Search for Gribov Copies Outside the First Griobov Region Using an Oriented Center Vortex Field Configuration

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Abstract

For making calculations in a gauge theory it is useful to fix a gauge. In non-abelian gauge theories, however, it does not suffice to impose only local conditions and fixing a gauge becomes much more complicated than for a commutative gauge group. For this purpose the Faddeev-Popov operator $M^{ab} = -\partial_\mu(\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu)$ is of interest, because the in Landau gauge $\partial_\mu A^a_\mu = 0$ arising ambiguity of the gauge fields can be resolved by restricting them to the first Gribov region, where the Faddeev-Popov operator is positive definite. Alternatively one could average over all Gribov copies in a suitable way to deal with the Gribov ambiguity. This would require to find solutions outside the first Gribov region - meaning solutions of the Faddeev-Popov operator with negative eigenvalues.
To look for such solutions in a $SU(2)$ Yang-Mills theory using an oriented center vortex field configuration ansatz is the purpose of this work.

Contents

1 Introduction

In the context of gauge theory the term gauge refers to controlling redundant degrees of freedom in the equations. Transformations between physically equivalent but mathematically distinct solutions are referred to as gauge transformations and form a group (more specifically a Lie Group) called gauge group, with the most well known example being the ambiguity of the electromagnetic potentials. In the classical example of electromagnetism this ambiguity is easily resolved by introducing a simple local condition - most commonly the Lorentz gauge condition

$$\boldsymbol{A} + \frac{1}{c^2}\partial_t V = 0 \tag{1}$$

which is usually written in relativistic notation as

$$\partial_\mu A^\mu = 0 \tag{2}$$

and in the more general context of gauge theory it is known as Landau gauge.
When dealing with theories where two gauge transformations do not necessarily commute, as is the case for the weak and strong interaction, the gauge group is said to be non commutative (or non abelian). In such non abelian gauge theories, the condition (2) is no longer sufficient. Instead, even when imposing (2), there are still gauge copies, in this context called Gribov copies, in what is called the residual gauge orbit, which are mathematically distinct from each other but describe the same physics.
To resolve this issue, the Faddeev-Popov operator was introduced as:

$$M^{ab} = -\partial_\mu D^{ab}_\mu = -\partial_\mu(\partial_\mu \delta^{ab} + g f^{abc} A^c_\mu) \tag{3}$$

The remaining ambiguity in the gauge fields A^c_μ describing a force field in analogy to the electromagnetic potentials can be resolved by restricting the values of these gauge fields such that the Faddeev-Popov operator is positive semi-definite [3]. The region in field configuration space, where this is the case, is called the first Gribov region. Correspondingly, Gribov copies outside the first Gribov region correspond to the Faddeev-Popov operator having at least 1 negative eigenvalue. Furthermore, when coupling the gauge fields A^c_μ to the quantum mechanical wave equations decribing matter, these matter fields will exhibit a corresponding symmetry under the gauge transformations. It turns out that this procedure can also be inversed in a way that derives the coupling of a force to a particle by demanding the symmetry of the equations describing its dynamics under a given transformation group, which is achieved by introducing the necessary counter terms into the equations. This is known as the gauge principle and relates to the gauge covariant derivative D_μ, which appears in so called adjoint representation in (3).
For introducing the gauge principle and with it the gauge covariant derivative, the classic electrodynamic field and its coupling to the non-relativistic Schrödinger equation and subsequently to the non-quantized Dirac equation serve as a good starting point. At the same time the used conventions for the following calculations will be defined. The rest of this section closely follows [1].

1.1 Electrodynamics

In the following natural Lorentz-Heaviside units will be used, which corresponds to setting $\epsilon_0 = \mu_0 = c = \hbar = 1$.

Therefore Maxwell's equations read:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho_{em} & \boldsymbol{\nabla} \times \boldsymbol{E} &= -\partial_t \boldsymbol{B} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 & \boldsymbol{\nabla} \times \boldsymbol{B} &= \partial_t \boldsymbol{E} + \boldsymbol{j_{em}} \end{aligned} \tag{4}$$

To write this system of equations in a mathematically more convenient way, the scalar potential V and vector potential $\boldsymbol{A}$ are introduced via the relations:

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \partial_t \boldsymbol{A} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{5}$$

While the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are observables and therefore unique, (5) does not specify the electromagnetic potentials uniquely. Instead the electric and magnetic fields are invariant under the simultaneous transformations:

$$V \to V' = V - \partial_t \chi \qquad \boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla} \chi \tag{6}$$

With an arbitrary differentiable function $\chi(t, \boldsymbol{x})$.

