero. It will be shown below that this state has to be unique. It should be noted that there is a peculiarity associated with the fact that the Hamilton operator is just the square of the generator of the symmetry. Since if supersymmetry is broken

$$Q_i|0\rangle \neq 0 \Rightarrow Q_i^2|0\rangle \neq 0 \tag{2.12}$$

the ground state can no longer have energy zero, but must have a non-zero energy. Therefore it is also doubly degenerate. Hence, the fact whether the ground state is degenerate or not is therefore indicative of whether supersymmetry is broken or not. This is described by the Witten index

$$\Delta = \mathrm{tr}(-1)^{N_F}.$$

N_F is the particle number operator of fermions

$$N_F = b^\dagger b.$$

Therefore, this counts the difference in total number of bosonic and fermionic states

$$\begin{aligned}\Delta &= \sum \left(\langle n_F = 0|(-1)^{N_F}|n_F = 0\rangle + \langle n_F = 1|(-1)^{N_F}|n_F = 1\rangle\right) \\ &= n_B - n_F\end{aligned}$$

where n_B is the total number of bosonic states and n_F is the total number of fermionic states. All states appear doubly degenerate except at zero energy. Furthermore, these degenerate states are related by a supersymmetry transformation, and thus have differing fermion number. Hence, the Witten index is reduced to the difference in number of states at zero energy,

$$\Delta = n_B(0) - n_F(0).$$

It is therefore only nonzero if supersymmetry is intact, and zero if not. However, there are cases where a zero Witten index does not necessarily imply that supersymmetry is broken, just that it can be broken.

2.4 Interactions and the superpotential

To obtain a non-trivial theory, it is necessary to introduce interactions. The easiest way to preserve the previous results, and obtain a supersymmetric theory, is to implement that $Q_\pm$ are necessarily nilpotent and fermionic in the simplest possible way. This can be obtained by replacing the definition (2.5) by

$$\begin{aligned} Q_+ &= Ab^\dagger \\ Q_- &= A^\dagger b, \end{aligned}$$

where the operators A and $A^\dagger$ are arbitrary bosonic functions of the original operators a and $a^\dagger$. In principle, they may also depend on bosonic quantities formed from b and $b^\dagger$, like $b^\dagger b$. These would not spoil the nilpotency. For simplicity, this possibility will be ignored here.

In this case N_B is in general no longer a good quantum number, but N_F remains trivially one. So all states can be split into states with either $N_F = 0$ or $N_F = 1$, and can furthermore be labeled by their energy,

$$\begin{aligned} H|En_F\rangle &= E|En_F\rangle \\ N_F|En_F\rangle &= n_F|En_F\rangle. \end{aligned}$$

It is therefore useful to rewrite the states as two-dimensional vectors

$$|En_F\rangle \sim \begin{pmatrix} |E0\rangle \\ |E1\rangle \end{pmatrix}.$$

In this basis, fermion operators are just matrices,

$$\begin{aligned} b^\dagger &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ b &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which act as, e. g.,

$$b^\dagger \begin{pmatrix} |x\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ |x\rangle \end{pmatrix}.$$

As a consequence any supersymmetric Hamilton operator has to be diagonal in this basis, and takes the form

$$H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \frac{1}{2}\{A, A^\dagger\} - \frac{1}{2}[A, A^\dagger]\sigma_3, \tag{2.13}$$

where σ_3 is the third of the Pauli matrices.

Non-trivial contributions in quantum theories usually only involve the coordinates, and not the momenta. Proceeding with the simplest way to obtain a non-trivial supersymmetric model is therefore to replace the definition of the bosonic creation and annihilation operators

$$\begin{aligned} a^\dagger &= \sqrt{\frac{m}{2}}\left(q - i\frac{p}{m}\right) \\ a &= \sqrt{\frac{m}{2}}\left(q + i\frac{p}{m}\right) \end{aligned}$$

with more general ones

$$\begin{aligned} A^\dagger &= \sqrt{\frac{1}{2}}\left(W(q) - i\frac{p}{\sqrt{m}}\right) \\ A &= \sqrt{\frac{1}{2}}\left(W(q) + i\frac{p}{\sqrt{m}}\right). \end{aligned}$$

The quantity $W(q)$ is called the superpotential, although it is not a potential in the strict sense, e. g. its dimension is not that of energy.

Using the relation (2.13) it is directly possible to calculate the corresponding Hamilton operator. The anti-commutator yields

$$\begin{aligned} \{A, A^\dagger\} &= \frac{1}{2}\left(\left(W - i\frac{p}{\sqrt{m}}\right)\left(W + i\frac{p}{\sqrt{m}}\right) + \left(W + i\frac{p}{\sqrt{m}}\right)\left(W - i\frac{p}{\sqrt{m}}\right)\right) \\ &= W^2 + \frac{p^2}{m}. \end{aligned}$$

For the commutator the general relation

$$[p, F(q)] = -i\frac{dF}{dq}$$

and the antisymmetry of the commutator is sufficient to immediately read off

$$[A, A^\dagger] = i\left[\frac{p}{\sqrt{m}}, W\right] = \frac{1}{\sqrt{m}}\frac{dW}{dq}.$$

The general supersymmetric Hamilton operator therefore takes the form

$$H = \frac{1}{2}\left(\frac{p^2}{m} + W^2\right) - \frac{\sigma_3}{2\sqrt{m}}\frac{dW}{dq}.$$

Supersymmetry therefore enforces a very restricted structure, since the terms proportional to the unit matrix and to σ_3 cannot be chosen independently, as would be possible in a non-supersymmetric theory.

It is actual possible to deduce from the asymptotic shape of the superpotential whether supersymmetry is intact or broken. To see this, use the fact that the Hamilton operator can be written as

$$H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{2.14}$$

Acting on the ground state wave function $(|00\rangle, |01\rangle)$ with energy zero, this implies

$$\begin{aligned} H_1|00\rangle &= A^\dagger A|00\rangle = 0 \\ H_2|01\rangle &= AA^\dagger|01\rangle = 0. \end{aligned}$$

Because $A^\dagger$ is adjoint to A and therefore

$$0 = \langle 00|A^\dagger A|00\rangle = (\langle 00|A)(A|00\rangle) \tag{2.15}$$

and analogue for the second state, this implies

$$A|00\rangle = 0 \text{ and } A^\dagger|01\rangle = 0.$$

It is therefore sufficient to solve the x-space differential equations

$$\left(\frac{1}{\sqrt{m}}\frac{d}{dx} \pm W\right)\left\langle x|0\begin{matrix}0\\1\end{matrix}\right\rangle = 0.$$

This can be solved by direct integration, yielding

$$\left\langle x|0\begin{matrix}0\\1\end{matrix}\right\rangle \sim \exp\left(\mp\sqrt{m}\int\limits_0^x W(y)dy\right).$$

Since the wave functions must be normalizable they have to vanish when $x \to \pm\infty$. Due to the appearance of the exponential this implies that

$$\mp\sqrt{m}\int\limits_0^{\pm\infty} W(y)dy \to -\infty \tag{2.16}$$

For the corresponding wave function to vanish sufficiently fast. This is only possible for one of the states. Thus, there exists at most only one ground state, as was discussed before. Furthermore, there may exist none at all, if the condition (2.16) cannot be fulfilled at all for a given potential. In particular, the condition (2.16) can only be fulfilled if the superpotential at infinity has a different sign for x going either to positive or negative infinity. For the same sign, the condition cannot be fulfilled. Therefore, it is simple to deduce for a given superpotential whether supersymmetry is exact or broken. Note that, since the superpotential is not the potential it is not possible to shift it arbitrarily, nor does it have an interpretation as a potential.

It is an interesting consequence that in the unbroken case it is possible to determine the superpotential from the ground-state wave function. For this, it is only necessary to investigate the upper component Schrödinger equation, the one with fermion number zero. It takes the form

$$H_1\langle x|00\rangle = A^\dagger A\langle x|00\rangle = -\frac{1}{2m}\frac{d^2}{dx^2}\langle x|00\rangle + \frac{1}{2}\left(W^2 - \frac{1}{\sqrt{m}}\frac{dW}{dx}\right)\langle x|00\rangle = 0.$$

Since the ground-state wave function does not have any nodes, it is possible to divide by it, obtaining

$$\begin{aligned} mW^2 - \sqrt{m}\frac{dW}{dx} &= \frac{1}{\langle x|00\rangle}\frac{d^2}{dx^2}\langle x|00\rangle \\ &= \left(\frac{1}{\langle x|00\rangle}\frac{d}{dx}\langle x|00\rangle\right)^2 + \frac{d}{dx}\left(\frac{1}{\langle x|00\rangle}\frac{d}{dx}\langle x|00\rangle\right) \end{aligned}$$

By explicit calculations, it can be shown that this differential equation of first order for W is solved by

$$W = -\frac{1}{\sqrt{m}}\frac{d}{dx}\ln\left(\langle x|00\rangle\right). \tag{2.17}$$

It should be noted that for every ordinary Hamilton operator H, a superpotential can be constructed by solving

$$V = \frac{1}{2}\left(W^2 - \frac{\hbar}{\sqrt{m}}W'\right),$$

with V the potential of H, for the superpotential W. The latter is then used to construct to $H_1 = H$ the H_2, creating a supersymmetric system. Likewise, it can be shown that any Hamilton oeprator can be written as $H = B^+B^-$ for two adjoint operators $(B^+)^\dagger = B^-$, and by setting $H_2 = B^-B^+$ a supersymmetric operator is obtained. Thus, any quantum-mechanical system can be supersymmetrized. Alternatively, of course, (2.17) can be used to construct the supersymmetric version.

2.5 A specific example: The infinite-well case

Given the relation (2.17), it is possible to construct for all quantum mechanical systems a supersymmetric version. This will be exemplified by the infinite well potential. The potential is given by

$$V_1 = \begin{cases} -\frac{\pi^2}{2mL^2} \text{ for } 0 \leq x \leq L \\ \infty \text{ otherwise} \end{cases}.$$

The normalization of the potential is chosen such that the ground-state energy is zero. The eigenvalues and eigenstates for this potential are

$$\begin{aligned}\langle x|E_n\rangle &= \sqrt{\frac{2}{L}}\sin\left(\frac{(n+1)\pi x}{L}\right)\\ E_n &= \frac{\pi^2}{2mL^2}(n+1)^2-\frac{\pi^2}{2mL^2}.\end{aligned} \tag{2.18}$$

To generate a supersymmetric system which has for states with fermion number zero exactly these properties is now straightforward. The superpotential can be constructed using the prescription (2.17). By direct calculation it becomes

$$W=-\frac{1}{\sqrt{m}}\frac{d}{dx}\ln\left(\langle x|00\rangle\right)=-\frac{1}{\sqrt{m}}\frac{1}{\langle x|00\rangle}\frac{d}{dx}\langle x|00\rangle=-\frac{\pi}{L\sqrt{m}}\cot\left(\frac{\pi x}{L}\right).$$

By construction, the Hamilton operator H_1 is just the ordinary one. The second operator looks much different,

$$\begin{aligned}H_2 &= \frac{p^2}{m}+\frac{1}{2}\left(W^2+\frac{1}{\sqrt{m}}\frac{dW}{dx}\right)\\ &= \frac{p^2}{m}+\frac{1}{2m}\left(\frac{\pi^2}{L^2}\cot^2\left(\frac{\pi x}{L}\right)+\frac{\pi^2}{L^2}\csc^2\left(\frac{\pi x}{L}\right)\right)\\ &= \frac{p^2}{m}+\frac{\pi^2}{2mL^2}\left(\frac{2}{\sin^2\left(\frac{\pi x}{L}\right)}-1\right).\end{aligned}$$

Although this potential is rather complicated, it is possible to write down the eigenspectrum immediately. By construction, there is no zero-mode, as this one is already contained in the spectrum of H_1. Furthermore, since the system is supersymmetric, the spectrum must be doubly degenerate. Since the conventional infinite-well potential is not degenerate, the operator H_2 must have the same spectrum for non-vanishing energy. This also exemplifies the possibilities supersymmetric methods may have outside supersymmetric theories. If it is possible to find a simple super-partner for a complicated potential, it is simple to determine the eigenspectrum of the original operator. Thus, the potential term in H_2 is also called partner-potential.

Also the eigenstates can now be constructed directly using the operators $A^\dagger$ and A. To obtain the ground-state of H_2, it is possible to start with the first excited state of H_1 and act with A on it. This can be seen as follows by virtue of (2.14)

$$H_2(A|n\rangle)=AA^\dagger A|n\rangle=A(A^\dagger A|n\rangle)=E_n(A|n\rangle).$$

Here, it has been used that the Hamilton operator H_1 is given by $A^\dagger A$. A similar relation can also be deduced for the eigenstates of H_1. Hence acting with A on an eigenstate of H_1

makes out of an eigenstate of H_1 an eigenstate of H_2 with the same energy. This yields for the ground-state of H_2 with the first excited state $|1\rangle$ of H_1

$$\begin{aligned}
\langle x|A|1\rangle &= -\frac{1}{\sqrt{2m}}\left(\frac{\pi}{L}\cot\left(\frac{\pi x}{L}\right)-\frac{d}{dx}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) \\
&= -\frac{1}{\sqrt{2m}}\left(\sqrt{\frac{2\pi^2}{L^{\frac{3}{2}}}}\cot\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)-\sqrt{\frac{8\pi^2}{L^{\frac{3}{2}}}}\cos\left(\frac{2\pi x}{L}\right)\right) \\
&= \sqrt{\frac{4\pi^2}{mL^{\frac{3}{2}}}}\sin^2\left(\frac{2\pi x}{L}\right).
\end{aligned}$$

Which has no nodes, and therefore is in fact the ground state. Note that the result needs still to be normalized. In this way, it is possible to generate the whole set of eigenstates of the second operator. Given these, the states

$$\begin{pmatrix} |n0\rangle \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ |n1\rangle \end{pmatrix}$$

represent eigenstates of the full supersymmetric Hamilton operator with zero and one fermionic quanta, respectively, and energy (2.18). $|n0\rangle$ is an eigenstate of H_1 with energy E_n, and H_2 is an eigenstate of H_2 with energy E_n.

This concludes the introduction of supersymmetry in conventional quantum mechanics.

Chapter 3

Grassmann and super mathematics

Before introducing supersymmetry in field theory, it is necessary to discuss the properties of Grassmann and supernumbers and the associated calculus. This will be necessary to introduce fermionic fields and also to introduce later the so-called superspace formalism which permits to write many quantities in supersymmetry in a very compact way. However, the superspace formalism is at times, due to its compactness, not simple to follow, and many basic mechanisms are somewhat obscured. Therefore, both formalisms will be introduced in parallel.

To some extent, this is a repetition of the machinery required for fermions in quantum field theory. However, the addition of supernumbers justifies here a little redundancy.

3.1 Grassmann numbers

Bosonic operators and fields of the same type commute,

$$[a, a] = 0.$$

They can therefore be described with ordinary (complex) numbers. However, fermionic operators and fields anticommute,

$$\{b, b\} = 0.$$

Hence a natural description should be by means of anti-commuting numbers. These numbers, the Grassmann numbers α^a, are defined by this property, yielding the so-called Grassmann algebra

$$\{\alpha^a, \alpha^b\} = 0$$

where the indices a and b serve to distinguish the numbers, and may therefore be themselves denumerable. Note that the Grassmann algebra is different from the so-called Clifford algebra

$$\{\beta^a, \beta^b\} = 2\delta^{ab}$$

which is obeyed, e. g., by the γ-matrices appearing in the Dirac-equation, and therefore also in the context of the description of fermionic fields.

A particular consequence of the Grassmann algebra is that all its elements are nilpotent,

$$(\alpha^a)^2 = 0.$$

Hence, the set $\mathcal{S}$ of independent Grassmann numbers with $a = 1, ..., N$ base numbers are

$$\mathcal{S} = \{1, \alpha^a, \alpha^{a_1}\alpha^{a_2}, ..., \alpha^{a_1} \times ... \times \alpha^{a_N}\},$$

where all a_i are different. This set contains therefore only 2^N elements. There are no more elements, as the square of every Grassmann number vanishes, and by anti-commuting thus any product containing twice the same Grassmann number vanishes. Of course, each element of $\mathcal{S}$ can be multiplied by ordinary complex numbers c, and can be added. This is very much like the case of ordinary complex numbers. Such combinations z are called supernumbers, and take the general form

$$z = c_0 + c_a\alpha^a + \frac{1}{2!}c_{ab}\alpha^a\alpha^b + ... + \frac{1}{N!}c_{a_1...a_N}\alpha^{a_1} \times ... \times \alpha^{a_N} = c_0 + c_S. \tag{3.1}$$

Here, the factorials have been included for later simplicity, and the coefficient matrices can be taken to be antisymmetric in all indices, as the product of α^as are antisymmetric. For $N = 2$ the most general supernumber is therefore

$$z = c_0 + c_1\alpha^1 + c_2\alpha^2 + c_{12}\alpha^1\alpha^2,$$

where the antisymmetry has already been used. Sometimes the term c_0 is also called body and the remaining part soul. It is also common to split the supernumber in its odd and even (fermionic and bosonic) part. Since any product of an even number of Grassmann numbers commutes with other Grassmann numbers, this association is adequate. For $N = 2$, e. g., the odd or fermionic contribution is

$$c_1\alpha^1 + c_2\alpha^2,$$

while the even or bosonic contribution is

$$c_0 + c_{12}\alpha^1\alpha^2.$$

Since the prefactors can be complex, it is possible to complex conjugate a supernumber. The conjugate of a product of Grassmann-numbers is defined as

$$(\alpha^a...\alpha^b)^* = \alpha^b...\alpha^a \tag{3.2}$$

Note that this implies that a product of an even number of Grassmann numbers is imaginary while an odd number is real,

$$\begin{aligned} \alpha^* &= \alpha \\ (\alpha\beta)^* &= \beta\alpha = -\alpha\beta, \end{aligned}$$

due to the anti-commutation when bringing the product back to its original order.

An important property of a supernumber z is its Grassmann parity $\pi(z)$. It differentiates between numbers which commute or anti-commute, and thus takes the values 0 or 1. Hence, for two supernumbers

$$z_1 z_2 = (-1)^{\pi(z_1)\pi(z_2)} z_2 z_1,$$

the Grassmann parity can be used to determine the sign of permutations. Note that only supernumbers with only even or odd numbers of Grassmann numbers have a definite Grassmann parity. Hence, supernumbers with definite Grassmann parity 0 or 1 are therefore called even or odd.

Finally, a norm can be defined as

$$|z|^2 = |c_0|^2 + \sum_{k=1}^{\infty} \sum_{\text{Permutations}} \frac{1}{k!} |c_{a_1 \dots a_k}|^2,$$

such that it is possible to give meaning to the statement that a supernumber is small.

3.2 Superspace

3.2.1 Linear algebra

In superspace, each coordinate is a supernumber instead of an ordinary number. Alternatively, this can be regarded as a product space of an ordinary vector space times a vector space of supernumbers without body. This is very much like the case of a complex vector space, which can be considered as a real vector space and one consisting only of the imaginary parts of the original complex vector space.

A more practical splitting is the one in bosonic and fermionic coordinates. In bosonic coordinates only even products (including none at all) of Grassmann numbers appear, while in the fermionic case there is always an odd number of Grassmann numbers. Denoting bosonic coordinates by β and fermionic ones by ϕ, vectors take the form

$$\begin{pmatrix} \beta^i \\ \hline \phi^j \end{pmatrix}.$$

In terms of the coefficients of a supernumber in a space with two fermionic and two bosonic coordinates, based on the supernumber (3.1), this takes the form

$$\begin{pmatrix} c_0 \\ c_{12} \\ \hline c_1 \\ c_2 \end{pmatrix}.$$

In the latter writing, it is important to notice that the unit vectors are not just bosonic ones, but rather the fermionic α_1 and α_2. This is like introducing the vector space of complex numbers with unit vectors 1 and i - the latter unit vector squares to -1 instead of one, as is the standard version. Similarly, the fermionic coordinates here still anti-commute.

Of course, this permits immediately to construct tensors of higher rank, in particular matrices. To have the same rules for matrix multiplication, it follows that a $(L+K)$-supermatrix must have the composition

$$\begin{pmatrix} A = \text{bosonic } L\times L & B = \text{fermionic } K\times L \\ C = \text{fermionic } L\times K & D = \text{bosonic } K\times K \end{pmatrix}$$

where bosonic and fermionic refers to the fact whether the entries are bosonic or fermionic (or even and odd, respectively). However, these matrices do have a number of properties which make them different from ordinary ones. Also, operations like trace have to be modified.

First of all, the transposition operation is different. The sub-product of two fermionic sub-matrices B and C behaves as

$$(BC)_{ik} = B_{ij}C_{jk} = -C_{jk}B_{ij} = -(C^TB^T)_{ki} = (BC)^T_{ki}.$$

Therefore, there appears an additional minus-sign when transposing products of B- and C-type matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T = \begin{pmatrix} A^T & -C^T \\ -B^T & D^T \end{pmatrix}.$$

As a consequence, applying a supertransposition twice does still return the original matrix.

Though it is possible to define the inverse of a Grassmann numbers, there is no direct possibility to determine it explicitly, but products of an even number of Grassmann numbers are ordinary numbers and can therefore be inverted. Therefore, the inverse matrix is rather complicated,

$$M^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}.$$

That this is the correct prescription can be checked by explicit calculation,

$$MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} A(A-BD^{-1}C)^{-1} - BD^{-1}C(A-BD^{-1}C)^{-1} & -B(D-CA^{-1}B)^{-1} + B(D-CA^{-1}B)^{-1} \\ C(A-BD^{-1}C)^{-1} - C(A-BD^{-1}C)^{-1} & D(D-CA^{-1}B)^{-1} - CA^{-1}B(D-CA^{-1}B)^{-1} \end{pmatrix}$$

and accordingly for $M^{-1}M$.

This result implies that a supernumber defined as in (3.1), which is formally a 1×1 matrix, has only an inverse if its pure complex part c_0 is non-zero. If it has, it can be explicitly determined as the infinite series

$$\frac{1}{z} = \frac{1}{c_0} \sum \left(-\frac{c_S}{c_0} \right)^n,$$

though this is rarely needed in practice.

Also the standard operations of trace and determinant get modified. The trace changes to the supertrace

$$\text{str}M = \text{tr}A - \text{tr}D. \tag{3.3}$$

The minus sign is necessary to preserve the cyclicity of the trace

$$\begin{aligned} \text{str}M_1M_2 &= \text{tr}(A_1A_2 + B_1C_2) - \text{tr}(C_1B_2 + D_1D_2) \\ &= (A_1)_{ij}(A_2)_{ji} + (B_1)_{ij}(C_2)_{ji} - (C_1)_{ij}(B_2)_{ji} + (D_1)_{ij}(D_2)_{ji}. \end{aligned}$$

The products of A and D matrices are ordinary matrices, and therefore are cyclic. However, the products of the fermionic matrices acquire an additional minus sign when permuting the factors, and renaming the indices,

$$\begin{aligned} & (A_2)_{ij}(A_1)_{ji} + (B_2)_{ij}(C_1)_{ji} - (C_2)_{ij}(B_1)_{ji} + (D_2)_{ij}(D_1)_{ji} \\ = \; & \text{tr}(A_2A_1 + B_2C_1) - \text{tr}(C_2B_1 + D_2D_1) = \text{str}M_2M_1. \end{aligned}$$

For the definition of the determinant the most important feature is to preserve the fact that the determinant of a product of matrices is a product of the respective determinants. This can be ensured when generalizing the identity

$$\det A = \exp(\text{tr} \ln A)$$

of conventional matrices for the definition of the superdeterminant

$$\text{sdet}M = \exp(\text{str} \ln M)$$

This can be proven by the fact that the determinant should be the product of all eigenvalues. Since the trace is the sum of all eigenvalues λ_i, and these are also for a super-matrix bosonic, it follows

$$\exp\left(\mathrm{str}\ln M\right)=\exp\left(\sum_{i\varepsilon A}\ln\lambda_i-\sum_{i\varepsilon D}\ln\lambda_i\right)=\exp\ln\left(\frac{\Pi_{i\varepsilon A}\lambda_i}{\Pi_{i\varepsilon D}\lambda_i}\right)=\frac{\Pi_{i\varepsilon A}\lambda_i}{\Pi_{i\varepsilon D}\lambda_i}. \tag{3.4}$$

Note the important fact that the eigenvalues of the fermionic dimension part D appears in the denominator rather than the numerator. This will play a crucial role in dealing with fermions in quantum theories.

The product rule for diagonalizable matrices follows then immediately. To prove that the superdeterminant of the product is the product of the individual determinants also in general requires the Baker-Campbell-Hausdorff formula

$$\exp F\exp G=\exp\left(F+G+\frac{1}{2}[F,G]+\frac{1}{12}\left([[F,G],G]+[F,[F,G]]\right)+...\right).$$

Set $F=\ln M_1$ and $G=\ln M_2$. Then it follows that

$$\mathrm{str}\ln(M_1M_2)=\mathrm{str}\ln\left(\exp F\exp G\right)=\mathrm{str}\left(F+G\right)=\mathrm{str}(\ln M_1+\ln M_2).$$

Here, it was invested that the trace of any commutator of two matrices vanishes due to the cyclicity of the trace. It then follows immediately that

$$\mathrm{sdet}(M_1M_2)=\exp\left(\mathrm{str}\ln(M_1M_2)\right)=\exp\mathrm{str}\left(\ln M_1+\ln M_2\right)=\mathrm{sdet}M_1\mathrm{sdet}M_2 \tag{3.5}$$

where the last step was possible as the supertraces are ordinary complex numbers.

To evaluate the superdeterminant explicitly, it is useful to rewrite a super-matrix as

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A & 0\\ C & 1\end{pmatrix}\begin{pmatrix}1 & A^{-1}B\\ 0 & D-CA^{-1}B\end{pmatrix}.$$

It then follows immediately by the product rule for determinants that

$$\mathrm{sdet}M=\det A\det\left(D-CA^{-1}B\right).$$

A similar formula an be obtained by isolating the superdeterminant of D rather than A. Since in both cases both factors are purely bosonic, they can be evaluated, and yield a bosonic number, as anticipated from a product of bosonic eigenvalues.

3.2.2 Analysis

To do analysis, it is necessary to define functions on supernumbers. First, start with analytic functions. This is rather simple, due to the nilpotency of supernumbers. Hence, for a function of one supervariable

$$z = b + f$$

only, with b bosonic and f fermionic, the most general function is

$$F(z) = F(b) + \frac{dF(b)}{db} f.$$

Any higher term in the Taylor series will vanish, since $f^2 = 0$. Since Grassmann numbers have no inverse, all Laurent series in f are equivalent to Taylor series. For a function of two variables, the most general form is

$$F(z_1, z_2) = f(b_1, b_2) + \frac{\partial F(b_1, b_2)}{\partial b_1} f_1 + \frac{\partial F(b_1, b_2)}{\partial b_2} f_2 + \frac{\partial^2 F(b_1, b_2)}{\partial b_1 \partial b_2} f_1 f_2.$$

There are no other terms, as any other term would have at least a square of the Grassmann variables, which therefore vanishes.

This can therefore be extended to more general functions, which are no longer analytical in their arguments,

$$F(b, f) = F_0(b) + F_1(b) f \tag{3.6}$$

and correspondingly of two variables

$$F(b_1, b_2, f_1, f_2) = F_0(b_1, b_2) + F_i(b_1, b_2) f_i + F_{12}(b_1, b_2) f_1 f_2,$$

and accordingly for more than two.

The next step is to differentiate such functions. Differentiating with respect to the bosonic variables occurs as with ordinary functions. For the differentiation with respect to fermionic numbers, it is necessary to define a new differential operator by its action on fermionic variables. As these can appear at most linear, it is sufficient to define

$$\begin{aligned} \frac{\partial}{\partial f_i} 1 &= 0 \\ \frac{\partial}{\partial f_i} f_j &= \delta_{ij} \end{aligned} \tag{3.7}$$

Since the result should be the same when $f_1 f_2$ is differentiated with respect to f_1 irrespective of whether f_1 and f_2 are exchanged before derivation or not, it is necessary to declare that the derivative anti commutes with Grassmann numbers:

$$\frac{\partial}{\partial f_1} f_2 f_1 = -f_2 \frac{\partial}{\partial f_1} f_1 = -f_2 = \frac{\partial}{\partial f_1}(-f_1 f_2) = \frac{\partial}{\partial f_1} f_2 f_1.$$

Alternatively, it is possible to introduce left and right derivatives. This will not be done here. As a consequence, the product (or Leibnitz) rule reads

$$\frac{\partial}{\partial f_i}(f_j f_k) = \left(\frac{\partial}{\partial f_i} f_j\right) f_k - f_j \frac{\partial}{\partial f_i} f_k.$$

Likewise, the integration needs to be constructed differently. In fact, it is not possible to define integration (and also differentiation) as a limiting process, since it is not possible to divide by infinitesimal Grassmann numbers. Hence it is necessary to define integration. As a motivation for how to define integration the requirement of translational invariance is often used. This requires then

$$\begin{aligned} \int df &= 0 \\ \int f df &= 1 \end{aligned} \tag{3.8}$$

Translational invariance follows then immediately as

$$\int F(b, f_1 + f_2) df_1 = \int (h(b) + g(b)(f_1 + f_2)) df_1 = \int (h(b) + g(b) f_1) df_1 = \int F(b, f_1) df_1$$

where the second definition of (3.8) has been used. Note that also the differential anti-commutes with Grassmann numbers. Hence, this integration definition applies for $f df$. If there is another reordering of Grassmann variables, it has to be brought into this order. In fact, performing the remainder of the integral using (3.8) yields $g(b)$. Hence, this definition provides translational invariance. It is an interesting consequence that integration and differentiation thus are the same operations for Grassmann variables, as can be seen from the comparison of (3.7) and (3.8).

A further consequence, which will be useful later on, is that multiple integrations always projects out the coefficient of a superfunction with the same number of Grassmann variables as integration variables, provided the same set appears. In particular,

$$\begin{aligned} &\int (g_0(b_1, b_2) + g_1(b_1, b_2) f_1 + g_2(b_1, b_2) f_2 + g_{12}(b_1, b_2) f_1 f_2) df_1 df_2 \\ = &\int (g_2(b_1, b_2) f_2 + g_{12}(b_1, b_2) f_2) df_1 = g_{12}(b_1, b_2) \end{aligned}$$

These integral relations will be useful in chapter 6 to introduce the so-called superspace formulation of supersymmetric field theories later in such a way that supersymmetry is directly manifest.

It is useful that also the Dirac-δ function can be expressed for Grassmann variables. It also takes a very simple form,

$$\delta(f_1 - f_2) = f_1 - f_2.$$

This can be proven by direct application,

$$\begin{aligned} &\int F(f_1)\delta(f_1 - f_2)df_1 = \int ((g_0(b_1) + g_1(b_1)f_1)f_1 - (g_0(b_1) + g_1(b_1)f_1)f_2)df_1 \\ = \;& g_0(b_1) + g_1(b_1)f_2 = F(f_2). \end{aligned}$$

Here, it was necessary to use the anticommutation relation for Grassmann variables on the last term to bring this into the form for which the definition applies.

Chapter 4

Non-interacting supersymmetric quantum field theories

Supersymmetric quantum mechanics is a rather nice playground to introduce the concept of supersymmetry. Also, it is a helpful technical tool for various problems. E. g., it was useful to solve a very complicated Hamilton operator by relating it to the partner problem of the infinite-well potential. However, its real power as a physical concept is only unfolded in a field theoretical context. In this case, no longer bosonic and fermionic operators are the quantities affected by supersymmetric transformation, but particles themselves, bosons and fermions.

4.1 Fermions

While bosons can be incorporated in supersymmetric theories rather straightforwardly, a little more is needed in case of fermions. As has been seen in the quantum mechanical case, supersymmetry requires the same number of bosonic and fermionic operators to appear in the theory. The translation to field theory will be that there is the same number of bosons and fermions in the theory. Now, fermions in particle physics are encountered, e. g., in the form of electrons, which are described by Dirac spinors. These spinors include not only the electron, but also its antiparticle. As both have the possibility to have spin up or down, these are four degrees of freedom. This would require at least four bosons to build a supersymmetric theory. This is already quite a number of particles. However, it is also possible to construct fermions which are their own antiparticles. Therefore the number of degrees of freedom is halved. These are called Majorana fermions. Since these work quite a little differently than ordinary fermions, these will be introduced in this section. However, as yet this is a purely theoretical concept. No Majorana fermions have been observed in nature so far, although there are speculations that neutrinos, which are usually described by ordinary fermions, may be Majorana fermions, but there is no clear experimental evidence for this. These Majorana fermions, with identical particle and anti-particle, can mathematically also described as only one particle. This the so-called Weyl-fermion formulation[1]. It is this formulation, which will be used predominantly here. However, also the Majorana formulation is useful, and will be introduced briefly.

Note that fermions always have to have at least spin 1/2 as a consequence of the so-called CPT-theorem (or, equivalently, Lorentz invariance). These are two degrees of freedom. Hence, it is not possible to construct a supersymmetric theory with less than two fermionic and two bosonic degrees of freedom, at least in four dimensions.

As the spinors describing fermions are actually complex, and only by virtue of the equations of motion are reduced to effectively two degrees of freedom, in principle also four bosonic degrees of freedom are needed off-shell, that is without imposing the equations of motions. This will be ignored for now, and will only be taken up later, when it becomes necessary to take this distinction into account when quantizing the theory.

4.1.1 Fermions, spinors, and Lorentz invariance

Supersymmetry transformations δ_ξ will relate bosons ϕ and fermions ψ, i. e. in infinitesimal form

$$\delta_\xi \phi \sim \xi\psi \tag{4.1}$$

with an infinitesimal parameter ξ. There is a number of observations to be made from this seemingly innocent relation. First of all, the quantity on the left-hand side is Grassmann-even, it is an ordinary (complex) number. The spinor ψ, describing a fermion, however, is Grassmann-odd, it is a fermionic number. Hence, also the parameter of the supersymmetry transformation ξ must be Grassmann-odd[2], such that the combination can give a Grassmann-even number. Secondly, a bosonic field, once more by virtue of the CPT-theorem, can have only integer spin. But since the fermion has non-integer spin, both sides would transform differently under Lorentz transformations. Since Lorentz symmetry should certainly not be broken by introducing supersymmetry (as this was the reason to introduce it at all), the parameter ξ can also not transform trivially under Lorentz trans-

[1] Note that in the more general case of non-supersymmetric theories there are subtle differences between Weyl fermions and Majorana fermions, especially if the number of dimensions differ from 4. This will be taken up later.

[2] Note that for Grassmann numbers infinitesimal is not a sensible notion, since $\xi^2 = 0$, and thus the statement $\xi^3 \ll \xi^2$ is wrong. When speaking of an infinitesimal transformation, it is always assumed that ξ is given by $\xi = \epsilon\xi'$ with ϵ an infinitesimal real parameter and ξ' a pure Grassmann number.

formations. It must also be a spinor of some kind, as is the fermion field. To be able to exactly identify which type, a detour on spinors and Lorentz transformations will be necessary.

4.1.1.1 Weyl spinors

The starting point is the Dirac equation

$$(i\partial_\mu \gamma^\mu + m)\,\Psi = 0,$$

where Ψ is a four-component (complex) spinor, m is the mass, and γ_μ are the Dirac matrices. The representation for the latter employed here is

$$\begin{aligned}\gamma_\mu &= \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix} \\ \sigma_\mu &= \begin{pmatrix} \sigma_0 \\ \sigma_i \end{pmatrix} \qquad \bar{\sigma}_\mu = \begin{pmatrix} \sigma_0 \\ -\sigma_i \end{pmatrix}\end{aligned} \tag{4.2}$$

where σ_i are the Pauli matrices and σ_0 is the unit matrix. The spinor Ψ can be divided into two two-component objects ψ and χ,

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}.$$

These two components fulfill a set of coupled Dirac equations

$$(E - \sigma_i p_i)\psi = \sigma^\mu p_\mu \psi = m\chi \tag{4.3}$$

$$(E + \sigma_i p_i)\chi = \bar{\sigma}^\mu p_\mu \chi = m\psi. \tag{4.4}$$

The coupling is only mediated by the mass. If the mass is zero, both equations decouple, and the spinors ψ and χ become eigenstates of the helicity operator $\sigma_i p_i/\sqrt{p_i p_i}$ with eigenvalues 1 and -1, respectively.

This is no longer true for finite masses. In this case, however, they are still eigenstates of γ_5, which is given by

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the zero-mass limit four-component spinors are not only eigenstates to γ_5 but also of the helicity operator. Those with γ_5 eigenvalue 1 and helicity eigenvalue 1 are called right-handed. Those with eigenvalues -1 are called left-handed. For a general spinor Ψ it follows

$$\begin{aligned}\frac{1}{2}(1+\gamma_5)\Psi &= \begin{pmatrix}\psi\\0\end{pmatrix} = \Psi_R\\ \frac{1}{2}(1-\gamma_5)\Psi &= \begin{pmatrix}0\\\chi\end{pmatrix} = \Psi_L.\end{aligned}$$

The importance of the subspinors ψ and χ is that they have a definite behavior under Lorentz transformations. These are therefore called Weyl spinors. To see this, take an infinitesimal Lorentz transformation

$$\begin{aligned}E\to E' &= E-\eta_i p_i\\ p\to p' &= p-\epsilon\times p-\eta E,\end{aligned}$$

where the vectors ϵ parametrizes a rotation and η a boost. With the general transformation behavior of Dirac spinors under infinitesimal Lorentz transformations

$$\Psi' = \Psi + \frac{i}{2}\begin{pmatrix}\epsilon_i\sigma_i-\eta_i\sigma_i & 0\\ 0 & \epsilon_i\sigma_i+\eta_i\sigma_i\end{pmatrix}\Psi \tag{4.5}$$

the transformation rules

$$\begin{aligned}\psi\to\psi' &= \psi+(i\epsilon_i\sigma_i/2-\eta_i\sigma_i/2)\psi\\ \chi\to\chi' &= \chi+(i\epsilon_i\sigma_i/2+\eta_i\sigma_i/2)\chi\end{aligned}$$

result. The important observation is that both transform the same under rotations, as spin 1/2 particles, but differently under boosts. Defining

$$V = 1+i\epsilon_i\sigma_i/2-\eta_i\sigma_i/2$$

the transformation rules simplify to

$$\begin{aligned}\psi' &= V\psi\\ \chi' &= V^{-1\dagger}\chi = (1+i\epsilon_i\sigma_i/2-\eta_i\sigma_i/2)^{-1\dagger}\chi\\ &= (1-i\epsilon_i\sigma_i/2+\eta_i\sigma_i/2)^{\dagger}\chi = (1+i\epsilon_i\sigma_i/2+\eta_i\sigma_i/2)\chi.\end{aligned}$$

Furthermore, from equations (4.3) and (4.4) it follows then that a multiplication with $\sigma^\mu p_\mu$ exchanges (up to a factor m) a ψ- and a χ-type-spinor, and thus changes the respective transformation properties under Lorentz transformations.

This is already important: Because the aim is to construct a Lorentz scalar for the supersymmetry transformation (4.1), and to do this with only two degrees of freedom, it is necessary to form a scalar out of Weyl spinors. For a Dirac spinor this is simple,

$$\Psi^\dagger\gamma_0\Psi = \begin{pmatrix}\psi^\dagger & \chi^\dagger\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \psi^\dagger\chi + \chi^\dagger\psi,$$

and thus scalars are products of ψ- and χ-type spinors. So there is already one possibility to construct a ψ from a χ, but this involves the Dirac equation and a momentum. A more general possibility is

$$\sigma_2\psi^{*'} = \sigma_2 V^*\psi^* = \sigma_2(1 - i\epsilon_i\sigma^{i*}/2 - \eta_i\sigma^{i*}/2)\psi^* = (1 + i\epsilon_i\sigma^i/2 + \eta_i\sigma^i/2)\sigma_2\chi = V^{-1\dagger}\sigma_2\psi^*,$$

where in the last step it was used that σ_2 anti-commutes with the real σ_1 and σ_3 matrices, but commutes with itself and is itself purely imaginary. Hence multiplying the complex-conjugate by σ_2, a process which may be recognized as charge conjugation, turns the Weyl-spinor ψ into one of χ-type, with the corresponding changed Lorentz-transformation. Hence, e. g., the quantity

$$(-i\sigma_2\psi^*)^\dagger\psi = (-i\sigma_2\psi)^T\psi = \psi^T(i\sigma_2)\psi$$

is a scalar, just as desired. Similar $i\sigma_2\chi^*$ transforms like a ψ, and so on. Hence transformations rules can be introduced, and corresponding translations between left-handed and right-handed spinors.