Moreover Maxwell's equations can be written in Lorentz covariant form. For this purpose the following 4-vectors are defined using the signature $(+,-,-,-)$ for the Minkowski product:

$$(x^\mu) = (t, \boldsymbol{x}) \quad (A^\mu) = (V, \boldsymbol{A}) \quad (\partial^\mu) = (\partial_t, -\boldsymbol{\nabla}) \quad (j^\mu_{em}) = (\rho_{em}, \boldsymbol{j_{em}}) \tag{7}$$

Unbolded vectors will always refer to the corresponding 4-vector. The difference between 4-vectors and scalars should be clear from context.

With the new 4-potential (A^μ) a gauge transformation as in (6) can be written as:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi \tag{8}$$

By plugging (5) into (4) and using the definition of the 4-potential, Maxwell's equations can be expressed in gauge inpendent form as:

$$\Box A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu_{em} \tag{9}$$

Where $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian operator.

To write Maxwell's equations in relativistic covariant form with gauge independent quantities a relativistic analogue to the electric and magnetic fields is required and it is obvious that this object would have to depend on both. From (9) a Lorentz tensor of rank 2 can readily be identified as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{10}$$

This is the electromagnetic field strength tensor. In matrix form it reads:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11}$$

From the definition of the field strength tensor and (9) it follows immediately that Maxwell's equations can now be written as:

$$\partial_\mu F^{\mu\nu} = j^\nu_{em} \tag{12}$$

1.2 Quantum Mechanics

In classical mechanics the Hamiltonian of a charged particle with mass m and charge q in an external electromagnetic field is:

$$H(\boldsymbol{p}, \boldsymbol{x}) = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + qV \tag{13}$$

The corresponding Hamilton operator for a quantum particle can be found by substituting $\boldsymbol{p} \to -i\boldsymbol{\nabla}$ as well as $E \to i\partial_t$ and leads to the following Schrödinger equation:

$$\left(\frac{1}{2m}(-i\boldsymbol{\nabla} - q\boldsymbol{A})^2 + qV\right)\psi(t, \boldsymbol{x}) = i\partial_t\psi(t, \boldsymbol{x}) \tag{14}$$

This corresponds to the Schrödinger equation of a free particle after the substitution:

$$\begin{aligned} \partial_t &\to D^0 \equiv \partial_t + iqV \\ \boldsymbol{\nabla} &\to \boldsymbol{D} \equiv \boldsymbol{\nabla} - iq\boldsymbol{A} \end{aligned} \tag{15}$$

However, as seen in the previous section, the electromagnetic potentials are not unique and therefore a gauge transformation must change the wave function as well. Since the particle's distributions for all observables must be gauge invariant, then in particular $|\psi(t, x)|^2$, defining the probability distribution in position space, must be gauge invariant. This can only be achieved, if a gauge transformation of the wave function changes it by only some phase factor $e^{i\alpha}$. Though this phase may depend on $(t, \boldsymbol{x})$:

$$\psi(t, \boldsymbol{x}) \to \psi'(t, \boldsymbol{x}) = e^{i\alpha(t,\boldsymbol{x})}\psi(t, \boldsymbol{x}) \tag{16}$$

Applying the ansatz (16) to (14) leads directly to:

$$\left.\begin{aligned} V \to V' &= V - \partial_t\chi(t, \boldsymbol{x}) \\ \boldsymbol{A} \to \boldsymbol{A}' &= \boldsymbol{A} + \boldsymbol{\nabla}\chi(t, \boldsymbol{x}) \end{aligned}\right\} \implies \psi(t, \boldsymbol{x}) \to \psi'(t, \boldsymbol{x}) = e^{iq\chi(t,\boldsymbol{x})}\psi(t, \boldsymbol{x}) \tag{17}$$

1.3 Quantum Electrodynamics

In the previous subsection a quantum particle in an external electromagnetic field was considered. In doing so, the effect of this moving charged particle on the electromagnetic field was neglected. To gain a fully interacting quantum theory of the electromagnetic interaction, the electromagnetic field must be quantized and the mutual dependece of it and charged particles have to be considered.
Moreover, while Maxwell's equations are relativistic, the Schrödinger equation is derived from the Hamiltonian of a classical particle and as such non relativistic. The suitable relativistic equation describing the behavior of a free fermion was found by Dirac:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{18}$$

where γ^μ are the 4×4 Dirac matrices and ψ is a four component field with special transformation behavior, called a bispinor. This bispinor would usually be subjected to a process of second quantization by reinterpreting the four component wave function in (18) as an annihilation operator of a fermion at spacetime coordinates x. For this work, however, it is sufficient to consider classical non-quantized field theory.
For further examination of the Dirac equation it is most useful to be able to derive it from a variational principle. For this purpose the action functional

$$S[\phi^a] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi^a(x), \partial_\mu \phi^a(x)) \tag{19}$$

is considered. Here the ϕ^a denote the components of a multivalued field ϕ.
From the Lagrangian density $\mathcal{L}$ the equations of motions can be derived by demanding $\delta S = 0$ leading to the Euler-Lagrange equations [4]:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^a} = 0 \tag{20}$$