4.1.1.2 Indices and dotted indices

However, in introducing these quantities, things became rather messy. It is therefore useful to introduce a compact index notation. This will be done, similarly to the case of special relativity, by the position of the indices. This notation is essentially based on the structure of the Lorentz group.

The Lorentz group consists out of rotations J and boosts K. In general, commutators of J and K do not vanish. However, defining skew versions of these operators

$$\begin{aligned} A &= \frac{1}{2}(J + iK)\\ B &= \frac{1}{2}(J - iK) \end{aligned}$$

this is the case. The Lorentz algebra becomes then a direct product of two SU(2) algebras

$$\begin{aligned} [A_i, A_j] &= \epsilon_{ijk}A_k\\ [B_i, B_j] &= \epsilon_{ijk}B_k\\ [A_i, B_j] &= 0. \end{aligned} \quad (4.6)$$

Hence, any representation of the Lorentz group can be assigned two independent quantum numbers, which are either integer or half-integer. E. g. scalars are then just twice the trivial case. Left-handed and right-handed fermions, however, belong to the $(1/2, 0)$ and $(0, 1/2)$ representations, vectors like the momentum belong to the $(1/2, 1/2)$ representation, and antisymmetric tensors like the generators of angular momentum to the $(1, 0) + (0, 1)$ representation. The simplification will now be based on distinguishing indices of the two different representations.

For this purpose, define the meaning of the index position for a χ-type spinor by

$$\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} = i\sigma_2\chi = \begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix}. \tag{4.7}$$

Hence, given an ordinary χ-type spinor with components χ_1 and χ_2, the corresponding ψ-type spinor has components χ^1 and χ^2.

Since scalars are obtained by multiplying χ- and ψ-type spinors, these can now be simply obtained from two χ-type spinors α and β by

$$\alpha^T\beta = (\alpha^1\alpha^2)\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha^1\beta_1 + \alpha^2\beta_2 = \alpha^a\beta_a.$$

This is very similar to the case of special relativity. Note that spinors are usually Grassmann-valued. Hence the order is relevant. The common convention is that the indices appear from top left to bottom right. Otherwise a minus-sign appears in the case of Grassmann-spinors,

$$\alpha^a\beta_a = -\beta_a\alpha^a,$$

and correspondingly for more elements

$$\alpha^a\beta^b\gamma_a\delta_b = -\alpha^a\gamma_a\beta^b\delta_b = -\gamma_a\alpha^a\delta_b\beta^b.$$

From the definition (4.7), it is also possible to read-off a 'metric' tensor, which can be used to raise and lower an index, the totally anti-symmetric rank two tensor ϵ^{ab}, yielding

$$\chi^a = \epsilon^{ab}\chi_b.$$

where $\epsilon^{12} = 1$ and $\epsilon_{12} = -1$.

This fixes the notation for χ-type spinors. Since there are also ψ-type spinors, it is necessary to also introduce a corresponding notation for them. However, in general the same notation could quickly lead to ambiguities. Therefore, a different convention is used: ψ-type spinors receive also upper and lower indices, but these in addition have a dot,

$$\begin{pmatrix} \psi_{\dot{1}} \\ \psi_{\dot{2}} \end{pmatrix} = -i\sigma_2\psi = \begin{pmatrix} -\psi^{\dot{2}} \\ \psi^{\dot{1}} \end{pmatrix}.$$

It is then also possible to contract these two indices analogously to obtain a scalar, but this time the ordering will be defined to be from bottom left to top right

$$\alpha^T\beta = \alpha_{\dot{a}}\beta^{\dot{a}}.$$

From the fact how these spinors can be contracted to form scalars, it can be read off directly that χ^a transforms by multiplication with the Lorentz transformation V^*, since χ_a transforms with $V^{-1\dagger}$. On the other hand, since $\psi^{\dot{a}}$ transforms with V, $\psi_{\dot{a}}$ has to transform with V^{-1T}.

Otherwise the same applies as previously for the χ-type spinors. Given this index notation, there are no ambiguities left in case of expressions with explicit indices. To be able to separate these also without using the indices explicitly, usually ψ-type spinors are written as $\bar{\psi}$. This is not the same as the conventional Dirac-bar, and the equalities

$$\psi^{\dot{1}} = \bar{\psi}^{\dot{1}*}$$
$$\psi^{\dot{2}} = \bar{\psi}^{\dot{2}*}$$

hold. However, since complex conjugation is involved when it comes to treating χ-type spinors, here the definition is

$$\bar{\chi}_{\dot{a}} = \chi^*_{\dot{a}}$$

Therefore, a scalar out of χ-type spinors can now be written as

$$\chi^\dagger\chi = \bar{\chi}\chi$$

and similarly

$$\psi^\dagger\psi = \psi\bar{\psi}.$$

Another scalar combination, which will be often used, is

$$\bar{\psi}\cdot\bar{\chi} = \bar{\psi}^\dagger\bar{\chi} = \epsilon_{ab}\psi^*_a\chi^{b*} = -\psi^*_1\chi^*_2 - \psi^*_2\chi^*_1.$$

Having now available a transformation which makes from an electron-type half spinor a positron-like half-spinor, it is natural to investigate what happens if both are combined into one single 4-component spinor, i. e., combining two Weyl spinors. To obtain the correct transformation properties under Lorentz transformation, this object is

$$\Psi = \begin{pmatrix} \psi^{\dot{1}} \\ \psi^{\dot{2}} \\ -\psi^{\dot{2}*} \\ \psi^{\dot{1}*} \end{pmatrix}.$$

Since there are only two independent degrees of freedom, the spinor Ψ cannot describe, e. g., an electron. Its physical content is made manifest by performing a charge conjugation

$$C\Psi = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \psi^* \\ -i\sigma_2\psi \end{pmatrix} = \begin{pmatrix} \psi \\ -i\sigma_2\psi^* \end{pmatrix} = \Psi,$$

i. e., it is invariant under charge conjugation and thus describes a particle which is its own antiparticle, like the photon. Spin 1/2-particles with this property are called Majorana fermions, and thus this is a Majorana spinor. Note that this combination is not possible for arbitrary dimensions (and arbitrary space-time manifolds), but is correct in four dimensional Minkowski space-time. In this case, which covers almost all of this lecture, Weyl and Majorana fermions can be used synonymously.

4.2 The simplest supersymmetric theory

This is sufficient to set the scene for a first supersymmetric quantum field theory.

As discussed previously, it will be necessary to have the same number of fermionic and bosonic degrees of freedom. This requires at least two degrees of freedom, since it is not possible to construct a fermion with only one. Consequently, two scalar degrees of freedom are necessary. The simplest system with this number of degrees of freedom is a non-interacting system of a complex scalar field ϕ and a free Weyl fermion χ, which will be described by the undotted spinor. The corresponding Lagrangian is given by

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi. \tag{4.8}$$

Note that here already with the fully quantized theory will be dealt. From this the corresponding action is constructed as

$$S = \int \mathcal{L} d^4x.$$

The corresponding physics will be invariant under a supersymmetry transformation if the action is invariant[3]. Since it is assumed that the fields vanish at infinity this requires invariance of the Lagrangian under the supersymmetry transformation up to a total derivative.

The according supersymmetry transformations can be constructed by trial and error. Here, they will be introduced with hindsight of the results, and afterwards their properties will be analyzed. The transformation

$$A' = A + \delta A$$

[3]Actually, only up to anomalies. This possibility will be disregarded here, and is no problem for the cases presented.

takes for the scalar field the form

$$\delta\phi = \xi^a \chi_a = (-i\sigma_2\xi)^T \chi. \tag{4.9}$$

Herein, ξ is a constant, Grassmann-valued spinor. It thus anticommutes with χ^a. This is necessary in order to form a scalar, complex number from χ. By dimensional analysis, ξ has units of $1/\sqrt{\text{mass}}$. The corresponding transformation law for the spinor is

$$\delta\chi = -i\sigma^\mu \bar{\xi}\partial_\mu\phi = \sigma^\mu\sigma_2\xi^*\partial_\mu\phi. \tag{4.10}$$

The pre-factor is fixed by the requirement that the Lagrangian is invariant under the transformation. The combination of ξ with σ_μ guarantees the correct transformation behavior of the expression under Lorentz transformation in spinor space. The derivative, which appears, is necessary to construct a scalar under Lorentz transformation in space-time, and to obtain the correct mass-dimension. It is the only object which can be used for this purpose, as it is the only one which appears in the Lagrangian (4.8), besides the scalar field. The general structure is therefore fixed by the transformation properties under Lorentz transformation. That the pre-factors are in fact also correct can be shown by explicit calculation,

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu((\delta\phi)^\dagger)\partial^\mu\phi + \partial_\mu\phi^\dagger\partial^\mu(\delta\phi) + (\delta\chi)^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu(\delta\chi) \\ &= i\partial_\mu\chi^\dagger\sigma_2\xi^*\partial^\mu\phi + i\chi^\dagger\bar{\sigma}^\nu\sigma^\mu\sigma_2\xi^*\partial_\nu\partial_\mu\phi - i\partial_\mu\phi^\dagger\partial^\mu(\xi^T\sigma_2\chi) - i\xi^T\sigma_2\sigma^\mu\bar{\sigma^\nu}\chi\partial_\nu\partial_\mu\phi^\dagger\end{aligned} \tag{4.11}$$

Herein partial integrations have been performed, as necessary to obtain this form. There are two linearly independent terms, one proportional to ξ^* and one to ξ in this expression. Both have therefore to either individually vanish or be total derivatives. To show this, it is helpful to note that

$$\bar{\sigma}^\nu\partial_\nu\sigma^\mu\partial_\mu = (\partial_0 - \sigma^j\partial_j)(\partial^0 + \sigma_i\partial^i) = \partial_0\partial^0 - \partial_i\partial^i = \partial^\mu\partial_\mu, \tag{4.12}$$

where it has been used that $\sigma_i^2 = 1$. Taking now only the terms proportional to ξ^* yields

$$i\partial_\mu\chi^\dagger\sigma_2\xi^*\partial^\mu\phi + i\chi^\dagger\sigma_2\xi^*\partial_\mu\partial^\mu\phi = \partial_\mu(\chi^\dagger i\sigma_2\xi^*\partial^\mu\phi). \tag{4.13}$$

This term is therefore indeed a total derivative. Likewise, also the term proportional to ξ^T can be manipulated to yield a pure total derivative. However, this is somewhat more complicated, as the combination (4.12) is not appearing. The last term can be rewritten as

$$-i\xi^T\sigma_2\sigma^\mu\bar{\sigma^\nu}\chi\partial_\nu\partial_\mu\phi^\dagger = \partial_\mu(\phi^\dagger i\xi^T\sigma_2\sigma^\mu\bar{\sigma^\nu}\partial_\nu\chi) + \phi^\dagger i\xi^T\sigma_2\sigma^\mu\bar{\sigma^\nu}\partial_\mu\partial_\nu\chi$$

It is then possible to use (4.12) on the last term to obtain

$$\partial_\mu(\phi^\dagger i\xi^T\sigma_2\sigma^\nu\bar{\sigma}^\mu\partial_\nu\chi) + \phi^\dagger i\xi^T\sigma_2\partial_\mu\partial_\mu\chi.$$

The first term is already a total derivative. The second term combines with the second-to-last term of (4.11) to a total derivative. Hence, the total transformation of the Lagrangian reads

$$\delta\mathcal{L} = \partial_\mu(\chi^\dagger i\sigma_2\xi^*\partial^\mu\phi + \phi^\dagger i\xi^T\sigma_2\sigma^\nu\bar{\sigma}^\mu\partial_\nu\chi + \phi^\dagger i\xi^T\sigma_2\partial^\mu\chi)$$

which is a total derivative.

Therefore, this theory is indeed supersymmetric. The set of fields ϕ and χ is called a supermultiplet. To be more precise, it is a left-chiral supermultiplet, because the spinor has been taken to be of χ-type. The χ-type spinor could be replaced with a ψ-type spinor, yielding a right-chiral supermultiplet, without changing the supersymmetry of the theory, although, of course, the transformation is modified.

There should be a note of caution here. Unfortunately, it will turn out that this demonstration is insufficient to show supersymmetry of the quantized theory, and it will be necessary to modify the Lagrangian (4.8). This problem will become apparent when discussing the supersymmetry algebra. However, most of the calculations performed so far can be used unchanged.

4.3 Supersymmetry algebra

It turns out that the supersymmetry transformations (4.9) and (4.10) will form an algebra, similar to the algebra (2.9) of the quantum-mechanical case. This algebra can be used to systematically construct supermultiplets, and is useful for many other purposes. Therefore, this algebra will be constructed here, based first on the simplest examples of supersymmetry transformations (4.9) and (4.10) and will be generalized thereafter.

4.3.1 Constructing an algebra from a symmetry transformation

Such an algebra has already been encountered in the quantum-mechanical case, and was given by (2.9). To obtain this algebra, it is necessary to construct the supersymmetry charges, which in turn generate the transformation. This is constructed as follows. In general, any symmetry transformation is a unitary transformation which thus preserves the values of observables. Thus for any field f, in general a transformation can be written as

$$f' = UfU^\dagger \tag{4.14}$$

with U a unitary operator. Any unitary operator can be written as

$$U = \exp(i\xi Q)$$

with Q hermitian, $Q = Q^\dagger$. If the transformation parameter ξ becomes infinitesimal, the relation (4.14) can be expanded to yield

$$\begin{aligned} f' &= (1 + i\xi Q)f(1 - i\xi Q) = f + [i\xi Q, f] \\ \to \delta f &= [i\xi Q, f]. \end{aligned}$$

In a quantum field theory, Q must again be a function of the f. To construct it, it is necessary to analyze the transformation properties of the Lagrangian under the infinitesimal transformation, which is given by

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{4.15}$$

The terms on the right-hand side can only be at most total derivatives. For simplicity, it will be assumed that the transformation U is not inducing such terms, but it will be necessary to include them below when returning to supersymmetry transformations, as there such terms are present, see (4.13). If then ∂K is dropped, the variation can be written as

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial f}\delta f + \frac{\partial\mathcal{L}}{\partial\partial^\mu f}\partial^\mu(\delta f).$$

This can be simplified using the equation of motion for f,

$$\frac{\partial\mathcal{L}}{\partial f} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu f}\right)$$

to yield

$$0 = \left(\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu f}\right)\right)\delta f + \frac{\partial\mathcal{L}}{\partial\partial^\mu f}\partial^\mu(\delta f) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu f}\delta f\right) = \partial_\mu j^\mu. \tag{4.16}$$

The current which has been thus defined is the symmetry current, and it is conserved. If a total derivative $\partial_\mu K^\mu$ exists, the definition of the symmetry current becomes

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu f}\delta f - K^\mu \tag{4.17}$$

Therefore, the charge defined as

$$q = \int d^d x j_0(x)$$

is also conserved. This derivation is also known as Noether's theorem, stating that for any symmetry there exists a conserved charge. Note that this charge is an operator, build from the field variables. It can be identified with the charge Q, which in general can be shown by an expansion of its constituents in power-series in f and $\partial_\mu f$ (ensuring the invariance of $\mathcal{L}$) and the usage of the commutation relations of f. This will not been done here, but below an explicit example for the case of supersymmetry will be discussed.

This completed, and thus with an explicit expression for generators of the transformation Q at hand, it is possible to construct the corresponding algebra by evaluating the (anti-)commutators of Q.

4.3.2 The superalgebra

The conserved supercurrent j^μ can be constructed using (4.15) and (4.16). Noting that $\partial_\mu \chi^\dagger$ is not appearing in the Lagrange density, only three derivatives plus the boundary term remain to yield

$$\begin{aligned} j^\mu &= -K^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger}\delta\phi^\dagger + \frac{\partial \mathcal{L}}{\partial \partial_\mu \chi}\delta\chi \\ &= -\chi^\dagger i\sigma_2 \xi^* \partial^\mu \phi + \partial_\nu \phi^\dagger i\xi^T \sigma_2 \sigma^\nu \bar{\sigma}^\mu \chi + \phi^\dagger i\xi^T \sigma_2 \partial_\mu \chi \\ &\quad -\partial^\mu \phi^\dagger \xi^T i\sigma_2 \chi + \chi^\dagger i\sigma_2 \xi^* \partial^\mu \phi + \chi^\dagger \bar{\sigma}^\mu \sigma_\nu i\sigma_2 \xi^* \partial^\nu \phi \end{aligned}$$

Here, and in the following, the necessary contribution from the hermitian conjugate contribution are not marked explicitly. This result can be directly reduced, since some terms cancel, to

$$\begin{aligned} j^\mu &= \chi^\dagger \bar{\sigma}^\mu \sigma_\nu i\sigma_2 \xi^* \partial^\nu \phi - \partial_\nu \phi^\dagger i\xi^T \sigma_2 \sigma^\nu \bar{\sigma}^\mu \chi \\ &= \xi^T(-i\sigma_2)J^\mu + \xi^* i\sigma_2 J^{\mu*} \\ J^\mu &= \sigma^\nu \bar{\sigma}^\mu \chi \partial_\nu \phi^\dagger. \end{aligned}$$

J^μ is the so-called supercurrent, which forms the conserved current by a hermitian combination, similar to the probability current in ordinary quantum mechanics, $i\psi^\dagger\partial_i\psi + i\psi\partial_i\psi^\dagger$. To write the complex-conjugate part of the current, it has been used that

$$\sigma_2 \sigma^\nu \bar{\sigma}^\mu = \sigma^\nu \bar{\sigma}^\mu \sigma_2$$

which follows by the anti-commutation rules for the Pauli matrices. This permits to construct the supercharge

$$Q = \int d^3x \sigma^\nu \chi \partial_\nu \phi^\dagger.$$

This indeed generates the transformation for the fields ϕ and χ. To show this, the canonical commutation relations

$$\begin{aligned} [\phi(x,t), \partial_t \phi(y,t)] &= i\delta(x-y) \\ \{\chi_a(x,t), \chi_b^\dagger(y,t)\} &= \delta_{ab}\delta(x-y), \end{aligned} \tag{4.18}$$

and all other (anti-)commutators vanishing, can be used.

For the bosonic fields this follows as

$$\begin{aligned} i[\xi Q,\phi(x)] &= i\int dy[\xi(\sigma^\nu\chi(y))\partial_\nu\phi^\dagger(y),\phi(x)] \\ &= i\int dy\xi\sigma^\nu\chi(y)[\partial_\nu\phi^\dagger(y),\phi(x)] \end{aligned} \tag{4.19}$$

Since the commutator of all spatial derivatives with the field itself vanishes, only the component $\nu=0$ remains,

$$i\int dy\xi\chi(y)[\partial_0\phi^\dagger(y),\phi(x)] = \int dy\xi\chi(y)\delta(x-y) = \xi\chi$$

which is exactly the form (4.9). The calculation for the transformation of $\phi^\dagger$ has to be performed with $\bar{\xi}\bar{Q}$, and yields the transformation rule with $\bar{\chi}$. Consequently, the transformation law for χ is obtained from

$$i[\xi Q+\bar{\xi}\bar{Q},\chi(x)] = -i\sigma^\mu(i\sigma_2\xi^*)\partial_\mu\phi,$$

and correspondingly for $\chi^\dagger$ from the complex conjugate version. To explicitly show this, it is necessary to note that

$$[\xi_a\chi^{\dagger a},\chi_b] = \xi_a\chi^{\dagger a}\chi_b - \chi_b\xi_a\chi^a = \xi_a\chi^{\dagger a}\chi_b + \xi_a\chi_b\chi^a = \xi_a\{\chi^{\dagger a},\chi_b\},$$

since Grassmann fields and variables anticommute.

It is now possible to construct the algebra.

First of all, the (anti-)commutators

$$[\bar{Q},\bar{Q}]=0 \qquad [Q,Q]=0 \qquad \{Q,Q\}=0 \qquad \{\bar{Q},\bar{Q}\}=0$$

all vanish, since in all cases all appearing fields (anti-)commute. There are thus, at first sight, one non-trivial commutator and one non-trivial anti-commutator.

For the non-vanishing cases, it is simpler to evaluate two consecutive applications of SUSY transformations. To perform this, note first that

$$[q,[p,f]]+[p,[f,q]]+[f,[q,p]]=0.$$

This can be shown by direct expansion. It can be rearranged to yield

$$[[q,p],f] = [q,[p,f]]-[p,[q,f]].$$

If q and p are taken to be ξQ and $\bar{\eta}\bar{Q}$, and f taken to be ϕ, this implies that the commutator of two charges can be obtained by determining the result from two consecutive applications of the SUSY transformations. Using (4.9) and (4.10), it is first possible to obtain the result for this double application. It takes the form

$$[\xi Q+\bar{\xi}\bar{Q},[\eta Q+\bar{\eta}\bar{Q},\phi]]=-i[\xi Q+\bar{\xi}\bar{Q},\eta^T(-i\sigma_2)\chi]=i\eta^T(-i\sigma_2)\sigma^\mu(-i\sigma_2\xi^*)\partial_\mu\phi. \quad (4.20)$$

Subtracting both possible orders of application yields then the action of the commutator

$$[[\xi Q+\bar{\xi}\bar{Q},\eta Q+\bar{\eta}\bar{Q}],\phi]=i(\xi^T(-i\sigma_2)\sigma^\mu(-i\sigma_2\eta^*)-\eta^T(-i\sigma_2)\sigma^\mu(-i\sigma_2\xi^*))\partial_\mu\phi.$$

Here it has been used that $\bar{Q}$ is commuting with ϕ, as it does not depend on $\phi^\dagger$.

Aside from a lengthy expression $f(\eta,\xi)$, which gives the composition rule for the parameters, there is one remarkable result: The appearance of $-i\partial_\mu\phi$, which is the action of P_μ on ϕ, the momentum or generator of translations. Hence, the commutator is given by

$$[\xi Q+\bar{\xi}\bar{Q},\eta Q+\bar{\eta}\bar{Q}]=f(\eta,\xi)P_\mu. \quad (4.21)$$

In fact, this is not all, due to aforementioned subtlety involving the fermions. This will be postponed to later.

Though anticipated in the introductory section 1.4, the appearance of the momentum operator seems at first surprising. However, in quantum mechanics it has been seen that the super-charges are something like the squareroot of the Hamilton operator, which is also applying, in the free case, to the momentum operator. This relation is hence less exotic than might be anticipated. Still, this implies that the supercharges are also something like the squareroot of the momentum operator, which leads to the notion of the supercharge being translation operators in fermionic dimensions. This idea will be taken up later when the superspace formulation will be discussed. Another important side remark is that this result implies that making supersymmetry a local gauge symmetry this connection yields automatically a connection to general relativity, the so-called supergravity theories.

Hence, the algebra for the SUSY-charges will not only contain the charges themselves, but necessarily also the momentum operator. However, the relations are rather simple, as the supercharges do not depend on space-time and thus (anti-)commute with the momentum operator, as does the latter with itself.

Thus, the remaining item is the anti-commutator of Q with $\bar{Q}$. For this, again the commutator is useful, as it can be expanded as

$$\begin{aligned}[\eta Q,\bar{\xi}Q^\dagger] &= \eta_1\xi_1^*(Q_2Q_2^\dagger+Q_2^\dagger Q_2)-\eta_1\xi_2^*(Q_2Q_1^\dagger+Q_1^\dagger Q_2)\\ &\quad -\eta_2\xi_1^*(Q_1Q_2^\dagger+Q_2^\dagger Q_1)+\eta_2\xi_2^*(Q_1Q_1^\dagger+Q_1^\dagger Q_1)\end{aligned}$$

Thus, all possible anticommutators appear in this expression. The explicit expansion of (4.21) is

$$(\eta_2\xi_2^*(\sigma_\mu)_{11}-\eta_2\xi_1^*(\sigma_\mu)_{12}-\eta_1\xi_2^*(\sigma_\mu)_{21}+\eta_1\xi_1^*(\sigma_\mu)_{22})P^\mu.$$

Thus, by coefficient comparison the anti-commutator is directly obtained as

$$\{Q_a, Q_b^\dagger\} = (\sigma^\mu)_{ab} P_\mu. \tag{4.22}$$

This completes the algebra for the supercharges, which is in the field-theoretical case somewhat more complicated by the appearance of the generator of translations, but also much richer.

Again, a subtlety will require to return to these results shortly. Before this, it is useful to collect a few more generic properties of the superalgebra, as well as of possible extensions of this superalgebra.

4.3.3 General properties of superalgebras

The superalgebra is what is called a graded Lie algebra. In contrast to an ordinary Lie algebra, which is characterized by its commutator relations

$$\left[t^a, t^b\right] = i f^{abc} t_c$$

with algebra elements t^a and structure constants f^{abc}, it has the commutator relations

$$\left[t^a, t^b\right\} = t^a t^b - (-1)^{\eta_a \eta_b} t^a t^b = i f^{abc} t_c,$$

where the η_i are known as gradings of the elements t_a, and are 0 for bosonic and 1 for fermionic generators. Because of the symmetry properties of the left-hand-side, this implies that f^{abc} is zero except when $\eta_c = \eta_a + \eta_b$. Still, if the t_a are Hermitian it follows that $f^*_{abc} = -f_{bac}$. Also a super-Jacobi identity follows

$$(-1)^{\eta_c \eta_a} \left[\left[t_a, t_b\right\}, t_c\right\} + (-1)^{\eta_a \eta_b} \left[\left[t_b, t_c\right\}, t_a\right\} + (-1)^{\eta_b \eta_c} \left[\left[t_c, t_a\right\}, t_b\right\} = 0.$$

It has the usual, but graded, implication for the relation of the structure constants. Note that the grading of a composite operator is in general given by $(\sum \eta_i) \mod 2$ with the gradings of the constituents operators η_i. Furthermore, any transformation based on this algebra involves necessarily a mixture of ordinary complex numbers and Grassmann numbers, and therefore the parameters are also a graded.

However, the algebra obtained in the previous section looks somewhat different from the quantum mechanical case (2.9), where there have been two supercharges, instead of one, which then did not anticommute as is the case here. Also in field theories this may happen, if there is more than one supercharge. Since in the case of multiple supercharges there is still only one momentum operator, the corresponding superalgebras are coupled. In general, this requires the introduction of another factor δ^{AB}, where A and B count the supercharges. In total, it is shown in the Haag-Lopuszanski-Sohnius theorem that the most general superalgebra, up to rescaling of the charges, is

$$\{Q_a^A, \bar{Q}_b^B\} = 2\delta_{AB}\sigma^{\mu}_{ab}P_{\mu} \tag{4.23}$$

$$\{Q_a^A, Q_b^B\} = \epsilon_{ab}Z^{AB}, \tag{4.24}$$

and there is a corresponding anti-commutator for $\bar{Q}^A$ and $\bar{Q}^B$. The symbol ϵ_{ab} is anti-symmetric, and thus this anti-commutator couples the algebra of different supercharges. Furthermore, there appear an additional anti-symmetric operator Z^{AB} called the central charges. This additional operator can be shown to commute with all other operators, especially of internal symmetries, and must belong therefore to an Abelian U(1) group. Still, this quantum number characterize states, if a theory contains states with non-trivial representations. However, this can only occur if there is more than one supercharge. Finally, the number of independent supercharges is labeled by $\mathcal{N}$.

The theory with only one supercharge is thus an $\mathcal{N} = 1$ theory. Cases with $\mathcal{N} > 1$ are called $\mathcal{N}$-extended supersymmetries. Since the full algebra also involves the Poincare group generators, it turns out that it is not possible to have an arbitrary number of independent supercharges. This number depends on the size of the Poincare algebra, and thus on the number of dimensions. Furthermore, it also depends on the highest spins of particles involved. Especially, theories above a certain $\mathcal{N}$, 4 in four dimensions, require necessarily gravitons or spin 3/2 particles. Above a certain $\mathcal{N}$, 8 in four dimensions, even objects of still higher spins are required. The latter case is not particularly interesting, as it can be shown that a four-dimensional interacting quantum-field theory cannot include non-trivially interacting particles of spin higher than 2. Including gravitons requires to include gravity, a possibility which will for the moment not be considered. There is furthermore no physical evidence (yet) for spin 3/2 particles, so this option will also be ignored. In addition, it also requires a form of gravity. Hence, for any non-gravitational theory in four dimensions, the maximum is $\mathcal{N} = 4$. The odd values 5 and 3 generate in four dimensions only the particle and anti-particle content of theories with larger $\mathcal{N}$, and therefore do not provide different theories: The particle and anti-particle content has to be included to satisfy the CPT theorem, and hence other possibilities are only relevant from a mathematical point of view, but not from a quantum-field-theoretical one.

Hence, in four dimensions there are thus besides $\mathcal{N} = 1$ theories $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. The additional charges provide more constraints, so theories with more supercharges become easier to handle. However, currently there seems to exist no hint that any other than $\mathcal{N} = 1$-theories could be realized at energy scales accessible in the foreseeable future. Thus, here primarily this case will be treated. However, given the importance of such more complicated theories, especially in the context of string theory, it is worthwhile to gather here some more conceptual points about superalgebras.

All of this applies quantitatively to four dimensions. The number of independent supercharges can actually be larger or smaller for different dimensionalities, e. g. $\mathcal{N}=16$ without gravity is possible in two dimensions.

First of all, it should be noted that any theory with $\mathcal{N}>1$ necessarily contains also $\mathcal{N}=1$ supersymmetry. Hence, any theory with $\mathcal{N}>1$ can only be a special case of the most general form of a $\mathcal{N}=1$ theory. The appearance of higher symmetry is then obtained by restrictions on the type of interactions and the type of particles in the theory.

If, for a given theory, all central charges vanish, the algebra is invariant under a $\mathrm{U}(\mathcal{N})$ rotation of the supercharges, which is true especially for $\mathcal{N}=1$. In fact, in the case of $\mathcal{N}=1$, this R symmetry, or R parity, is just a (global) U(1) group, i. e. an arbitrary phase of the supersymmetry charges, which is part of the full algebra by virtue of

$$[T_R, Q_\alpha] = -i(\gamma_5)_\alpha^\beta Q_\beta.$$

This R symmetry forms an internal symmetry group with generator T_R. However, this symmetry may be explicitly, anomalously, or even spontaneously broken, without breaking the supersymmetry itself[4]. A broken R symmetry indicates merely that the relative orientation (and size) of the supersymmetry charges is (partly) fixed.

From the algebra (4.23-4.24) it can be read off that supercharges must have dimension of mass$^{\frac{1}{2}}$, and hence central charges of mass. This observation is of significance, as it embodies such theories with an inherent mass-scale. In fact, it can be shown that the mass of massive particles which form a supermultiplet in an extended supersymmetry must obey the constraint

$$M \geq \frac{1}{\mathcal{N}}\mathrm{tr}\sqrt{Z^\dagger Z},$$

and thus there is a minimum mass. If the mass satisfies the bound, such particles are called Bogomol'nyi-Prasad-Sommerfeld (BPS) states[5].

The structure of the superalgebra involving the momentum has a large number of consequences. This is particularly the case in theories which are scaleless, at least in the classical case. Such theories do not have a dimensionful parameter at the classical level, but may develop one by dimensional transmutation at the quantum level. This is

[4]This is not true in theories which satisfy the conformal version of the Poincare group, i. e. conformal theories. In this case, the R symmetry needs necessarily to be intact, and the combination of supersymmetry, conformal symmetry, and R symmetry forms together the superconformal symmetry.

[5]That this is the same name as for certain topological excitations in gauge theories is not coincidental. In extended supersymmetric gauge theories both quantities are related.

essentially an anomaly, as it stems from the non-invariance of the path integral measure under scale transformations. If the scalelessness survives the quantization, the combination of the scale symmetry and the supersymmetry can yield a very large symmetry group, the superconformal group. This topic will be relegated to later.

Even if the scale symmetry is broken, the superalgebras imposes constraints. Especially, in theories with R parity, this yields a connection between the R current and the trace of the energy-momentum tensor, which is intimately connected to the scale violation. Especially, various relations between R parity and scale operators remain intact even if both the R parity and the scale symmetry are broken at the quantum level. However, because of the Higgs sector, the standard model of particle physics is classically not scaleless, and therefore these relation do not apply in particle physics, until an extension of the standard model is found and experimentally supported which has both symmetries.

4.3.4 Supermultiplets

As already noted, supersymmetry requires different multiplets of particles to be present in a theory. This is a central concept, and requires further scrutiny. To simplify the details, once more mainly $\mathcal{N} = 1$ superalgebras will be considered.

A supermultiplet is a collection of fields which transform into each other under supersymmetry transformations. The naming convention is that a fermionic superpartner of a field a is called a-ino, and a bosonic super-partner s-a.

One result of the algebra obtained in the previous subsections was that the momentum operator (anti-)commutes with all supercharges. Consequently, also P^2 (anti-)commutes with all supercharges,

$$[Q_a, P^2] = [\bar{Q}, P^2] = \{Q, P^2\} = \{\bar{Q}, P^2\} = 0.$$

Since the application of P^2 just yields the mass of a pure state, the masses of a particle s and its super-partner ss must be degenerate, symbolically

$$\begin{aligned} P^2|s\rangle &= m^2|s\rangle \\ P^2|ss\rangle = P^2Q|s\rangle = QP^2|s\rangle = Qm^2|s\rangle = m^2Q|s\rangle &= m^2|ss\rangle. \end{aligned}$$

This is very similar to the degenerate energy levels, which were encountered in the quantum mechanical case.

Further insight can be gained from considering the general pattern of the spin in a supermultiplet. The first step should be to show the often assured statement that the number of bosonic and fermionic degrees of freedom equals. Here, this will be done only for massless states. The procedure can be generalized to massive states, but this only complicates matters without adding anything new.

As a starting point consider some set of massless states. Every state is then characterized by its four-momentum p^μ, with $p^2 = 0$, it spin s, and its helicity h, which for a massless particle can take only the two values $h = \pm s$ if $s \neq 0$ and zero otherwise.

Taking the trace of the spinor indices in (4.23) yields

$$Q_{i\alpha}Q^{\dagger j\alpha} + Q^{\dagger j\alpha}Q_{i\alpha} = \delta_{ij}P^0.$$

Applying to this a rotation operator by 2π and taking the trace over states with the same energy but different spins and helicities yields

$$\sum_{sh} \left\langle p, s, h \left| \left(Q_{i\alpha}Q^{\dagger j\alpha} + Q^{\dagger j\alpha}Q_{i\alpha}\right) e^{-2\pi i J_3} \right| p, s, h \right\rangle = \delta^{ij} \sum_{sh} \left\langle p, s, h \left| P^0 e^{-2\pi i J_3} \right| p, s, h \right\rangle .$$

Since the supercharges are fermionic they anticommute with the rotation operator. Furthermore, the trace is cyclic, as it is a finite set of states. Thus, the expression can be rewritten as

$$\sum_{sh} \left\langle p, s, h \left| Q_{i\alpha}Q^{\dagger j\alpha}e^{-2\pi i J_3} - Q_{i\alpha}Q^{\dagger j\alpha}e^{-2\pi i J_3} \right| p, s, h \right\rangle = 0,$$

and thus also the right-hand-side must vanish. However, the right-hand-side just counts the number of states, weighted with 1 or -1, depending on whether the states are bosonic or fermionic. The number of helicity states differ, depending on whether the states are massive or not, yielding

$$\sum_{s\geq 0}(-1)^{2s}(2s+1)n_s = 0 \tag{4.25}$$

$$n_0 + 2\sum_{s\geq 0}(-1)^{2s}n_s = 0, \tag{4.26}$$

for the massive and the massless case, respectively. n_s is the number of states with the given spin, and n_0 the number of massless spin-0 particles. The factor 2 actually does not arise from the formula, as the trace is also well-defined when taking only one helicity into account. It is the CPT theorem which requires to include both helicity states for a physical theory.

In case of the massive Wess-Zumino model, the numbers are $n_0 = 2$ and $n_{1/2} = 1$, yielding $n_0 - 1(2)n_{1/2} = 0$. Note that this therefore counts on-shell degrees of freedom. For the massless case, the n_s are the same, but this time the 2 comes from a different place, $2 + 2(-1)1 = 0$.

So far, it is only clear that in a supermultiplet bosons and fermions must be present with the same number of degrees of freedom, but not necessarily their relative spins, as long as the conditions (4.25-4.26) are fulfilled. In the free example, one was a spin-0 particle and one a spin-1/2 particle. This pattern of a difference in actual spin of 1/2 is general.

This can be seen by considering the commutation relations of the supercharges under rotation. Since the supercharges are spinors, they must behave under a rotation δ_ϵ as

$$\delta_\epsilon Q = -\frac{i\epsilon\sigma}{2}Q = i\epsilon[J,Q],$$

where J is the generator of rotations. In particular, for the third component follows

$$[J_3,Q] = -\frac{1}{2}\sigma_3 Q$$

and thus

$$[J_3,Q_1] = -\frac{1}{2}Q_1 \qquad [J_3,Q_2] = \frac{1}{2}Q_2 \tag{4.27}$$

for both components of the spinor Q. It then follows directly that the super-partner of a state with total angular momentum j and third component m has third component $m \pm 1/2$, depending on the transformed spinor component. This can be seen as

$$J_3 Q_1|jm\rangle = (Q_1 J_3 - [Q_1,J_3])|jm\rangle = \left(Q_1 m - \frac{1}{2}Q_1\right)|jm\rangle = Q_1\left(m - \frac{1}{2}\right)|jm\rangle.$$

Likewise, the other spinor component of Q yields the other sign, and thus raises instead of lowers the third component. Of course, this applies vice-versa for the hermitian conjugates.

To also determine the value of j, assume a massless state with momentum $(p,0,0,p)$. The massive case is analogous, but more tedious. Start with the lowest state with $m=-j$. Then, of course, the state is annihilated by Q_1 and $Q_2^\dagger$, as they would lower m further.

Also $Q_1^\dagger$ annihilates the state, which is a more subtle result. The anti-commutator yields the result

$$Q_1^\dagger Q_1 + Q_1 Q_1^\dagger = (\sigma_\mu)_{11}P^\mu = p^0 - p^3,$$

where the minus-sign in the second term appears due to the metric. Thus

$$\left\langle pj-j\left|Q_1^\dagger Q_1 + Q_1 Q_1^\dagger\right|pj-j\right\rangle = p^0 - p^3 = p - p = 0 \tag{4.28}$$

but also

$$\langle pj-j|Q_1^\dagger = (Q_1|pj-j\rangle)^\dagger = 0,$$

as discussed above. Hence the first term in (4.28) vanishes, and leaves

$$\left\langle pj-j\left|Q_1 Q_1^\dagger\right|pj-j\right\rangle = 0.$$

But this is just the norm of $Q_1^\dagger|pj-j\rangle$. A zero-norm state is however not appropriate to represent a particle state[6], and thus $Q_1^\dagger$ has to annihilate the state as well, the only alternative to obtain the same result.

This leaves only $Q_2|pj-j\rangle$ as a non-zero state. This state has to be proportional to a state of type $|pj-j+1/2\rangle$. Since Q_2 is Grassmann-valued, and therefore nil-potent, a second application of Q_2 yields again zero. Furthermore, since Q_1 and Q_2 anticommute, its application also yields zero,

$$Q_1Q_2|pj-j\rangle = -Q_2Q_1|pj-j\rangle = 0.$$

The application of $Q_1^\dagger$ can be calculated as in the case of (4.28). But the appearing momentum combination is p_1+p_2, being zero for the state. This leaves only $Q_2^\dagger$. Applying it yields

$$Q_2^\dagger Q_2|pj-j\rangle = ((\sigma_\mu)_{22}P^\mu - Q_2Q_2^\dagger)|pj-j\rangle = (p^0+p^3+0)|pj-j\rangle = 2p|pj-j\rangle.$$

Hence, this returns the original state. Thus, the value of j in a supermultiplet can differ only by one half, and there are only two (times the number of internal quantum number) states in each supermultiplet. It does not specify the value of j, so it would be possible to have a supermultiplet with $j=0$ and $j=1/2$, as in the example above.