In the case of the Dirac equation a valid Lagrangian density can be found to be:

$$\mathcal{L}_D = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \tag{21}$$

With the Dirac conjugate field $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$.
This Lagrangian is obviously invariant under multiplication of $\psi(x)$ with any phase factor. It is said to exhibit a global $U(1)$ symmetry.
Now the procedure that was used to identify the correct gauge transformation from the known interaction, can be reversed to obtain an iteracting theory by explicitly demanding the Lagrangian to be invariant under multiplication of the wave function with a spacetime dependant phase factor. For the Lagrangian to then be invariariant under this local $U(1)$ transformation, additional terms in the Dirac equation have to be included to compensate the terms coming from the differentiation of the exponential. While this could theoretically be achieved in many ways, the simplest one is to make a substitution equivalent to (15) which is known as minimal coupling:

$$\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu \tag{22}$$

In this context the modified differential operator D^μ is called (gauge) covariant derivative.
Adding to that the well known Lagrangian of the electromagnetic field

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{23}$$

yields the full Lagrangian of quantum electrdynamics:

$$\mathcal{L}_{QED} = \overline{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{24}$$

The variables by which to vary are then $\overline{\psi}$ and A and result in a Dirac equation coupled to the electromagnetic potentials via the substition in (22) and Maxwell's equations that contain a current density coupling it to the Dirac equation by $j^\mu_{em} = q\overline{\psi}(x)\gamma^\mu\psi(x)$ leading to a coupled system of equations describing the fully interacting theory.

1.4 Gauge Theory

In the previous sections it was shown that given a Lagrangian of a free theory exhibiting a global $U(1)$ symmetry one can enforce this symmetry to be local instead through the substitution (22) by introducing a new gauge field that transforms just like the electromagnetic 4-potential. In fact, it turns out that interpreting this gauge field as the electromagnetic 4-potential is indeed correct. This procedure of taking a global continuous symmetry and enforcing it to be a local one by introducing a sufficient number of gauge fields is generally known as the gauge principle. The number of gauge fields to be introduced is then given by the dimension of the continuous group (Lie group) which was $dim(U(1)) = 1$ for the former case.
A Lie group element h can be written in terms of an exponential function:

$$h = exp(X) \tag{25}$$

With X being an element of the associated Lie algebra. The Lie algebra $\mathfrak{g}$ of an abstract Lie group G is its tangent space at the identity equipped with a so-called Lie bracket as a sort of commutator. But the most commonly encountered Lie groups are matrix groups, in which case the Lie algebra is a real subspace of some matrix space $\mathbb{C}^{n\times n}$ with the actual matrix commutator as the Lie bracket and the exponential function is then just the matrix exponential defined via its convergent power series. Such is the case for the groups $SU(2)$ and $SU(3)$, which are the symmetry groups associated with the weak and strong force respectively. Both of these fall under the category of Yang-Mills theories, which are based on $SU(n)$ gauge groups. Since these lie at the heart of the Standard Model of particle physics, further discussion will mostly be restricted to $SU(n)$. To apply the gauge principle in a similar fashion as in the previous section, equation (25) above can be rewritten by choosing a basis in $\mathfrak{g}$:

$$h = exp(ig\chi^a X^a) \tag{26}$$

The X^a here are called the generators of the Lie group, since they do indeed generate it via the exponential function. The factor ig is just a convention that is used mostly in physics and does not appear in more mathematical leaning literature. But the g here is important in physics, since it determines how strongly the gauge fields couple to an equation and is therefore callled coupling constant.

The transformation (26) can then be turned into a local one by making the coefficients χ^a into spacetime dependent functions $\chi^a(x)$. Differentiaton of the matrix exponential follows the same rule as for scalars and the covariant derivative can be derived as:

$$D_\mu = \partial_\mu + igA_\mu \tag{27}$$

Because a Lagrangian in general does only depend on the fields and their first derivatives, minimal coupling in the Lagrangian with the covariant derivative (27) leads to the interaction terms.

2 Faddeev-Popov Operator in the Vacuum

The approach taken here follows [2] and revolves around finding solutions of the Faddeev-Popov operator in Euclidean space time. Therefore there is no difference between co- and contravariant vectors and from here on the coordinate vector is written as

$$(x^\mu) = (x_\mu) = (x, y, z, t) \tag{28}$$

The Faddeev-Popov Operator was defined in (3).
The quantities f^{abc} are the so-called structure constants and are defined as:

$$-i[X^a, X^b] = f^{abc} X^c \tag{29}$$

which is a meaningful definition since the generators X^a constitute a basis of the Lie algebra and therefore all Lie algebra elements have a unique expansion in them. The structure constants do, however, depend on the generators and are not unique for a given Lie algebra.

To apply this equation to the gauge group $SU(2)$, a suitable set of generators has to be found first. Conveniently the well-known Pauli matrices do generate $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{30}$$

For the commutator of the Pauli matrices one gets:

$$[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k \tag{31}$$

where ϵ^{ijk} of course denotes the Levi-Civita symbol.
To simplify the following equations by a bit, it is more convenient to choose $\frac{\sigma_i}{2}$ as generators and with these, the structure constants for $SU(2)$ are:

$$f^{abc} = \epsilon^{abc} \tag{32}$$

Unlike for spacetime indices where co- and contravariant tensors usually have to be distinguished clearly, no distinction has to be made for the indices enumerating the gauge fields in general.

Now given this rather convenient set of generators, the Faddeev-Popov operator in $SU(2)$ reads:

$$M^{ab} = -\partial_\mu(\delta^{ab}\partial_\mu + g\epsilon^{abc}A^c_\mu) \tag{33}$$

And in the vacuum $A^c_\mu = 0$ this obviously simplifies to:

$$M^{ab} = -\delta^{ab}\partial_\mu\partial_\mu = -\delta^{ab}\Delta \tag{34}$$

So in the vacuum the equation simplifies to 3 decoupled 4-dimensional Laplace operators.
The spectrum of the negative of the Laplace operator is, of course, non-negative and so no eigenfunctions for negative eigenvalues can exist. But solving the equation for non-negative eigenvalues is still instructive and useful and the solutions which will be respresented in bipolar coordinates will be used to analyze the equation in the oriented center vortex field ansatz.
Bipolar coordinates are just a pair of normal 2-dimensional polar coordinates defined as:

$$x = r\sin\theta \qquad y = r\cos\theta \qquad z = \rho\sin\eta \qquad t = \rho\cos\eta \tag{35}$$

The Laplace Operator in this coordinate system is then

$$\begin{aligned} \Delta &= \Delta_{r,\theta} + \Delta_{\rho,\eta} \\ \Delta_{r,\theta} &= \frac{1}{r}\partial_r r + \frac{1}{r^2}\partial_\theta^2 \\ \Delta_{\rho,\eta} &= \frac{1}{\rho}\partial_\rho \rho + \frac{1}{\rho^2}\partial_\eta^2 \end{aligned} \tag{36}$$

And the divergence of a vector field $(A_\mu) = A_r e^r + A_\theta e^\theta + A_\rho e^\rho + A_\eta e^\eta$ is

$$div(A) = \partial_\mu A_\mu = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \frac{1}{\rho}\partial_\rho(\rho A_\rho) + \frac{1}{\rho}\partial_\eta A_\eta \tag{37}$$

The eigenvalue equation to the Faddeev-Popov operator in the vacuum is then

$$-\Delta \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} = \omega^2 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} \tag{38}$$

Since these equations are completely decoupled, they can be solved individually. A straight forward seperation ansatz $\phi^a(r,\theta,\rho,\eta) = \Psi^a(r,\theta)\Omega^a(\rho,\eta)$ yields

$$-\Delta_{r,\theta}\,\Psi^a = C\,\Psi^a \tag{39a}$$
$$-\Delta_{\rho,\eta}\,\Omega^a = (\omega^2 - C)\,\Omega^a \tag{39b}$$

Equation (39a) now implies that $C > 0$ since the spectrum of the negative of the Laplacian is strictly positive in all dimensions. But then (39b) restricts C further, since now $(\omega^2 - C) > 0$ as well for the same reason. With $\omega^2 > 0$ one can substitute $C = s^2\omega^2$, with the conditions for C translating to $0 < s < 1$. The solution to the 2 dimensional Laplacian in polar coordinates can be explicitly found by applying another seperation ansatz. Since this is, however, a well-known result the solution to (39) shall directly be given here as:

$$\Psi(r,\theta) = J_{|n|}(s\omega r)e^{in\theta} \tag{40a}$$
$$\Omega(\rho,\eta) = J_{|m|}(\sqrt{1-s^2}\,\omega\rho)e^{im\eta} \tag{40b}$$