Note that only the states $m=0$ and $m=-1/2$, but not $m=+1/2$, the anti-particle state, are contained in the supermultiplet. This is called a chiral supermultiplet. Alternatively, it would be possible to have $j=1/2$ and $j=1$, the vector supermultiplet, or $j=2$ and $j=3/2$, the gravity supermultiplet appearing in supergravity.

While algebraically the anti-state is not necessary, any reasonable quantum field theory is required to have CPT-symmetry. Thus for any supermultiplet also the corresponding antiparticles, the antimultiplet, have to appear in the theory as well. By this, the missing $m=1/2$ state above is introduced into the theory.

To have supermultiplets which include more states requires to work with an $\mathcal{N}>1$ algebra, where the additional independent supercharges permit further rising and lowering. It is then possible to have supermultiplets, which include more different spin states. Also, the absence of central charges was important, since the anti-commutator of the fields has been used. E.g. in $\mathcal{N}=2$ SUSY, the supermultiplet contains two states with $j=0$ and two with $m=\pm1/2$. This is also the reason why theories with $\mathcal{N}\neq1$ seem to be no good candidates for an extension of the standard model: SUSY, as will be seen below, requires that all superpartners transform the same under other transformations, like e. g. gauge transformations. In an $\mathcal{N}=2$ theory, there would be a left-handed and a right-handed

[6]In gauge theories, this statement has to be refined.

electron, both transforming under all symmetries in the same way. But in the standard model, the weak interactions couple differently to left-handed and right-handed electrons, and thus it is not compatible with $\mathcal{N} > 1$ SUSY. For $\mathcal{N} = 1$ SUSY, however, independent chiral multiplets (and, thus, more particles) can be introduced to solve this problem. Since also the quarks (and the superpartners of the gauge bosons therefore should also) are coupled differently in the weak interactions, this applies to all matter-fields, and thus it is not possible to enhance some particles with $\mathcal{N} = 1$ and others with $\mathcal{N} > 1$ SUSY.

One of the key quantities in the above discussion has been spin. However, spin is considered usually a good quantum number because it is a well-defined observable, as it commutes with the Hamiltonian. This is no longer precisely true when supersymmetry is involved, and a more general concept is needed.

To find this concept, it is first important to understand what spin is in a quantum field theory. In fact, a particle species is specified as a representation of the Poincare group or of one of its subgroup, i. e. fields are orbits in the Poincare group. Such orbits can be classified using Casimir invariants, as group theory shows.

One of the Casimirs is P^2, the square of the momentum operator, yielding the rest mass of a particle[7].

The spin should also appear as a second Casimir of the Poincare group, requiring again a Poincare-invariant operator. It is obtained using the Pauli-Lubanski vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}, \tag{4.29}$$

being, due to the Levi-Civita tensor, orthogonal to the momentum vector, and thus linearly independent. To show that its square W^2 is actually a Casimir operator requires to show that it commutes with both the momentum operator and the generator of rotation. Since

$$[P_\lambda, W_\mu] = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P_\nu [P_\lambda, M^{\rho\sigma}] = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma} P_\nu (\delta^\lambda_\rho P^\sigma - \delta^\sigma_\lambda P^\rho) = 0,$$

already the momentum operator and the Pauli-Lubanski vector commute, so does the square of any of the two vectors with the other vector. Since W^2 is a scalar, it also commutes with the generator of rotations, which can be shown explicitly in the same way. Together with the linear independence, this is sufficient that W^2 is an independent Casimir.

[7]It should be noted that in a full axiomatic quantum field theory particle species may be classified according to these invariants, but any interacting state, especially in theories with massless particles, will not be an eigenstate of the mass operator, or other Casimirs, as any state also contains virtual contributions from multiparticle states. Idealized 'sharp' eigenstates of the mass operators are actually not compatible with Poincare symmetry in an interacting quantum field theory. In practice, this fine distinction is often irrelevant, especially when using perturbation theory only.

To show that this vector is indeed associated with the usual spin consider its commutator

$$[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} P^\rho W^\sigma,$$

which can again be obtained by direct evaluation. In the rest frame only those commutators remain with $\mu \neq 0$ and $\nu \neq 0$, yielding

$$[W_i, W_j] = im\epsilon_{ijk} W_k \tag{4.30}$$

which is, up to a normalization, just the spin algebra. This especially implies that its eigenvalues behave, up to a factor of m, like the ones of a spin, and indeed the eigenvalues of W^2 are thus spin eigenvalues. A more explicit proof can be obtained by using the explicit form of the Pauli-Lubanski vector (4.29), if desired.

In a supersymmetric theory, this is no longer true. The commutator of W^μ with a supercharge Q yields

$$\begin{aligned} [W^\mu, Q] &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu [M_{\rho\sigma}, Q] = i\sigma^{\mu\nu} Q P_\nu \qquad (4.31)\\ \sigma_{\mu\nu} &= \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \end{aligned}$$

and thus

$$[W^2, Q] = W^\mu [W_\mu, Q] + [W^\mu, Q] W_\mu = 2iP^\mu \sigma_{\mu\nu} P^\nu Q$$

and therefore W^2 is no longer a suitable operator to characterize a particle, as it no longer belongs to the maximal set of commuting operators.

The problem arises, because of the connection of the superalgebra and the momentum operators. A suitable solution is therefore to generalize the Pauli-Lubanski-vector to a quantity also involving the supercharges. As it will turn out a suitable choice is the superspin, defined as

$$S^\mu = W^\mu - \frac{1}{4} Q\sigma^\mu \bar{Q}.$$

To check this, consider the commutation relation of the second part with a supercharge

$$\left[-\frac{1}{4} Q\sigma^\mu \bar{Q}, Q\right] = 2Q\sigma^\mu \bar{\sigma}^\nu P_\nu = 2\sigma^\nu \bar{\sigma}^\mu Q P_\nu,$$

which directly follows from the superalgebra. This implies, together with (4.31),

$$[S^\mu, Q] = -\frac{1}{2} Q P^\mu, \tag{4.32}$$

where it has been used that $(i\sigma^{\mu\nu} + \sigma^\nu\bar{\sigma}^\mu/2) = g^{\mu\nu}/2$, as can be shown by explicit calculation. Since S^μ is Hermitian, the commutation relation with $\bar{Q}$ follows directly. Furthermore, S^μ commutes with P_μ as the Pauli-Lubanski vector does and so does the supercharge. Combining all of this together yields

$$[S^\mu, S^\nu] = i\epsilon_{\mu\nu\rho\sigma}P^\rho S^\sigma,$$

which reduces in the rest system in the same way as in (4.30) to

$$[S^i, S^j] = im\epsilon^{ijk}S^k.$$

This is again the same type of spin-algebra, characterized by eigenvalues s, as before. Thus, the superspin acts indeed as a spin.

It then only remains to construct an adequate Casimir operator. Define for this the antisymmetric matrix

$$C^{\mu\nu} = S^\mu P^\nu - S^\nu P^\mu,$$

which commutes with the supercharges

$$[C^{\mu\nu}, Q] = [S^\mu, Q]P^\nu - [S^\nu, Q]P^\mu = \frac{1}{2}\left(-QP^\mu P^\nu + QP^\nu P^\mu\right) = 0.$$

Squaring this operator $C^2 = C^{\mu\nu}C_{\mu\nu}$, which also commutes with the supercharges and also with P^μ, as it is a scalar. It remains to show that this Casimir is indeed different from P^2. An explicit calculation shows

$$C^2 = 2m^2S^2 - 2(SP)^2$$

which in the rest frame for a massive particle reduces to

$$C^2 = -2m^4 s(s+1)$$

where S is the superspin in the rest frame and zero, integer, or half-integer. The situation for massless particles is as before, yielding for every spin only the corresponding helicities. Therefore, any particle can be assigned to representations of the Poincare group with the continuous parameters mass $m^2 \geq 0$ and the discrete superspin S, where the latter coincides in the rest frame, but only there, with the usual spin. Due to this coincidence, usually no difference is made between superspin and spin in name, and both expressions are used synonymously. Note that

$$[S^3, W^3] = 0,$$

and therefore the eigenvalues of both operators can be used to characterize the magnetic quantum number, especially in the rest frame. Thus, the above discussed counting in the rest frame is indeed legit.

4.3.5 Off-shell supersymmetry

Now, it turns out that the results so far are not complete. If in (4.20) instead of ϕ the superpartner χ is used, it turns out that complications arise. Thus, something has to be modified.

To see this inconsistency, start by first performing two supersymmetry transformations on χ

$$\begin{aligned}\delta_\eta\delta_\xi\chi_a=[\eta Q+\bar{\eta}\bar{Q},[\xi Q+\bar{\xi}\bar{Q},\chi_a]] &= -i[\eta Q+\bar{\eta}\bar{Q},\sigma^\mu(i\sigma_2\xi^*)_a\partial_\mu\phi]\\ &= -i(\sigma^\mu(i\sigma_2\xi^*))_a\partial_\mu[\eta Q+\bar{\eta}\bar{Q},\phi]\\ &= -i(\sigma^\mu(i\sigma_2\xi^*))_a(\eta^T(-i\sigma_2)\partial_\mu\chi). \qquad (4.33)\end{aligned}$$

To simplify this further, note that for any three spinors η, ρ, λ

$$\lambda_a\eta^b\rho_b+\eta_a\rho^b\lambda_b+\rho_a\lambda^b\eta_b=0 \qquad (4.34)$$

holds. This follows by explicit calculation. E.g., for $a=1$

$$\lambda_1\eta_1\rho_2-\lambda_1\eta_2\rho_1+\eta_1\rho_1\lambda_2-\eta_1\rho_2\lambda_1+\rho_1\lambda_1\eta_2-\rho_1\lambda_2\eta_1=0.$$

To rearrange the terms such that they cancel always an even number of transpositions are necessary, and thus the Grassmann nature is not changing the signs. The case $a=2$ can be shown analogously. Now, identify $\lambda=\sigma_\mu(-i\sigma_2)\xi^*$, $\eta=\eta$, and $\rho=\partial_\mu\chi$. Thus, (4.33) can be rewritten as

$$\delta_\eta\delta_\xi\chi=-i\left(\eta_a\partial_\mu\chi^T(-i\sigma_2)\sigma^\mu(-i\sigma_2)\xi^*+\partial_\mu\chi_a(\sigma^\mu(-i\sigma_2\xi^*))^T(-i\sigma_2)\eta\right).$$

Using

$$(-i\sigma_2)\sigma^\mu(-i\sigma_2)=(-\bar{\sigma}^\mu)^T, \qquad (4.35)$$

which can be checked by explicit calculation, shortens the expressions significantly. Performing also a transposition, it then takes the form

$$\begin{aligned}\delta_\eta\delta_\xi\chi &= -i\eta_a(\xi^+\bar{\sigma}^\mu\partial_\mu\chi)+\partial_\mu\chi_a(\sigma^\mu(-i\sigma_2\xi^*)^T(-i\sigma_2)\eta)\\ &= -i\eta_a(\xi^T\bar{\sigma}^\mu\partial_\mu\chi)-i\eta^T(-i\sigma_2)\sigma^\mu(-i\sigma_2)\xi^*\partial_\mu\chi_a,\end{aligned}$$

where also the second term became transposed. To construct the transformation in reverse order is achieved by exchanging η and ξ, thus yielding

$$\begin{aligned}(\delta_\eta\delta_\xi-\delta_\xi\delta_\eta)\chi_a &= i(\xi^T(-i\sigma_2)\sigma^\mu(-i\sigma_2)\eta)^*-\eta^T(-i\sigma_2)\sigma^\mu(-i\sigma_2)\xi^*)\partial_\mu\chi_a\\ &\quad +i\xi_a(\eta^T\bar{\sigma}^\mu\partial_\mu\chi)-i\eta_a(\xi^T\bar{\sigma}^\mu\partial_\mu\chi). \qquad (4.36)\end{aligned}$$

The first term is exactly the same as in (4.20), but with η and ξ exchanged. Hence, if this term would be the only one, the commutator of two SUSY transformations would be, in fact, the same irrespective of whether it acts on ϕ or χ, as it should. But it is not. The two remaining terms seem to make this impossible.

However, on closer inspection it becomes apparent that in both terms the expression $\bar{\sigma}^\mu\partial_\mu\chi$ exists. This is precisely the equation of motion for the field χ, the Weyl equation. Thus the two terms vanish, if the field satisfies its equation of motion. In a classical theory, this would be sufficient. However, in a quantum theory exist virtual particles, i. e., particles which not only not fulfill energy conservation, but also not their equations of motions. Hence such particles, which are called off (mass-)shell, are necessary. Thus the algebra so far is said to close only on-shell. Hence, although the theory described by the Lagrangian (4.8) is classically supersymmetric, it is not so quantum-mechanically. Quantum effects break the supersymmetry of this model.

Therefore, it is necessary to modify (4.8), to change the theory, to obtain one which is also supersymmetric on the quantum level. Actually, this result is already an indication of how this can be done. Off-shell, the number of degrees of freedom for a Weyl-fermion is four, and not two, as there are two complex functions, one for each spinor component. Thus, the theory cannot be supersymmetric off-shell, as the scalar field has only two degrees of freedom. To make the theory supersymmetric off-shell, more (scalar) degrees of freedom are necessary, which, however, do not contribute at the classical level.

This can be done by the introduction of an auxiliary scalar field F, which has to be complex to provide two degrees of freedom. It is called auxiliary, as it has no consequence for the classical theory. The later can be most simply achieved by giving no kinetic term to this field. Thus, the modified Lagrangian takes the form

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^\dagger F. \tag{4.37}$$

It should be noted that this field has mass-dimension two, instead of one as the other scalar field ϕ. The equation of motion for this additional field is

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu F} - \frac{\delta\mathcal{L}}{\delta F} = F^\dagger = 0(= F).$$

Thus, indeed, at the classical level it does not contribute.

Of course, if it should contribute at the quantum level, it cannot be invariant under a SUSY transformation. The simplest (and correct) guess is that this transformation should only be relevant off-shell. As it must make a connection to χ, the ansatz is

$$\begin{aligned}\delta_\xi F &= -i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi\\ \delta_\xi F^\dagger &= i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\xi,\end{aligned}$$

where ξ has been inserted as its transpose to obtain a scalar. The appearance of the derivative is also enforced to obtain a dimensionally consistent equation.

This induces a change in the Lagrangian under a SUSY transformation as

$$\delta\mathcal{L}_F = Fi\partial_\mu\chi^\dagger\bar{\sigma}^\mu\xi - F^\dagger i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi.$$

This expression is not a total derivative. Hence, to obtain a supersymmetric theory additional modifications for the transformation laws of the other fields are necessary. However, since part of the fermion term already appears, the modifications

$$\begin{aligned}\delta\chi &= \sigma_\mu\sigma_2\xi^*\partial_\mu\phi + \xi F\\ \delta\chi^\dagger &= i\partial_\mu\phi^\dagger\xi^T(-i\sigma_2)\sigma^\mu + F^\dagger\xi^\dagger\end{aligned}$$

immediately lead to cancellation of the newly appearing terms, and one additional total derivative,

$$\begin{aligned}\delta\mathcal{L}_F &= F^\dagger\xi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \chi^\dagger i\bar{\sigma}^\mu\xi\partial_\mu F + Fi\partial_\mu\chi^\dagger\bar{\sigma}^\mu\xi - F^\dagger\xi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi\\ &= \partial_\mu(i\chi^\dagger\bar{\sigma}^\mu\xi F) + F^\dagger\xi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - F^\dagger\xi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi\\ &= \partial_\mu(i\chi^\dagger\bar{\sigma}^\mu\xi F)\end{aligned}$$

Thus, without modifying the transformation law for the ϕ-field, the new Lagrangian (4.37) indeed describes a theory which is supersymmetric on-shell. To check this also off-shell, the commutator of two SUSY transformations has to be recalculated. For this, as everything is linear, it is only necessary to calculate the appearing additional terms. This yields

$$\delta(\delta_\eta\delta_\xi\chi) = \delta_\eta(\xi_a F) = -i\xi_a(\eta^\dagger\bar{\sigma}^\mu\partial_\mu\chi).$$

That is indeed one of the offending terms in (4.36). Similarly,

$$\delta(\delta_\xi\delta_\eta\chi) = \delta_\xi(\eta_a F) = -i\eta_a(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi),$$

and thus also the second term is canceled. Thus, the theory described by the Lagrangian (4.37) is also off-shell supersymmetric. Of course, this requires a check for the transformation of F, but this is also passed, and is not providing anything new. It is therefore skipped.

A full proof requires the recalculation of all (anti-)commutators. This is a tedious work, but finally it turns out that the theory indeed has the same commutation relations, which hold also off-shell. In particular, the supercurrent is not modified at all, as no kinetic term for F appears, and the surface term and the one coming from the transformation of χ exactly cancel. Note that the presence of these auxiliary fields require the modifications of the anticommutation relations for the fermions to include an additional factor $(1+F)$. Also, the field F must have a non-trivial commutation relation with ϕ. These are consequences of the off-shell nature of the auxiliary fields, and on-shell all this anomalous contributions will cancel, since F vanishes. This is, however, not much different from an interacting theory, where also the ordinary commutation relations break down.

Chapter 5

Interacting supersymmetric quantum field theories

5.1 The Wess-Zumino model

The theory treated so far was non-interacting and, after integrating out F using its equation of motion, had a very simple particle content. Of course, any relevant theory should be interacting. The simplest case will be constructed in this section. It is an extension of the free theory.

The starting point is the free theory discussed previously, and hence is based on the Lagrangian (4.37), but supplemented with a yet unspecified interaction

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F + \mathcal{L}_i(\phi, \phi^\dagger, \chi, \chi^\dagger, F, F^\dagger) + \mathcal{L}_i^\dagger.$$

The last term is just the hermitian conjugate of the second-to-last term, necessary to make the Lagrangian hermitian. It is now necessary to find an interaction Lagrangian $\mathcal{L}_i$ such that supersymmetry is preserved.

Since the theory should be renormalizable, the maximum dimension (for the relevant case of 3+1 dimensions) of the interaction terms is 4. The highest dimensional fields are F and χ. Furthermore, the interaction terms should be scalars. Thus the possible form is restricted to

$$\mathcal{L}_i = U(\phi, \phi^\dagger) F - \frac{1}{2} V(\phi, \phi^\dagger) \chi^a \chi_a.$$

The $-1/2$ is introduced for later convenience. On dimensional grounds, with F having dimension mass^2 and χ mass$^{\frac{3}{2}}$, no other terms involving these fields are possible. In principle it would appear possible to also have a term $Z(\phi, \phi^\dagger)$, depending only on the scalar fields. However, such a term could not include derivative terms. Under a SUSY transformation ϕ is changed into χ. Hence, the SUSY transformation will yield a term with three ϕ fields and one χ field for dimensional reasons, and no derivatives or F fields. But all kinetic terms will include at least one derivative, as well as any transformations of the F field. The only possible term would be the one proportional to χ^2. But its SUSY transformation includes either an F field or the derivative of a ϕ-field. Hence none could cancel the transformed field. Since no derivative is involved, it can also not be changed into a total derivative, and thus such a term is forbidden.

Further, on dimensional grounds, the interaction term U can be at most quadratic in ϕ and V can be at most linear.

The free part of the action is invariant under a SUSY transformation. It thus suffices to only investigate the interacting part. Furthermore, if $\mathcal{L}_i$ is invariant under a SUSY transformation, so will be $\mathcal{L}_i^\dagger$. Thus, to start, consider only the second term. Since a renormalizable action requires the potentials U or V to be polynomial in the fields, the transformation rule is for either term, called Z here,

$$\delta_\xi Z = \frac{\delta Z}{\delta\phi}\delta_\xi\phi + \frac{\delta Z}{\delta\phi^\dagger}\delta_\xi\phi^\dagger.$$

The contribution from the SUSY transformation acting on the potential V yields

$$\frac{\delta V}{\delta\phi}(\xi^a\chi_a)(\chi^b\chi_b) + \frac{\delta V}{\delta\phi^\dagger}(\xi^{\dot{a}}\chi_{\dot{a}})(\chi^a\chi_a).$$

Again, none of these terms can be canceled by any other contribution appearing, since none of these can involve three times the χ field. Also, both cannot cancel each other, since the contributions are independent. As noted, the first term can be at maximum of the form $a + b\phi$, and thus only a common factor. By virtue of the identity (4.34) setting $\lambda = \rho = \eta = \chi$ it follows that this term is zero: In the case of all three spinors equal, the terms are all identical. This, however, does not apply to the second term, as no identity exist if one of the indices is dotted. There is no alternative other than to require that V is not depending on $\phi^\dagger$. Hence, the function V can be, and in four dimensions for renormalizable theories is, a holomorphic function[1] of the field ϕ. Being holomorphic is a quite strong constraint on a function, and hence it is quite useful in practice to obtain various general results for supersymmetric theories.

As an aside, this fact is of great relevance, as it turns out that it is not possible to construct the standard model Yukawa interactions due to this limitation with only one Higgs doublet as it is done in the non-supersymmetric standard model, but instead

[1] In theories with both left and right chiral supermultiplets, e. g. $\mathcal{N} = 2$ theories, it is possible to also form a further potential which depends on both, the so-called Kähler potential. Kähler potentials are, in a sense, generalizations of holomorphic potentials, and hence also quite constraining.

requires at least two doublets. The more detailed reason is once more the necessity to have two independent supermultiplets to represent the left-right asymmetry of the electroweak interaction of the standard model, and this requires ultimately the doubling of Higgs particles in the minimal supersymmetric standard model. However, it is not required that both have the same mass.

This is already sufficient to restrict the function V to the form

$$V = M + y\phi.$$

The first term gives a mass to the fermionic fields, and the second term provides a Yukawa interaction. It is convenient, as will be shown latter, to write a generating functional for this term V as

$$V = \frac{\delta W}{\delta\phi\delta\phi}$$

with

$$W = B + A\phi + \frac{1}{2}M\phi^2 + \frac{1}{6}y\phi^3. \tag{5.1}$$

This function W is called the superpotential for historically reasons, and will play a central role, as will be seen later. The linear term is not playing a role here, but can be important for the breaking of SUSY, as will be discussed later. The constant is essentially always irrelevant.

The next step is to consider all terms which produce a derivative term upon a SUSY transformation. These are

$$\begin{aligned} & -iU\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\frac{1}{2}V\chi^T\sigma_2\sigma^\mu\sigma_2\xi^*\partial_\mu\phi - i\frac{1}{2}V\xi^\dagger\sigma_2\sigma^{\mu T}\sigma_2\chi\partial_\mu\phi \\ = & -iU\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - i\frac{1}{2}V\xi^\dagger\sigma_2\sigma^{\mu T}\sigma_2\chi\partial_\mu\phi - i\frac{1}{2}V\xi^\dagger\sigma_2\sigma^{\mu T}\sigma_2\chi\partial_\mu\phi \\ = & -iU\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - i\frac{1}{2}V\xi^\dagger\bar{\sigma}^\mu\chi\partial_\mu\phi - i\frac{1}{2}V\xi^\dagger\bar{\sigma}^\mu\chi\partial_\mu\phi \\ = & -iU\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - iV\xi^\dagger\bar{\sigma}^\mu\chi\partial_\mu\phi, \end{aligned}$$

where (4.35) was used twice. The combination of derivatives of ϕ and χ cannot be produced by any other contribution, since all other terms will not yield derivatives. Since neither U nor V are vectors, there are also no possibilities for a direct cancellation of both terms. The only alternative is thus that both terms could be combined to a total derivative. This is possible, if

$$U = \frac{\delta W}{\delta\phi}$$

as can be seen as follows

$$
\begin{aligned}
& -i\xi^\dagger\bar{\sigma}^\mu\left(U\partial_\mu\chi+\chi V\partial_\mu\phi\right) \\
= & -i\xi^\dagger\bar{\sigma}^\mu\left(U\partial_\mu\chi+\chi\partial_\mu\frac{\delta W}{\delta\phi}\right) \\
= & -i\xi^\dagger\bar{\sigma}^\mu\left(\frac{\delta W}{\delta\phi}\partial_\mu\chi+\chi\partial_\mu\frac{\delta W}{\delta\phi}\right) \\
= & -i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\left(\frac{\delta W}{\delta\phi}\chi\right).
\end{aligned}
$$

This is indeed a total derivative. Hence, by virtue of the form of the superpotential (5.1)

$$U=\frac{\delta W}{\delta\phi}=A+M\phi+\frac{1}{2}y\phi^2.$$

This fixes the interaction completely. It remains to check that also all the remaining terms of the SUSY transformation cancel or form total derivatives. These terms are

$$
\begin{aligned}
& \frac{\delta U}{\delta\phi}F\xi^a\chi_a-\frac{1}{2}V(\xi^a\chi_aF+\chi^a\xi_aF) \\
= & \frac{\delta^2W}{\delta\phi\delta\phi}F\xi^a\chi_a-\frac{1}{2}\frac{\delta^2W}{\delta\phi\delta\phi}(\xi^a\chi_aF+\xi^a\chi_aF)=0,
\end{aligned}
$$

and hence the theory is, in fact, supersymmetric. Here it has been used how spinor scalar-products of Grassmann numbers can be interchanged, giving twice a minus-sign in the second term.

This permits to write down the full, supersymmetric Lagrangian of the Wess-Zumino model. It takes the form

$$
\begin{aligned}
\mathcal{L}=\partial_\mu\phi^\dagger\partial^\mu\phi+\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi+F^\dagger F & + M\phi F-\frac{1}{2}M\chi\chi+\frac{1}{2}y\phi^2F-\frac{1}{2}y\phi\chi\chi \\
& + M^*\phi^\dagger F^\dagger-\frac{1}{2}M^*(\chi\chi)^\dagger+\frac{1}{2}y^*\phi^{\dagger 2}F^\dagger-\frac{1}{2}y^*\phi^\dagger(\chi\chi)^\dagger,
\end{aligned}
\tag{5.2}
$$

where linear and constant terms have been dropped, and which is now the full, supersymmetric theory.

The appearance of three fields makes this theory already somewhat involved. However, the field F appears only quadratically, and without derivatives in the Lagrangian remains an auxiliary field, just as

$$\mathcal{L}_F=F^\dagger F+\frac{\delta W}{\delta\phi}F+\frac{\delta W^\dagger}{\delta\phi^\dagger}F^\dagger.$$

Thus, it is directly possible to select F such that it satisfies its equations of motion at the operator level, which read[2]

$$\begin{aligned}\frac{\delta\mathcal{L}_F}{\delta F^\dagger} &= F+\frac{\delta W^\dagger}{\delta\phi^\dagger}=0\\ \frac{\delta\mathcal{L}_F}{\delta F} &= F+\frac{\delta W^\dagger}{\delta\phi^\dagger}=0\end{aligned}$$

This can be used to replace F in the original Lagrangian. This exchanges $F\delta W/\delta\phi$ by $\delta W/\delta\phi\delta W^\dagger/\delta\phi^\dagger$, and makes the term thus equal to its hermitian conjugate. The total result is therefore

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^\dagger\partial^\mu\phi+\chi^\dagger i\bar\sigma^\mu\partial_\mu\chi-MM^*\phi^\dagger\phi-\frac{1}{2}M\chi^T(-i\sigma_2)\chi-\frac{1}{2}M^*\chi^\dagger(i\sigma_2)\chi^{\dagger T}\\ &-\frac{y}{2}(M\phi\phi^{\dagger 2}+M^*\phi^\dagger\phi^2)-yy^*\phi^2\phi^{\dagger 2}-\frac{1}{2}(y\phi\chi^a\chi_a+y^*\phi^\dagger\chi^{a\dagger}\chi_a^\dagger)\end{aligned}\quad(5.3)$$

There are a number of interesting observations to be made in this Lagrangian. First of all, at tree-level it is explicit that the fermionic and the bosonic field have the same mass. Of course, SUSY guarantees this also beyond tree-level. Secondly, the interaction-structure is now surprisingly the one which was originally claimed to be inconsistent with SUSY. The reason for this is that of course also in the SUSY transformations the equations of motions for F and $F^\dagger$ have to be used. As a consequence, these transformations are no longer linear, thus making such an interaction possible. Finally, although two masses and three interactions terms do appear, there are only two independent coupling constants, M and y. That couplings for different interactions are connected in such a non-trivial way is typical for SUSY. It was one of the reasons for hoping that SUSY would unify the more than thirty independent masses and couplings appearing in the standard model. Unfortunately, as will be discussed below, the necessity to break SUSY jeopardizes this, leading, in fact, for the least complex theories to many more independent couplings and masses, about three-four times as much.

Before discussing further implications of these results, it should be remarked that this result generalizes directly to an arbitrary number of internal degrees of freedom of ϕ and χ (and, of course, F), such as flavor or gauge indices, as long as for each boson there is also a fermion with the same index. Of course, also the coupling constants will then carry indices. The superpotential takes the form

$$W=B+A_i\phi_i+\frac{1}{2}M_{ij}\phi_i\phi_j+\frac{1}{6}y_{ijk}\phi_i\phi_j\phi_k$$

[2]In a path integral formulation, this would be equivalent to performing the integral over F explicitly, and thereby remove it from the theory.

and thus the general Lagrangian then takes the form

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi_i^\dagger\partial^\mu\phi_i + \chi_i^\dagger i\bar\sigma^\mu\partial_\mu\chi_i - M_{ij}M_{ik}^*\phi_j^\dagger\phi_k - \frac{1}{2}M_{ij}\chi_i^T(-i\sigma_2)\chi_j - \frac{1}{2}M_{ij}^*\chi_i^\dagger(i\sigma_2)\chi_j^{\dagger T}\\ &-\frac{y_{ijk}}{2}(M_{il}\phi_l\phi_j^\dagger\phi_k^\dagger + M_{il}^*\phi_l^\dagger\phi_j\phi_k) - y_{ijk}y_{ilm}\phi_j\phi_k\phi_l^\dagger\phi_m^\dagger - \frac{1}{2}(y_{ijk}\phi_i\chi_j^a\chi_{ka} + y_{ijk}^*\phi_i^\dagger\chi_j^{a\dagger}\chi_{ak}^\dagger),\end{aligned}$$

and all transformation rules change accordingly.

5.1.1 Majorana form

For many actual calculations the form (5.3) of the Wess-Zumino model is actually somewhat inconvenient. It is often more useful to reexpress it in terms of Majorana fermions

$$\Psi = \begin{pmatrix}(i\sigma_2)\chi^* \\ \chi\end{pmatrix},$$

and the bosonic fields

$$\begin{aligned}A &= \frac{1}{\sqrt{2}}(\phi+\phi^\dagger)\\ B &= \frac{1}{\sqrt{2}}(\phi-\phi^\dagger).\end{aligned}$$

It can then be verified by direct expansion that the new Lagrangian in terms of these fields for a single flavor becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\bar\Psi(i\gamma^\mu\partial_\mu - M)\Psi + \frac{1}{2}\partial^\mu A\partial_\mu A - \frac{1}{2}M^2A^2 + \frac{1}{2}\partial^\mu B\partial_\mu B - \frac{1}{2}M^2B^2\\ &-MgA(A^2+B^2) + \frac{1}{2}g^2(A^2+B^2)^2 - g(A\bar\Psi\Psi + iB\bar\Psi\gamma_5\Psi)\end{aligned}$$

where M and $g = y$ have been chosen real for simplicity. In this representation, A and B do no longer appear on equal footing: A is a scalar field while B is necessarily pseudoscalar, due to its coupling to the fermions. Therefore, it can also only appear quadratic and not linear in the three-scalar term, explaining the absence of a BA^2 term. Still, despite these differences, this is a standard Lagrangian for which the Feynman rules are known. It will now be used to demonstrate the convenient behavior of a supersymmetric theory when it comes to renormalization.

5.2 Feynman rules for the Wess-Zumino model

For the demonstration that supersymmetric theories have in fact better renormalization properties than ordinary quantum field theories, it is useful to make an explicit example calculation at the one-loop level. For this purpose, of course, the Feynman rules are needed, i. e. the propagators and vertices.

The propagator D of the field A is

$$D^{AA}(p) = \frac{i}{p^2 - M^2 + i\epsilon}.$$

Since the quadratic term for the B has the same form, the same propagator appears

$$D^{BB}(p) = \frac{i}{p^2 - M^2 + i\epsilon}.$$

The calculation for the fermionic field is more complicated, since it is necessary to invert the matrix-valued kinetic term. The result is, however, just the same as for any other fermion, delivering

$$D^{\chi\chi}(p) = \frac{i(\gamma_\mu p^\mu + M)}{p^2 - M^2 + i\epsilon}.$$

This concludes the list of propagators.

All in all, there are four three-point vertices and two four-point vertices in the Wess-Zumino model. The first three-point vertex couples three A-fields,

$$\Gamma^{AAA} = -6Mg.$$

The other three-point vertices always couple different fields,

$$\begin{aligned}
\Gamma^{ABB} &= -2Mg \\
\Gamma^{A\chi\chi} &= ig \\
\Gamma^{B\chi\chi} &= ig\gamma_5.
\end{aligned}$$

The four-point vertices are

$$\begin{aligned}
\Gamma^{AAAA} &= 12ig^2 \\
\Gamma^{BBBB} &= 12ig^2 \\
\Gamma^{AABB} &= 4ig^2.
\end{aligned}$$

This completes the list of the vertices, and thus the Feynman rules.

5.3 The scalar self-energy to one loop

Also in a field theory the vacuum energy in a supersymmetric theory is canceled very much like in supersymmetric quantum mechanics. However, the corresponding explicit calculations are cumbersome. Therefore, the explicit calculations will be postponed until a more general result than just a perturbative leading-order calculation can be obtained.

Another benefit of supersymmetric theories is that they solve the so-called naturalness problem. What this explicitly means, and how it is solved, will be discussed here.

The naturalness problem is simply the observation that the Higgs particle is rather light[3], although the theory would easily permit it to be much heavier, of the order of almost the Planck scale, without loosing its internal consistency. The reason for this is the unconstrained nature of quantum fluctuations. Supersymmetric theories make it much harder for the Higgs to be very heavy, in fact, its mass becomes exponentially reduced compared to a non-supersymmetric theory. To see this explicitly, it is simplest to perform a perturbative one-loop calculation of the scalar self-energy in a theory with a very similar structure as the Wess-Zumino model, but with a fermion-boson coupling which is instead chosen to be h for the moment. This can be regarded as a simple mock-up of the electroweak sector of the standard model, dropping the gauge fields.

At leading order, there are three classes of diagrams appearing in the one-particle irreducible set of Feynman diagrams. The first is a set of tadpole diagrams, the second a set of one-loop graphs with internal bosonic particles, and the third the same, but with fermionic particles. These will be calculated in turn here. The calculation will be performed for the case of the A boson. It is similar, but a little more tedious, for the B boson.

The mathematically most simple ones are the tadpole diagrams. There are two of them, one with an A boson attached, and one with a B boson attached. Their contribution Π_t to the self-energy is

$$\Pi_t = -\frac{12}{2}g^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-M^2+i\epsilon}-\frac{4}{2}g^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-M^2+i\epsilon}, \tag{5.4}$$

where the first term stems from the A-tadpole and the second one from the B-tadpole, and the factors $1/2$ are symmetry factors. The integration over p_0 can be performed first by contour-integration and using the Cauchy theorem, since

$$\begin{aligned}\Pi_t &= -8g^2\int\frac{d^3\vec{p}}{(2\pi)^4}\int dp_0\frac{1}{p_0^2-\vec{p}^2-M^2+i\epsilon}\\ &= -8g^2\int\frac{d^3p}{(2\pi)^4}\int\limits_{-\infty}^{\infty}dp_0\frac{1}{(p_0+\sqrt{\vec{p}^2+M^2})(p_0-\sqrt{\vec{p}^2+M^2})+i\epsilon}\end{aligned}$$

[3]The mass of the Higgs is actually not a perfectly well-defined concept, and some field-theoretical subtleties appear. However, this just recasts the problem at a higher level of complexity. Therefore, for the sake of simplicity, here this somewhat simplified view will be used.

This has a pole in the upper half-plane, and vanishes sufficiently fast on a half-circle at infinity. The residue at the simple poles $p_0 = \pm\sqrt{\vec{p}^2 + M^2}$ is $1/(p_0 \mp \sqrt{\vec{p}^2 + M^2})$, dropping the small contribution of $i\epsilon$, which only served to not have the pole on the axis. The Cauchy theorem then yields, using polar coordinates in the final expression,

$$\Pi_t = \frac{ig^2}{\pi^2} \int \vec{p}^2 d|\vec{p}| \frac{1}{\sqrt{\vec{p}^2 + M^2}}.$$

This integral is divergent. Regularizing it by a cut-off Λ^2 turns it finite. This expression can then be calculated explicitly to yield

$$\begin{aligned} \Pi_t &= \frac{ig^2}{\pi^2} \left(\Lambda^2 \sqrt{1 + \frac{M^2}{\Lambda^2}} - M^2 \ln \left(\frac{\Lambda + \Lambda\sqrt{1 + \frac{M^2}{\Lambda^2}}}{M} \right) \right) \\ &\approx \frac{ig^2}{\pi^2} \left(\Lambda^2 - M^2 \ln \left(\frac{\Lambda}{M} \right) + \mathcal{O}(1) \right). \end{aligned} \tag{5.5}$$

This result already shows all the structures which will also appear in the more complicated diagrams below. First of all, the result is not finite as the cutoff is removed, i.e., by sending Λ to infinity. In fact, it is quadratically divergent. In general, thus, this expression would need to be renormalized to make it meaningful. Since the leading term is momentum-independent, this will require a renormalization of the mass, which is thus quadratically divergent. This is then the origin of the naturalness problem: In the process of renormalization, the first term will be subtracted by a term $-\Lambda^2 + \delta m^2$, where the first term will cancel the infinity, and the second term will shift the mass to its physical value. However, even a slight change in Λ or g^2 would cover even a large change in δm, if the final physical mass is small. There is no reason why it should be small therefore, and thus the mass is not protected. If, e.g., the first would not be present, but only the logarithmic second one, the cancellation would be of type $M^2 \ln\left(\frac{\Lambda}{M}\right) + \delta m^2$. Now, even large changes in Λ will have only little effect, and thus there is no fine-tuning involved to obtain a small physical mass. This will be exactly what will happen in a supersymmetric theory: The quadratic term will drop out in contrast to a non-supersymmetric one, and thus will provide a possibility to obtain a small physical mass without fine tuning of g, Λ and M.

The next contribution stems from the loop graphs involving a boson splitting in two. With an incoming A boson, it can split in either two As, two Bs, or two χs. The contribution from the bosonic loops are once more identical, up to a different prefactor due to the different coupling. Their contribution is

$$-\frac{1}{2}((6Mg)^2 + (2Mg)^2) \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\epsilon} \frac{1}{(p-q)^2 - M^2 + i\epsilon},$$

where the factor $1/2$ is a symmetry factor and q is the external momentum of the A particle. An explicit evaluation of this expression is possible, and discussed in many texts on perturbation theory. This is, in particular when using a cutoff-regularization, a rather lengthy exercise. However, to explicitly show how the naturalness problem is solved, it is only interesting to keep the quadratically divergent piece of the contribution. However, the integrand scales as $1/p^4$ for large momenta. Thus, the integral is only logarithmically divergent, and will thus only contribute at order $M^2\ln(\Lambda/M)$, instead of at Λ^2. For the purpose at hand, this contribution may therefore be dropped.