The product of (40a) and (40b) then gives a solution to (38) for all values of $n, m \in \mathbb{Z}$ and $0 < s < 1$. Since the Faddeev-Popov operator is linear, any sum or integral (with respect to n, m and s) over these families of solutions, will again yield a solution to the equation. Moreover it should be noted that with the Laplacian having a purely continuous spectrum, none of the solutions obtained in (40) can be normalizable. Instead one would obtain normalizable wave functions by superposing solutions of different eigenvalues in the same way that normalizable wave functions of the free Hamiltonian in Quantum Mechanics are constructed from plane wave solutions via Fourier synthesis. Therefore the primary boundary conditions set for solutions in the next chapter will not be for them to be normalizable, but only to be bounded and vanishing at infinity.

3 Faddeev-Popov Operator in an Oriented Center Vortex Field

In the previous section it was shown that in the vacuum the Faddeev-Popov operator reduces to the Laplacian and as such does not admit eigenfunctions for negative eigenvalues (neither proper nor improper). Leaning on the approach taken in [2], an oriented center vortex field configuration will be used to examine the Faddeev-Popov operator and different profile functions will be used to attempt to find solutions for negative eigenvalues. Working in bipolar coordinates the ansatz for the vector fields $(A^a_\mu) = A^a_r e^r + A^a_\theta e^\theta + A^a_\rho e^\rho + A^a_\eta e^\eta$ in the eigenvalue equation will be:

$$A^a_\eta = \delta^{3a}\frac{1}{g}\frac{\mu(\rho)}{\rho} \qquad\qquad A^a_r = A^a_\theta = A^a_\rho = 0 \tag{41}$$

where for an oriented center vortex the profile function $\mu(\rho)$ must have the asymptotic behavior

$$\mu(\rho) \xrightarrow[\rho\to 0]{} K \qquad\qquad \mu(\rho) \xrightarrow[\rho\to\infty]{} 2n+1 \tag{42}$$

with n being a non-negative integer or $n = -\frac{1}{2}$ and some real constant K.
To apply this ansatz the structure constants for $SU(2)$ given by the Levi-Civita symbol are now inserted into the equation yielding:

$$-\Delta\phi^1 - \frac{\mu}{\rho^2}\partial_\eta\phi^2 = -|\omega|^2\,\phi^1 \tag{43a}$$

$$-\Delta\phi^2 + \frac{\mu}{\rho^2}\partial_\eta\phi^1 = -|\omega|^2\,\phi^2 \tag{43b}$$

$$-\Delta\phi^3 = -|\omega|^2\,\phi^3 \tag{43c}$$

Now equation (43c) is equivalent to the vacuum case and does not admit a non-trivial solution for negative eigenvalues, but the remaining two equations are now coupled through the vortex (41).
To then simplify the coupled system the following ansatz is chosen:

$$\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} = R(r,\theta)\begin{pmatrix}\psi^1(\rho,\eta)\\ \psi^2(\rho,\eta)\end{pmatrix} \tag{44}$$

It should be noted that this ansatz might in general not contain all solutions, because demanding that ϕ^1 and ϕ^2 have the same behavior in (r,θ) is certainly a restriction. But when only trying to find any solution at all to (43), this is not particularly problematic and the ansatz makes further analysis of the equations much simpler.
Applying (44) to (43) gives

$$-\psi^1(\Delta_{r,\theta}R) - R(\Delta_{\rho,\eta}\psi^1) - R\frac{\mu}{\rho^2}\partial_\eta\psi^2 = -|\omega|^2\,R\psi^1 \tag{45a}$$

$$-\psi^2(\Delta_{r,\theta}R) - R(\Delta_{\rho,\eta}\psi^2) + R\frac{\mu}{\rho^2}\partial_\eta\psi^1 = -|\omega|^2\,R\psi^2 \tag{45b}$$

Now both of these equations are seperable individually and after some simple algebraic manipulations one can bring both of them into the form:

$$-\frac{1}{R}\Delta_{r,\theta}R = F(\rho,\eta) \tag{46}$$

As per usual in a seperation ansatz, such an equation can only be fulfilled for all arguments if both sides are identical to some constant. Therefore the equation for R can be written as

$$-\frac{1}{R(r,\theta)}\Delta_{r,\theta}R(r,\theta) = C \tag{47}$$

After multiplying with R, this is once again the Laplace eigenvalue equation and will only lead to admissable solutions for $C > 0$. Dividing (45) by R also leaves the left hand side of (47) as the only expression dependant on (r,θ) which can then be substituted by C, yielding

$$\psi^1 C - \Delta_{\rho,\eta}\psi^1 - \frac{\mu}{\rho^2}\partial_\eta\psi^2 = -|\omega|^2\,\psi^1 \tag{48a}$$

$$\psi^2 C - \Delta_{\rho,\eta}\psi^2 + \frac{\mu}{\rho^2}\partial_\eta\psi^1 = -|\omega|^2\,\psi^2 \tag{48b}$$