This leaves the contribution with a fermion loop. It reads

$$-2h^2\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{tr}((\gamma_\mu p^\mu+M)(\gamma_\nu(p^\nu-q^\nu)+M))}{(p^2-M^2+i\epsilon)((p-q)^2-M^2+i\epsilon)}.$$

The factor of 2 in front stems from the fact that for a Majorana fermion particle and anti-particle are the same. Thus, compared to an ordinary fermion, which can only split into particle and anti-particle to conserve fermion number, the Majorana fermion can split into two particles, two anti-particles, or in two ways in one particle and an anti-particle, in total providing a factor four. This cancels the symmetry factor $1/2$ and lets even a factor of 2 standing. Using the trace identities $\mathrm{tr}1=4$, $\mathrm{tr}\gamma_\mu$=0, and $\mathrm{tr}\gamma_\mu\gamma_\nu=4g_{\mu\nu}$ this simplifies to

$$-8h^2\int\frac{d^4p}{(2\pi)^4}\frac{p(p-q)+M^2}{(p^2-M^2+i\epsilon)((p-q)^2-M^2+i\epsilon)}.$$

Since the numerator scales with p^2, the integral is quadratically divergent. Again, it suffices to isolate this quadratic piece. Suppressing the $i\epsilon$, and rewriting the expression by introducing a zero as

$$\begin{aligned}&-4h^2\int\frac{d^4p}{(2\pi)^4}\frac{(p^2-M^2)+((p-q)^2)-M^2)-q^2+4M^2}{(p^2-M^2)((p-q)^2-M^2)}\\ =&\ -4h^2\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p-q)^2-M^2}+\frac{1}{p^2-M^2}+\frac{4M^2-q^2}{(p^2-M^2)((p-q)^2-M^2)}\right)\\ =&\ -4h^2\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-M^2}+\frac{1}{p^2-M^2}+\frac{4M^2-q^2}{(p^2-M^2)((p-q)^2-M^2)}\right),\end{aligned}$$

where in the first term a change of variables $p\to p+q$ was made[4]. The last term is only logarithmically divergent, and thus irrelevant for the present purpose. The first two terms are identical, and of the same type as (5.4). Thus, they can be integrated in the same manner, yielding

$$-\frac{ih^2}{\pi^2}\Lambda^2+\mathcal{O}\left(M^2\ln\frac{\Lambda^2}{M^2}\right).$$

[4]This requires some precautions, but is permitted in this case.

This cannot cancel the previous contribution, unless $g = h$. However, for a supersymmetric theory, supersymmetry dictates $g = h$. But then this is just the negative of (5.5), and thus cancels exactly this contribution. Thus, all quadratic divergences appearing have canceled exactly, and only the logarithmic divergence remains. As has been allured to earlier, this implies a solution of the naturalness problem. In fact, it can be shown that this result also holds in higher order perturbation theory, and only logarithmic divergences appear, thus lower than just the superficial degree of divergence.

In fact, in the present case it is possible to reduce the number of divergences even further. For simplicity, the calculation above has been performed with the F field integrated out. Keeping this field explicitly, it is found (after a more tedious calculation) that the mass of the bosons (and fermions) become finite, and the divergences are all pushed into a wave-function renormalization. Hence, the masses of the particles become fixed, making supersymmetric theories much more predictive (and 'natural') than non-supersymmetric ones.

This feature of canceling quadratic divergences is no accident, but is a general feature of supersymmetric theories. Since fermions and bosons contribute with opposite sign, the fact that supersymmetry requires a precise match between both species leads always to cancellations which lower the degree of divergences. This is one of the most striking benefits of supersymmetric theories.

Chapter 6

Superspace formulation

6.1 Supertranslations

After having now a first working example of a supersymmetric theory and seen its benefits, it is necessary to understand a bit more of the formal properties of supersymmetry. The possibly most striking feature is the somewhat mysterious relation (4.22). It is still not very clear what the appearance of the momentum operator in the SUSY algebra signifies. It will turn out that this connection is not accidental, and lies at the heart of a very powerful, though somewhat formal, formulation of supersymmetric theories in the form of the superspace formulation. This formulation will permit a more direct understanding of why the supermultiplet is as it is, and will greatly aid in the construction of more supersymmetric theories. For that reason, it has become the preferred formulation used throughout the literature.

To start with the construction of the super-space formulation, note that the supercharge Q is a hermitian operator. Thus, it is possible to construct a unitary transformation from it by exponentiating it, taking the form[1]

$$U(\theta,\theta^*) = \exp(i\theta Q)\exp(i\bar{\theta}\bar{Q}).$$

Note that the expressions in the exponents are of course appropriate scalar products, and that θ and θ^* are independent, constant spinors.

Acting with U on any operator ϕ yields thus a new operator ϕ dependent on θ and θ^*

$$U(\theta,\theta^*)\phi U(\theta,\theta^*)^{-1} = \phi(\theta,\theta^*)$$

by definition. This is reminiscent of ordinary translations. Given the momentum operator P_μ, and the translation operator $V(x) = \exp(ixP)$, a field ϕ at some point, say 0, acquires

[1]The order of Q and $\bar{Q}$ is purely conventional, see below.

a position dependence by the same type of operation,

$$V(x)\phi(0)V(x)^\dagger = \phi(x).$$

Thus, in a sense, the operator U provides a field with an additional fermionic coordinate. Of course, this interpretation is done with hindsight, as any unitary operator is providing a field acted upon an additional degree of freedom.

That this interpretation actually makes sense can be most easily seen by applying the operator U twice. To evaluate the operator

$$U(\xi,\xi^*)U(\theta,\theta^*) = \exp(i\xi Q)\exp(i\bar{\xi}\bar{Q})\exp(i\theta Q)\exp(i\bar{\theta}\bar{Q})$$

it is most convenient to use the Baker-Campbell-Hausdorff formula

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{6}[[A,B],B] + ...\right).$$

The commutator of ξQ and $\bar{\xi}\bar{Q}$ for the first two factors can be reduced to the known commutation relation (4.22) in the following way

$$\begin{aligned}
[i\xi Q, i\bar{\xi}\bar{Q}] &= i^2[\xi^a Q_a, -\xi^{b*}Q_b^\dagger] \\
&= i^2(-\xi^a Q_a \xi^{b*} Q_b^\dagger + \xi^{b*}Q_b^\dagger \xi^a Q_a) \\
&= i^2\xi^a\xi^{b*}(Q_a Q_b^\dagger + Q_b^+ Q_a) \\
&= i^2\xi^a\xi^{b*}\{Q_a, Q_b^\dagger\} \\
&= i^2\xi^a\xi^{b*}(\sigma^\mu)_{ab}P_\mu,
\end{aligned}$$

where (4.22) has already been used in the end. This is an interesting result. First of all, it is practical. Since P_μ commutes with both Q and $Q^\dagger$ all of the higher terms in the Baker-Campbell-Hausdorff formula vanish. Secondly, the appearance of the momentum operator, though not unexpected, lends some support to the idea of interpreting the parameters θ and ξ as fermionic coordinates and U as a translation operator in this fermionic space. But to make this statement more definite, the rest of the product has to be analyzed as well. Note that since all of the formulation has been covariant throughout the expression $i\xi^a\xi^{b*}(\sigma^\mu)_{ab}$, though looking a bit odd at first sight, has actually to be a four vector to form again a Lorentz invariant together with P_μ. Of course, this quantity, as a product of two Grassmann numbers, is an ordinary number, so this is also fine. Note further that since P commutes with Q, this part can be moved freely in the full expression.

Combining the next term is rather straight-forward,

$$\begin{aligned}
U(\xi,\xi^*)U(\theta,\theta^*) &= \exp(i^2\xi^a\xi^{b*}(\sigma^\mu)_{ab}P_\mu)\exp(i(\xi Q+\bar{\xi}\bar{Q}))\exp(i\theta Q)\exp(i\bar{\theta}\bar{Q})\\
&= \exp(i^2\xi^a\xi^{b*}(\sigma^\mu)_{ab}P_\mu)\times\\
&\quad\times\exp\left(i\left(\xi Q+\bar{\xi}\bar{Q}+\theta Q-\frac{1}{2}[\xi Q+\bar{\xi}\bar{Q},\theta Q]\right)\right)\exp(i\bar{\theta}\bar{Q})\\
&= \exp(i^2(\xi^a\xi^{b*}+\xi^a\theta^{b*})(\sigma^\mu)_{ab}P_\mu)\exp(i(\xi Q+\bar{\xi}\bar{Q}+\theta Q))\exp(i\bar{\theta}\bar{Q})
\end{aligned}$$

where it has been used that Q commutes with itself. Also, an additional factor of i^2 has been introduced. The next step is rather indirect,

$$\begin{aligned}
&\exp(i^2(\xi^a\xi^{b*}+\xi^a\theta^{b*})(\sigma^\mu)_{ab}P_\mu)\exp(i((\xi+\theta)Q+\bar{\xi}\bar{Q}))\exp(i\bar{\theta}\bar{Q}) \qquad (6.1)\\
=\ &\exp(i^2(\xi^a\xi^{b*}+\xi^a\theta^{b*})(\sigma^\mu)_{ab}P_\mu)\exp(i(\xi+\theta)Q)\times\\
&\times\exp(i\bar{\xi}\bar{Q})\exp(-i(\xi^a\xi^{b*})(\sigma^\mu)_{ab}P_\mu)\exp(i\bar{\theta}\bar{Q})\\
=\ &\exp(i^2\xi^a\theta^{b*}(\sigma^\mu)_{ab}P_\mu)\exp(i(\xi+\theta)Q)\exp(i(\bar{\xi}+\bar{\theta})\bar{Q}).
\end{aligned}$$

In the second-to-last step, the identity $\exp i(A+B)=\exp iA\exp iB\exp([A,B]/2)$ has been used, which follows from the Baker-Campbell-Hausdorff formula in this particular case by moving the commutator term on the other side. Also, it has been used that $\bar{Q}$ commutes with itself to combine the last two factors.

Hence, in total two consecutive operations amount to

$$U(\xi,\xi^*)U(\theta,\theta^*)=\exp(i^2\xi^a\theta^{b*}(\sigma^\mu)_{ab}P_\mu)\exp(i(\xi+\theta)Q)\exp(i(\bar{\xi}+\bar{\theta})\bar{Q}).$$

This result is not of the form $U(f(\xi,\xi^*),g(\theta,\theta^*))$, and thus the individual supertransformations do not form a group. This is not surprising, as the algebra requires that also the momentum operator must be involved. A better ansatz is thus

$$U(a_\mu,\xi,\xi^*)=\exp(iPa)\exp(i\xi Q)\exp(i\bar{\xi}\bar{Q}).$$

Since the momentum operator commutes with both Q and $\bar{Q}$, it follows directly that

$$\begin{aligned}
U(a_\mu,\xi,\xi^*)U(b_\mu,\theta,\theta^*) &= \exp(i(a_\mu+b_\mu+i\xi^a\theta^{b*}(\sigma^\mu)_{ab})P_\mu)\exp(i(\xi+\theta)Q)\exp(i(\bar{\xi}+\bar{\theta})\bar{Q})\\
&= U(a^\mu+b^\mu+i\xi^a\theta^{b*}(\sigma^\mu)_{ab},\xi+\theta,\xi^*+\theta^*),
\end{aligned}$$

and consequently

$$U(x_\mu,\xi,\xi^*)U(a_\mu,\theta,\theta^*)\phi(0)U(x_\mu,\xi,\xi^*)^{-1}U(a_\mu,\theta,\theta^*)^{-1}=\phi(x^\mu+a^\mu+i\xi^a\theta^{b*}(\sigma^\mu)_{ab},\xi+\theta,\xi^*+\theta^*).$$

This then forms a group, as it should be, the group of supertranslations. The group is not Abelian, as a minus sign appears if θ and ξ are exchanged in the parameter for the momentum operator. However, the ordinary translations form an Abelian subgroup of this group of supertranslations. It is now also clear why there is a similarity to ordinary translations, compared to other unitary transformations: The latter only form a simple direct product group with ordinary translations, which is not so in case of the supertranslations. In this case, it is a semidirect product. This also justifies to call the parameters ξ and ξ^* supercoordinates in an abstract superspace.

The construction of these supertranslations, and by this the definition of fermionic supercoordinates and thus an abstract superspace, will now serve as a starting point for the construction of supersymmetric theories using this formalism.

6.2 Coordinate representation of supercharges

6.2.1 Differentiating and integrating spinors

Before proceeding, it is necessary to discuss how to differentiate and integrate various scalar products of spinors. A typical derivative, which will appear throughout, is

$$\frac{\partial}{\partial\theta^1}(\theta^b\theta_b) = \frac{\partial}{\partial\theta^1}(-\epsilon_{bc}\theta^b\theta^c) = \frac{\partial}{\partial\theta^1}(-\theta^1\theta^2 + \theta^2\theta^1) = \frac{\partial}{\partial\theta^1}(-2\theta^1\theta^2) = -2\theta^2 = 2\theta_1,$$

where it has been used that $\theta^a = \epsilon^{ab}\theta_b$. Similarly

$$\frac{\partial}{\partial\theta^2}(\theta^b\theta_b) = 2\theta^1 = 2\theta_2.$$

Combining both expressions yields

$$\frac{\partial}{\partial\theta^a}(\theta^b\theta_b) = 2\theta_a.$$

Using that $\bar\chi_{\dot a} = \chi_a^*$, it is possible to obtain a similar expression for a derivative of $\bar\chi\bar\chi$ with respect to $\bar\chi_{\dot a}$,

$$\frac{\partial}{\partial\bar\chi_{\dot a}}(\bar\chi_{\dot b}\bar\chi^{\dot b}) = \frac{\partial}{\partial\chi_a^*}(\chi^{b*}\chi_b^*) = \frac{\partial}{\partial\chi_a^*}(\epsilon^{bc}\chi_b^*\chi_c^*) = \epsilon^{bc}(\delta^{ab}\chi_c^* - \delta_{ac}\chi_c^*) = 2\epsilon^{ac}\chi_c^* = 2\chi^{a*}.$$

Therefore, a derivative with respect to θ^a acts like a χ^a spinor, in the language of section 4.1, and a derivative with respect to θ_a^* like a $\bar\theta^{\dot a}$-type spinor.

Integrating is very much similar to this, as integration and differentiation is the same for Grassmann variables. Denoting $d^2\theta = d\theta_1 d\theta_2$ and $d^2\bar\theta = d\bar\theta^{\dot 2} d\bar\theta^{\dot 1}$ and finally $d^4\theta = d^2\bar\theta d^2\theta$, where the order must be kept, it is simple to integrate scalar products:

$$\int d^2\theta\frac{1}{2}\theta^a\theta_a = \frac{1}{2}\int d\theta_1 d\theta_2(-\theta_1\theta_2 + \theta_2\theta_1) = \frac{1}{2}\int d\theta_1(\theta_1 + \theta_1) = 1,$$

and analogously for the $\bar\theta$ integral.

6.2.2 Coordinate representation

Ordinarily, infinitesimal translations $U(\epsilon_\mu)$ can be written in terms of a derivative

$$U(\epsilon_\mu) = \exp(-i\epsilon_\mu P^\mu) \approx 1 - i\epsilon_\mu P^\mu = 1 + \epsilon^\mu \partial_\mu. \tag{6.2}$$

If the correspondence of θ and θ^* should be taken seriously, such a differential representation for supertranslations should be possible. Hence for

$$U(\epsilon_\mu, \xi, \xi^*)\phi(x_\mu, \theta, \theta^*)U(\epsilon_\mu, \xi, \xi^*)^{-1} = \phi(x_\mu, \theta, \theta^*) + \delta\phi$$

with ϵ_μ and ξ and ξ^* all infinitesimal it should be possible to write for $\delta\phi$

$$\delta\phi = (\epsilon_\mu - i\theta^a \xi^{b*}(\sigma^\mu)_{ab})\partial_\mu \phi + \xi^a \frac{\partial\phi}{\partial\theta^a} + \xi_a^* \frac{\partial\phi}{\partial\theta_a^*}.$$

In analogy to (6.2), this implies that in the expression

$$\delta\phi = ((\epsilon^\mu \partial_\mu - i\theta^a \xi^{b*}(\sigma^\mu)_{ab})\partial_\mu - i\xi^a Q_a - i\xi_a^* Q^{a\dagger})\phi$$

it is necessary to identify

$$\begin{aligned} Q_a &= i\frac{\partial}{\partial\theta^a} && (6.3)\\ Q^{a\dagger} &= i\frac{\partial}{\partial\theta_a^*} + \theta^b(\sigma^\mu)_{ba}\partial_\mu. && (6.4) \end{aligned}$$

To form the second part of the $Q^\dagger$ part, it is necessary to note that $\xi_a^* Q^{a\dagger} = -\xi^{a*}Q_a^\dagger$, giving the overall sign.

This representation of Q and $Q^\dagger$ is only making sense, if these operators fulfill the corresponding algebra. In particular, Q and $Q^\dagger$ must commute with themselves. That is trivial in case of Q. Since θ and θ^* are independent variables, this is also the case for $Q^\dagger$. Furthermore, the anticommutation relation (4.22) has to be fulfilled[2]. This can be checked explicitly

$$\begin{aligned} \{Q_a, Q_b^\dagger\} &= \left\{ i\frac{\partial}{\partial\theta^a}, i\frac{\partial}{\partial\theta_b^*} + \theta^c(\sigma^\mu)_{cb}\partial_\mu \right\} \\ &= \left\{ i\frac{\partial}{\partial\theta^a}, i\frac{\partial}{\partial\theta_b^*} \right\} + \left\{ i\frac{\partial}{\partial\theta^a}, \theta^c(\sigma^\mu)_{cb}\partial_\mu \right\} \\ &= \left\{ i\frac{\partial}{\partial\theta^a}, \theta^c \right\} i(\sigma^\mu)_{cb}\partial_\mu \\ &= i(\sigma^\mu)_{ab}\partial_\mu = (\sigma^\mu)_{ab}P_\mu. \end{aligned}$$

[2] This is sufficient, as the commutator can be constructed from the anti-commutator, as has been done backwards in section 4.3.2.

where the fact that derivatives with respect to Grassmann variables anticommute has been used in going from the second to the third line and furthermore that

$$\partial_{\theta^a}(\theta^b\phi)+\theta^b\partial_{\theta^a}\phi=(\partial_{\theta^a}\theta^b)\phi+(\partial_{\theta^a}\phi)\theta^b+\theta^b\partial_{\theta^a}\phi=\delta^{ab}\phi+(\partial_{\theta^a}\phi)\theta^b-(\partial_{\theta^a}\phi)\theta^b,$$

using the anticommutativity of Grassmann numbers. Hence, the operators (6.3) and (6.4) are in fact a possible representation of the supersymmetry algebra, and there indeed exists a derivative formulation for these operators. This again emphasizes the strong similarity of the supersymmetry algebra and the translation algebra, once for fermionic and once for bosonic coordinates in the super space.

6.3 Supermultiplets

Now, given this superspace, the first question is what the vectors in this superspace represent. The simplest vector will have only components along one of the coordinates, which will be taken to be θ for now. Furthermore, these vectors are still functions. But their dependence on the Grassmann variables is by virtue of the properties of Grassmann numbers rather simple. Thus such a vector $\Phi(x,\theta)$ can be written as

$$\Phi(x,\theta)=\phi(x)+\theta\chi(x)+\frac{1}{2}\theta\theta F(x).$$

The θ-variables are still spinors, and the appearing products are still scalar products. Due to the antisymmetry of the scalar product, the last term does not vanish, though of course quantities like θ_a^2 vanish. The names for the component fields have been selected suggestively, but at the moment just represent arbitrary (bosonic or fermionic, complex) functions. Not withstanding, the set of the field (ϕ,χ,F) is called a chiral supermultiplet[3].

To justify the notation, it is sufficient to have a look at the infinitesimal transformation properties of Φ under $U(0,\xi,\xi^*)$. Of course, a non-zero translation parameter would just shift the x-arguments of the multiplet, and is thus only a notational complication. The change in the field is thus, as usual, given by

$$\begin{aligned}\delta\Phi &= (-i\xi^a Q_a - i\xi^*_a Q^{a\dagger})\Phi = (-i\xi^a Q_a + i\xi^{a*}Q^\dagger_a)\Phi\\ &= \left(\xi^a\frac{\partial}{\partial\theta^a}+\xi^{a*}\frac{\partial}{\partial\theta^{a*}}+i\xi^{a*}\theta^b(\sigma^\mu)_{ba}\partial_\mu\right)\left(\phi(x)+\theta^c\chi_c+\frac{1}{2}\theta^c\theta_c F\right)\\ &= \delta_\xi\phi+\theta^a\delta_\xi\chi_a+\frac{1}{2}\theta^a\theta_a\delta_\xi F,\end{aligned}$$

[3]Actually a left one, as only the left-type spinor χ appears.

where the last line is by definition the change in the individual components of the superfield. Since Φ is not depending on θ^{a*}, the derivative with respect to this variable can be dropped. It thus only remains to order the result by powers in θ.

At order zero, a contribution can only come from the action of the θ-derivative on the χ term. This yields

$$\delta_\xi\phi = \xi\chi$$

for the transformation rule of the ϕ-field.

At order one appears

$$\delta_\xi\chi_a = \xi_a F - i\xi^{b*}(\sigma^\mu)_{ab}\partial_\mu\phi = \xi_a F - i(i\sigma_2\xi^*)_b(\sigma^\mu)_{ab}\partial_\mu\phi,$$

giving the transformation for the χ-field.

Finally, because θ^a are Grassmann variables, order two is the highest possible order, yielding

$$\begin{aligned}
\frac{1}{2}\theta\theta\delta_\xi F &= i\xi^{b*}\theta^a(\sigma^\mu)_{ab}\theta^c\partial_\mu\chi_c = -\frac{1}{2}\theta\theta\epsilon^{ac}i\xi^{b*}(\sigma^\mu)_{ab}\partial_\mu\chi_c \\
&= -\frac{1}{2}\theta\theta i\xi^{a*}(\sigma^{T\mu})_{ab}\epsilon^{bc}\partial_\mu\chi_c \\
&= -\frac{1}{2}\theta\theta i(i\sigma_2\xi^*)_a(\sigma^{T\mu})\epsilon^{bc}\partial_\mu\chi_c \\
&= -\frac{1}{2}\theta\theta i\xi^*_d(i\sigma_2)_{da}(\sigma^{T\mu})_{ab}(i\sigma_2)_{bc}\partial_\mu\chi_c \\
&= \frac{1}{2}\theta\theta(-i)\xi^*_d(\bar\sigma^\mu)_{dc}\partial_\mu\chi_c \\
&= \frac{1}{2}\theta\theta(-i)\xi^\dagger(\bar\sigma^\mu)\partial_\mu\chi
\end{aligned}$$

where it has been used that

$$\theta^a\theta^b = -\frac{1}{2}\epsilon^{ab}\theta^c\theta_c. \tag{6.5}$$

This can be shown simply by explicit calculation, keeping in mind that $\theta^{a2} = 0$ and $\epsilon^{12} = 1 = -\epsilon_{12}$, as well as $(i\sigma_2)_{cd} = \epsilon^{cd}$ and $\bar\sigma^\mu = i\sigma_2\sigma^{T\mu}i\sigma_2$.

Thus, the transformation rules for the field components are

$$\begin{aligned}
\delta_\xi\phi &= \xi\chi \\
\delta_\xi\chi_a &= \xi_a F - i(i\sigma_2\xi^*)_b(\sigma^\mu)_{ab}\partial_\mu\phi \\
\delta_\xi F &= -i\xi^\dagger(\bar\sigma^\mu)\partial_\mu\chi.
\end{aligned} \tag{6.6}$$

These are exactly the transformation rules obtained in section 4.3.5. Thus, a chiral supermultiplet under a supertranslation transforms in exactly the same way as the field content of a free (or interacting, in case of the Wess-Zumino model) supersymmetric theory. This result already indicates that supersymmetric theories can possibly be formed by using scalars with respect to supertranslation in much the same way as ordinary theories are built from scalars with respect to Lorentz transformations. The following will show that this is indeed the case. Actually, this is not exactly the way it turns out, but the idea is similar.

6.4 Other representations

All of this can be repeated, essentially unchanged, for a right chiral multiplet, where the dependence on θ is replaced by the one on θ^*. Including both, θ and θ^* actually does not provide something new, but just leads to a reducible representation. That is most easily seen by considering the condition

$$\frac{\partial}{\partial\theta_a^*}\Phi(x,\theta,\theta^*) = 0.$$

If this condition applies to the field Φ then so does it to the transformed field $\delta\Phi$, since in the latter expression

$$\frac{\partial}{\partial\theta_c^*}\delta\Phi = \frac{\partial}{\partial\theta_c^*}\left(-i\theta^a(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu\Phi + \xi^a\frac{\partial\Phi}{\partial\theta^a} + \xi_a^*\frac{\partial\Phi}{\partial\theta_a^*}\right)$$

at no point an additional dependence on θ^* is introduced. Hence, the fields $\Phi(x,\theta)$ form an invariant subgroup of the SUSY transformation, and likewise do $\Phi(x,\theta^*)$. Any representation including a dependence on θ and θ^* can thus be only a reducible one. Nonetheless, this representation is useful, as it will permit to construct a free supersymmetric theory.

Before investigating this possibility, one question might arise. To introduce the supervectors Φ, the particular supertranslation operator

$$U_I(x,\theta,\theta^*) = \exp(ixP)\exp(i\theta Q)\exp(i\bar\theta\bar Q),$$

called type I, has been used. Would it not also have been possible to use the operators

$$\begin{aligned} U_{II}(x,\theta,\theta^*) &= \exp(ixP)\exp(i\bar\theta\bar Q)\exp(i\theta Q) \\ U_r(x,\theta,\theta^*) &= \exp(ixP)\exp(i\theta Q + i\bar\theta\bar Q)? \end{aligned}$$

The answer to this question is, in fact, yes. Both alternatives could have been used. And these would have generated different translations in the field. In fact, when using the expression $\Phi_i = U_i\Phi(0,0,0)U_i$ to generate alternative superfields the relations

$$\Phi_r(x,\theta,\theta^*) = \Phi_I\left(x^\mu - \frac{1}{2}i\theta^a\sigma^\mu_{ab}\theta^{b*},\theta,\theta^*\right) = \Phi_{II}\left(x^\mu + \frac{1}{2}i\theta^a\sigma^\mu_{ab}\theta^{b*},\theta,\theta^*\right) \tag{6.7}$$

would have been found. An explicit expression, e. g. in case of the left supermultiplet would be

$$\Phi_r(x,\theta,\theta^*) = \phi+\theta\chi+\frac{1}{2}\theta\theta F-\frac{1}{2}i\theta^a\sigma^\mu_{ab}\theta^{b*}\partial_\mu\phi+\frac{1}{4}i(\theta\theta)(\partial_\mu\chi^a\sigma^\mu_{ab}\theta^{b*})-\frac{1}{16}(\theta\theta)(\bar\theta\bar\theta)\partial^2\phi, \quad (6.8)$$

which is obtained by the Taylor-expansion in θ from the relation (6.7).

I. e., the different supertranslations lead to unitarily equivalent supervectors, which can be transformed into each other by conventional translations, and thus a unitary transformation. This already suggests that these are just unitarily equivalent representations of the same algebra. This can be confirmed: For each possibility it is possible to find a representation in terms of derivatives which always fulfills the SUSY algebra. Hence, all of these representations are equivalent, and one can choose freely the most appropriate one, which in the next section will be the type-I version.

6.5 Constructing interactions from supermultiplets

For now, lets return to the (left-)chiral superfield, and the type-I transformations.

Inspecting the transformation rules under a supertranslation of a general superfield one thing is of particular importance: The transformation of the F-component of the superfield, (6.6), corresponds to a total derivative. As a supertranslation is nothing else than a SUSY transformation this implies that any term which is constructed from the F-component of a superfield will leave the action invariant under a SUSY transformation. Hence, the question arises how to isolate this F-term, and whether any action be constructed out of it.

The first question is already answered: This can be done by twofold integration. Due to the rules for analysis of Grassmann variables, it follows that

$$\int d\theta_1 \int d\theta_2 \Phi = \int d\theta_1 \int d\theta_2 \left(\phi + \theta^a\chi_a + \frac{1}{2}\theta^a\theta_a F\right) = \int d\theta_1(\chi_2 + \theta_1 F) = F.$$

Thus twofold integration is isolating exactly this part.

The answer to the second question is less obvious. A single superfield will only contribute a field F. That is not producing a non-trivial theory. However, motivated by the construction of Lorentz invariants by building scalar products, the simplest idea is to consider a product of two superfields. This yields a superscalar

$$\Phi\Phi = \left(\phi + \theta\chi + \frac{1}{2}\theta\theta F\right)^2 = \phi^2 + 2\phi\theta\chi + \theta\theta\phi F + (\theta\chi)(\theta\chi),$$

and all other terms vanish, as they would be of higher order in θ. Using (6.5), the last term can be rearranged to yield

$$\theta^a\chi_a\theta^b\chi_b = -\theta^a\theta^b\chi_a\chi_b = \frac{1}{2}\theta\theta\epsilon^{ab}\chi_a\chi_b = -\frac{1}{2}\theta\theta\chi^a\chi_a = \frac{1}{2}(\theta\theta)(\chi\chi), \tag{6.9}$$

and thus

$$\Phi\Phi = \phi^2 + 2\phi\theta\chi + \frac{1}{2}\theta\theta(2F\phi - \chi\chi).$$

Isolating the F-term and multiplying with a constant $M/2$, which is not changing the property of being a total derivative under SUSY transformations, yields

$$W_2 = M\phi F - \frac{M}{2}\chi\chi.$$

However, this is a well-known quantity, it is exactly the terms quadratic in the fields in the Lagrangian (5.2) which are not part of the free Lagrangian, and to which one has to add, of course, also the hermitian conjugate to obtain a real action, even though both are separately SUSY invariant.

The fact that only terms of order two in the fields could have been obtained is clear, as only a square was evaluated. However, it is surprising at first sight that in fact all terms quadratic to this order have been obtained. However, in the construction of the Wess-Zumino Lagrangian the minimal set has been searched for, and this here is then the minimal set. Note then that this implies that when stopping at this point, the Lagrangian for a massive, but otherwise free, supersymmetric theory has been obtained, as can be checked by integrating out F. Note that obtaining the free part of the Lagrangian, or a massless supersymmetric theory, can also be performed by the present methods, but requires some technical complications to be discussed later.

This suggests that it should be possible to obtain the full interaction part of the Wess-Zumino model by also inspecting the product of three superfields, as this is the highest power in fields appearing in (5.2). And in fact, evaluating the F-component of such a product yields

$$\begin{aligned}\int d^2\theta\Phi^3 &= 3\phi^2F - \phi\chi\chi + 2\phi\int d^2\theta(\theta\chi)(\theta\chi)\\ &= 3\phi^2F - \phi\chi\chi + 2\phi\int d^2\theta(-\frac{1}{2}\theta\theta)(\chi\chi) = 3\phi^2F - 3\phi\chi\chi.\end{aligned}$$

Multiplying this with $y/6$ exactly produces the terms which are cubic in the fields in the Wess-Zumino Lagrangian (5.2). Thus, this Lagrangian could equally well be written as

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^\dagger + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^\dagger F + \int d^2\theta(W + W^\dagger)$$

with

$$W = \frac{M}{2}\Phi\Phi + \frac{y}{6}\Phi\Phi\Phi,$$

being the superpotential. In contrast to the one introduced in section 5.1 this superpotential now includes all interactions of the theory, not only the ones involving the ϕ terms. Thus, it was indeed possible to construct the supersymmetric theory just by usage of products of superfields. This construction principle carries over to more complex situations, and can be considered as the construction principle for supersymmetric theories.

The generalization to fields with an internal degree of freedom, like flavor or charge, is straightforward, yielding a superpotential of type

$$W = \frac{M_{ij}}{2}\Phi_i\Phi_j + \frac{y_{ijk}}{6}\Phi_i\Phi_j\Phi_k,$$

just as in the case of the original treatment of the Wess-Zumino model.

In principle, the free part can be constructed in a similar manner. However, it turns out to be surprisingly more complicated in detail. While so far only the field $\Phi_I(x,\theta,0)$ has been used, for the free part it will be necessary to use in addition the quantity $\Phi_r(x,\theta,\theta^*)$. The free part can then be obtained from (6.8) by the expression

$$\begin{aligned}\int d^4\theta\Phi_r(x,\theta,\theta^*)^\dagger\Phi_r(x,\theta,\theta^*) &= -\frac{1}{4}\phi^\dagger\partial^2\phi - \frac{1}{4}\partial^2\phi\phi^\dagger + F^\dagger F \\ &+ \int d^4\theta\frac{1}{4}((i(\bar\chi\bar\theta)(\theta\theta)\partial_\mu\chi^{a\dagger}\sigma^\mu_{ab}\theta^{b*} - i\theta^a\sigma^\mu_{ab}\partial_\mu\chi^b(\bar\theta\bar\theta)(\theta\chi) \\ &+\partial_\mu\phi^\dagger\theta^a\sigma^\mu_{ab}\theta^{b*}\theta^c\sigma^\nu_{cd}\theta^{d*}\partial_\nu\phi)\end{aligned}$$

Since the action is the only quantity of interest, partial integrations are permitted. Thus $\phi^\dagger\partial^2\phi = -\partial\phi^\dagger\partial\phi$. Furthermore, applying (6.9) twice yields the relation

$$(\bar\theta\bar\chi)(\bar\theta\bar\chi) = -\frac{1}{2}(\bar\theta\bar\theta)(\bar\chi\bar\chi),$$

which, of course holds similarly for unbarred quantities. This can be used to reformulate the second line to isolate products of $\theta^2\bar\theta^2$, and also for the third line. These manipulations together yield

$$\begin{aligned}\int d^4\theta\Phi_r(x,\theta,\theta^*)^\dagger\Phi_r(x,\theta,\theta^*) &= \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi + F^\dagger F + i\bar\chi\bar\sigma^\mu\partial_\mu\chi + \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi \\ &= \partial_\mu\phi^\dagger\partial^\mu\phi + i\bar\chi\bar\sigma^\mu\partial_\mu\chi + F^\dagger F,\end{aligned}$$

which is exactly the Lagrangian of the free supersymmetric theory, or, more precisely, the integration kernel of the action. Hence, the complete Wess-Zumino model Lagrangian can be written as

$$\mathcal{L} = \int d^4\theta\Phi_r^\dagger\Phi_r + \int d^2\theta\left(\frac{M}{2}\Phi_I\Phi_I + \frac{y}{6}\Phi_I\Phi_I\Phi_I\right) + \int d^2\bar\theta\left(\frac{M}{2}\Phi_{II}\Phi_{II} + \frac{y}{6}\Phi_{II}\Phi_{II}\Phi_{II}\right),$$

which is in fact now an expression where supersymmetry is manifest, and the last term just creates the Hermitiean conjugate of the second-to-last term to have a hermitiean action.

Chapter 7

Supersymmetric gauge theories

All relevant theories in particle physics are actually not of the simple type consisting only out of scalars and fermions, but are gauge theories. Therefore, to construct the standard model it is necessary to work with supersymmetric gauge theories. Before this can be done, it is of course necessary to introduce ordinary gauge theories. The following will hence include a short reminder, or a crash course, in gauge theories, and is by no means in any respect complete.

Furthermore, in the following the superspace formalism is actually less practically useful, so it will not be used.

7.1 Supersymmetric Abelian gauge theory

7.1.1 Ordinary Abelian gauge theory

The simplest possible gauge theory is the quantum field theoretical generalization of electrodynamics. In classical electrodynamics, it was possible to transform the gauge potential A_μ by a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \omega,$$

where ω is an arbitrary function. A defining property of such gauge transformations is the fact that they do not alter any measurable quantities. In particular, the electric and magnetic fields $\vec{E}$ and $\vec{B}$, which are obtained from the gauge potential by

$$\begin{aligned}\vec{E}_i &= -\frac{\partial}{\partial t}A_i - \partial_i A_0 \\ \vec{B}_i &= (\vec{\nabla} \times \vec{A})_i,\end{aligned}$$

are invariant under such transformations. From the vector potential, it is possible to form the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which is also invariant under gauge transformations. The Maxwell equations can then be written in the compact form

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= j^\nu \\ \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} &= 0,\end{aligned}$$

where j^μ is the matter current. These are the equations of motions of the classical Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu.$$

Consistently quantizing this theory is actually highly non-trivial, and beyond the scope of this lecture. However, for the purpose of this lecture it is only interesting how the Lagrangian of the quantized version of this theory looks. It turns out that the first term proportional to $F_{\mu\nu}F^{\mu\nu}$ is already the Lagrangian of the quantized electromagnetic field. It then remains to construct the electric current. If an electron is represented by a spinor ψ, this spinor is actually no longer invariant under a gauge transformation. However, as in quantum mechanics, only the phase is affected by a gauge transformation. Therefore, under a gauge transformation the spinors change as

$$\psi \to \exp(-ie\omega)\psi,$$

where the same function ω appears as for the vector potential, which is now representing the field of the photon, and is called gauge field. Since ω is a function, the kinetic term for an electron is no longer invariant under a gauge transformation, and has to be replaced by

$$i\bar{\psi}(\gamma^\mu(\partial_\mu + ieA_\mu))\psi.$$

This replacement

$$\partial_\mu \to \partial_\mu + ieA_\mu = D_\mu$$

is called minimal coupling[1], and D_μ the covariant derivative. This is gauge invariant, as an explicit calculation shows,

$$\begin{aligned} i\bar{\psi}'(\gamma^\mu(\partial_\mu + ieA'_\mu))\psi' &= i\bar{\psi}\exp(ie\omega)\gamma^\mu(\partial_\mu(\exp(-ie\omega)\psi) + \exp(-ie\omega)(ieA_\mu\psi + ie\partial_\mu\omega\psi)) \\ &= i\bar{\psi}\exp(ie\omega)\gamma^\mu(\exp(-ie\omega)(\partial_\mu\psi - ie\partial_\mu\omega\psi) + \exp(-ie\omega)(ieA_\mu\psi + ie\partial_\mu\omega\psi)) \\ &= i\bar{\psi}(\gamma^\mu(\partial_\mu + ieA_\mu))\psi. \end{aligned}$$

[1]It is possible to also formulate consistent theories with non-minimal coupling. However, none of these have so far been compatible with experiment, and they will therefore be ignored here.

Thus, the (gauge-invariant) Lagrangian of QED is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

where a mass term has been added, which is trivially gauge-invariant. The second term is thus the quantum version of the $j^\mu A_\mu$ term.

This type of gauge theories is called Abelian, as the phase factor $\exp(i\omega)$ with which the gauge transformation is performed for the fermions is an element of the U(1) group. Thus, U(1) is called the gauge group of the theory. Furthermore, the vector potential, or gauge field, is given by a real function tensored with an element of the corresponding u(1) algebra.

7.1.2 Supersymmetric Maxwell theory

The simplest gauge theory to construct is a supersymmetric version of the Abelian one, neglecting the electrons for now. The photon is a boson with spin 1. Hence its superpartner, the photino, has to be a fermion. The photon has on-shell two degrees of freedom, so the photino has to be a Weyl fermion. Off-shell, however, the photon has three degrees of freedom, corresponding to the three different magnetic quantum numbers possible. So another bosonic degree of freedom is necessary to cancel all fermionic degrees of freedom. This other off-shell bosonic degree of freedom will be the so-called D field. This shows again the condition on the exact equality of the number of fermionic and bosonic degrees of freedom in a supersymmetric theory on-shell and off-shell.

Will this be a flavor of quantum electron dynamics then, just with the Dirac electron replaced by a Weyl one and one field added? The answer to this is a strict no. Since the supersymmetry transformation just acts on the statistical nature of particles it cannot change an uncharged photon into a charged photino. Thus, the photino has also to have zero charge, as does the D boson. The simplest supersymmetric gauge theory is then the free supersymmetric Maxwell theory, as there are no interactions possible between uncharged particles. Its form has thus to be of the type

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2. \tag{7.1}$$

Note that the absence of charge also implies that no covariant derivative can appear. Hence, the photino λ has to be invariant under a gauge transformation, as D has to be. Thus the gauge dynamics is completely contained in the photon field.

To construct the SUSY transformation for A_μ, it is necessary that is has to be real, as A_μ is a real field. Furthermore, it has to be a Lorentz vector. Finally, since it is an infinitesimal transformation it has to be at most linear in the transformation parameter ξ. The simplest quantity fulfilling these properties is

$$\delta_\xi A_\mu = \xi^\dagger \bar\sigma_\mu \lambda + \lambda^\dagger \bar\sigma_\mu \xi.$$

As noted, the photino is a gauge scalar. It can therefore not be directly transformed with the field A_μ, but a gauge-invariant combination of A_μ is necessary. The simplest such quantity is $F_{\mu\nu}$. To absorb the two indices, and at the same time provide the correct transformation properties under Lorentz transformation, the SUSY transformations should be of the form

$$\begin{aligned}\delta_\xi \lambda &= \frac{i}{2}\sigma^\mu \bar\sigma^\nu \xi F_{\mu\nu} + \xi D\\ \delta_\xi \lambda^\dagger &= -\frac{i}{2}\xi^\dagger \bar\sigma^\nu \sigma^\mu F_{\mu\nu} + \xi^\dagger D,\end{aligned}$$

where the pre-factor has been chosen with hindsight. In principle, it could also be determined a-posterior, provided that otherwise these simplest forms work. The terms containing the D field have been added in analogy with the Wess-Zumino model. Finally, the SUSY transformation for the D-boson has to vanish on-shell, and thus should be proportional to the equations of motion of the other fields. Furthermore, it is a real field, and thus its transformation rule has to respect this, similarly as for the photon. In principle, its transformation could depend on the equations of motions of both the photon and the photino. However, inspired by the properties of the F boson in the Wess-Zumino model the ansatz is one depending only on the equations of motion of the photino, which is indeed sufficient. The transformation rule such constructed is

$$\delta_\xi D = -i(\xi^\dagger \bar\sigma^\mu \partial_\mu \lambda - \partial_\mu \lambda^\dagger \bar\sigma^\mu \xi),$$

which has all of the required properties.

It remains to demonstrate that these are the correct transformation rules and that the theory is supersymmetric. Since ξ is taken to be infinitesimal and Grassmann, it is sufficient to evaluate the transformations once more only up to an order linear in ξ.

The transformation of the photon term yields

$$\begin{aligned}-\frac{1}{4}\delta_\xi(F_{\mu\nu}F^{\mu\nu}) &= -\frac{1}{4}((\delta_\xi F_{\mu\nu})F^{\mu\nu} + F_{\mu\nu}(\delta_\xi F^{\mu\nu})) = -\frac{1}{2}F_{\mu\nu}\delta_\xi F^{\mu\nu}\\ &= -\frac{1}{2}F^{\mu\nu}(\partial_\mu \delta_\xi A_\nu - \partial_\nu \delta_\xi A_\mu) = -F_{\mu\nu}\partial^\mu \delta_\xi A^\nu = -F_{\mu\nu}\partial^\mu(\xi^\dagger \bar\sigma^\nu \lambda + \lambda^\dagger \bar\sigma^\nu \xi),\end{aligned}$$

where the antisymmetry of $F_{\mu\nu}$ has been used. The only term which can cancel this is the one part from the transformation of the spinors being proportional to $F_{\mu\nu}$. This contribution is

$$\begin{aligned}\delta_\xi^{F_{\mu\nu}}(i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda) &= i(\delta_\xi\lambda^\dagger)\bar{\sigma}^\mu\partial_\mu\lambda + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu(\delta_\xi\lambda)\\ &= \frac{1}{2}\left(\xi^\dagger\bar{\sigma}^\nu\sigma^\mu F_{\mu\nu}\bar{\sigma}^\rho\partial_\rho\lambda + (\partial_\rho\lambda^\dagger)\bar{\sigma}^\rho\sigma^\mu\bar{\sigma}^\nu\xi F_{\mu\nu}\right).\end{aligned}$$

The structure of this term is already quite similar, but the product of three σs is different. However, since the σ are Pauli matrices, it is always possible to rewrite them in terms of single ones,

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho = g^{\mu\nu}\bar{\sigma}^\rho - g^{\mu\rho}\bar{\sigma}^\nu + g^{\nu\rho}\bar{\sigma}^\mu - i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma, \tag{7.2}$$

where ϵ is totally antisymmetric. This simplifies the expression at lot. The contraction $g^{\mu\nu}F_{\mu\nu}$ vanishes, since g is symmetric and $F_{\mu\nu}$ is antisymmetric. Because

$$F_{\mu\nu}\partial_\rho\lambda = -\lambda(\partial_\rho\partial_\mu A_\nu - \partial_\rho\partial_\nu A_\mu),$$

this expression is symmetric in two indices. Thus any contraction with the ϵ-tensor also vanishes. Thus, the expression reduces to

$$\begin{aligned}&\xi^\dagger F_{\mu\nu}(-g^{\nu\rho}\bar{\sigma}^\mu + g^{\mu\rho}\bar{\sigma}^\nu)\partial_\rho\lambda + (\partial_\rho\lambda^\dagger)(g^{\mu\rho}\bar{\sigma}^\nu - g^{\nu\rho}\bar{\sigma}^\mu)\xi F_{\mu\nu}\\ =\ &-2F_{\mu\nu}\xi^\dagger\bar{\sigma}^\mu\partial^\nu\lambda + 2(\partial^\mu\lambda^\dagger)\bar{\sigma}^\nu F_{\mu\nu} = 2F_{\mu\nu}(\xi^\dagger\bar{\sigma}^\nu\partial^\mu\lambda + \partial^\mu\lambda^\dagger\bar{\sigma}^\nu\xi).\end{aligned}$$

This precisely cancels the contribution from the photon transformation, when combined with the factor 1/2.

The expressions involving D are simpler. The contribution from the photino term is

$$\delta_\xi^D(i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda) = iD\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - i\partial_\mu\lambda^\dagger\bar{\sigma}^\mu\xi D,$$

which cancels against the contribution from the D term

$$\delta_\xi\frac{1}{2}D^2 = D\delta_\xi D = -iD(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - \partial_\mu\lambda^\dagger\bar{\sigma}^\mu\xi).$$

Thus the Lagrangian (7.1) describes indeed a supersymmetric theory, consisting of the non-interacting photon, the photino, and the D-boson.

Of course, it would once more be possible to construct the theory just from the D-component of a super-vector. This will not be done here.

7.2 Supersymmetric Yang-Mills theory

7.2.1 Ordinary Yang-Mills theory

Although U(1) gauge theory already provides an enormous host of interesting physical effects, e. g. solid state physics, its complexity is not sufficient to describe all the phenomena encountered in the standard model, e. g. nuclear physics. A sufficiently complex theory is obtained when the Abelian gauge algebra u(1) is replaced by the non-Abelian gauge algebra[2] su(N), where N is referred to often as the number of colors. In case of the strong interactions N is 3, and for the weak interactions it is 2.

In this case, all fields carry an additional index, a, which indicates the charge with respect to this gauge group. E. g., the gluon index runs from 1 to 8, while the quark index runs from 1 to 3, because the former have charges corresponding to the adjoint representation of SU(3), and the latter to the fundamental one.

In particular, a gauge field can now be written as $A_\mu = A_\mu^a \tau_a$, with τ_a the generators of the algebra of the gauge group and A_μ^a are the component fields of the gauge field for each charge. The gauge transformation of a fermion field is thus

$$\begin{aligned} \psi &\to g\psi \\ g &= \exp(i\tau^a\omega_a), \end{aligned}$$

with ω_a arbitrary functions and a takes the same values as for the gauge fields. The corresponding covariant derivative is thus

$$D_\mu = \partial_\mu + ieA_\mu^a \tau_a,$$

with the τ_a in the fundamental representation of the gauge group. The corresponding gauge transformation for the gauge fields has then to take the inhomogeneous form

$$A_\mu \to gA_\mu g^{-1} + g\partial_\mu g^{-1}.$$

The expression for $F_{\mu\nu}$ is then also no longer gauge invariant, and has to be generalized to

$$F_{\mu\nu} = F_{\mu\nu}^a \tau_a = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu].$$

This quantity is still not gauge invariant, and thus neither are magnetic nor electric fields. However, the expression

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

is. Hence, the Lagrangian for a non-Abelian version of QED, Yang-Mills theory without fermions and QCD with fermions, reads

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{7.3}$$

which at first looks simple, but just because the explicit form has been defined into appropriate quantities, and $F_{\mu\nu}$ and D_μ are now matrix-valued in the group space of the gauge group.

[2]In principle, a theory can be build for any Lie algebra, but only the su(N) algebra have been so far relevant to describe experiments, and will therefore be used here exclusively.

7.2.2 Supersymmetric version

A much more interesting theory will be the supersymmetric version of the non-Abelian gauge theory, again neglecting the matter part. The first thing to do is to count again degrees of freedom. The gauge field is in the adjoint representation. For SU(N) as a gauge group there are therefore N^2-1 independent gauge fields. On-shell, this has to be canceled by exactly the same number of fermionic degrees of freedom, so the same number of fermions, called gauginos. Quarks or electrons, like in the Lagrangian of the non-supersymmetric Yang-Mills theory (7.3), are however in the fundamental representation of the gauge group, and thus there are a different number, e. g. N for SU(N). This cannot match, and the super-partners of the gauge fields, the gauginos, have therefore to be also in the adjoint representation of the gauge group. Therefore, they are completely different from ordinary matter fields like quarks or leptons. Hence, for each field in the standard model below it will be necessary to introduce an independent superpartner.

Of course, to close the SUSY algebra also off-shell, it will again be necessary to introduce additional scalars. However, also these have to be in the adjoint representation of the gauge group. In this case, it is useful to write the Lagrangian explicitly in the index form. This yields the Lagrangian

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + i\lambda_a^\dagger\bar{\sigma}^\mu D_\mu^{ab}\lambda_b + \frac{1}{2}D^aD_a \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc}A^b_\mu A^c_\nu \\
D^{ab}_\mu &= \delta^{ab}\partial_\mu + ef^{abc}A^c_\mu, \qquad (7.4)
\end{aligned}$$

where f^{abc} are the structure constants of the gauge group and D^{ab}_μ is the covariant derivative in the adjoint representation. The gauge bosons transform in the usual way, but the gaugino and the D-boson transform under gauge transformations in the adjoint representation. I. e., they transform like the gauge field, except without the inhomogeneous term as $g\lambda g^{-1}$, when written as algebra elements. Thus, even the D^2 term is gauge invariant, as no derivatives are involved.

Making the ansatz

$$\begin{aligned}
\delta_\xi A^a_\mu &= \xi^\dagger\bar{\sigma}_\mu\lambda^a + \lambda^{a\dagger}\bar{\sigma}_\mu\xi \\
\delta_\xi\lambda^a &= \frac{i}{2}\sigma^\mu\bar{\sigma}^\nu\xi F^a_{\mu\nu} + \xi D^a \\
\delta_\xi\lambda^{a\dagger} &= -\frac{i}{2}\xi^\dagger\bar{\sigma}^\nu\sigma^\mu F^a_{\mu\nu} + \xi^\dagger D^a \\
\delta_\xi D^a &= -i(\xi^\dagger\bar{\sigma}^\mu D^{ab}_\mu\lambda^b - D^{ab}_\mu\lambda^\dagger\bar{\sigma}^\mu\xi)
\end{aligned}$$

as the most simple generalization of the Abelian case is actually working. The reason for this is simple: After expressing everything in components, all quantities (anti-)commute.

Furthermore, under partial integration

$$\begin{aligned} D_\mu^{ab}\lambda^b F^{\mu\nu a} &= (\partial_\mu\lambda^a + ef^{abc}A_\mu^c\lambda^b)F^{\mu\nu a} = -\lambda^a\partial_\mu F^{\mu\nu a} - ef^{bac}A_\mu^c\lambda^b F^{\mu\nu a} \\ &= -\lambda^a\partial_\mu F^{\mu\nu a} - ef^{abc}A_\mu^c\lambda^a F^{\mu\nu b} = -\lambda^a D_\mu^{ab}F^{\mu\nu b}. \end{aligned} \tag{7.5}$$

Thus, all manipulations performed in section 7.1 can be performed equally well in case of the non-Abelian theory. Therefore, only those terms which appear in addition to the Abelian case have to be checked.

The most simple is the modification of the D-term by the appearance of the covariant derivative. These terms trivially cancel with the corresponding gaugino term just as in the Abelian case, as there also a covariant derivative appears

$$\begin{aligned} \delta_\xi^D(i\lambda_a^\dagger\bar\sigma^\mu D_\mu^{ab}\lambda_b) &= iD_a\xi^\dagger\bar\sigma^\mu D_\mu^{ab}\lambda_b - iD_\mu^{ab}\lambda_a^\dagger\bar\sigma^\mu\xi D_b \\ \delta_\xi\frac{1}{2}D^2 = D_a\delta_\xi D^a &= -iD_a(\xi^\dagger\bar\sigma^\mu D_\mu^{ab}\lambda_b - D_\mu^{ab}\lambda_b^\dagger\bar\sigma^\mu\xi). \end{aligned}$$

The next contribution stems from the appearance of the covariant derivative in the gaugino term. The contribution form the gauge boson-gaugino coupling cancels with the contribution from the self-coupling of the gauge bosons. This contribution alone gives

$$\begin{aligned} &-\frac{1}{2}F_{\mu\nu}^a\delta_\xi^{AA}F_a^{\mu\nu} + \delta_\xi^\lambda(i\lambda_a^\dagger\bar\sigma_\mu ef^{abc}A_c^\mu\lambda_b) \\ &= \frac{ef^{abc}}{2}F_{\mu\nu}^a((\xi^\dagger\bar\sigma^\mu\lambda_b + \lambda_b^\dagger\bar\sigma^\mu\xi)A_\nu^c + A_\mu^b(\xi^\dagger\bar\sigma^\nu\lambda_c + \lambda_c^\dagger\bar\sigma^\nu\xi)) \\ &+\frac{ef^{abc}}{2}F_a^{\rho\sigma}(\xi^\dagger\bar\sigma_\rho\sigma_\sigma\bar\sigma_\mu A_c^\mu\lambda_b - \lambda_b^\dagger\bar\sigma_\mu\sigma_\rho\bar\sigma_\sigma\xi A_c^\mu) \end{aligned} \tag{7.6}$$

The last expression can be reformulated using (7.2). Since the following proceeds identically for the contributions proportional to λ and $\lambda^\dagger$, only the former will be investigated explicitly. Applying therefore (7.2) to the third contribution yields

$$\frac{ef^{abc}}{2}F_a^{\rho\sigma}\xi^\dagger(g_{\rho\sigma}\bar\sigma_\mu - g_{\rho\mu}\bar\sigma_\sigma + g_{\sigma\mu}\bar\sigma_\rho - i\epsilon_{\rho\sigma\mu\delta}\bar\sigma_\delta)\lambda_b A_c^\mu.$$

The first term yields zero, as the trace of F vanishes. The contribution with the ϵ-tensor vanishes, as $f^{abc}F_a^{\rho\sigma}A_c^\mu$ is symmetric in the three Lorentz indices, and therefore also vanishes upon contraction with the ϵ-tensor, as an explicit calculation shows. The remaining two terms then exactly cancel the two terms from the transformation of $F_{\mu\nu}^a$, just by relabeling the Lorentz indices, and shifting them appropriately up and down. Therefore, also this contribution is not violating supersymmetry.

Then, only the term from the transformation of the gauge boson in the covariant derivative coupling to the gauginos remains. Its transformation is

$$\delta_\xi^A(i\lambda_a^\dagger\bar\sigma_\mu ef^{abc}A_c^\mu\lambda_c) = ief^{abc}\lambda_a^\dagger\bar\sigma^\mu(\xi^\dagger\bar\sigma_\mu\lambda_c + \lambda_c^\dagger\bar\sigma_\mu\xi)\lambda_b.$$

This contribution contains λ cubed, and can therefore not be canceled by any other contribution. However, rewriting the first term in explicit index notation yields

$$ief^{abc}\lambda^*_{ia}\bar{\sigma}^\mu_{ij}\lambda_{bj}\lambda^*_{ck}\bar{\sigma}_{\mu kl}\xi_l.$$

The expression $\bar{\sigma}^\mu_{ij}\bar{\sigma}_{\mu kl}$ is symmetric in the first and third index for each term individually, and the expression $\lambda^*_{ia}\lambda^*_{ck}$ is antisymmetric in exact these two indices. Therefore, this contribution drops out, and similarly for the second contribution.

Thus, all in all this theory is supersymmetric and therefore the supersymmetric generalization of Yang-Mills theory, called often super-Yang-Mills theory, or SYM, for short.

7.3 Supersymmetric QED

The Abelian gauge theory contained only an uncharged fermion, the photino. To obtain a supersymmetric version of QED a U(1)-charged fermion is necessary. Since this cannot be introduced into the vector supermultiplet of the photon, the most direct way to introduce it is by the addition of a chiral supermultiplet which will be coupled covariantly to the vector supermultiplet. Of course, this introduces only a Majorana electron, but this will be sufficient for the beginning. Of course, compensating scalar fields to make the theory supersymmetric will be required. Hence, at least a combination of supersymmetric Maxwell theory and Wess-Zumino theory is required.

The minimally coupled version is

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi + i\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2. \tag{7.7}$$

This theory contains now the photon, its super-partner the photino, the (Majorana) electron, its super-partner the selectron, and the auxiliary bosons F and D. In essence, this Lagrangian is the sum of the non-interacting Wess-Zumino model and the supersymmetric Abelian gauge theory, where in the former the derivatives have been replaced by gauge covariant derivatives. This implies that all newly added fields, ϕ, χ, and F, are charged, and transform under gauge transformations. Only the photino and the D-boson remain uncharged.

Of course, also the derivatives appearing in the supersymmetry transformations of electron, the selectron, and the F-boson have to be replaced by their covariant counterparts. Thus the minimal set of rules for the matter sector reads

$$\begin{aligned}
\delta_\xi\phi &= \xi\chi \\
\delta_\xi\chi &= \sigma^\mu\sigma_2\xi^* D_\mu\phi + \xi F \\
\delta_\xi F &= -i\xi^\dagger\bar{\sigma}^\mu D_\mu\chi
\end{aligned}$$

while those for the gauge sector are left unchanged

$$\begin{aligned}\delta_\theta A_\mu &= \theta^\dagger\bar{\sigma}_\mu\lambda + \lambda^\dagger\bar{\sigma}_\mu\theta\\ \delta_\theta\lambda &= \frac{i}{2}\sigma^\mu\bar{\sigma}^\nu\theta F_{\mu\nu} + \theta D\\ \delta_\theta D &= -i(\theta^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - \partial_\mu\lambda^\dagger\bar{\sigma}^\mu\theta)\end{aligned}$$

However, even with (7.5), it can then be shown that this theory is not yet supersymmetric under these transformation. The reason is that supersymmetry requires additional interactions as will be discussed below. The Lagrangian (7.7) is indeed, in contrast to ordinary QED, not the most general[3] one which can be written with this field content and which is supersymmetric, gauge-invariant, and renormalizable. It is possible to add additional interactions which do have all of these properties. However, this will require not only a fixed constant of proportionality between ξ and θ, but also a minor change in the transformation of the F field.

In fact, it is possible to add further interactions between the matter and the gauge sector to (7.7). These have to be, of course, gauge-invariant, and will also be chosen to be renormalizable. These two requirements already limit the number of possible terms severely to the interaction terms $\phi^\dagger\chi\lambda$, $\lambda^\dagger\chi^\dagger\phi$, and $\phi^\dagger\phi D$. All other terms are either not gauge-invariant, not Lorentz-invariant or not at the same time (superficially) renormalizable, like terms involving F. Hence the interaction Lagrangian takes the form

$$A(\phi^\dagger\chi\lambda + \lambda^\dagger\chi^\dagger\phi) + B\phi^\dagger\phi D.$$

Checking all terms for their invariance under supersymmetry transformations is a long and tedious exercise, which will not be performed here. Only those elements will be presented, which influence either the values of A and B, or modify the supersymmetry transformations themselves.

The first step is to check all contributions from the transformed part of the interaction Lagrangian which are linear in the D-field. These will occur either from transformations of the photino λ or from the term proportional to B when either of the other fields are transformed. This yields

$$A(\phi^\dagger\chi\theta D + \theta^\dagger\chi^\dagger D\phi) + B(\chi^\dagger\xi^\dagger\phi D + \phi^\dagger\xi\chi D).$$

No other contribution in the Lagrangian will produce such terms which couple the matter fields to the D-boson. Thus, these have to cancel by themselves. This is only possible if

$$A\theta = -B\xi, \tag{7.8}$$

[3] Here, and in the following, Wess-Zumino-like couplings, which are in fact gauge-invariant and supersymmetric, will be ignored.

yielding already a constraint for the transformation parameters. Thus, in contrast to the case when two gauge sectors are coupled, coupling two supersymmetric sectors cannot be done independently. The reason is again that both supersymmetry transformations are tied to the momentum transformation, thus not permitting to leave them independent.

The transformation of the ϕ-fields in the A-term will yield terms having only fermionic degrees of freedom,

$$A((\chi^\dagger\xi^\dagger)(\chi\lambda)+(\lambda^\dagger\chi^\dagger)(\xi\chi)).$$

The only other term which can generate such a combination of the electron and the photino field stems from the photon-electron coupling term, which reads

$$-q\chi^\dagger\bar{\sigma}^\mu\chi(\delta_\theta A_\mu)=-q\chi^\dagger\bar{\sigma}^\mu\chi(\theta^\dagger\bar{\sigma}_\mu\lambda+\lambda^\dagger\bar{\sigma}_\mu\theta).$$

This has the same field content as the previous contribution, when the relation between ξ and θ is used, but not the same Lorentz structure. To recast the expression, the identity

$$\begin{aligned}&(\chi^\dagger\bar{\sigma}^\mu\chi)(\lambda^\dagger\bar{\sigma}_\mu\theta)=\chi_a^*\bar{\sigma}^\mu_{ab}\chi_b\lambda_c^*\bar{\sigma}_{\mu cd}\theta_d\\ =\;&\chi_a^*\chi_b\lambda_c^*\theta_d\sigma^\mu_{ab}\bar{\sigma}_{\mu cd}=-2\chi_a^*\chi_b\lambda_c^*\theta_d\delta_{ac}\delta_{bd}=2(\chi^\dagger\lambda^\dagger)(\chi\theta),\end{aligned}$$

where the involved identity for the σ-matrices follows from direct evaluation, can be used. Evaluating the previous expression then yields

$$-2q\theta((\chi^\dagger\theta^\dagger)(\chi\lambda)+(\chi^\dagger\lambda^\dagger)(\chi\theta)).$$

This implies the relation

$$A\xi=2q\theta. \tag{7.9}$$

Together with the condition (7.8) this implies that A and B, and θ and ξ have to be proportional to each other, the constant of proportionality involving the charge. However, a relative factor is still permitted, and is required to be fixed. As the covariant derivative already provided one constraint, it is not surprising that the selectron-photon coupling term provides another one. Taking the supersymmetry transformation of the interaction term between two selectrons and one photon yields, when taking only the transformation of the photon field,

$$\begin{aligned}&-iq(\phi^\dagger(\partial_\mu\phi)\delta_\theta A^\mu-(\partial_\mu\phi)^\dagger\phi\delta_\theta A^\mu)\\ =\;&iq((\partial_\mu\phi^\dagger)\phi(\theta^\dagger\bar{\sigma}^\mu\lambda+\lambda^\dagger\bar{\sigma}^\mu\theta)-\phi^\dagger(\partial_\mu\phi)(\theta^\dagger\bar{\sigma}^\mu\lambda+\lambda^\dagger\bar{\sigma}^\mu\theta)).\end{aligned} \tag{7.10}$$

Terms with such a contribution can also be generated by both interaction terms, if in the A case the electron and in the B case the D-boson is transformed. Specifically,

$$Ai(\phi^\dagger(\sigma^\mu\sigma_2\xi^*\partial_\mu\phi)\lambda+\lambda^\dagger(\partial_\mu\phi^\dagger)\xi^T\sigma_2\sigma^\mu\phi)-iB(\phi^\dagger\phi(\theta^\dagger\bar{\sigma}^\mu\partial_\mu\lambda-(\partial_\mu\lambda^\dagger)\bar{\sigma}^\mu\theta)).$$

To simplify this expression, the relation (4.35) can be used in both A-terms, yielding

$$Ai(\phi^\dagger(\partial_\mu\phi)\xi^\dagger\bar{\sigma}^\mu\lambda-(\partial_\mu\phi^\dagger)\phi\lambda^\dagger\bar{\sigma}^\mu\xi)-iB(\phi^\dagger\phi(\theta^\dagger\bar{\sigma}^\mu\partial_\mu\lambda-(\partial_\mu\lambda^\dagger)\bar{\sigma}^\mu\theta)).$$

Integrating further in the B-term by parts yields

$$\begin{aligned}&Ai(\phi^\dagger(\partial_\mu\phi)\xi^\dagger\bar{\sigma}^\mu\lambda-(\partial_\mu\phi^\dagger)\phi\lambda^\dagger\bar{\sigma}^\mu\xi)-iB(((\partial_\mu\phi^\dagger)\phi+\phi^\dagger(\partial_\mu\phi))\theta^\dagger\bar{\sigma}^\mu\lambda\\&-((\partial_\mu\phi^\dagger)\phi+\phi^\dagger(\partial_\mu\phi))\lambda^\dagger\bar{\sigma}^\mu\theta).\end{aligned}\tag{7.11}$$

Now, in both contributions, (7.10) and (7.11), terms appear proportional to λ and to $\lambda^\dagger$ of the same structure. So both will be vanishing independently, if the pre-factors combine in the same way. The prefactor of λ is

$$iq((\partial_\mu\phi^\dagger)\phi-\phi^\dagger\partial_\mu\phi)\theta_a+Ai\phi^\dagger(\partial_\mu\phi)\xi_a+iB((\partial_\mu\phi^\dagger)\phi+\phi^\dagger\partial_\mu\phi)\theta_a.$$

This will vanish, if the conditions

$$\begin{aligned}q\theta_a+B\theta_a&=0\\-q\theta_a+A\xi_a+B\theta_a&=0\end{aligned}$$

are met. Together with the condition (7.8) and (7.9), this yields the result

$$\begin{aligned}A&=-\sqrt{2}q\\B&=-q\\\theta&=-\frac{1}{\sqrt{2}}\xi.\end{aligned}$$

Actually, this result is not unique, and it would be possible to replace A by $-A$ and θ by $-\theta$, without problems. So this can be freely chosen, and the conventional choice is the one adopted here.

With these choices, all variations performed will be either total derivatives or will cancel each other. However, one contribution is not working out, which is the one involving the F-contribution from the variation of the electron in the A-term. It yields

$$-\sqrt{2}q(\phi^\dagger\xi\lambda F+\lambda^\dagger\xi^\dagger F^\dagger\phi)\tag{7.12}$$

Since there is no other term available which contains both the selectron and the photino, it is not possible to cancel this contribution. The only possibility is to modify the transformation rule for the F-boson as

$$\delta_\xi F=-i\xi^\dagger\bar{\sigma}^\mu D_\mu\chi+\sqrt{2}q\lambda^\dagger\xi^\dagger\phi.$$

The only consequence of this modification is that the $FF^\dagger$ term transforms, restricted to the photino contributions, as

$$\delta_\xi^\lambda(FF^\dagger)=\sqrt{2}q(\phi^\dagger\xi\lambda F+\lambda^\dagger\xi^\dagger F^\dagger\phi),$$

canceling exactly the offending contribution.

Thus, finally the Lagrangian for supersymmetric QED reads

$$\begin{aligned}\mathcal{L} &= (D_\mu\phi)^\dagger D^\mu\phi+i\chi^\dagger\bar\sigma^\mu D_\mu\chi+F^\dagger F-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda+\frac{1}{2}D^2\\ &\quad -\sqrt{2}q((\phi^\dagger\chi)\lambda+\lambda^\dagger(\chi^\dagger\phi))-q\phi^\dagger\phi D.\end{aligned}$$

A number of remarks are in order. First, though two essentially independent sectors have been coupled, this lead not to a product structure with two independent supersymmetry transformations, but only to one common transformation. The deeper reason for this is the appearance of the momentum operator in both supersymmetry algebras, inevitably coupling both. The second is that the combination of gauge symmetry and supersymmetry required the introduction of interaction terms between both sectors to yield a supersymmetric theory. This interaction is so strongly constrained that its structure is essentially unique, and besides no new coupling constants appear compared to the two original theories. Again, supersymmetry is tightly constraining. Third, the equation of motion for the D-boson yields $D=q\phi^\dagger\phi$. As a consequence, integrating out the D-boson lets a quadratic interaction term $-q^2/2(\phi^\dagger\phi)^2$ appear. This is a necessary ingredient for the possibility of a Brout-Englert-Higgs effect, driven by a selectron condensation. Thus, even the simplest non-trivial supersymmetric gauge-theory provides much more possibilities than ordinary QED.

It should be noted that the supersymmetry transformation derived here is not unique. It is possible to write down a set of transformations which only involve ordinary instead of covariant derivatives, and again a slightly modified transformation for the F boson. Both formulations yield identically the same physical results, and especially the Lagrangian is the same. However, for the purpose of generalizing to theories like supergravity, the present formalism, called the de Wit-Freedman formalism, is more useful. The difference between both sets of transformations is essentially only how gauge conditions transform under supersymmetry transformations. In the formalism using only ordinary derivatives, the gauge conditions are not transformed covariantly, and therefore any supersymmetry transformation must be accompanied by a gauge transformation to also maintain the gauge condition intact. Since gauge transformation do not change physics, it is thus rather a matter of convenience from a physical perspective.

7.4 Supersymmetric QCD

Again, having the standard model in mind, it is necessary to generalize supersymmetric QED to a non-Abelian version, the simplest of which is supersymmetric QCD. However, in the following the gauge group will not be made explicit, and thus the results are valid for any (semi-)simple Lie group as gauge group.

Since in QCD, and in the standard model in general, the fermions are in the fundamental representation, while the gauge fields are in the adjoint representation, it is not possible to promote somehow the matter fields to the super partners of the gauge bosons, despite these being charged in the supersymmetric version of Yang-Mills theory. It is therefore again necessary to introduce the matter fields as independent fields, together with their superpartners, and couple them minimally to the gauge fields. Therefore, besides the gauge-fields, the gluons, their superpartner, the gluinos, the D-bosons, there will be the quarks, their superpartners, the squarks, and the F-bosons.

Fortunately, the results of supersymmetric QED, together with those for supersymmetric Yang-Mills theory, can be generalized. It is thus possible to write down the transformation rules and the Lagrangian immediately. The only item which requires some more investigation are couplings between the fundamental and the adjoint sector. This applies in particular to the appearing quark-gluino-squark couplings. In general, uncontracted indices would imply a gauge-variant term, which may not appear in the Lagrangian. To obtain appropriate contractions, it is, e. g., necessary to write instead of $\phi_i^\dagger\phi_i D^\alpha$ the terms

$$\phi^\dagger D\phi = \phi^\dagger\tau^\alpha D^\alpha\phi = \phi_i^\dagger(\tau^\alpha D^\alpha)_{ij}\phi_j,$$

where the index i takes values in the fundamental representation, while the index α takes values in the adjoint representation. Such combinations are gauge-invariant, when traced. Of course, this has also to be applied to the coupling term appearing in the transformation rule for the F-boson. Taking thus the non-Abelian versions of the transformation rules as

$$\begin{aligned}
\delta_\xi\phi^i &= \xi\chi^i \\
\delta_\xi\chi^i &= \sigma^\mu\sigma_2\xi^* D_\mu^{ij}\phi^j + \xi F^i \\
\delta_\xi F^i &= -i\xi^\dagger\bar\sigma^\mu D_\mu^{ij}\chi^j - \sqrt{2}q\phi^i\tau^\alpha\lambda^{\alpha\dagger}\xi^\dagger \\
\delta_\xi A_\mu^\alpha &= -\frac{1}{\sqrt{2}}\left(\xi^\dagger\bar\sigma_\mu\lambda^\alpha + \lambda^{\alpha\dagger}\bar\sigma_\mu\xi\right) \\
\delta_\xi\lambda^\alpha &= -\frac{i}{2\sqrt{2}}\left(\sigma^\mu\bar\sigma^\nu\xi F_{\mu\nu}^\alpha + 2\xi D^\alpha\right) \\
\delta_\xi D^\alpha &= \frac{i}{\sqrt{2}}\left(\xi^\dagger\bar\sigma^\mu D_\mu^{\alpha\beta}\lambda^\beta - D_\mu^{\alpha\beta}\lambda^{\beta\dagger}\bar\sigma^\mu\xi\right)
\end{aligned}$$

it is possible to show that the non-Abelian generalization can be constructed just by covariantizing all derivatives, and replacing coupling terms by gauge-invariant ones. This yields

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha}+i\lambda^{\alpha\dagger}\bar{\sigma}^{\mu}D^{\alpha\beta}_{\mu}\lambda^{\beta}+\frac{1}{2}D^{\alpha}D^{\alpha}\\
&+D^{ij}_{\mu}\phi_j D^{\mu}_{ik}\phi^{\dagger}_k+\chi^{\dagger}_i i\bar{\sigma}^{\mu}D^{ij}_{\mu}\chi_j+F^{\dagger}_iF_i\\
&+M\phi_iF_i-\frac{1}{2}M\chi_i\chi_i+\frac{1}{2}y_{ijk}\phi_i\phi_jF_k-\frac{1}{2}y_{ijk}\phi_i\chi_j\chi_k+h.c.\\
&-\sqrt{2}e(\phi^{\dagger}\chi\lambda+\lambda^{\dagger}\chi^{\dagger}\phi)-g\phi^{\dagger}\phi D,
\end{aligned}
\tag{7.13}
$$

where it should be noted that inter-representation couplings equal, e. g.,

$$
\begin{aligned}
\phi^{\dagger}\chi\lambda &= \phi^{\dagger}\tau^{\alpha}\chi\lambda^{\alpha}=\phi^{\dagger}_i\tau^{\alpha}_{ij}\chi^a_j\lambda^{\alpha}_a\\
\phi^{\dagger}\phi D &= \phi^{\dagger}\tau^{\alpha}\phi D^{\alpha}=\phi^{\dagger}_i\tau^{\alpha}_{ij}\phi_jD^{\alpha}.
\end{aligned}
$$

Note that the coupling constants y acquired gauge indices in the fundamental representation. Similar to the Higgs-fermion coupling in the standard model, relations between the different elements of y ensure gauge invariance of these couplings, and these could be determined by explicit evaluation. Note further that the interaction between the matter sector in the fundamental representation and the gauge-sector in the adjoint representation is completely fixed by gauge symmetry and supersymmetry, and there is no room left for any other interaction. In particular, despite the fact that six fields interact with altogether 11 interaction vertices, there are only three independent parameters, the mass parameter M, the Yukawa coupling y, which is constrained by gauge symmetry, and the gauge-coupling e. E. g., the mixed term appearing from the supersymmetry transformation of the F coupling in the Yukawa term vanishes due to the antisymmetry of the y-matrix, which is necessary to ensure gauge invariance.

It is worthwhile to evaluate the terms including a D explicitly after using its equation of motion, which reads

$$
\frac{\delta\mathcal{L}}{\delta D^{\dagger}_{\alpha}}=D^{\alpha}-e\phi^{\dagger}\tau^{\alpha}\phi=0,
$$

and similarly for F. After integrating out both the D and F field, this yields the total self-interaction (or potential V) of the ϕ field,

$$
V=|M|^2\phi^{\dagger}\phi+\frac{1}{2}e^2(\phi^{\dagger}\tau^{\alpha}\phi)^2-y_{ijk}y^{*}_{lmk}\phi_i\phi_j\phi^{\dagger}_l\phi^{\dagger}_m.
$$

This potential is positive definite, and all of the couplings are uniquely defined. Thus, in contrast to the case of the standard model, the Higgs-potential, as this is the role the squark plays, is completely determined due to supersymmetry. This puts, at least perturbatively, strong constraints on the Higgs mass in the supersymmetric version of the standard model. This will be discussed in more detail below

One remark should be added now. It is in principle possible to add to the Lagrangian (7.13) a further term proportional to $\theta\epsilon_{\mu\nu\rho\sigma}F_a^{\rho\sigma}F_a^{\mu\nu}$, with θ a new coupling constant, a so-called topological term. Due to the antisymmetric tensor, any contribution of such a term drops out in perturbation theory, and it can only contribute beyond perturbation theory. It indeed does so, and plays an important role in topics like chiral symmetry breaking and anomalies. This is already true in the non-supersymmetric version. However, in nature it is experimentally known that for any such term in the standard model the parameter θ is very small, and only an upper bound of about 10^{-10} is known.

However, from the point of view of supersymmetry, this term is conceptually interesting. After rescaling the gauge fields with the coupling constant g, it is possible to combine this term with the term $F_{\mu\nu}^a F_a^{\mu\nu}$ in such a way as that the whole theory now depends entirely on the complex combination $G = g + i\theta$, the holomorphic coupling. Unbroken supersymmetry then ensures that the partition function, and thus any quantity, is holomorphic in G, which permits many highly non-trivial statements, also in the non-perturbative domain. However, the details of this are beyond the scope of this lecture.

7.5 Gauge theories with $\mathcal{N} > 1$

Gauge theories with more than one supercharge are only of very limited phenomenological use in the context of particle physics, as for intact supersymmetry parity cannot be broken; left-handed and right-handed fermions are treated on equal footing. Since the weak interactions do break parity, this is at odds with experiment. However, they are relevant for several reasons. One is that in some extensions of the standard model it is possible to start without parity breaking, and such a theory could be supersymmetrized. However, these are rather involved constructions, which do not appear very promising. Second, theories with larger supersymmetries are more constrained, which helps in obtaining results. They therefore can serve as better accessible, simplified models of ordinary theories. Third, gauge theories with extended supersymmetries play an important role in the context of string theory.

The simplest extension is the $\mathcal{N} = 2$ supersymmetric version of Yang-Mills theory. This requires the combination of a $\mathcal{N} = 1$ gauge supermultiplet with a chiral multiplet. However, because now the chiral and the gauge multiplet is related by the extended supersymmetry, also the members of the chiral multiplet must be in the adjoint representation, in contrast to the case of super QCD. Hence, there are the gauge field, the gauginos and the corresponding D field, as well as a complex adjoint scalar ϕ, a Majorana fermion ψ, and the corresponding complex F fields.

The Lagrangian of this theory for a simple Lie-algebra is[4]

$$\begin{aligned}\mathcal{L} &= -(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}\bar{\psi}\gamma_\mu D^\mu\psi + F^\dagger F - 2\sqrt{2}gf^{abc}\Re(\lambda_a^T\psi_c\phi_b^\dagger) + igf^{abc}\phi_b^\dagger\phi_c D_a \\ &+\frac{1}{2}D^2 - \frac{1}{4}F_\mu^a F_a^{\mu\nu} - \frac{1}{2}\bar{\lambda}\gamma_\mu D^\mu\lambda + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_{\rho\sigma}^a.\end{aligned}$$

The theory has no free parameters, besides the gauge coupling g and the θ parameter. The two supersymmetry transformations differ by the way on which supermultiplets they act. One acts on the conventional two sets, but the other acts on the mixed sets (ϕ, λ, F) and $(A, -\psi, D)$, though with the same supersymmetry transformation. Furthermore, this theory has a $SU(2)$ R-symmetry which transforms between the two sets.

This theory supports a Higgs effect, and indeed in this case the masses of the particles turn out to be unaffected by radiative corrections. I. e., the tree-level masses are already exact, and saturate the BPS bound. This will not be detailed further here, but should give an idea of how strongly the dynamics are constrained by supersymmetry.