Substituting $C = s^2|\omega|^2$ with $s > 0$, reshuffling the terms and expanding the Laplacians into its partial derivatives leads to:

$$-\partial_\rho^2\psi^1 - \frac{1}{\rho}\partial_\rho\psi^1 - \frac{1}{\rho^2}\partial_\eta^2\psi^1 - \frac{\mu}{\rho^2}\partial_\eta\psi^2 = -(1+s^2)|\omega|^2\,\psi^1 \tag{49a}$$

$$-\partial_\rho^2\psi^2 - \frac{1}{\rho}\partial_\rho\psi^2 - \frac{1}{\rho^2}\partial_\eta^2\psi^2 + \frac{\mu}{\rho^2}\partial_\eta\psi^1 = -(1+s^2)|\omega|^2\,\psi^2 \tag{49b}$$

Now since these are polar coordinates and η represents an angle, the functions need to be periodic in this variable and can be expanded into a Fourier series

$$\psi^a = \sum_{m=-\infty}^{\infty} c_m^a e^{im\eta} \tag{50}$$

When slotting this into (49) one can do a comparison of coefficients since the exponentials are linearly independant and this yields 2 equations in the ρ dependant Fourier coefficients c_m^a:

$$-\partial_\rho^2 c_m^1 - \frac{1}{\rho}\partial_\rho c_m^1 + \frac{1}{\rho^2}m^2 c_m^1 - \frac{im\mu}{\rho^2}c_m^2 + (1+s^2)|\omega|^2\,c_m^1 = 0 \tag{51a}$$

$$-\partial_\rho^2 c_m^2 - \frac{1}{\rho}\partial_\rho c_m^2 + \frac{1}{\rho^2}m^2 c_m^2 + \frac{im\mu}{\rho^2}c_m^1 + (1+s^2)|\omega|^2\,c_m^2 = 0 \tag{51b}$$

These equations decouple for $m = 0$ and reduce to the same equation in this case:

$$-\partial_\rho^2 c_m^1 - \frac{1}{\rho}\partial_\rho c_m^1 + (1+s^2)|\omega|^2\,c_m^1 = 0 \tag{52}$$

This is a modified Bessel equation with the modified Bessel functions I_α, K_α as solutions. However, the modified Bessel function of first kind I_α exponentially increases for large arguments and the modified Bessel function of second kind K_α has a singularity at 0 and this makes neither solution admissable. Therefore $c_0^1 = c_0^2 = 0$.

For further analysis of the remaining Fourier coefficients, it is useful to write out (51) as real and imaginary part. Setting $c_m^a = b_m^a + ie_m^a$ gives:

$$-\partial_\rho^2 b_m^1 - \frac{1}{\rho}\partial_\rho b_m^1 + \frac{1}{\rho^2}m^2 b_m^1 + \frac{m\mu}{\rho^2}e_m^2 + (1+s^2)|\omega|^2 b_m^1 = 0 \tag{53a}$$

$$-\partial_\rho^2 e_m^1 - \frac{1}{\rho}\partial_\rho e_m^1 + \frac{1}{\rho^2}m^2 e_m^1 - \frac{m\mu}{\rho^2}b_m^2 + (1+s^2)|\omega|^2 e_m^1 = 0 \tag{53b}$$

$$-\partial_\rho^2 b_m^2 - \frac{1}{\rho}\partial_\rho b_m^2 + \frac{1}{\rho^2}m^2 b_m^2 - \frac{m\mu}{\rho^2}e_m^1 + (1+s^2)|\omega|^2 b_m^2 = 0 \tag{53c}$$

$$-\partial_\rho^2 e_m^2 - \frac{1}{\rho}\partial_\rho e_m^2 + \frac{1}{\rho^2}m^2 e_m^2 + \frac{m\mu}{\rho^2}b_m^1 + (1+s^2)|\omega|^2 e_m^2 = 0 \tag{53d}$$