The only further non-trivial extension possible without adding gravity is the $\mathcal{N} = 4$ case. This is the combination of two $\mathcal{N} = 2$ theories, which therefore has an SU(4) R-symmetry, permitting the fields to give different supersymmetry transformations. This theory is somewhat involved. It contains besides the gauge supermultiplet a left-chiral supermultiplet with complex fields ψ and ϕ, and two more left-chiral, denoted by primes $'$, multiplets and their complex conjugates. The lengthy Lagrangian, after integrating out the auxiliary fields for brevity, reads

$$\begin{aligned}\mathcal{L} &= -(D_\mu\phi)^\dagger D^\mu\phi - (D_\mu\phi')^\dagger D^\mu\phi' - (D_\mu\phi'')^\dagger D^\mu\phi'' - \frac{1}{2}\bar{\psi}\gamma_\mu D^\mu\psi - \frac{1}{2}\bar{\psi}'\gamma_\mu D^\mu\psi' \\ &-\frac{1}{2}\bar{\psi}''\gamma_\mu D^\mu\psi'' - \frac{1}{2}\bar{\lambda}\gamma_\mu D^\mu\lambda - 2\sqrt{2}gf^{abc}\Re(\phi_a\psi_b'\psi_c'') - 2\sqrt{2}gf^{abc}\Re(\lambda_a^T\psi_c\phi_b^\dagger) \\ &-2\sqrt{2}gf^{abc}\Re(\phi_b'\psi_c^{''T}\psi_a) - 2\sqrt{2}gf^{abc}\Re(\phi_c''\psi_b^{'T}\psi_a) + 2\sqrt{2}gf^{abc}\Re(\psi_b^{'T}\lambda_a\phi_c^{'\dagger}) \\ &+2\sqrt{2}gf^{abc}\Re(\psi_b^{''T}\lambda_a\phi_c^{''\dagger}) - \frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_a^{\rho\sigma} \\ &+g^2 f^{ade}f^{bce}(\phi_a\phi_b^\dagger + \phi_b\phi_a^\dagger)(\phi_c'\phi_d' + \phi_c^{''\dagger}\phi_d^{''\dagger}) + \frac{g^2}{2}\left|f^{abc}(\phi_b^{'\dagger}\phi_c' - \phi_b''\phi_c^{''\dagger})\right|^2 \\ &-\frac{g^2}{2}f^{abc}f^{ade}\phi_b^\dagger\phi_c\phi_d^\dagger\phi_e + 2g^2\left|f^{abc}\phi_b'\phi_c''\right|^2\end{aligned}$$

Though it does not look like it, the potential in the scalars can be symmetrized, albeit of the expense of becoming even more lengthy. However, it is possible to rewrite the

[4] A term $-D$ has to be added, if the gauge group is U(1), a case which will not be considered here.

Lagrangian in a much shorter form by exploiting the SU(4) R symmetry. Collecting for each color a the right-handed fermions in an SU(4) vector $\Psi=(\psi_R,\lambda_R,\psi'_R,\psi''_R)$ and the scalars into an antisymmetric SU(4) tensor

$$\Phi=\begin{pmatrix} 0 & \phi^* & \phi'' & -\phi' \\ -\phi^* & 0 & -\phi'^\dagger & -\phi''^\dagger \\ -\phi'' & -\phi'^\dagger & 0 & \phi \\ \phi' & \phi''^\dagger & -\phi & 0 \end{pmatrix} \tag{7.14}$$

separately, this yields

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(D_\mu\Phi^{ij})_a(D^\mu\Phi^{ij})_a^\dagger-\frac{1}{2}\Psi^T_{Lia}(\gamma_\mu D^\mu\Psi_{Ri})_a+\frac{1}{2}\Psi^T_{Rai}(\gamma_\mu D^\mu\Psi_{Li})_a\\ &\quad -\sqrt{2}gf^{abc}\Re(\Phi_{ija}\Psi^T_{Lib}\Psi_{Ljc})-\frac{g^2}{8}\left|f^{abc}\Phi^{ij}_b\Phi^{kl}_c\right|^2-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a+\frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_aF^{\rho\sigma}_a,\end{aligned}$$

which is more compact, but treats the gaugino not explicitly different from the matter particles. However, the SU(4) R-parity is manifest. In both cases, $\mathcal{N}=2$ and $\mathcal{N}=4$, the proof of the supersymmetry is rather lengthy, and will be skipped here.

For $\mathcal{N}=4$, the potential is a sum of squares, and thus the vacuum energy is always zero. Hence, supersymmetry remains unbroken in this theory. Furthermore, there exists evidence that there is an exact mapping of the $\mathcal{N}=4$ theory at a given coupling to another $\mathcal{N}=4$ theory with the same structure but with inverse coupling, thus linking a strongly interacting and a weakly interacting theory. This is a so-called duality, which are quite useful, if truly existing.

7.6 The β-function of super-Yang-Mills theory

A remarkable fact, stated here without proof, of super-Yang-Mills theories is that for vanishing θ the only appearing infinity is in the one-loop correction to the β-function. As a consequence, the one-loop form is exact, and given by

$$\begin{aligned}\beta(g) &= -\frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1-\frac{1}{6}C_2^f-\frac{1}{12}C_2^s\right)\\ C_1\delta_{cd} &= f^{abc}f^{abd}\\ C_2^f\delta_{cd} &= \sum_{\text{fermions}}\mathrm{tr}\tau^c\tau^d\\ C_2^s\delta_{cd} &= \sum_{\text{scalars}}\mathrm{tr}\tau^c\tau^d,\end{aligned}$$

i. e. determined by the representations of the various involved particles. Most notably, for the $\mathcal{N}=4$ case the requirements on the involved fields balance the C_i such that the

β-function vanishes. Hence, this theory is finite, i. e. no renormalization is necessary. Moreover, as the β function vanishes, the coupling does not depend on the energy scale. As a consequence, the theory is scale-free, and hence conformal. But this also means that it lacks any kind of observables, and has thus no dynamics. However, such a behavior makes the possibility of a duality also more plausible.

It should be noted that many supersymmetric theories beyond super-Yang-Mills theory exhibit similar features, i. e. perturbative β-functions which contain only a finite number of terms. In some rare cases, supersymmetry provides strong enough constraints to show that this is true even non-perturbatively. However, such theories, which include $\mathcal{N} = 4$ super-Yang-Mills theory, usually have such strict constrains that they exhibit little or no dynamics.

7.7 Supergravity

The second important gauge theory, besides Yang-Mills theory with or without matter, is gravity. Gravity can be considered as a gauge theory for translations. Therefore, local supersymmetry will therefore create gravity. Without going into too much details, especially as many questions on off-shell supergravity have not been solved, here only a short introduction is made.

7.7.1 General relativity

Before talking about gravity in a particle physics setup, it seems appropriate to quickly repeat the basics of classical general relativity.

The basic dynamical variable is the metric, which describes the the invariant length element ds by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

The inverse of the metric is given by the contravariant tensor

$$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda.$$

As a consequence, for any derivative δ

$$\delta g^{\mu\nu} = -g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\rho} \tag{7.15}$$

holds. The metric is assumed to be non-vanishing and has a signature such that its determinant is negative,

$$g = \det g_{\mu\nu} < 0.$$

The covariant volume element dV is therefore given by

$$\begin{aligned} dV &= hd^4x \\ h &= \sqrt{-g} = \sqrt{-\det g_{\mu\nu}} > 0, \end{aligned}$$

implying that h is real (hermitian), and has derivative

$$\delta h = \frac{1}{2} h g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} h g_{\mu\nu} \delta g^{\mu\nu} \tag{7.16}$$

as a consequence of (7.15).

The most important concept of general relativity is the covariance (or invariance) under a general coordinate transformation $x_\mu \to x'_\mu$ (diffeomorphism) having

$$\begin{aligned} dx'^\mu &= \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = J^\mu_\nu dx^\nu \\ \det(J) &\neq 0, \end{aligned}$$

where the condition on the Jacobian J follows directly from the requirement to have an invertible coordinate transformation everywhere. Scalars $\phi(x)$ are invariant under such coordinate transformations, i. e., $\phi(x) \to \phi(x')$. Covariant and contravariant tensors of n-th order transform as

$$\begin{aligned} T'_{\mu\ldots\nu}(x') &= \frac{\partial x_\mu}{\partial x'_\alpha} \ldots \frac{\partial x_\nu}{\partial x'_\beta} T_{\alpha\ldots\beta}(x) \\ T'^{\mu\ldots\nu}(x') &= \frac{\partial x'^\mu}{\partial x^\alpha} \ldots \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\ldots\beta}(x) \end{aligned}$$

respectively, and contravariant and covariant indices can be exchanged with a metric factor, as in special relativity. As a consequence, the ordinary derivative ∂_μ of a tensor A_ν of rank one or higher is not a tensor. To obtain a tensor from a differentiation the covariant derivative must be used

$$\begin{aligned} D_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \qquad (7.17) \\ \Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \end{aligned}$$

where Γ are the Christoffel symbols. Only the combination hA_ν, yielding a tensor density, obeys

$$D_\mu(hA_\nu) = \partial_\mu(hA_\nu).$$

As a consequence, covariant derivatives no longer commute, and their commutator is given by the Riemann tensor $R_{\lambda\rho\mu\nu}$ as

$$\begin{aligned} [D_\mu, D_\nu] A^\lambda &= R^\lambda_{\rho\mu\nu} A^\rho \\ R^\lambda_{\rho\mu\nu} &= \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\rho}, \end{aligned}$$

which also determines the Ricci tensor and the curvature scalar

$$\begin{aligned} R_{\mu\nu} &= R^{\lambda}_{\nu\mu\lambda} \\ R &= R^{\mu}_{\mu}, \end{aligned}$$

respectively.

These definitions are sufficient to write down the basic dynamical equation of general relativity, the Einstein equation

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = -\kappa T_{\mu\nu}$$

which can be derived as the Euler-Lagrange equation from the Einstein-Hilbert Lagrangian[5]

$$\mathcal{L} = \frac{1}{2\kappa} hR - \frac{1}{\kappa} h\Lambda + h\mathcal{L}_M,$$

where $\mathcal{L}_M$ is the matter Lagrangian yielding the covariantly conserved energy momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} = \left(-\eta_{\mu\nu}\mathcal{L} + 2\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}(g_{\mu\nu} = \eta_{\mu\nu})\right), \tag{7.18}$$

and again $\kappa = 16\pi G_N$ is Newton's constant, and Λ gives the cosmological constant (with arbitrary sign).

For the purpose of quantization it is useful to rewrite the first term of the Lagrangian, the Einstein-Hilbert contribution $\mathcal{L}_E$, as

$$\begin{aligned} \mathcal{L}_E &= \frac{1}{2\kappa} h g^{\mu\nu}(\Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda}) + \partial_\mu V^\mu \\ V^\lambda &= \frac{1}{2\kappa} h(g^{\mu\lambda} g^{\sigma\tau} - g^{\mu\sigma} g^{\lambda\tau})\partial_\mu g_{\sigma\tau}. \end{aligned}$$

The second term is a total derivative, and therefore quite often can be dropped.

There is an important remark to be made about classical general relativity. The possibility of making a general coordinate transformation leaving physics invariant has the consequence that both energy and three momentum loose their meaning as physically meaningful concepts, just like charge in a non-Abelian gauge theory. Indeed it is possible to alter the energy of a system by performing a space-time coordinate transformation. Only the concept of total energy (or momentum) of a localized distribution of particles when regarded from far away in an otherwise flat space-time can be given an (approximate) physically meaning, similarly to electric charge. Therefore, many concepts which are usually taken to be physical lose this meaning when general relativity is involved. This carries over to any quantum version.

[5] In the following usually the cosmological constant term will be absorbed in the matter part.

7.7.2 The Rarita-Schwinger field

The metric, as a symmetric tensor, describes a spin 2 object. Supersymmetrizing gravity therefore requires a spin 3/2 field, which is the so-called Rarita-Schwinger field. This field will be the associated gauge field for the local supersymmetry, just as the metric field is the gauge field for the local translation symmetry.

In analogy to conventional gauge theories, the Rarita-Schwinger field is required to transform under a local transformation as

$$\Psi_\mu \to \Psi_\mu + \partial_\mu \epsilon$$

where ϵ is a spinor-valued function. This follows in essentially the same as for vector fields from teh representation theory of the Lorentz group. Thus, a Rarita-Schwinger field Ψ carries both, a vector index and a spinor index. This was to be expected, as this couples effectively a spin 1 and a spin 1/2 object to create spin 3/2, just as for the metric two spin 1 indices are coupled to spin 2. As the supercharges are Weyl/Majorana spinors, so are the Rarita-Schwinger components.

Since the transformation is linear, it is an Abelian gauge theory, and the corresponding field strength tensor

$$\Omega_{\mu\nu} = \partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu$$

is therefore gauge invariant, but carries also a spinor index.

It is still necessary to postulate a Lagrangian for the theory, which is gauge-invariant. Introducing $\bar{\Psi} = \Psi^\dagger \gamma_0$, a possibility is

$$\begin{aligned} \mathcal{L} &= -\bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho. \\ \gamma^{\mu\nu\rho} &= \frac{1}{2}\{\gamma^\mu, \gamma^{\rho\sigma}\} \\ \gamma^{\mu\nu} &= \frac{1}{2}[\gamma^\mu, \gamma^\nu] \end{aligned}$$

As for the Maxwell case, there are no gauge-invariant, perturbatively renormalizable further interaction terms possible. Without interactions, only non-interacting Rarita-Schwinger fields are possible. The equation of motion is, similar to the Dirac equation,

$$\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho = 0.$$

It follows that the Rarita-Schwinger field can have (classically) physical modes only for $d > 3$, similar like the vector potential only for $d > 2$. This equation of motion also implies

$$\gamma^\mu \Psi_{\mu\nu} = 0,$$

which is Rarita-Schwinger form of the Maxwell equations. The equations of motions can be solved in a similar way as the free Dirac equation, and creates the free-field solutions.

It is possible to add a mass term, yielding

$$\mathcal{L} = -\bar{\Psi}_\mu(\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho})\Psi_\rho,$$

in contrast to the vector gauge fields.

7.7.3 Supergravity

The actual supergravity action is somewhat involved. Here, only the situation will be considered without additional matter fields, as they would have to be supersymmetrized as well. As shown above, the coupling of different matter multiplets leads to intertwining of those, which leads to a rather involved result. Also, the cosmological constant will be set to zero in the following.

The coupling between fermions and gravity actually is not a straightforward exercise in itself[6]. The approach taken here is based on exchanging the metric in favor of a different type of dynamical variables, the so-called vierbeins, defined as

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$$

where η is the space-time-constant Minkowski metric, and also the indices a and b run therefore from 0 to 3. This relation implies

$$e^\mu_a g_{\mu\nu} e^\nu_b = \eta_{ab},$$

i. e. the vierbein is the matrix field which yields a transformation of some given metric to the Minkowski metric. This field is therefore sometimes also called a frame field, as it locally transforms the metric to a Minkowski frame. Both indices of the vierbein can be raised and lowered using the Minkowski metric.

The Lagrangian of the simplest $\mathcal{N} = 1$ supergravity is then

$$\begin{aligned}
\mathcal{L} &= \frac{\det e}{2\kappa}\left(e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu\Psi_\rho\right)\\
R_{\mu\nu ab} &= \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b\\
D_\nu &= \partial_\nu + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\\
\omega_{\nu ab} &= 2e_{\mu[a}\partial_{[\nu}e^{\mu]}_{b]} - e_{\mu[a}e^\sigma_{b]}e_{\nu c}\partial^\mu e^c_\sigma
\end{aligned}$$

[6]In fact, there is more than one possibility, and they differ at the quantum level. Without experimental input, it is at the current time not possible to decide which one is correct. Here, the most prevalent construction is chosen.

where the brackets around the indices indicates that the expression has to be antisymmetrized with respect to the same-type indices. It is seen that the covariant derivative couples gravity and the Rarita-Schwinger field. This theory is therefore coupled. The involvement of the vierbein also makes the Einstein-Hilbert part more involved, and modifies the Riemann tensor, which now involves different types of indices.

The, now local, supersymmetry transformations of the fields are, without proof,

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu \\ \delta\Psi_\mu &= D_\mu\epsilon, \end{aligned}$$

with the spinor 1/2 field $\epsilon(x)$.

Chapter 8

Supersymmetry breaking

All of the theories investigated so far have manifest supersymmetry. As a consequence, as it was shown generally, the masses of the particles and the sparticles have to be degenerate. This is not what is observed in nature: There is no scalar particle with unit electric charge and the mass of the electron in nature. This is an experimentally very well established fact. Nor has any superpartner for any known particle been found so far. Indeed, if they exist, most of them need to be extremely heavy, significantly above 100 GeV in mass. Otherwise the equal coupling strength to the known forces would have made them observable in experiments long ago. Supersymmetry can therefore not be a symmetry of nature. It is therefore necessary to break supersymmetry in some way.

For the breaking of symmetries two prominent mechanisms are available in quantum field theories. A breaking can either be by explicit breaking or by spontaneous breaking. There is also the breaking by quantum anomalies in the quantization process. However, so far no really attractive, consistent, and experimentally relevant mechanisms to break supersymmetry by anomalies has been found. This option will therefore not be followed here.

Explicit breaking refers to the case when some term is added to the Lagrangian which spoils the symmetry of the theory present without this term. A tree-level mass term for quarks in QCD is such a case, where chiral symmetry is broken by this. If the term is superrenormalizable, like a mass-term, this is not affecting the high-energy properties of an asymptotically free theory, and the breaking is said to be soft. However, low-energy properties may be qualitatively different. If the offending term is small compared to all other scales of the theory, its effect is possibly weak, and the symmetry is said to be broken mildly only. Relations due to the original symmetry may therefore be still approximately valid. However, since interacting quantum field theories are non-linear by nature, there is no guarantee for this.

Spontaneous breaking appears when the Lagrangian is invariant under a symmetry transformation, but the ground-state is not[1]. E. g., the magnetization of a ferromagnet with no external magnetic field is an example of such a case. In field theory, QCD with massless fermions is another example. Also there, the chiral symmetry is spontaneously broken, yielding (approximately) the known masses of the hadrons.

Unfortunately, adding an explicit breaking is not always possible. An example is the so-called breaking of electroweak gauge symmetry in the standard model. In this case, any explicit breaking term will spoil renormalizability[2]. Also, explicit breaking terms have in general less attractive properties, varying from theory to theory. Therefore, spontaneous breaking is more desirable. However, any spontaneous mechanism for supersymmetry breaking known so far is not consistent with the requirements of experiments. Therefore, it is required to introduce explicit breaking of supersymmetry. Unfortunately, no simple possibility is known to obtain acceptable results, and therefore many of the attractive properties of supersymmetry are lost. In particular, almost a hundred additional coupling constants and parameters will be necessary even for the simplest supersymmetric extension of the standard model. This will be detailed in the next chapter. In this chapter, only the underlying mechanisms will be discussed.

8.1 Dynamical breaking

As has already been described in the quantum-mechanical case, it is necessary for a spontaneous breaking of supersymmetry that some quantity ω', which is not invariant under supersymmetry transformations, $\delta\omega' \neq 0$, must develop a vacuum expectation value[3],

$$\langle 0|\omega'|0\rangle \neq 0.$$

Since this implies that ω' belongs to a supermultiplet of some kind, there exists a field ω such that

$$\omega' = i[Q,\omega].$$

As in the case of quantum mechanics, this implies

$$\langle 0|\omega'|0\rangle = \langle 0|i[Q,\omega]|0\rangle = \langle 0|iQ\omega - i\omega Q|0\rangle \neq 0.$$

[1]There are some subtleties involved here what is precisely meant by ground-state in a quantum field theory. This subtleties are often irrelevant, especially in the following discussion. Hence, they will be glossed over.

[2]Actually, not superficially, but still.

[3]There are once more field-theoretically subtleties involved with this statement, which will be glossed over here.

Since Q is hermitian, this implies that $Q|0\rangle \neq 0$, as otherwise this expectation value would vanish. Conversely, this implies that if supersymmetry is unbroken, the vacuum state is uncharged with respect to supersymmetry, $Q|0\rangle = 0$. It can be shown that this exhausts all possibilities.

The implications of this can be obtained when noting that, as in quantum mechanics, there exist a connection between supersymmetry generators and the Hamiltonian, and thus the energy. The commutation relations for the Q_a yield

$$\begin{aligned} \{Q_1, Q_1^\dagger\} &= (\sigma^\mu)_{11} P_\mu = & P_0 + P_3 \\ \{Q_2, Q_2^\dagger\} &= (\sigma^\mu)_{22} P_\mu = & P_0 - P_3. \end{aligned}$$

Thus the Hamiltonian $H = P_0$ is given by

$$H = \frac{1}{2}\left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right).$$

Taking the expectation value of H yields

$$\langle 0|H|0\rangle = \frac{1}{2}\sum_a \left((Q_a|0\rangle)^2 + \left(Q_a^\dagger|0\rangle\right)^2\right).$$

Hence, the ground-state energy for the case of unbroken supersymmetry is zero, as none of the right-hand terms can be different from zero, as announced earlier. Since the right-hand side is a sum of squares it also follows that in case of spontaneous supersymmetry breaking the vacuum energy is not zero, but is larger than zero.

To detect breaking, it is necessary to specify the object ω', which breaks the supersymmetry. In principle, this can also be a composite operator. Such mechanisms are known, e. g., in QCD or in technicolor models of electroweak symmetry breaking, where the Higgs is a composite object of two fermions. Here, it will be restricted to the case where ω' is an elementary field. It should be kept in mind that so far no method of spontaneous supersymmetry breaking has been found, which appears phenomenologically fully satisfactory. Still, it is possible in principle. The situation in the non-gauge and the gauge case are a little different, and will be treated in turn in the following.

8.1.1 The O'Raifeartaigh model

The O'Raifeartaigh model is a non-gauge model, essentially an extension of the Wess-Zumino model, which can exhibit spontaneous supersymmetry breaking. To study the possible elementary fields for developing a vacuum expectation value, as $\langle 0|\omega|0\rangle$ is exactly this, it is helpful to reconsider the transformations under supersymmetry in the Wess-Zumino model

$$\begin{aligned}\delta_\xi\phi &= i[\xi Q,\phi] = && \xi\chi\\ \delta_\xi\chi &= i[\xi Q,\chi] = && -i\sigma^\mu i\sigma_2\xi^*\partial_\mu\phi+\xi F\\ \delta_\xi F &= i[\xi Q,F] = && -i\xi^\dagger\bar\sigma^\mu\partial_\mu\chi.\end{aligned}$$

Phenomenologically, so far Lorentz-symmetry-respecting models are most interesting, and thus any condensates may not break this symmetry. This rules out already χ, $\partial_\mu\chi$ and $\partial_\mu\phi$, as all of these have a definite direction. Hence, the only field remaining is the scalar F field. The value of F is fixed by its equation of motion as

$$F=-\frac{\delta W^\dagger}{\delta\phi^\dagger}=-\left(M\phi+\frac{1}{2}y\phi^2\right)^\dagger. \tag{8.1}$$

The contribution of F to the Hamiltonian, and thus the interaction energy, is given by $FF^\dagger$. This is a positive definite contribution. Its lowest value is achieved, as can be seen from (8.1), exactly when all ϕ-fields vanish. In this case, the contribution of F to the ground-state energy is zero, and is thus not able to break supersymmetry. It is necessary to force F to a value different from zero. For this purpose, it is actually insufficient to just add a constant term to (8.1). Though this formally shifts F to a value different from zero for $\phi=0$, it is always possible to shift it back if ϕ is replaced by a non-zero, constant value. A non-zero, constant value for all the fields will not produce kinetic energy. So, the only other contribution could come from the Yukawa coupling to the fermions. However, it is still permitted to set these to zero. Thus, the ground state energy becomes once more zero, and supersymmetry is intact, despite the non-vanishing value of F and ϕ. This is in fact not a contradiction: The shift in the ϕ-field can then be taken to be a renormalization of the field, and the resulting theory is in fact supersymmetric.

It thus requires a more complicated approach. However, including a linear term is already a good possibility, but it turns out to be impossible with just one flavor. O'Raifeartaigh showed that it is possible, if there are at least three flavors. The parameters of the superpotential are then chosen as

$$\begin{aligned}A_i &= -gM^2\delta_{i2}\\ M_{ij} &= \frac{m}{2}(\delta_{i1}\delta_{j3}+\delta_{i3}\delta_{j1})\\ y_{ijk} &= \frac{g}{3}(\delta_{i2}\delta_{j3}\delta_{k3}+\delta_{i3}\delta_{j2}\delta_{k3}+\delta_{i3}\delta_{j3}\delta_{k2}),\end{aligned}$$

with g, m, and M real and positive, yielding a superpotential

$$W=m\phi_1\phi_3+g\phi_2(\phi_3^2-M^2).$$

As the Wess-Zumino-Lagrangian is supersymmetric for any form of the superpotential, this choice is not breaking supersymmetry explicitly. However, even at tree-level the minimum energy is non-zero, thus supersymmetry is spontaneously broken. This can be seen as follows. The equations of motion for the three $F_i^\dagger$ fields take the form

$$\begin{aligned} F_1^\dagger &= -m\phi_3 \\ F_2^\dagger &= -g(\phi_3^2 - M^2) \\ F_3^\dagger &= -m\phi_1 - 2g\phi_2\phi_3, \end{aligned}$$

and correspondingly for the F_i. From these equations of motion it follows that, at least at tree-level, either the field F_1 or the field F_2 will have a vacuum expectation value. This cannot be shifted away by a wave-function renormalization of ϕ_3, since anything shifting F_1 to zero will shift F_2 to a non-zero value. Putting it differently, these equations of motions force F_1 and F_2 to have different values. Since the contribution of F to the vacuum energy is a sum of squares of type $F_iF_i^\dagger$, at least the contribution from one flavor is always non-zero.

The contribution of the Yukawa term may at first seem to be a tempting possibility to change the situation. However, this would require that the field χ acquires a vacuum expectation value. This would require that the vacuum has to have a non-zero spinor component, as then $\langle 0|\chi_i|0\rangle$ would be non-zero. This would clearly break Lorentz invariance, and is thus not admissible.

It therefore remains to minimize the potential energy with respect to the fields ϕ_i and F_i. Writing the potential explicitly yields

$$V = \sum_i F_iF_i^\dagger = m^2|\phi_3|^2 + g^2|\phi_3^2 - M^2|^2 + |m\phi_1 + 2g\phi_2\phi_3|^2. \tag{8.2}$$

This has to be minimal for the vacuum state. Since ϕ_1 can always be chosen such that the third term vanishes, it remains to check the first two terms. Rewriting the expression in terms of the real part A and the imaginary part B of ϕ_3 yields the expression

$$V = g^2M^4 + (m^2 - 2g^2M^2)A^2 + (m^2 + 2g^2M^2)B^2 + g^4(A^2 + B^2)^2.$$

If the first expression is positive, i. e. $m^2 \geq 2g^2M^2$, the lowest values of V is achieved for $A = B = 0$. Otherwise, a solution with A and B non-zero exists. This provides little qualitative new results for the present purpose, and so only the first case will be treated. In this case, $\phi_3 = 0$, and consequently thus $\phi_1 = 0$. The value of ϕ_2 is not constrained at all, and ϕ_2 could take, in principle, any value. Therefore, it is called a flat direction of the potential. Under certain circumstances, this may pose a problem in the form of an instability, but this is of no interest here.

With this result, F_2 acquires a vacuum expectation value of size gM^2, and the vacuum energy is the positive value g^2M^4. Since M has the dimension of mass, this vacuum energy at the same time gives the scale of supersymmetry breaking. If g would be of order one, $M = 1$ TeV would, e. g., signal a breaking of supersymmetry at the scale 1 TeV, which would be accessible at the LHC.

It is noteworthy that

$$0 \neq \langle 0|[Q,\chi]|0\rangle = \sum_n (\langle 0|Q|n\rangle\langle n|\chi|0\rangle - \langle 0|\chi|n\rangle\langle n|Q|0\rangle).$$

Since Q and χ are both fermionic, this implies that there exists a state $|n\rangle$, which must also be fermionic, which couples to the vacuum by the generator Q such that $\langle 0|Q|n\rangle$ is non-zero. Since it couples in such a way to the vacuum, it can be shown that it is massless. This is the SUSY version of the Goldstone theorem, which differs by the appearance of a massless fermionic mode from the conventional one. That is, of course, due to the fact that the supercharge is fermionic.

Since the mass-matrix of the superpotential has only entries in the (13)-submatrix, this implies that the flavor 2 fermion is massless, and can take this role. Due to the relation to the Goldstone theorem, it is called goldstino, though there is no Goldstone boson in the theory. Consequently, also the boson ϕ_2 is not having a mass. This correlates to the fact that ϕ_2 is the flat direction in the superpotential: Moving in this direction is not costing any energy, similar to a Goldstone excitation in conventional theories.

That supersymmetry is indeed broken can also be seen explicitly by the masses of the remaining two flavors. By diagonalizing the mass-term for the other two fermion flavors, it is directly obtained that there exists two linear combinations, both with masses m. The mass-terms for the scalars are obtained by taking the quadratic terms of the potential (8.2), yielding

$$m^2\phi_3\phi_3^\dagger + gM^2(\phi_3^2 + \phi_3^{\dagger 2}) + m^2\phi_1\phi_1^\dagger.$$

This confirms the masslessness of the ϕ_2 boson. Furthermore, real and imaginary part of the flavor 1 have mass m^2. The flavor 3 has real and imaginary parts with different masses, $m^2 \pm gM^2$. This already implies that the supermultiplets are no longer mass-degenerate, and supersymmetry is indeed broken.

However, when summing up everything, it turns out that the relation

$$\sum_{\text{scalars}} m_s^2 = 0+0+m^2+m^2+m^2+gM^2+m^2-gM^2 = 2\sum_{\text{fermions}} m_f^2 = 2(0+m^2+m^2) \quad (8.3)$$

holds. Note that the complex scalars correspond to two scalar particles, while each Weyl fermion represents one particle with two different spin orientation. It turns out that this relations holds generally for this (F-type) spontaneous breaking of supersymmetry. However, this implies that the masses of the particles and their super-partners have to be quite similar. As a consequence, such a mechanism is not suitable for the standard model, since otherwise already super-partners would have been observed[4]. More fundamentally, when constructing the minimal supersymmetric standard model, it will turn out that there is no gauge-invariant scalar field which could play the role of the second flavor in this model. Since the superpotential has to be gauge-invariant term-by-term, it is thus not possible to have a linear term in that case in the superpotential, and this type of spontaneous supersymmetry breaking is not permitted.

8.1.2 Spontaneous breaking of supersymmetry in gauge theories

Interactions are necessary for a spontaneous breaking of supersymmetry. The simplest non-trivial case of a gauge theory is that of supersymmetric QED. In the matter sector, the breaking of supersymmetry can only proceed once more due to a Wess-Zumino-like interaction, which is not present in supersymmetric QED. Therefore, the field F cannot develop a vacuum expectation value, and neither the electron nor the selectron are possible candidates, due to the arguments in the preceding section. Inspecting the remaining transformation rules

$$\begin{aligned}
i[\xi Q, A_\mu] &= -\frac{1}{\sqrt{2}}(\xi^\dagger\bar{\sigma}_\mu\lambda + \lambda^\dagger\bar{\sigma}_\mu\xi) \\
i[\xi Q, \lambda] &= -\frac{i}{2\sqrt{2}}\sigma^\mu\bar{\sigma}^\nu F_{\mu\nu}\xi + \frac{1}{\sqrt{2}}\xi D \\
i[\xi, D] &= \frac{i}{\sqrt{2}}(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - (\partial_\mu\lambda)^\dagger\bar{\sigma}^\mu\xi),
\end{aligned}$$

then, by the same reasoning as before, suggests that the only field which can provide a scalar condensate is the D-field. However, the contribution of the D-boson to the potential is just $eD\phi^\dagger\phi$, and the equation of motion is $D = e\phi^\dagger\phi$. Thus, there is no sum of terms, as in the Wess-Zumino case, which can be exploited to construct a potential which offers the possibility for supersymmetry breaking.

However, there is another possibility, the addition of the so-called Fayet-Iliopoulos term. In this case, the D-sector of the Lagrangian is replaced by

$$\mathcal{L}_D = M^2 D + \frac{1}{2}D^2 - gD\phi^\dagger\phi.$$

[4]Adding an additional heavy fourth generation may seem at first sight a tempting possibility to evade this constraint. However, it can be shown that for the charge structure of the standard model further constraints exist which forces always at least one super-partner to be light enough to be already detected.

The D-field is gauge-invariant, thus this Lagrangian is also still gauge-invariant. Furthermore, the SUSY transformation of D is a total derivative, and thus any linear term in D is also not spoiling supersymmetry, and this Lagrangian is therefore a perfectly valid extension of supersymmetric QED. The new equation of motion for D is then

$$D = -M^2 + g\phi\phi,$$

yielding the contribution

$$\frac{1}{2}(-M^2 + g\phi^\dagger\phi)^2 \tag{8.4}$$

to the Hamiltonian's potential energy. The sign of g can be selected freely. If g is greater than zero, then a minimum of this potential energy[5] is obtained at $|\phi| = M/\sqrt{g}$. In this case, the potential energy contribution is zero, and supersymmetry is unbroken. However, the scalar condensate is equivalent to a Higgs-effect, giving the bosons a mass. But if $g < 0$, then the minimum is obtained for $\langle\phi\rangle = 0$, with a potential energy contribution which is non-zero, $M^4/2$. Thus, supersymmetry is broken in this case, and the D field acquires the expectation value $-M^2$.

As a consequence, the selectron field acquires a mass by its interaction with the D-field, but the other fields remain massless, in particular the photon, the photino, and the electron. Thus, the degeneration in the mass spectrum of the matter fields is indeed broken, signaling consistently the breakdown of supersymmetry. Note that the sum-rule (8.3) cannot be applied here, as this supersymmetry breaking proceeds by a different mechanism.

Unfortunately, this mechanism cannot be extended to non-Abelian gauge groups, as in this case the D field is no longer gauge-invariant, and neither is its supersymmetry transformation anymore a total derivative. Thus, in the non-Abelian case, supersymmetry breaking by the Fayet-Iliopoulos mechanism is not possible.

Also utilizing then just the QED sector of the standard model is not an option: In the standard model the single $\phi^\dagger\phi$ in (8.4) is replaced with a sum over all fields carrying electric charge, and with their respective positive and negative charges. Thus, the corresponding minimum would be obtained by some of the squark and slepton fields developing a vacuum expectation value, and some not. As a consequence, in the standard model case, the breaking of supersymmetry with a Fayet-Iliopoulos term would imply the breaking of the electromagnetic and color gauge symmetries, which is not compatible with experiment.

[5] Note that the arguments on which fields can acquire such a vacuum expectation values are no longer valid anymore, since instead of the supersymmetry transformations for this case the gauge transformation have to be investigated.

Therefore, also this second mechanism of spontaneous breaking of supersymmetry is not viable for the standard model, and it is necessary to turn to explicit breaking.

The only alternative would be if there would exist another, yet unknown, additional interaction, coupling weakly to the standard model, a so-called hidden sector. This could be used to obtain supersymmetry breaking by this mechanism. However, this requires a model which goes significantly beyond the minimal supersymmetric standard model discussed below.

8.2 Explicit breaking

Thus, at the present time, no satisfactory mechanism exists for the breaking of supersymmetry. Therefore, it is necessary to parametrize this lack of knowledge in the form of explicitly supersymmetry breaking terms[6]. However, if some of the specific properties of supersymmetric theories, in particular the better renormalization properties and the protection of the scalar masses, should be retained, it is not possible to add arbitrary terms to the Lagrangian.

To ensure the survival, or at least only minor modification, of these properties, it is necessary to restrict the explicit supersymmetry breaking contributions to soft contributions. Soft contributions are such contributions which become less and less relevant with increasing energy in an asymptotically free theory. This ensures that supersymmetry becomes effectively restored at large energies. To ensure such a property, it is necessary that the terms contain coupling constants which have a positive energy dimensions. In case of the theories discussed so far, these can appear in the form of two types of terms.

One type of such terms are masses which do not emerge from a superpotential. In particular, such terms are allowed even without integrating out the F-bosons. Such masses are possible for both, the bosonic and the fermionic fields, but not for the auxiliary fields, as their mass dimensions are not permitting renormalizable mass terms. However, these cannot appear for gauge bosons, as in standard gauge theories. Thus, e. g., in the supersymmetric QCD, such terms would be of type

$$-\frac{1}{2}\left(m_\lambda\lambda\lambda + m_\chi\chi\chi + m_\lambda^\dagger\lambda^\dagger\lambda^\dagger + m_\chi^\dagger\chi^\dagger\chi^\dagger + 2m_\phi^2\phi^\dagger\phi\right)$$

Note that all mass terms are gauge-invariant. Also, since this is an effective parametrization, all masses, m_λ, m_χ, and m_ϕ, are independent parameters. Furthermore, these masses may be complex. When the F-field would be integrated out, the additional mass terms for the electrons and the selectrons would mix with those introduced above. Not only is

[6] Of course, it cannot be excluded that supersymmetry in nature is in fact explicitly broken.

such a Lagrangian not explicitly supersymmetric anymore, but, since the super-partners are no longer mass-degenerate, also the spectrum is manifestly no longer supersymmetric.

Furthermore, an additional possibility are three-boson couplings. These couplings would be of type

$$a_{ijk}\phi_i\phi_j\phi_k + b_{ijk}\phi_i^\dagger\phi_j\phi_k + c_{ijk}\phi_i^\dagger\phi_j^\dagger\phi_k + d_{ijk}\phi_i^\dagger\phi_j^\dagger\phi_k^\dagger,$$

where, of course, not all couplings are independent, but are constrained by gauge-invariance. That such couplings break supersymmetry is evident, as they not include the particles and their super-partners equally.

All in all, to the three independent parameters of the supersymmetric QCD, the gauge coupling, and the two Wess-Zumino parameters g and M, four more have emerged, three masses, and at least one three-boson coupling. All of these additions are in fact supersymmetry breaking. Inside such a model, there is no possibility to predict the values of the additional coupling constants. However, in cases where the (broken) supersymmetric theory is just the low-energy limit of a unified theory at some higher scale, this underlying theory can provide such predictions, at least partially. E. g., in the cases above, the expectation values of fields have been obtained in terms of the masses and the coupling constants of the fields, or vice versa.

This already closes the list of possible soft supersymmetry breaking terms, although for specific models much more possibilities may exist.

8.3 Breaking by mediation

As has been seen, non-explicit supersymmetry-breaking faces the problem that the relation between particles are inconsistent with experiment. Explicit breaking, however, introduces numerous additional parameters. A compromise is supersymmetry-breaking by mediation. In this case, dynamical breaking of supersymmetry is made possible by additional particles and interactions. Thereby the problem of experimentally unwanted breaking of electromagnetism or the strong interaction as well as relations like (8.3) can be circumvented. Furthermore, this usually requires much less parameters as an explicit breaking, and provides a dynamical origin.

Of course, just adding more generations or variously charged particles will not solve the problem. Though this may circumvent the simple sumrule (8.3), this will lead to other problems, like too strong Yukawa-interactions for the additional generations to be easily compatible with the observed Higgs particle, or conflicts with further sum-rules specific to the standard model. Also, breaking of the electromagnetic and strong interaction can usually not be avoided in this way.

The alternative is then the addition of a complete new sector of particles, including their own interactions. This sector is arranged such that supersymmetry breaking is possible. To communicate this to the standard model requires some kind of interaction. This is performed by so-called messengers. These are further, usually again additional, particles, which are made quite heavy, significantly above the electroweak scale. In this way, even small breakings will be able to introduce substantial effects. This is usually done by resolving the explicit breaking parameters into effective vertices of interactions with this messenger particles. This can be achieved in a similar way as in the Higgs and weak interaction effects at low-energy, where both effects only appear as point-interaction, in the form of the fermion masses and the effective four-fermion interaction of the effective Fermi theory.

This procedure seems still to be an ad hoc resolution of the problem. To embed this in a less ad-hoc framework, two particular possibilities have been pursued.

One is gauge mediation. In this case it is assumed that the three standard-model gauge interactions are just part of one unified gauge-interaction at high scales, and so is the supersymmetry-breaking sector. The breaking of this master gauge theory has then separated the standard-model, and fractured it into three interactions, and the breaking sector at some high-scale. The surplus gauge-bosons at this high scale then usually have masses of the breaking scale, but still interact with both sectors. In this way they can act as messengers. While this scenario is in general very attractive, as it solves many problems of the standard model, it also has its own problems. Especially, the larger mass hierarchies emerging in this case are usually accidental, and not well understood.

An alternative is gravity-mediated supersymmetry-breaking, where the gravitino acts as messenger. As this setup requires a full super-gravity theory, it will not be detailed here any further. However, it appears phenomenologically somewhat more appealing, as it is compatible with experimental results with rather little effort, mainly due to the weak interaction of gravity. It usually leads to keV-scale gravitinos, but since they couple gravitationally, and thus very weak, this is not at odds with phenomenology.

Chapter 9

A primer on the minimal supersymmetric standard model

9.1 The Higgs sector of the standard model

Before developing its supersymmetric extension, there is another part of the standard model, besides gauge theories, which should be briefly repeated. This is the Higgs sector, with the associated Brout-Englert-Higgs (BEH) effect. One additional effect is the mixing between the weak interactions, or weak isospin, an SU(2) Yang-Mills theory, with QED, the U(1) gauge theory. Though independent, both parts are intertwined with the BEH effect in the standard model, and should therefore be discussed in parallel.

Begin by considering the SU(2)×U(1) part of the standard model with one scalar field that transforms as an isodoublet under the weak isospin group SU(2) and under the hypercharge group U(1). The covariant derivative is given by

$$
\begin{aligned}
iD_\mu &= i\partial_\mu - g_i W_\mu^a Q_a - g_h B_\mu \frac{y}{2} \\
&= i\partial_\mu - g_i W_\mu^+ Q^- - g_i W_\mu^- Q^+ - g_i W_\mu^3 Q^3 - g_h B_\mu \frac{y}{2}
\end{aligned}
$$

with the charge basis expressions

$$
\begin{aligned}
Q^\pm &= \frac{(Q^1 \pm iQ^2)}{\sqrt{2}} \\
W_\mu^\pm &= \frac{W_\mu^1 \pm iW_\mu^2}{\sqrt{2}}.
\end{aligned}
$$

Note there are two gauge coupling constants, g_i and g_h for the subgroups SU(2) and U(1), respectively, which are independent. The hypercharge y of the particles are, in the standard model, an arbitrary number, and have to be fixed by experiment. The Q^a are the generators of the gauge group SU(2), and satisfy the algebra

$$[Q^a, Q^b] = i\epsilon_{abc}Q^c$$

within the representation t of the matter field on which the covariant derivative acts. In the standard model, these are either the fundamental representation $t = 1/2$, i. e. doublets, and thus the $Q^a = \tau^a$ are just the Pauli matrices, or singlets $t = 0$, in which case it is the trivial representation with the $Q^a = 0$.

Returning to the gauge bosons, linear combinations

$$\begin{aligned} W_\mu^3 &= Z_\mu \cos\theta_W + A_\mu \sin\theta_W \\ B_\mu &= -Z_\mu \sin\theta_W + A_\mu \cos\theta_W \end{aligned}$$

can be written where Z_μ (A_μ) is the Z-boson (photon). Then the coupling constant e is defined as

$$g_i \sin\theta_W = e = g_h \cos\theta_W, \tag{9.1}$$

implying the relation

$$\frac{1}{e^2} = \frac{1}{g_i^2} + \frac{1}{g_h^2}.$$

This definition (9.1) introduces the weak mixing, or Weinberg, angle

$$\tan\theta_W = \frac{g_h}{g_i}.$$

The conventional electric charge, determining the strength of the coupling to the photon field A_μ, is thus defined as

$$eQ = e\left(Q^3 + \frac{y}{2}\underline{1}\right), \tag{9.2}$$

where $\underline{1}$ is the unit matrix in the appropriate representation of the field, i. e. either the number one or the two-dimensional unit matrix.

The total charge assignment for all the standard model particles is then

- Left-handed neutrinos: $t = 1/2$, $t_3 = 1/2$, $y = -1$ ($Q = 0$), color singlet
- Left-handed leptons: $t = 1/2$, $t_3 = -1/2$, $y = -1$ ($Q = -1$), color singlet
- Right-handed neutrinos: $t = 0$, $y = 0$ ($Q = 0$), color singlet
- Right-handed leptons: $t = 0$, $y = -2$ ($Q = -1$), color singlet
- Left-handed up-type (u, c, t) quarks: $t = 1/2$, $t_3 = 1/2$, $y = 1/3$ ($Q = 2/3$), color triplet

- Left-handed down-type (d, s, b) quarks: $t = 1/2$, $t_3 = -1/2$, $y = 1/3$ $(Q = -1/3)$, color triplet
- Right-handed up-type quarks: $t = 0$, $y = 4/3$ $(Q = 2/3)$, color triplet
- Right-handed down-type quarks: $t = 0$, $y = -2/3$ $(Q = -1/3)$, color triplet
- W^+: $t = 1$, $t_3 = 1$, $y = 0$ $(Q = 1)$, color singlet
- W^-: $t = 1$, $t_3 = -1$, $y = 0$ $(Q = -1)$, color singlet
- Z: $t = 1$, $t_3 = 0$, $y = 0$ $(Q = 0)$, color singlet
- γ: $t = 0$, $y = 0$ $(Q = 0)$, color singlet
- Gluon: $t = 0$, $y = 0$ $(Q = 0)$, color octet
- Higgs: $t = 1/2$, $t_3 = \pm 1/2$, $y = 1$ $(Q = 0, +1)$, color singlet

Note that the right-handed neutrinos have no charge and participate in the gauge interactions only by neutrino oscillations, i. e., by admixtures due to the leptonic PNMS matrix an their interaction with the Higgs boson. Any theory beyond the standard model, including supersymmetry, has to reproduce this assignment at low energies.

It is now possible to discuss the BEH effect in more detail. The complex doublet of scalar Higgs-bosons can be written as

$$H = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$$

and the Lagrangian for H takes the form

$$\mathcal{L}_H = (D_\mu H)^\dagger (D^\mu H) - V(H^\dagger H) \tag{9.3}$$

with some (renormalizable) potential V. To generate the masses in the standard model it must be assumed that (the quantum version of) the Higgs potential has an unstable extremum for $H = 0$ and a nontrivial minimum, e. g.

$$V(H) = \frac{\lambda}{2}(H^\dagger H - v^2)^2 \tag{9.4}$$

The Higgs boson then develops[1] a vacuum expectation value v. It is always possible to find a gauge in which v is real and oriented along the upper component, and thus to be

[1]The following are, in fact, gauge-dependent statements. It is always possible to find a gauge in which $v = 0$. However, in such gauges, e. g. the W boson will have zero mass to all orders in perturbation theory. If one wants to use perturbation theory to describe experiments, these gauges are not suitable. Hence, in the following it will always be understood that a gauge has been chosen where a non-zero vacuum expectation value is possible.

annihilated by the electric charge to make it neutral,

$$\langle H\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

In the conventions used here, the value of v is $v = (2G_F)^{-1/2} \approx 250$ GeV, where G_F is Fermi's constant. Note that the operator Q defined by (9.2) acting on the Higgs vacuum expectation value yields zero, which implies that the condensate is uncharged, and this implies that the photon remains massless.

Inserting the decomposition of H into vacuum expectation value v and quantum fluctuations $h = H - v$ into (9.3) generates the masses of the weak gauge bosons as

$$\begin{aligned}
\mathcal{L} &= 1/2(\partial h)^\dagger \partial h + 1/2M_W^2 W_\mu^+ W^{\mu -} + 1/2M_Z^2 Z_\mu Z^\mu - 1/2M_H^2 hh^\dagger \\
&\quad -\frac{\sqrt{\lambda}}{2}M_H(h^2h^\dagger + (h^\dagger)^2 h) - \frac{1}{8}\lambda(hh^\dagger)^2 \\
&\quad +1/2\left(hh^\dagger + \frac{M_H}{\lambda}(h+h^\dagger)\right)(g_i^2 W_\mu^+ W^{\mu -} + (g_h^2 + g_i^2)Z_\mu Z^\mu) + ... \\
M_H &= v\sqrt{2\lambda} \\
M_W &= \frac{g_i v}{2} \\
M_Z &= \frac{v}{2}\sqrt{g_2^2 + g_1^2} = \frac{M_W}{\cos\theta_W}.
\end{aligned}$$

Here, the electromagnetic interaction and other terms have been dropped for clarity. This Lagrangian also exhibits the coupling of the Higgs field h to itself and to the W and Z fields. It implies that the Higgs mass is just a rewriting of the four-Higgs coupling, and either has to be measured to fix the other. Of course, measuring both independently is a check of the theory.

The matter fields couple with maximal parity violation to the weak gauge fields, i. e. their covariant derivatives have the form, for, e. g. the left-handed weak isospin doublet of bottom and top quark $\Psi_L = (t,\ b)_L$

$$\begin{aligned}
\bar{\Psi}_L i\gamma^\mu D_\mu \Psi_L &= \bar{\Psi}_L i\gamma^\mu \partial_\mu \Psi_L - \frac{1}{\sqrt{2}}\bar{t}\gamma_\mu \frac{1-\gamma_5}{2} bW^{\mu +} - \frac{1}{\sqrt{2}}\bar{b}\gamma_\mu \frac{1-\gamma_5}{2} tW^{\mu -} \\
&\quad -\frac{2e}{3}\bar{t}\gamma^\mu \frac{1-\gamma_5}{2} tA_\mu + \frac{e}{3}\bar{b}\gamma^\mu \frac{1-\gamma_5}{2} bA_\mu - \bar{\Psi}_L e\tan\theta\gamma^\mu \Psi_L Z_\mu,
\end{aligned}$$

The problem with a conventional mass term would be that it contains the combination $\bar{\Psi}_L \Psi_R$, with Ψ_R being the sum of the right-handed bottom and top, which is not a singlet under weak isospin transformation, and thus would make the Lagrangian gauge-dependent, yielding a theory which is not physical.

This can be remedied by the addition of a Yukawa-type interaction between the fermions and the Higgs of the form:

$$g_t \bar{\Psi}_L \cdot H t_R + g_b \bar{\Psi}_L \cdot H^\dagger b_R,$$

where $\cdot$ indicates a scalar product in isospin space, and which couples the left- and right-handed fermions to the Higgs field. This combination is gauge-invariant and physically sensible for arbitrary Yukawa couplings g_b and g_t. When the Higgs develops its vacuum expectation value, masses $m_t = g_t v$ and $m_b = g_b v$ arise for the top and bottom quarks, respectively. This mechanism is replicated for both the other quarks and all leptons, though it is experimentally yet unclear whether one of the neutrinos is massless. Either possibility is consistent with the standard model and this mechanism.

9.2 The supersymmetric minimal supersymmetric standard model

From the previous examples of simple theories it is clear that a supersymmetry transformation cannot change any quantum number of a field other than the spin. In the standard model, however, none of the bosons has the same quantum numbers as any of the fermions. Hence, to obtain a supersymmetric version of the standard model, it will be necessary to construct for each particle in the standard model a new super-partner. Of course, additional fields need also be included, like appropriate F-bosons and D-bosons. Here, the supersymmetric version of the standard model with the least number of additional fields will be introduced, the so-called minimal supersymmetric standard model (MSSM). Furthermore, since no viable low-energy supersymmetry breaking mechanism for such a theory is (yet) known, supersymmetry will be broken explicitly. This will require roughly 100 new parameters. This may seem a weakness at first. However, the advantage is that any kind of supersymmetry at high energies can be accommodated by such a theory. Hence, if there are no additional particles or sectors, for which no experimental evidence exists so far, any supersymmetric high-energy theory will look at accessible energies like this minimal-supersymmetric standard model.

The MSSM requires the following new particles

- The photon is uncharged. Therefore, its super-partner has also to be uncharged, and to be a Weyl fermion, to provide two degrees of freedom. It is called the photino
- The eight gluons carry adjoint color charges. This requires eight Weyl fermions carrying adjoint color charges as well, and are called gluinos

- Though massive, the same is true for the weak bosons, leading to the charged superpartners of the W-bosons, the winos, and of the Z-boson, the zino, together the binos. Except for the photino all gauginos, the gluinos and the binos, interact through covariant derivatives with the original gauge-fields

- Assuming that all neutrinos are massive, no distinction between left-handed and right-handed leptons is necessary, except for the parity-violating weak interactions. However, no new interactions should be introduced due to the superpartners, and hence the superpartners cannot be spin-1 bosons, which would be needed to be gauged. Therefore these superpartners are taken to be scalars, called the sneutrinos, the selectron, the smuon, and the stau, together the sleptons

- The same applies to quarks, requiring the fundamentally charged squarks

- The Higgs requires a fermionic superpartner, the higgsino. However, requiring supersymmetry forbids that the Higgs has the same type of coupling to all the standard model fields. Therefore, a second set of Higgs particles is necessary, with their corresponding super-partners. These are the only new particles required which are not introduced as superpartners of existing particles

- Of course, a plethora of auxiliary D and F bosons will be necessary

It is necessary to discuss the formulation of these fields, and the resulting Lagrangian, in more in detail.

The electroweak interactions act differently on left-handed and right-handed fermions. It thus fits naturally to use independent left- and right-handed chiral multiplets to represent the fermions, together with their bosonic superpartners, the sfermions. However, both Weyl-spinors can be combined into a single Dirac-spinor, as the total number of degrees of freedom match. The electroweak interaction can then couple by means of the usual $1 \pm \gamma_5$ coupling asymmetrically to both components. In this context, it is often useful to use the fact that a charge-conjugated right-handed spinor is equivalent to a left-handed one, as discussed previously, permitting to use exclusively left-handed chiral multiplets, where appropriate charge-conjugations are included.

Furthermore, for each flavor it is necessary to introduced two chiral multiplets, giving a total of 12 supermultiplets for the quarks and the leptons each. The mixing of quark and lepton flavors proceeds in the same way as in the standard model. This requires thus already the same number of parameters for two CKM-matrices as in the standard model.

The gauge-bosons are much simpler to introduce. Again, a gauge-multiplet is needed for the eight gluons. As electroweak symmetry breaking is not yet implemented, and actually cannot without breaking supersymmetry as well as noted below, there is actually a SU(2) gauge multiplet and an U(1) gauge multiplet, which do not yet represent the weak gauge bosons and the photon. Instead, before mixing, these are called $W^{\pm}$ and W^0, and B. Only after mixing, the W^0 and the B combine to the Z and the conventional photon.

Concerning interactions, there are, of course, the three independent gauge couplings of the strong, weak, and electromagnetic forces, to be denoted by g, g' and e. After electroweak symmetry breaking, g' and e will mix, just as in the standard model. Note that the coupling of the gauge multiplets and the chiral multiplets will induce additional interactions, as in case of supersymmetric QED and QCD. However, no additional parameters are introduced by this.

It remains to choose the superpotential for the 24 chiral multiplets, and to introduce the Higgs fields. The parity violating weak interactions imply that a mass-term, the component of the superpotential proportional to $\chi\chi$, is not gauge-invariant, since a product of two Weyl-spinors of the type $\chi\chi$ is not, if only the left-hand-type component is transformed under such a gauge transformation. This is just as in the standard model. Therefore, it is not possible, also in the MSSM, to introduce masses for the fermions by a tree-level term. Again, the only possibility will be one generated dynamically by the coupling to the Higgs field[2]. There, however, a problem occurs. In the standard model this is mediated by a Yukawa-type coupling of the Higgs field to the fermions. However, it is necessary, for the sake of gauge-invariance, to couple the two weak charge states of the fermions differently, one to the Higgs field, and one to its complex conjugate. This is not possible in a supersymmetric theory, as the holomorphic superpotential can only depend on the field, and not its complex conjugate. It is therefore necessary to couple both weak charge states to different Higgs fields. This makes it possible to provide a gauge-invariant Yukawa-coupling for both states, but requires that there are two instead of one complex Higgs doublets in the MSSM. Furthermore, also these fields need to be part of chiral supermultiplets, and therefore their partners, the higgssinos, are introduced as Weyl fermions. However, for the Higgs bosons, which have no chirality, the limitations on a mass-term do not apply, and one can therefore be included in the superpotential.

Before writing down the superpotential explicitly, it is necessary to fix the notation. Left-handed Quark fields will be denoted by $u, d, ...$, and squark fields by $\tilde{Q} = \tilde{u}, \tilde{d}, ...$ Note that always a u and d-type quark form a doublet with respect to the weak interactions, and equivalently the squarks. Right-handed quarks are denoted as $\bar{u}, \bar{d}, ...$ and the corresponding squarks as $\tilde{\bar{q}} = \tilde{\bar{u}}, \tilde{\bar{d}}, ...$ These form singlets with respect to the weak

[2]Actually, a gaugino condensation would also be possible, if a way would be known how to trigger it and reconcile the result with the known phenomenology of the standard model.

interactions. This notation corresponds to left-handed fields, which are obtained from the original right-handed fields, to be denoted by, e. g. u_R, by charge conjugation. The index L for left-handed will always be suppressed. Likewise, the left-handed leptonic doublets, including an electron-type and a neutrino-type fermion, are denoted as $\nu, e, ...$, and the corresponding sleptons as $L = \tilde{\nu}, \tilde{e}, ...$ Consequently, the right-handed leptons are denoted as $\bar{\nu}, \bar{e}, ...$ and the singlet sleptons as $\tilde{\bar{l}} = \tilde{\bar{\nu}}, \tilde{\bar{e}}, ...$ The two Higgs doublets are denoted by H_u and H_d, denoting to which type of particle they couple. The corresponding higgsinos are denoted by $\tilde{H}_u$ and $\tilde{H}_d$. In contradistinction to the quarks, where the doublet consists out of a component of u and d-quarks, the doublets for the Higgs are formed in the form H_u^+ and H_u^0, and H_d^0 and H_d^-. The reason for this is that after giving a vacuum expectation value to the Higgs-field and expanding all terms of the potentials, effective mass-terms for the u-type quarks will then finally couple only to H_u-type Higgs fields, and so on. The remaining fields are the gluons g with super-partner gluinos $\tilde{g}$, the gauge W-bosons W with super-partner winos $\tilde{W}$, and the gauge field B with its super-partner bino $\tilde{B}$. After mixing, the usual W-bosons $W^\pm$ with superpartner winos $\tilde{W}^\pm$, the Z-boson with superpartner zino $\tilde{Z}$ and the photon γ with the photino $\tilde{\gamma}$ will be obtained.

With this notation, it is possible to write down the superpotential for the MSSM as

$$W = \tilde{y}_u^{ij}\tilde{\bar{u}}_i\tilde{Q}_jH_u - \tilde{y}_d^{ij}\tilde{\bar{d}}_i\tilde{Q}_jH_d + \tilde{y}_\nu^{ij}\tilde{\bar{e}}_i\tilde{L}_jH_u - \tilde{y}_e^{ij}\tilde{\bar{e}}_i\tilde{L}_jH_d + \mu H_uH_d, \tag{9.5}$$

where gauge-indices have been left implicit on both, the couplings and the fields, and the auxiliary fields have already been integrated out. The appearing Yukawa-couplings y are the same as in the standard model. In particular, vanishing neutrino masses would be implemented by $y_\nu^{ij} = 0$. If only the masses of the heaviest particles, the top, bottom, and the τ should be retained, this requires that all y-components would be zero, except for $y_u^{33} = y_b$, $y_d^{33} = y_t$ and $y_e^{33} = y_\tau$, leaving gauge indices implicit. The choice of this potential is not unique, but the one for which the MSSM is most similar to the standard model.

In principle, it would also be possible to add contributions like

$$\lambda_e^{ijk}\tilde{L}_i\tilde{L}_j\tilde{\bar{e}}_k + \lambda_L^{ijk}\tilde{L}_i\tilde{Q}_j\tilde{\bar{d}}_k + \mu_L^i\tilde{L}_iH_u \tag{9.6}$$

and

$$\lambda_B^{ijk}\tilde{\bar{u}}_i\tilde{\bar{d}}_j\tilde{\bar{d}}_k. \tag{9.7}$$

In all cases, these are couplings of squarks and/or sleptons. Since these carry the same charges as their counterparts, they will also carry the same lepton and baryon numbers. As a consequence, the squarks $\tilde{Q}$ from the multiplets including the particle-like left-handed quarks carry baryon number $B = 1/3$, while those from the anti-particle-like right-handed chiral multiplets carry baryon number $-1/3$. Similarly, the fields $\tilde{L}$ carry lepton number

$L = 1$ and $\bar{e}$ -1. Thus the interaction vertices in (9.6) violate lepton number conservation and the ones of (9.7) baryon number conservation. On principal grounds, there is nothing wrong with this, as both quantities are violated by non-perturbative effects also in the standard model. However, these violations are very tiny, even below nowadays experimental detection limit. Interaction vertices like (9.6) and (9.7), on the other hand, would provide very strong, and experimentally excluded, violations of both numbers, as long as the coupling constants λ_i and μ_L would not be tuned to extremely small values. Such a fine-tuning is undesirable.

However, such direct terms could be excluded, if all particles in the MSSM would carry an additional multiplicatively conserved quantity, called R-parity, which is defined as

$$R = (-1)^{3B+L+2s},$$

with s the spin. This is just the discrete Z_2 subgroup of the U(1) R-symmetry of an $\mathcal{N} = 1$ supersymmetry, which is retained even after supersymmetry breaking in the MSSM.

Such a quantum number would be violated by interaction terms like (9.6) and (9.7), and these are therefore forbidden[3]. The contribution $2s$ in the definition of R implies that particles and their super-partners always carry opposite R-parity. This has some profound consequences. One is that the lightest superparticle (LSP) cannot decay into ordinary particles. It is thus stable. This is actually an unexpected bonus: If this particle would be electromagnetically uncharged, it is a natural dark matter candidate. However, it has also to be uncharged with respect to strong interactions, as otherwise it would be bound in nuclei. This is not observed, at least as long as its mass is not extremely high, which would be undesirable, as then all superparticles would be very massive, preventing a solution of the naturalness problem by supersymmetry. Therefore, it interacts only very weakly, and is thus hard to detect. Not surprisingly, it has not be detected so far, if it exists.

Summarizing, the assumption of R-parity restricts the superpotential to the form (9.5), and thus fixes the MSSM completely.

Returning then to this remaining possible superpotential (9.5), the parameter μ characterizes the masses of the Higgs and higgsinos. In comparison to the standard model with two independent parameters in the Higgs sector, this is the only new parameter entering the theory when including the Higgs sector and keeping supersymmetry unbroken. The self-interaction of the Higgs field will be entirely determined, due to supersymmetry, just by the already included parameters, thus having potentially less parameters than in case of the standard model.

[3] A non-perturbative violation of R-parity notwithstanding, as for baryon and lepton number in the standard model.

Unfortunately, there is a catch. Writing all gauge components explicitly, the mass term for the Higgs takes the form

$$\mu H_u H_d = \mu^{ab} H_u^a H_d^b = \mu\epsilon_{ab} H_u^a H_d^b,$$

where the index structure of the parameter μ is dictated by gauge invariance. The corresponding interaction terms in the Lagrangian are then of the form

$$\frac{\delta W}{\delta H_i} F^i + h.c. = \mu(H_u^1 F_d^2 + H_d^2 F_u^1 - H_u^2 F_d^1 - H_d^1 F_u^2) + h.c..$$

Integrating out all F-bosons in the Higgs sector yields the mass term for the Higgs, just as in the Wess-Zumino model. This produces

$$|\mu|^2 (H_u^i H_u^{i\dagger} + H_d^i H_d^{i\dagger}).$$

This implies firstly that both Higgs doublets are mass-degenerate. More seriously, it implies that this common mass is positive. In the conventional treatment of electroweak symmetry, however, a negative mass is mandatory to obtain a perturbatively valid description of electroweak symmetry. Since the appearance of the positive mass is a direct consequence of supersymmetry, as was seen in case of the Wess-Zumino model, there seems to be no possibility to have at the same time perturbative symmetry breaking and intact supersymmetry. One alternative would be non-perturbative effects, which has not been excluded so far. However, it seems somewhat more likely that it is not possible to have unbroken supersymmetry but so-called broken gauge symmetry simultaneously, and this has lead to a search for a common origin of both phenomena.

After analyzing the features which make the MSSM different from the standard model, it is instructive to see how the standard-model-type interactions are still present.

This is straightforward in the case of the gauge-boson self-interactions, as these are automatically included in the non-Abelian field-strength tensors appearing already in the supersymmetric version of Yang-Mills theory, (7.4). It is more interesting to investigate how the usual Dirac-type quarks and gluons and their coupling to the strong and weak interactions are recovered.

The simpler case is the parity-preserving strong interactions. Due to the gauge-covariant derivative, the coupling of a quark-type left-handed Weyl-fermion is

$$-\frac{1}{2} g \chi_q^\dagger \bar{\sigma}^\mu A_\mu \chi_q = -\frac{1}{2} g \chi_{qi}^\dagger \bar{\sigma}_\mu A_{ij}^\mu \chi_{qj}, \tag{9.8}$$

where in the second expression the gauge indices are made explicit, keeping in mind that $A_\mu = \tau^\alpha A_\mu^\alpha$ is matrix-valued. The right-handed contribution can be rewritten as a left-handed Weyl fermion by virtue of charge conjugation,

$$\begin{aligned}\psi_u &= \chi^c_{\bar{q}} = i\sigma_2\chi^*_{\bar{q}}.\\ \chi_{\bar{q}} &= -i\sigma_2\chi^{c*}_{\bar{q}}\end{aligned}$$

The original coupling of the right-handed fermion, which will be the anti-particle, is

$$\begin{aligned}\frac{1}{2}g\chi^\dagger_{\bar{q}}\bar{\sigma}^\mu A^*_\mu\chi_{\bar{q}} &= \frac{1}{2}g(-i\sigma_2\chi^{c*}_{\bar{q}})^\dagger\bar{\sigma}^\mu A^*_\mu(-i\sigma_2\chi^{c*}_{\bar{q}})\\ = -\frac{1}{2}g\chi^{cT}_{\bar{q}}\sigma_2\bar{\sigma}_\mu\sigma_2 A^*_\mu\chi^{c*}_{\bar{q}} &= -\frac{1}{2}g\chi^{c\dagger}_{\bar{q}}\sigma^\mu A_\mu\chi^c_{\bar{q}}.\end{aligned} \tag{9.9}$$

In the last step, it has been used that $\sigma_2\bar{\sigma}^\mu\sigma_2 = -\sigma^{\mu T}$, and that $A^*_\mu = A^T_\mu$, since the A_μ are hermitian. The remaining step is just rewriting everything in indices and rearranging. Combining (9.8) and (9.9) yields

$$-\frac{1}{2}g(\chi^{c\dagger}_{\bar{q}}\sigma^\mu A_\mu\chi^c_{\bar{q}} + \chi^\dagger_q\bar{\sigma}^\mu A_\mu\chi_q) = -\frac{1}{2}g\bar{\Psi}\gamma^\mu A_\mu\Psi$$

with $\Psi^T = (\chi^c_{\bar{q}}\chi)$. This is precisely the way an ordinary Dirac quark Ψ would couple covariantly to gluons. Hence, by combining two chiral multiplets and one gauge multiplet, it is possible to recover the couplings of the standard model strong interactions.

The electromagnetic interaction, which is also parity-preserving, emerges in the same way, just that the photon field is not matrix-valued.

The weak interaction violates parity maximally by just coupling to the left-handed components. In the standard model, its coupling is given by

$$-\frac{1}{2}g'\bar{\Psi}_e\frac{1-\gamma_5}{2}\gamma^\mu W_\mu\frac{1-\gamma_5}{2}\Psi.$$

However, the action of $(1-\gamma_5)/2$ on any spinor is to yield

$$\frac{1-\gamma_5}{2}\begin{pmatrix}\chi_{\bar{e}}\\ \chi_e\end{pmatrix} = \begin{pmatrix}0\\ \chi_e\end{pmatrix},$$

and thus reduces the Dirac-spinor to its left-handed component. Thus, only the coupling

$$-\frac{1}{2}g\chi^\dagger_e\bar{\sigma}^\mu W_\mu\chi_e$$

remains. This is precisely the coupling of a left-handed chiral multiplet to a gauge multiplet. It is therefore sufficient just to gauge the left-handed chiral multiplets with the weak interactions to obtain the standard-model-type coupling. In addition also inter-generation mixing has to be included, but this proceeds in the same way as in the the standard-model, and is therefore not treated explicitly. This concludes the list of standard model couplings.

9.3 Breaking the supersymmetry in the MSSM

This concludes the list of additional particles and interactions required for the MSSM. As discussed previously, unbroken supersymmetry requires the particles and their superpartners to have the same mass. This implies that the superpartners should have been observed if this is the case, as the selectron should also have only a mass of about 511 keV, as the electron, and must couple in the same way to electromagnetic interactions. Therefore, it would have to be produced at the same rate in, e. g., electromagnetic pion decays as electrons. This is not the case, and therefore supersymmetry must be (completely) broken. Explicit searches in particle physics experiments actually imply that the superpartners are much heavier than their partners of the standard model, as non has been discovered so far. This puts limits on most masses above several hundred of GeV. Hence, supersymmetry is necessarily strongly broken in nature. This also removes F-term breaking as a possibility in the standard model.

Since no mechanism is yet known how to generate supersymmetry breaking in a way which would be in a accordance with all observations, and without internal contradictions, this breaking is only parametrized in the MSSM, and requires a large number of additional free parameters. Together with the original about 30 parameters of the standard model, these are then more than a 130 free parameters in the MSSM. Exactly, there are 105 additional parameters in the MSSM, not counting contributions from massive neutrinos.

These parameters include masses for all the non-standard model superpartners, that is squarks and sleptons, as well as photinos, gluinos, winos, and binos. Only for the Higgsinos it is not possible to construct a gauge-invariant additional mass-term, due to the chirality of the weak interactions, in much the same way as for quarks and leptons in the standard model. One advantage is, however, offered by the introduction of these free mass parameters: It is possible to introduce a negative mass for the Higgs particles, reinstantiating the same way to describe the breaking of electroweak symmetry as in the standard model. That again highlights how electroweak symmetry breaking and supersymmetry breaking may be connected. These masses also permit to shift all superpartners to such scales as they are in accordance with the observed limits so far. However, if the masses would be much larger than the scale of electroweak symmetry breaking, i. e., 250 GeV, it would again be very hard to obtain results in agreement with the observations without fine-tuning. However, such mass-matrices should not have large off-diagonal elements, i. e., the corresponding CKM-matrices should be almost diagonal. Otherwise, mixing would produce flavor mixing also for the standard model particles exceeding significantly the observed amount.

In addition, it is possible to introduce triple-scalar couplings. These can couple at will the squarks, sleptons, and the Higgs bosons in any gauge-invariant way. Again, observational limits restrict the magnitude of these couplings. Furthermore, some couplings, in particular certain inter-family couplings, are very unlikely to emerge in any kind of proposed supersymmetry breaking mechanism, and therefore are usually omitted.

Finally, the mass-matrices in the Higgs sector may have off-diagonal elements, without introducing mixing in the quark sector. This are contributions of the type

$$b(H_u^1 H_d^2 - H_u^2 H_d^1) + h.c.,$$

where the mass parameter b may also be complex.

The arbitrariness of these enormous amount of coupling constants can be reduced, if some model of underlying supersymmetry breaking is assumed. One, rather often, invoked one is that the minimal supersymmetric standard model is the low-energy limit of a supergravity theory, the so-called mSUGRA scenario. Though by now experimentally ruled out, it is still interesting as the simplest representative of such types of models.

In the mSUGRA case, it is, e. g., predicted, that the masses of the superpartners of the gauge-boson superpartners should be degenerate,

$$M_{\tilde{g}} = M_{\tilde{W}} = M_{\tilde{B}} = m_{1/2}, \tag{9.10}$$

and also the masses of the squarks and sleptons should be without mixing and degenerate

$$m_{\tilde{Q}}^2 = m_{\tilde{q}}^2 = m_{\tilde{L}}^2 = m_{\tilde{l}}^2 = m_0^2 \mathbf{1}, \tag{9.11}$$

and these masses also with the ones of the Higgs particles,

$$m_H^2 = m_0^2.$$

Furthermore, all trilinear bosonic couplings would be degenerate, with the same value A_0. If the phase of all three parameters, A_0, $m_{1/2}$, and m_0 would be the same, also CP violation would not differ from the one of the standard model. The latter is an important constraint, as the experimental limits on such violations are very stringent, although not yet threatening to rule out the MSSM proper.

Of course, as the theory interacts, all of these parameters are only degenerate in such a way at the scale where supersymmetry breaks. Since the various flavors couple differently in the standard model, and thus in the minimal supersymmetric standard model, the parameters will again differ in general at the electroweak scale, or any lower scale than the supersymmetry breaking scale.

9.4 MSSM phenomenology

9.4.1 Coupling unification and running parameters

In the electroweak theory it is found that at some energy scale the electromagnetic and the weak coupling become both of the same value. This is known as electroweak unification. Of course, if also the strong coupling would become of the same value at the same energy, this would strongly indicate that all three couplings originate from one coupling at this unification scale, and become different at low energies because the gauge group to which the unified coupling corresponds becomes broken at this unification scale. This is also the idea behind so-called grand-unified theories (GUT) that at large energies there exists one gauge-group, say SU(5), which becomes broken by a Higgs effect, similar to that one in the standard model, to yield the product gauge group SU(3)×SU(2)×U(1) of the standard model at the unification scale. This unification is of course also interesting, as any evidence of it would support the idea of gauge-mediated supersymmetry-breaking.

To check, whether such a unification actually occurs, it is necessary to determine the running of the effective coupling constants with the energy (renormalization) scale. There are three couplings[4], the strong one g, the weak one g', and the electromagnetic one. It is always possible to write the electromagnetic one as $e/\sin\theta_W$. The angle θ_W is known as the Weinberg angle. In the standard model its value originates from the electroweak symmetry breaking. Its measured value is 28.7°. The relevant couplings are then

$$\alpha_1(Q) = \tfrac{g(Q)^2}{4\pi} =_{Q=m_Z} 0.119 \tag{9.12}$$

$$\alpha_2(Q) = \tfrac{g'(Q)^2}{4\pi} =_{Q=m_Z} 0.0338 \tag{9.13}$$

$$\alpha_3(Q) = \tfrac{5}{3}\tfrac{e(Q)^2}{4\pi\cos^2\theta_W} =_{Q=m_Z} 0.0169 \tag{9.14}$$

In this case the appropriate mixed combination for the electromagnetic coupling has been used. The factor 5/3 appears as in an Abelian gauge theory there is a certain freedom in redefining the charge and the generator of gauge transformation not present in non-Abelian gauge theories. It has been set here to a conventional value, which would be expected if the standard model product gauge groups would indeed originate from a common one at high energies.

To one-loop order, all coupling constants evolve according to the renormalization group equation

$$\frac{d\alpha_i}{d\ln Q} = -\frac{\beta_i}{2\pi}\alpha_i^2, \tag{9.15}$$

[4]Other couplings, like the Higgs-self-coupling or the Yukawa couplings do not unify at this scale.

where β_i is the first coefficient in a Taylor expansion of the so-called β-function for the coupling i. For a non-Abelian coupling its value is given by

$$\beta_i = \frac{11}{3}C_A - \frac{2}{3}C_A n_f^a - \frac{1}{3}n_f - \frac{1}{6}n_s,$$

where C_A is the adjoint or second Casimir of the gauge group with value N for a SU(N) group. The numbers n_f^a, n_f, and n_s are the number of adjoint fermionic chirality states, fundamental fermionic chirality states and the number of complex scalars charged under this coupling, respectively. In the Abelian case, the value of β_i is given by

$$\beta_i = -\frac{2}{3}\sum_f Y_f^2 - \frac{1}{3}\sum_s Y_s^2,$$

with Y_f and Y_s the charge of fermions and scalars with respect to the interaction. This stems from the fact that the charge for matter minimally coupled to an Abelian gauge-field can be chosen freely, while this is not the case for a non-Abelian case. The fact that the Y_f and Y_s are rational numbers in the standard model is another indication for the U(1) part of the standard model to be emerged from some other gauge group by symmetry breaking.

As the coefficients of the β-functions do not depend on the couplings themselves, the differential equation (9.15) is readily integrated to yield

$$\alpha_i(Q)^{-1} = \alpha_i(Q_0)^{-1} + \frac{\beta_i}{2\pi}\ln\frac{Q}{Q_0},$$

where $\alpha_i(Q_0)$ is the initial condition, the value of the coupling at some reference scale Q_0. As already indicated in (9.12-9.14), this reference scale will be the Z-boson mass, as all three couplings have been measured with rather good precision at this scale at LEP and LEP2.

The question to be posed is now whether the three couplings $\alpha_i(Q)$ have at some energy Q the same value. To obtain a condition, it is most simple to use the linear system of equations to eliminate Q and $\alpha_i(Q)$ in favor of the known measured values. This yields the conditional equation

$$B_x = \frac{\alpha_3(m_Z)^{-1} - \alpha_2(m_Z)^{-1}}{\alpha_2(m_Z)^{-1} - \alpha_1(m_Z)^{-1}} = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_2} = B_t.$$

With the values given in (9.12-9.14) the left-hand side is readily evaluated to be $B_x = 0.72$. The right-hand side depends only on the β-functions, and therefore to this order only on the particle content of the theory.

In the standard model, there are no scalars charged under the $N = 3$ strong interactions, but 12 chirality states of fermions, yielding $\beta_1 = 7$. The positive value indicates that the strong sector is described by an asymptotically free theory. For the $N = 2$ weak interactions, there are 12 left-handed chiral states (6 quarks and 6 leptons), and one Higgs field, yielding $\beta_2 = 19/6$. Taking all electromagnetically charged particles, and the normalization factor 5/3, into account, $\beta_3 = -41/10$. Note that also some components of the Higgs field are electromagnetically charged. Evaluating B_t yields $115/218 \approx 0.528$. Thus, in the standard model, to this order the couplings will not match. This is also not changed in higher orders of perturbation theory.

The situation changes in the MSSM. For β_1, there are now in addition the gluinos, giving one species of adjoint fermionic chiralities, and 12 squarks, yielding $\beta_1 = 3$. The same applies to the weak case with one species of adjoint winos and zinos, 12 sleptons and squarks, two higgsinos and one additional Higgs doublet. Altogether this yields $\beta_2 = -1$. The change of sign is remarkably, indicating that the supersymmetric weak sector is no longer asymptotically free, as is the case in the standard model. The anti-screening contributed from the additional degrees of freedom is sufficient to change the behavior of the theory qualitatively. The case of β_3 finally changes to $-33/5$, after all bookkeeping is done. Together, this yields $B_t = 5/7 = 0.714$. This is much closer to the desired value of 0.72, yielding support for the fact that the MSSM emerges from a unified theory. Taking this result to obtain the unification scale, it turns out to be $Q_U \approx 2.2 \times 10^{16}$ GeV, which is an enormously large scale, though still significantly below the Planck scale of 10^{19} GeV. This result is not changing qualitatively, if the calculation is performed to higher order nor if the effects from supersymmetry breaking and breaking of the presumed unifying gauge group is taken into account. Thus the MSSM, without adding any constraint, seems a natural candidate for a theory emerging from a grand unified one, being one reason for its popularity and of supersymmetry in general. However, exact unification is never achieved in the MSSM, and therefore requires further contributions to occur.

These results suggest to also examine the running of other parameters as well. In particular, the parameters appearing due to the soft breaking of supersymmetry are particularly interesting, as their behavior will be the effect which obscures supersymmetry in nature, if it exists.