Now the two pairs of equations ((53a), (53d)) and ((53b), (53c)) are decoupled from each other and identical. So one can restrict the discussion to ((53a), (53d)). Furthermore (53a) and (53d) only differ in that b_m^1 and e_m^2 have switched positions in the equation. It is then obvious that adding and subtracting these 2 equations from each other, yields similar equations for $b_m^+ = b_m^1 + e_m^2$ and $b_m^- = b_m^1 - e_m^2$ respectively. Specifically one gets:

$$-\partial_\rho^2 b_m^+ - \frac{1}{\rho}\partial_\rho b_m^+ + \frac{1}{\rho^2}m^2 b_m^+ + \frac{m\mu}{\rho^2}b_m^+ + (1+s^2)|\omega|^2 b_m^+ = 0 \tag{54a}$$

$$-\partial_\rho^2 b_m^- - \frac{1}{\rho}\partial_\rho b_m^- + \frac{1}{\rho^2}m^2 b_m^- - \frac{m\mu}{\rho^2}b_m^- + (1+s^2)|\omega|^2 b_m^- = 0 \tag{54b}$$

Finally, these 2 equations only differ by the sign of m, which means that $b_m^- = b_{-m}^+$ and only one singular ordinary differential equation has to be solved. For convenience (54a) is chosen in the form:

$$\rho^2\partial_\rho^2 b_m^+ + \rho\partial_\rho b_m^+ - m(m+\mu(\rho))b_m^+ - \rho^2(1+s^2)|\omega|^2 b_m^+ = 0 \tag{55}$$

3.1 Asymptotic Analysis

Having reduced the problem to a single ordinary differential equation (55) one can still do some further analysis before specifying the profile function. Since the asymtotic behavior of admissable profile functions $\mu(\rho)$ is known for small and large ρ according to (42), one can use this to find asymptotic solutions of (55).

First looking at the asymptotic equation for small ρ, one can neglect terms proportional to lower powers of ρ if they contain derivatives of the same order and set $\mu(\rho) = K$. This transforms (55) into:

$$\rho^2 \partial_\rho^2 b_m^+ + \rho \partial_\rho b_m^+ - m(m+K) b_m^+ = 0 \tag{56}$$

This is an Euler Equation and can be solved by defining:

$$\begin{aligned} \rho &= e^z \\ d_m(z) &= b_m^+(e^z) \end{aligned} \tag{57}$$

This ansatz transforms (56) into:

$$\partial_z^2 d_m(z) - m(m+K) d_m(z) = 0 \tag{58}$$

Now simply solving the characteristic polynomial for this linear Differential equation with constant coefficients, yields the asymptotic solutions to b_m^+ as:

$$b_m^+(\rho) \xrightarrow[\rho \to 0]{} A \cos\Big(\sqrt{m(m+K)}\log(\rho) + \phi\Big),\ m\ between\ 0\ and\ -K \tag{59a}$$

$$b_m^+(\rho) \xrightarrow[\rho \to 0]{} A + B\log(\rho),\ m = 0\ or\ m = K \tag{59b}$$

$$b_m^+(\rho) \xrightarrow[\rho \to 0]{} A\rho^{\sqrt{m(m+K)}} + B\rho^{-\sqrt{m(m+K)}},\ else \tag{59c}$$

The first solution here always has an essential singularity around 0, which means that for a given K the Fourier indices have to be restricted to avoid this case. No further restrictions towards m or K can be derived from this.

Now looking at the asymptotic behavior for large values of ρ by the same procedure applied for small arguments, one finds that (55) can be reduced to:

$$\rho^2 \partial_\rho^2 b_m^+ + \rho \partial_\rho b_m^+ - \rho^2(1+s^2)|\omega|^2 b_m^+ = 0 \tag{60}$$

Now by simply substituitng $z = \sqrt{(1+s^2)|\omega|^2}\rho$, one transforms the above equation into

$$z^2 \partial_z^2 d_m(z) + z \partial_z d_m(z) - z^2 d_m(z) = 0 \tag{61}$$

where $d_m(z) = b_m^+(\rho)$. So the factor $\alpha = \sqrt{(1+s^2)|\omega|^2}$ is revealed to be just a scale parameter.
The equation (61) is a modified Bessel differential equation with the two modified Bessel functions of degree 0 as solutions. Therefore one can write:

$$b_m^+(\rho) \xrightarrow[\rho \to \infty]{} A\, I_0(\alpha\rho) + B\, K_0(\alpha\rho) \tag{62}$$

Here I_0 is the modified Bessel function of first kind and degree 0. These functions all have the property that $I_0(0) = 1$ and $I_n(0) = 0$ for $n \neq 0$ and grow exponential like for large arguments.
K_0 is the modified Bessel function of second kind and degree 0. These all have a logarithmic singularity at 0 like the Bessel function of second kind and they drop to 0 asymptotically for large arguments in a similar fashion to an exponential function.

Now, as the asymptotic solution for large arguments, can for any choice of the parameters in the equation have a normalizable solution, no further restriction can be found from it.