The first interesting quantity is the mass of the gauge-boson superpartners, the gauginos. The running is given by a very similar expression as for the coupling constants,

$$\frac{dM_i}{d\ln Q} = -\frac{\beta_i}{2\pi}\alpha_i M_i.$$

The β-function can be eliminated using the equation for the running of the coupling (9.15). This yields the equation

$$0 = \frac{1}{\alpha_i}\frac{dM_i}{d\ln Q} - \frac{M_i}{\alpha_i^2}\frac{d\alpha_i}{d\ln Q} = \frac{d}{d\ln Q}\frac{M_i}{\alpha_i}.$$

This implies immediately that the ratio of the gaugino mass for the gauge group i divided by the corresponding gauge coupling is not running, i. e., it is renormalization group-invariant.

If there exists a scale m_U at which the theory unifies, like suggested by the running of the couplings, then also the masses of the gauginos should be equal. As discussed previously, this would be the case in supergravity as the origin of the minimal supersymmetric standard model. This would yield

$$\frac{M_i(m_U)}{\alpha_i(m_U)} = \frac{m_{1/2}}{\alpha_U(m_U)},$$

and since the ratios are renormalization-group invariant it follows that

$$\frac{M_1(Q)}{\alpha_1(Q)} = \frac{M_2(Q)}{\alpha_2(Q)} = \frac{M_2(Q)}{\alpha_2(Q)}.$$

Since the α_i are known, e. g., at $Q = m_Z$, it is possible to deduce the ratio of the gaugino masses at the Z-mass, yielding

$$\begin{aligned} M_3(m_Z) &= \tfrac{\alpha_3(m_Z)}{\alpha_2(m_Z)} M_2(m_Z) \approx \frac{1}{2} M_2 \\ M_1(m_Z) &= \tfrac{\alpha_1(m_Z)}{\alpha_2(m_Z)} M_2(m_Z) \approx \frac{7}{2} M_2. \end{aligned} \tag{9.16}$$

This implies that the masses of the gluinos to the winos to the bino behave as $7:2:1$. This implies that the gluino is the heaviest of the gauginos. Note that the masses are not necessarily the masses of the original gauge bosons in the electroweak sector. The bino and the winos will together with the higgsinos form the final mass eigenstates. Therefore, the running of the parameters will provide already guiding predictions on how supersymmetry breaking is provided in nature, if the observed mass pattern of these particles matches this prediction, provided once more that higher-order and non-perturbative corrections are small.

An even more phenomenologically relevant result is obtained by the investigation of the scalar masses, i. e., the masses of the Higgs, the squarks, and the sleptons. Retaining only the top-quark Yukawa coupling y_t, which dominates all other Yukawa couplings at one loop, and a unified three-scalar coupling A_0 the corresponding third-generation and

Higgs evolution equations are given by

$$\frac{dm_{H_u}^2}{d\ln Q} = \frac{1}{4\pi}\left(\frac{3X_t}{4\pi} - 6\alpha_2 M_2^2 - \frac{6}{5}\alpha_3 M_3^2\right) \tag{9.17}$$

$$\frac{dm_{H_d}^2}{d\ln Q} = \frac{1}{4\pi}\left(-6\alpha_2 M_2^2 - \frac{6}{5}\alpha_3 M_3^2\right)$$

$$\frac{dm_{\tilde{t}_L}^2}{d\ln Q} = \frac{1}{4\pi}\left(\frac{X_t}{4\pi} - \frac{32}{3}\alpha_1 M_1^2 - 6\alpha_2 M_2^2 - \frac{2}{15}\alpha_3 M_3^2\right) \tag{9.18}$$

$$\frac{dm_{\tilde{t}_R}^2}{d\ln Q} = \frac{1}{4\pi}\left(\frac{2X_t}{4\pi} - \frac{32}{3}\alpha_1 M_1^2 - \frac{32}{15}\alpha_3 M_3^2\right) \tag{9.19}$$

$$X_t = 2|y_t|^2\left(m_{H_u}^2 + m_{\tilde{t}_l}^2 + m_{\tilde{t}_r}^2 + A_0^2\right).$$

A number of remarks are in order. The quantity X_t emerges from the Yukawa couplings to the top quark. Therefore, the down-type Higgs doublet will, to leading order, not couple, and therefore its evolution equation will not depend on this strictly positive quantity. Furthermore, the Higgs fields couple to leading order not to the strong interaction, while the squarks do. Hence the former have no term depending on α_1, while the latter do. Similarly, only squarks from the left-handed chiral multiplet couple directly to the weak interactions, and therefore receive contributions from the weak interaction proportional to α_2. Still, all of these particles couple electromagnetically, and therefore receive contributions proportional to α_3.

Since all couplings α_i are strictly positive, the values of the masses can only decrease by the top-quark contribution X_t when lowering the scale Q. In particular, this implies that the mass of H_d will, at this order, only increase or at best stay constant. The strongest decrease will be observed for the H_u contribution, as the factor three in front of X_t magnifies the effect. Furthermore, the largest counteracting contribution ($7^2 = 49$ and $2^2 = 4$ and the largest α_1!) due to the gluino term is absent for the Higgs fields. Therefore, the mass of the H_u will decreases fastest from its unification value $\sqrt{\mu^2 + m_0^2}$ at the unification scale. In fact, this decrease may be sufficiently strong to drive the mass parameter negative at the electroweak scale. This would trigger electroweak symmetry breaking by the same mechanism as in the ordinary standard model. Using parameters which prevent the squark masses from becoming negative, and thus preserving vitally the color gauge symmetry, this indeed happens. Again, explicitly broken, but unified, supersymmetry provides the correct low-energy phenomenology of the standard model.

9.4.2 The electroweak sector

One of the most interesting questions in contemporary standard model physics is the origin of the value of the mass of the Higgs boson. In the standard model, this is an essentially independent parameter. However, in the minimal supersymmetric extension of the standard model, this parameter is less arbitrary, due to the absence of the hierarchy problem, and the fact that part of the Higgs couplings are determined by supersymmetry, like the four-Higgs coupling.

To determine the Higgs mass, it is first necessary to determine the electroweak symmetry breaking pattern. To do this, the first step is an investigation of the Higgs potential. This is more complicated than in the standard model case, due to the presence of a second Higgs doublet.

At tree-level, the quadratic term for the Higgs fields is determined by the contribution from the supersymmetric invariant term, and the two contributions from explicit breaking, yielding together

$$V_1 = (|\mu|^2+m_{H_u}^2)(H_u^+H_u^{+\dagger}+H_u^0H_u^{0+})+(|\mu|^2+m_{H_d}^2)(H_d^0H_d^{0\dagger}+H_d^-H_u^{-\dagger})+b(H_u^+H_d^- -H_u^0H_d^0)+h.c.,$$

where the Higgs fields are labeled by their electric instead of the hypercharge. The parameters $m_{H_i}^2$ can have, despite their appearance, both signs.

The quadratic part of the potential originates from two contributions. Both are from the D and F couplings for the groups under which the Higgs fields are charged, the weak isospin group SU(2) and the hypercharge group U(1). Since these contributions are four-point vertices, there are no contributions from explicit supersymmetry breaking. Thus, the quartic part of the potential takes the form

$$V_2 = \frac{e^2+g'^2}{8}(H_u^+H_u^{+\dagger}+H_u^0H_u^{0+}-H_d^0H_d^{0\dagger}-H_d^-H_d^{-\dagger})^2+\frac{g'^2}{8}|H_u^+H_d^{0\dagger}+H_u^0H_d^{-\dagger}|^2,$$

where g' is the gauge coupling of the weak isospin gauge group and e the one of the weak hypercharge group.

To obtain the experimentally measured electroweak phenomenology, this potential has to have a non-trivial minimum, especially if all other fields vanish. The latter also implies that the cubic coupling appearing in the Lagrangian due to the supersymmetry breaking are not relevant for this question, as they always involve at least one other field.

It is possible to simplify this question. The expressions are invariant under a local gauge transformation, as is the complete Lagrangian. If therefore any of the fields has a non-vanishing value at the minimum, it is always possible to perform a gauge transformation such that a specific component has this, and the other ones vanish. Choosing then H_u^+ to be zero, the potential must be extremal at this value of H_u^+, thus

$$\frac{d(V_1+V_2)}{dH_u^+}|_{H_u^+=0}=0.$$

Performing this derivative yields the condition

$$bH_d^- + \frac{g^2}{2}H_d^{0\dagger}H_u^{0\dagger}H_d^- = 0,$$

which can be fulfilled by either $H_d^-=0$ or a condition on the neutral component. Already from the fact that the electric symmetry may not be broken, it follows that the first possibility is adequate. However, the second alternative would in fact also not lead to a satisfactory formation of an extremum. Therefore, only the electrically neutral components of both Higgs doublets matter.

In this case, the potential simplifies to

$$V=(|\mu|^2+m_{H_u}^2)|H_u^0|^2+(|\mu|^2+m_{H_d}^2)|H_d^0|^2-(bH_u^0H_d^0+b^*H_u^{0\dagger}H_d^{0\dagger})+\frac{e^2+g'^2}{8}(|H_u^0|^2-|H_d^0|^2)^2. \tag{9.20}$$

In contrast to the standard model the four-Higgs coupling is fixed, and rather small, about 0.065. The latter is already indicative for a rather light Higgs boson, as will be seen below. The phase of b can be included in the Higgs fields, making itself real. Furthermore, if the Higgs fields would have a complex expectation value, the masses of the squarks and sleptons would become complex as well, which implies CP violation. Such an effect would have been measured, and therefore should not be permitted. Therefore, H_u^0 and H_d^0 must also be real. If there should exist a non-trivial minimum, this implies that the product $H_u^0H_d^0$ must be positive, as otherwise all terms would be positive. By a global U(1) gauge transformation, both fields can then be chosen to be positive.

The direction $H_u^0=H_d^0$ is pathologically, as the highest-order term vanishes, a so-called flat direction. For the potential to be still bounded from below, thus providing a perturbatively stable vacuum state, requires

$$2|\mu|^2+m_{H_u}^2+m_{H_d}^2>2b(>0).$$

Therefore, not the masses of both H_u^0 and H_d^0 can be negative simultaneously.

Studying furthermore systematically the derivatives yields

$$\frac{\partial V}{\partial H_u} = 2(|\mu|^2 + m_{H_u}^2)H_u^0 - bH_d^0 + \frac{e^2 + g^{'2}}{4}H_u^0(H_u^{02} - H_d^{02}) \tag{9.21}$$

$$\frac{\partial V}{\partial H_d} = 2(|\mu|^2 + m_{H_d}^2)H_d^0 - bH_u^0 - \frac{e^2 + g^{'2}}{4}H_d^0(H_u^{02} - H_d^{02}) \tag{9.22}$$

$$\frac{\partial^2 V}{\partial H_u^2} = 2(|\mu|^2 + m_{H_u}^2) + \frac{e^2 + g^{'2}}{4}(3H_u^{02} - H_d^{02})$$

$$\frac{\partial^2 V}{\partial H_d^2} = 2(|\mu|^2 + m_{H_d}^2) + \frac{e^2 + g^{'2}}{4}(H_u^{02} - 3H_d^{02})$$

$$\frac{\partial^2 V}{\partial H_u \partial H_d} = -b - \frac{e^2 + g^{'2}}{2}H_u^0 H_d^0 = \frac{\partial^2 V}{\partial H_d \partial H_u}. \tag{9.23}$$

If both masses would have the same sign, $H_u^0 = H_d^0 = 0$ could be a minimum, which is not helpful for the present purpose. Hence, either must be negative such that not all second derivatives have the same sign. This behavior is supported by the fact that only one of the masses is driven to potentially negative values by the renormalization group flow, as discussed previously.

It is not possible to restrict the solution and the parameters further just from the Lagrangian at this point. However, it is experimentally possible to fix at least a particular combination of the associated vacuum expectation values $\langle H_u^0 \rangle$ and $\langle H_d^0 \rangle$. Studying the coupling to the electroweak gauge bosons, the term which will produce the mass is the quartic one originating from the covariant derivative terms

$$(D_\mu H_u)^\dagger (D^\mu H_u) + (D_\mu H_d)^\dagger (D^\mu H_d)$$
$$D_\mu = \partial_\mu + ig' W_\mu + ieB_\mu,$$

where the coupling matrices have been included into the gauge fields. This yields the quartic contributions

$$(ig'W_\mu + ieB_\mu)H_u(ig'W^\mu + ieB^\mu)H_u + (ig'W_\mu + ieB_\mu)H_d(ig'W^\mu + ieB^\mu)H_d.$$

Introducing the combination $W_\mu^\pm = W_\mu^1 \pm W_\mu^2$, $Z_\mu = (-eB_\mu + g'W_3^\mu)/(e^2 + g^{'2})$ (and correspondingly the photon $\gamma_\mu = (eB_\mu + g'W_3^\mu)/(e^2 + g^{'2})$), performing the algebra, and setting the Higgs fields to their vacuum expectation values yields[5]

$$\left(\frac{1}{2}(e^2 + g^{'2})Z_\mu Z^\mu + \frac{1}{2}g^2(W_\mu^+ W^{\mu+} + W_\mu^- W^{\mu-})\right)(\langle H_u^0 \rangle^2 + \langle H_d^0 \rangle^2). \tag{9.24}$$

Thus, this yields the tree-level masses for the electroweak gauge bosons. The value for the combined condensate is therefore, according to experiment, $(174\ \mathrm{GeV})^2$.

[5]Which should be again understood to choosing a particular gauge.

It is furthermore possible to determine mass-bounds for the Higgs[6]. This is a bit more complicated than in the standard model case, as both Higgs fields will mix. Is is therefore useful to discuss this first for the example of two scalar fields $\phi_{1,2}$, which both condense with vacuum expectation values $v_{1,2}$, and which interact through some potential $V(\phi_1,\phi_2)$. The corresponding Lagrangian is then

$$\mathcal{L}=\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2-V(\phi_1,\phi_2).$$

Since the setting is perturbative, it is possible to expand V around the minima $\phi_i=v_i$ to obtain the masses and keeping only terms quadratic, yielding

$$\begin{aligned}\mathcal{L}_2 &= \partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2-\frac{1}{2}\frac{\partial^2V}{\partial\phi_1^2}(\phi_1-v_1)^2\\ &\quad -\frac{1}{2}\frac{\partial^2V}{\partial\phi_2^2}(\phi_2-v_2)^2-\frac{\partial^2V}{\partial\phi_1\partial\phi_2}(\phi_1-v_1)(\phi_2-v_2). \qquad (9.25)\end{aligned}$$

This can be simplified by setting

$$\Phi_i=\sqrt{2}(\phi_i-v_i)$$

as

$$\mathcal{L}_2=-\frac{1}{2}\Phi^T\partial^2\Phi-\Phi^TM\Phi$$

with

$$M=\frac{1}{2}\begin{pmatrix}\frac{\partial^2V}{\partial\phi_1^2} & \frac{\partial^2V}{\partial\phi_1\partial\phi_2}\\ \frac{\partial^2V}{\partial\phi_1\partial\phi_2} & \frac{\partial^2V}{\partial\phi_2^2}\end{pmatrix}.$$

Since the mass-matrix M is real and symmetric, it can be diagonalized by some orthogonal matrix

$$U=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix},$$

which can be used to change the basis for the fields as

$$\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}=U\Phi.$$

If the eigenvalues of M are given by $m_\pm^2$, the resulting Lagrangian takes the form

$$\mathcal{L}_2=-\frac{1}{2}(\phi_+\partial^2\phi_++\phi_-\partial^2\phi_-+m_+^2\phi_+^2+m_-^2\phi_-^2),$$

[6] It should be noted that in the following implicitly unitary gauge has been chosen. For the calculations performed, this is not relevant, except that it permits to expand H_u and H_d around their vacuum expectation value. However, the results are therefore not necessarily gauge-invariant, and should be interpreted with caution.

thus making the masses for the newly defined fields explicit. It should be noted that the matrix M has to be evaluated at $\phi_i = v_i$.

Returning to the Higgs fields in the minimal supersymmetric standard model, there are eight instead of two fields, but only two out of the eight have a non-vanishing vacuum expectation value. Examples of the relevant derivatives are given in (9.21-9.23). Using furthermore that the vanishing of the first derivatives implies certain relations for the values of $\langle H_u \rangle$ and $\langle H_d \rangle$, it is possible to simplify the ensuing expressions algebraically. Finally, defining

$$\tan\beta = \frac{\langle H_u \rangle}{\langle H_d \rangle},$$

the resulting expressions become relatively simple. It is furthermore useful that the mass-matrix assumes a block-diagonal form, so that it is possible to reduce the complexity further by studying only doublets at each time.

The first doublet to be studied are the (non-condensing) imaginary parts of H_u^0 and H_d^0. The corresponding mass-matrix takes the form

$$M_{i0} = \begin{pmatrix} b\cot\beta & b \\ b & b\tan\beta \end{pmatrix},$$

which has eigenvalues $m_+^2 = 0$ and $m_-^2 = 2b/\sin(2\beta)$. The massless mode, as it is uncharged, will become effectively the longitudinal component of the Z^0. The combination

$$\sqrt{2}(\cos\beta\Im(H_u^0) + \sin\beta\Im(H_D^0))$$

is the eigenvector to the second eigenvalue. This mass-eigenstate is commonly referred to as the A^0, and is a pseudo-scalar. This is an uncharged second Higgs field (the first one will be one of the condensed ones). It is one of the extra Higgs particles not present in the standard model. In practical calculations, the parameter b is often traded for the mass of this particle, $m_- = m_{A^0}$.

The next pair is H_u^+ and H_d^+. The corresponding mass-matrix is given by

$$M_{+-} = \begin{pmatrix} b\cot\beta + \frac{g'^2\langle H_d^0\rangle^2}{2} & b + \frac{g'^2\langle H_u^0\rangle\langle H_d^0\rangle}{2} \\ b + \frac{g'^2\langle H_u^0\rangle\langle H_d^0\rangle}{2} & b\tan\beta + \frac{g^2\langle H_u^0\rangle^2}{2} \end{pmatrix}.$$

The eigenvalues are 0 and $m_W^2 + m_{A^0}^2$, where m_W can be read off from (9.24). The positively charged and massless field combination of H_u^+ and $H_d^{-\dagger}$, the latter having positive charge as an anti-particle, will therefore become the longitudinal component of the W^+. The other state, usually just called H^+, corresponds to a positively charged Higgs particle, which is not appearing in the standard model. In much the same way the doublet $H_u^{+\dagger}$

and H_d^- yield the longitudinal component of the W^- and another negatively charged Higgs particle H^- also absent from the standard model.

Finally, the mass-matrix for the condensing real parts of H_u^0 and H_d^0 is given by

$$M = \begin{pmatrix} b\cot\beta + m_Z^2 \sin^2\beta & -b - \frac{m_Z^2 \sin(2\beta)}{2} \\ -b - \frac{m_Z^2 \sin(2\beta)}{2} & b\tan\beta + m_Z^2\cos^2\beta \end{pmatrix}$$

with eigenvalues

$$\begin{aligned} m_{h^0}^2 &= \frac{1}{2}\left(m_{A^0}^2 + m_Z^2 - \sqrt{(m_{A^0}^2 + M_Z^2)^2 - 4m_{A^0}^2 m_Z^2 \cos^2(2\beta)}\right) \\ m_{H^0}^2 &= \frac{1}{2}\left(m_{A^0}^2 + m_Z^2 + \sqrt{(m_{A^0}^2 + M_Z^2)^2 - 4m_{A^0}^2 m_Z^2 \cos^2(2\beta)}\right) \end{aligned}$$

for the two eigenstates h^0 and H^0, giving two more neutral, scalar Higgs particles. So instead of the one of the standard model, there are two in the minimal supersymmetric standard model.

The masses for the fields A^0, $H^\pm$ and H^0 are all containing a contribution $m_{A^0}\sqrt{2b/sin(2\beta)}$, which is unconstrained, and could, in principle, become arbitrarily large. This is not the case for the mass of h^0. If the A^0 mass would be small, it is possible to expand the root, yielding

$$m_{h^0}^2 \approx m_{A^0}^2 \cos^2(2\beta) \leq m_{A^0}^2.$$

For large masses of the A^0 it becomes

$$m_{h^0}^2 \approx m_Z^2 \cos^2(2\beta) \leq m_Z^2, \tag{9.26}$$

where only the experimental known mass of the Z-boson enters. Thus, though the bounds are not known, the mass is constrained. Since the experimental bounds for the A^0 indicate a rather large mass, the second expression (9.26) is more appropriate. Unfortunately, such a low mass for the lightest neutral Higgs boson is excluded experimentally. Even if the more precise formulas would be used, instead of the approximate ones, the situation is not improving qualitatively.

That could have been already a dismissal of the minimal version of a supersymmetric standard model. However, the leading quantum correction to this bound yields

$$m_{h^0}^2 \leq m_Z^2 + \frac{3m_t^4}{2\pi^2(\langle H_u\rangle^2 + \langle H_d\rangle^2)} \ln \frac{\sqrt{m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2}}{\sqrt{2}m_t}, \tag{9.27}$$

where m_t is the known top quark mass and $m_{\tilde{t}_i}$ are the masses of the staus after mixing between the left and right multiplets occurred, which, in principle, can be different. Already for masses of the order of the experimentally excluded stau masses the bound is increased by these radiative corrections above the Higgs mass. On the one hand, this is good for the minimal supersymmetric standard model, but on the other hand this implies that leading order corrections are large, and subleading corrections may be relevant. This reduces the predictiveness of the bound, as then the other parameters enter in various ways.

There is a further problem with the corrections (9.27) to the Higgs mass. Consider once more the conditions (9.21-9.22) for the appearance of a Higgs condensate. Combining both equations to remove the parameter b, and use the tree-level masses for the weak gauge bosons (9.24), yields the condition for condensation as

$$\frac{1}{2}m_Z^2 = -|\mu|^2 + \frac{m_{H_d}^2 - m_{H_u}^2 \tan^2\beta}{\tan^2\beta - 1}.$$

For example let $\tan\beta$ become large, i. e., the condensate $\langle H_u\rangle$ is much larger than the condensate $\langle H_d\rangle$. Then the condition becomes

$$\frac{1}{2}m_Z^2 = -|\mu|^2 - m_{H_u}^2.$$

Both parameters on the right-hand side are not constrained immediately by physics. However, if both parameters would be much larger than the mass of the Z boson, it would be necessary for them to cancel almost exactly. If this would be the case, it would immediately raise the same questions as in the original fine-tuning problem of the standard model if the Higgs mass would be much larger than the ones of the weak bosons. This is therefore also called the little fine-tuning or hierarchy problem. The experimentally established mass of the Higgs is, in fact, borderline. Together with the large lower limits for the masses of the other sparticles, the MSSM, though still consistent with experiment, becomes in fact increasingly fine-tuned, therefore abolishing its original motivation. Still, it is not excluded, and may yet describe nature.

That this is indeed somewhat of a problem becomes apparent when considering the renormalization constant for the mass of the Higgs boson, which is given by the integral of (9.17), and is approximately

$$\delta m_{H_u}^2 \approx -\frac{3y_t^2}{8\pi^2}(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2)\ln\left(\frac{\sqrt{2}\Lambda_U}{\sqrt{m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2}}\right), \qquad (9.28)$$

where it was assumed that the stau masses give the dominant contributions, due to the condition (9.27). The value of the unification scale is still of the order of 10^{15} GeV. If therefore the mass shift due to the leading quantum corrections should be small, this yields approximately that the geometric average of the stau masses should not be larger than about 150 GeV. Otherwise the quantum corrections alone would produce a fine-tuning problem. This condition is, however, in violation of the bound of 500 GeV necessary to shift the h^0 Higgs boson out of the current experimental reach. Therefore, it cannot be permitted. Shifting then the stau masses to the required value yields a mass correction large compared to the Z-mass, actually, it becomes exponentially worse due to the logarithmic dependence. Thus a fine-tuning problem arises. Whether this constitutes a problem, or just an aesthetic displeasure, is not only a question of personal taste. It also is a challenge to understand whether nature prefers for what reasons theories with or without finetuning.

9.4.3 Mass spectrum

So far, the masses of the Higgs bosons and the electroweak bosons have been calculated. The gluons also remain massless, in accordance with observation. The remaining mass spectrum for the minimal supersymmetric standard model at tree-level will be discussed in the following.

The first particles are the remaining ones from the standard model. These have to acquire, of course, their observed masses. Due to the parity violating nature of the weak interactions, these masses can effectively arise only due to the Yukawa interaction with the Higgs particles. In unitary gauge, these couplings can be split into a contribution which contains only the Higgs vacuum expectation values

$$y\chi\chi H = y\chi\chi\langle H\rangle + y\chi\chi(H - \langle h\rangle)$$

and behave therefore as mass terms. Thus, the masses are given by

$$\begin{aligned}
m_{uct} &= y_{uct}\langle H_u\rangle = y_{uct}\frac{\sqrt{2}m_W\sin\beta}{g'}\\
m_{dsb} &= y_{dsb}\langle H_d\rangle = y_{dsb}\frac{\sqrt{2}m_W\cos\beta}{g'}\\
m_{e\mu\tau} &= y_{e\mu\tau}\langle H_d\rangle = y_{e\mu\tau}\frac{\sqrt{2}m_W\cos\beta}{g'}\\
m_{\nu_e\nu_\mu\nu_\tau} &= y_{\nu_e\nu_\mu\nu_\tau}\langle H_u\rangle = y_{\nu_e\nu_\mu\nu_\tau}\frac{\sqrt{2}m_W\sin\beta}{g'}.
\end{aligned}$$

The twelve Yukawa couplings are all free parameters of the minimal supersymmetric standard model. As is visible, it strongly depends on the value of β, whether these couplings are strong, preventing perturbative descriptions, or weak enough that at sufficiently high energies perturbation theory is adequate.

The situation for the gluinos is actually simpler. Since the color symmetry is unbroken, no other fermions exist with the quantum numbers of the gluinos, and they do not couple to the parity violating weak interactions. Thus, the only contribution comes from the explicit supersymmetry breaking term

$$-\frac{1}{2}M_1\tilde{g}\tilde{g} - \frac{1}{2}M_1^*\tilde{g}^\dagger\tilde{g}^\dagger.$$

The mass parameter can be complex in general, but the corresponding tree-level mass will just be its absolute value. Therefore, the value of the mass of the gluinos is unconstrained in the minimal supersymmetric standard model, but by virtue of relation (9.16) it is tied to the masses of the other gauginos in case of unification.

The situation becomes more complicated for the so-called neutralinos, the superpartners of W^0 and B, the wino $\tilde{W}^0$ and the bino $\tilde{B}$. These fermions are electrically uncharged, and can both mix, similar to their standard-model versions. Furthermore, the neutral superpartners of the two Higgs fields, the higgsinos $\tilde{H}_d^0$ and $\tilde{H}_u^0$ are both also fermionic and uncharged. Hence, these can mix with the binos and the winos as well. The gauge eigenstate is therefore

$$\tilde{G}^{0T} = (\tilde{B}, \tilde{W}^0, \tilde{H}_d^0, \tilde{H}_u^0).$$

The most direct mixing is due to the interaction mediated by the weak F-boson in the Wess-Zumino-like contribution to the superpotential, which leads to the contributions

$$\frac{1}{2}\mu(\tilde{H}_u^0\tilde{H}_d^0 + \tilde{H}_d^0\tilde{H}_u^0) + \frac{1}{2}\mu^*(\tilde{H}_u^{0\dagger}\tilde{H}_d^{0\dagger} + \tilde{H}_d^{0\dagger}\tilde{H}_u^{0\dagger}).$$

Furthermore, the weak gauge symmetry and supersymmetry demand the existence of couplings of the generic type

$$-\sqrt{2}g'H_u\tilde{H}_u\tilde{W} = -\sqrt{2}g'(H_u + \langle H_u\rangle)\tilde{H}_u\tilde{W} \tag{9.29}$$

to ensure supersymmetry in the super Yang-Mills part of the weak symmetry. If the Higgs fields condense, these yield a mixing term proportional to the condensates. The condensate is uncharged, and therefore this mixing can only combine two neutral fields or two of opposite charge. Hence, only the fields $\tilde{H}_d^0$ and $\tilde{H}_u^0$ will mix with the neutral wino and the bino by this mechanism. Finally, the wino and the bino can have masses due to the explicit supersymmetry breaking, similar to the gluinos,

$$-\frac{1}{2}M_2\tilde{W}^0\tilde{W}^0 - \frac{1}{2}M_3\tilde{B}\tilde{B} + h.c..$$

No such contribution exist for the higgsinos, as it is not possible to write down such a term while preserving explicitly the weak gauge symmetry. Hence, the total mass-matrix, with all mixing terms, takes the form

$$
\begin{aligned}
&-\frac{1}{2}\tilde{G}^0 M_{\tilde{G}^0}\tilde{G}^0 + h.c. \\
M_{\tilde{G}^0} &= \begin{pmatrix} M_3 & 0 & -\cos\beta\sin\theta_W m_Z & \sin\beta\sin\theta_W m_Z \\ 0 & M_2 & \cos\beta\cos\theta_W m_Z & -\sin\beta\cos\theta_W m_Z \\ -\cos\beta\sin\theta_W m_Z & \cos\beta\cos\theta_W m_Z & 0 & -\mu \\ \sin\beta\sin\theta_W m_Z & -\sin\beta\cos\theta_W m_Z & -\mu & 0 \end{pmatrix}.
\end{aligned}
$$

Thus, all four fields mix. The mass eigenstates are denoted by $\tilde{\chi}_i$ with $i = 1, ..., 4$, and are called neutralinos. These particles interact only weakly, like the neutrinos, and hence their name. Since the a-priori unknown parameters M_2, M_3 and μ, as well as β, enter their mass matrix, their masses cannot be predicted. However, if any of the neutralinos would be the lightest supersymmetric particles, then by virtue of R-parity conservation it would be stable. Since it interacts so weakly, it would be a perfect candidate for dark matter, which cannot be provided by neutrinos since their mass is too small. In fact, at least for some range of parameters the masses of the neutralinos would be such that they are perfectly compatible with the properties required for cold, non-baryonic, dark matter.

A similar situation arises for the charged counterpart of the neutralinos, the charginos. These stem from the mixing of the charged higgsinos and the charged winos. Since the positively and negatively charged particles cannot mix, two doublets appear instead of one quartet,

$$
\begin{aligned}
\tilde{G}^+ &= \begin{pmatrix} \tilde{W}^+ \\ \tilde{H}_u^+ \end{pmatrix} \\
\tilde{G}^- &= \begin{pmatrix} \tilde{W}^- \\ \tilde{H}_d^- \end{pmatrix}
\end{aligned}
\tag{9.30}
$$

where the charged winos are linear combinations $\tilde{W}^1 \pm i\tilde{W}^2$ of the off-diagonal winos. Similar as in the case of the neutralinos, there is a contribution from the weak F-term and the condensation of the neutral Higgs fields from the super Yang-Mills action, which mixes both components of each doublet. Finally, there is again the explicit supersymmetry breaking mass of the winos. All in all, the corresponding mass term is

$$
-\frac{1}{2}(\tilde{G}^{+T} M_{\tilde{G}^\pm}^T \tilde{G}^- + \tilde{G}^{-T} M_{\tilde{G}^\pm}\tilde{G}^+) + h.c. \tag{9.31}
$$

$$
M_{\tilde{G}^\pm} = \begin{pmatrix} M_2 & \sqrt{2}\sin\beta m_W \\ \sqrt{2}\cos\beta m_W & \mu \end{pmatrix}.
$$

The structure of the mass term is dictated by the (hypercharge) gauge symmetry. To obtain the masses, it is necessary to diagonalize the mass-matrix, and identify the actual masses. Unitary matrices V and U can be taken to transform the charged gauge eigenstates into the chargino mass eigenstates

$$\begin{aligned}\tilde{\chi}^{+} &= V\tilde{G}^{+}\\ \tilde{\chi}^{-} &= U\tilde{G}^{-}.\end{aligned}$$

To obtain the mass eigenstates, U and V will be chosen such that $U^{*}M_{\tilde{G}^{\pm}}V^{-1}$ is diagonal, as is required to have a diagonal structure in both terms in (9.31). However, when determining the associated propagators, from which the pole mass can be read off, not the quantity $U^{*}M_{\tilde{G}^{\pm}}V^{-1}$ appears, but rather the quadratic version $VX^{\dagger}XV^{-1} = UXX^{\dagger}U^{T}$, as a consequence of the inversion and the structure of a fermion propagator. The equality follows trivial from linear algebra. Performing the algebra leads to masses of the charginos

$$|m_{\tilde{\chi}_i^{\pm}}|^2 = \frac{1}{2}\left(M_2^2 + |\mu|^2 + 2m_W^2) \mp \sqrt{(M_2^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin(2\beta)|^2}\right)$$

where the upper and lower sign refer to the two members of each charged doublet. Therefore there is always one pair of oppositely charged charginos having the same mass, as would be expected from CPT.

Thus remains the largest group of additional particles, the sfermion superpartners of the fermionic standard model particles. These contain the squarks and sleptons. Fortunately, due to the presence of the strong charge, squarks and sleptons will not mix. However, the three families within each sector could in principle mix. Since such mixings would have to be very small to be not in conflict with flavor changing currents observed in experiments, these will be neglected[7]. Furthermore, compared to the scale of the supersymmetric parameters, only the Yukawa couplings of the third family can give any significant contribution, and will only be considered in this case. Therefore, the first two and the third family will be treated in turns.

If a common mass at some unification scale can be expected then the masses will run

[7]In fact, the absence of strong mixing sets strong limits on the properties of the mass matrices in the squark and slepton sector. Since these are only parameters in the MSSM, this raises the questions why the should be so specifically shaped. It is often assumed that whatever mechanism drives the flavor physics of the standard model will also be responsible for this feature of the MSSM.

according to

$$
\begin{aligned}
m^2_{\tilde{q}L} &= m_0^2 + K_1 + K_2 + \frac{1}{9}K_3 \\
m^2_{\tilde{u}_R,\tilde{c}_R} &= m_0^2 + K_1 + \frac{16}{9}K_3 \\
m^2_{\tilde{d}_R,\tilde{s}_R} &= m_0^2 + K_1 + \frac{4}{9}K_3 \\
m^2_{\tilde{l}_L} &= m_0^2 + K_2 + K_3 \\
m^2_{\tilde{e}_R,\tilde{\mu}_R} &= m_0^2 + 4K_3 \\
m^2_{\nu_{\tilde{e}R},\nu_{\tilde{\mu}R}} &= m_0^2.
\end{aligned}
$$

The contributions K_i are the ones from the various interactions, weighted accordingly with the corresponding weak isospin and hypercharge quantum numbers. Since to this order the Yukawa interactions can be neglected, there are only the contributions from the interactions which all tend to increase the mass, except for the right handed sneutrinos to this order. The strong interaction K_1 will give the largest contribution. Hence, the squark masses will tend to be larger than the slepton masses.

On top of these contributions stemming purely from the soft breaking of the supersymmetry, there are again additional effects from the coupling to the Higgs fields. This coupling arises, since the equation of motion for the (weak isospin) D boson yields

$$D = g'(\tilde{q}^\dagger\tilde{q}^\dagger + \tilde{l}^\dagger\tilde{l}^\dagger + H_u^\dagger H_u + H_d^\dagger H_d),$$

which by virtue of the term D^2 in the Lagrangian yield terms of the type

$$(\tilde{q}^\dagger\tilde{q})g'^2(\langle H_u^0\rangle^2 - \langle H_d^0\rangle^2).$$

Similarly, the hypercharge D boson contributes

$$e^2\frac{-y}{2}(\tilde{q}^\dagger\tilde{q})(\langle H_u^0\rangle^2 - \langle H_d^0\rangle^2),$$

where y is the corresponding weak hypercharge. In total the contribution $\Delta_{\tilde{f}}$ for each sfermion with weak isospin quantum number t_3 and charge $c = y/2 + t_3$ is

$$\Delta_{\tilde{f}} = m_Z^2\cos(2\beta)\left(t_3 - c\sin^2\theta_W\ .\right)$$

As a consequence the weak isospin splittings give a measure of β as

$$m^2_{\tilde{d}_L} - m^2_{\tilde{u}_L} = m^2_{\tilde{e}_L} - m^2_{\tilde{\nu}_e} = -m_W^2\cos(2\beta),$$

and in the same manner for the second family. This constraint could be used to experimentally verify and/or predict parameters of the theory. The present experimental constraints favor a rather large β, such that the down-type sfermions would be heavier.

There is a third source for the masses of the sfermions. These are due to contributions from the F-bosons of chiral multiplets and the associated superpotential. However, when neglecting all but third-family Yukawa couplings, this will only contribute in case of the third family, which will be considered now. The contributions in the superpotential are, e. g., for the stop from the right-handed top

$$W = y_t \tilde{t}_R \tilde{t}_L H_u^0.$$

Integrating out the associated F field by its equation of motion yields a term

$$-y_t^2 \tilde{t}_L^\dagger \tilde{t}_L H_u^0 H_u^{0\dagger} = -y_t^2 \tilde{t}_L^\dagger \tilde{t}_L \langle H_u^0 \rangle^2 - y_t^2 \tilde{t}_L^\dagger \tilde{t}_L (H_u^0 H_u^{0\dagger} - \langle H_u^0 \rangle^2).$$

Thus, the stop gets a further contribution $y_t \langle H_u^0 \rangle$ to its mass, which is exactly the same mass as for the top. Thus, the stop has at least as much mass as the top. An equivalent contribution also appears for the stop from the right-handed multiplet, and actually for all other multiplets. That this mass is the same as for the super-partners is not surprising, as it emerges from the supersymmetric part of the action, and thus will not spoil the mass degeneracy of the super-partners. However, due to the term $\mu H_u^0 H_d^0$, there is also a mixing between the left-handed and right-handed stops,

$$\mu m_t \cot\beta (\tilde{t}_R^\dagger \tilde{t}_L + \tilde{t}_L^\dagger \tilde{t}_R) - A_0 m_t (\tilde{t}_R^\dagger \tilde{t}_L + \tilde{t}_L^\dagger \tilde{t}_R) \tag{9.32}$$

and for all other squarks and sleptons as well. The second term appears from the soft supersymmetry breaking trilinear scalar couplings. Hence, the mass eigenstates are not the charge eigenstates for the third top. Within this approximation, this is not the case for the first two families. These effects come on top of the mass contributions discussed previously for the first two families.

The total mass term for the stop is therefore

$$-(\tilde{t}_L^\dagger \tilde{t}_R^\dagger) M_{\tilde{t}} \begin{pmatrix} \tilde{t}_L \\ \tilde{t}_R \end{pmatrix}$$

with the mass matrix

$$M_{\tilde{t}_L} = \begin{pmatrix} m_{\tilde{t}}^2 + m_t^2 + \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right) m_Z^2 \cos(2\beta) & m_t^2 (A_0 - \mu\cos\beta) \\ m_t^2 (A_0 - \mu\cot\beta) & m_{\tilde{t}_R}^2 + m_t^2 - \frac{2}{3}\sin^2\theta_W m_Z^2 \cos(2\beta) \end{pmatrix}.$$

In contrast to the first two families, the $m_{\tilde{t}}$ terms are now given by

$$\begin{aligned} m_{\tilde{t}_L}^2 &= x_t + m_0^2 + K_1 + K_2 + \frac{1}{9} K_3 \\ m_{\tilde{t}_R}^2 &= x_t + m_0^2 + K_1 + \frac{16}{9} K_3 \end{aligned}$$

where the x_t term stems from the X_t contribution in the evolution equations (9.18) and (9.19). As in the case of the Higgs fields, the mass-matrix has to be diagonalized. Since m_t appears in the off-diagonal entries the mixing will be very strong, as long as A_0 and $\mu\cos\beta$ are not surprisingly similar. In particular, the off-diagonal elements may be negative, and together with the contribution x_t this effect tends to decrease the stop masses. In fact, in most scenarios the stop will be the lightest of the squarks, and in some regions of the parameter space the mass would be driven to negative values and thus initiate the breaking of color symmetry, excluding these values for the parameters.

The situation for the sbottom, stau, and stau sneutrino is similar, just that all parameters are exchanged for their equivalent of that type. The mixing in these cases will be much smaller, since m_b, m_τ and m_{ν_τ} are all much smaller, and could be neglected. In particular, also the contribution x_t will be essentially negligible, and therefore the masses of these super-partners will be larger than the ones of the stops.

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