3.2 Solutions for Different Profile Functions

Having concluded the asymtotic analysis of the equation, it is time to specify a profile function and attempt to find solutions for it.
But for that, it is easier to consider a transformation of (55) equivalent to the one in (61) with $d_m(z) = b_m^+(\rho)$ and $z = \sqrt{(1+s^2)|\omega|^2}\rho = \alpha\rho$ which yields

$$z^2\partial_z^2 d_m(z) + z\partial_z d_m(z) - m(m+\mu(z/\alpha))d_m(z) - z^2 d_m(z) = 0 \tag{63}$$

This equation now describes the by α scaled solution to 55. The parameter α in this equation only serves to modify the profile function.

The following somewhat arbitrarily selected profile functions will be investigated

$$\begin{aligned}
\mu_1(\rho) &= \mu(z/\alpha) = (2n+1)\cdot\frac{(z/\alpha)^6}{1+(z/\alpha)^6} \\
\mu_2(\rho) &= \mu(z/\alpha) = (2n+1)\cdot e^{-(z/\alpha)^3} \\
\mu_3(\rho) &= \mu(z/\alpha) = K\cdot\tanh(\alpha/z)
\end{aligned} \tag{64}$$

where the first 2 functions have flux $2n+1$ and $\mu(0) = 0$, and the third is of flux 0 with $\mu(0) = K$.

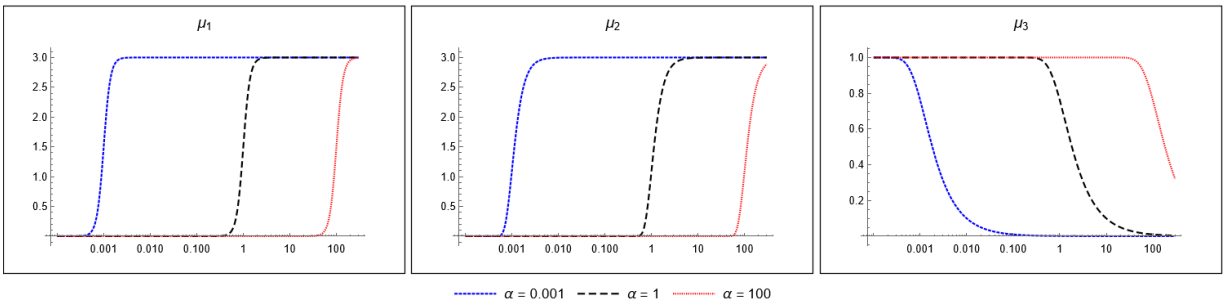

Fig. 1: Chosen profile functions for different values of α shown on logarithmic abscissa.

To investigate the solutions of (63) for the above profile functions, a direct approach is very difficult. The corresponding differential equations have no easily accesible solutions and a good approach in such a case is to either expand the solution into a power series or solve numerically. In this case, however, the behavior for small arguments is somewhat problematic and there is no guarantee that the solution will be expandable as a power series around 0. Indeed this is evident already for the case $m = 0$ and $\mu = 0$, which is solved by the modified Bessel functions, where the modified Bessel function of second kind is not analytical at $\rho = 0$. Also, in general, solving the equations for the coefficients of a power series, usually yields complicated recursion relations for the coefficients, from which it is very difficult to extract any information about the solutions.
Therefore only numerical solutions for some selected sets of parameters will be presented in the following. The question to answer by this approach, is whether the at 0 convergent solution also stays bounded for large arguments, which was not the case for the modified Bessel differential equation. To answer this, one has to be aware that when solving the equation numerically, no initial conditions will be able to suppress one of the linearly independant solutions completely due to numerical errors. So at least 2 solutions for the same equation have to be compared, with one solution obtained by choosing initial conditions that suppress the unbounded solution for large arguments and the other solution resulting from initial conditions that suppress the solution with singularity at 0. Appropriate initial conditions for this can be obtained from the asymptotic analysis in the previous subsection.
The in the following plots used initial conditions are:

$$d_m(10^{-3}) = 0.01, \qquad d'_m(10^{-3}) = 0.01 \tag{65a}$$
$$d_m(30) = 0.01, \qquad d'_m(30) = 0.01 \tag{65b}$$

(65a) to suppress the fast growing solutions at 0 and (65b) to suppress the fast growing solutions for $z \to \infty$.
The plots are all on logarithmic abcissa and the quantity on the ordinate is actually $(\mathrm{Arsinh}(\mathrm{d_m(z)}) - \log(2))/\log(10)$, which is almost identical to the decadic logarithm for larger values, but causes no issues with negative values resulting in a kind of pseudo double logarithmic representation.

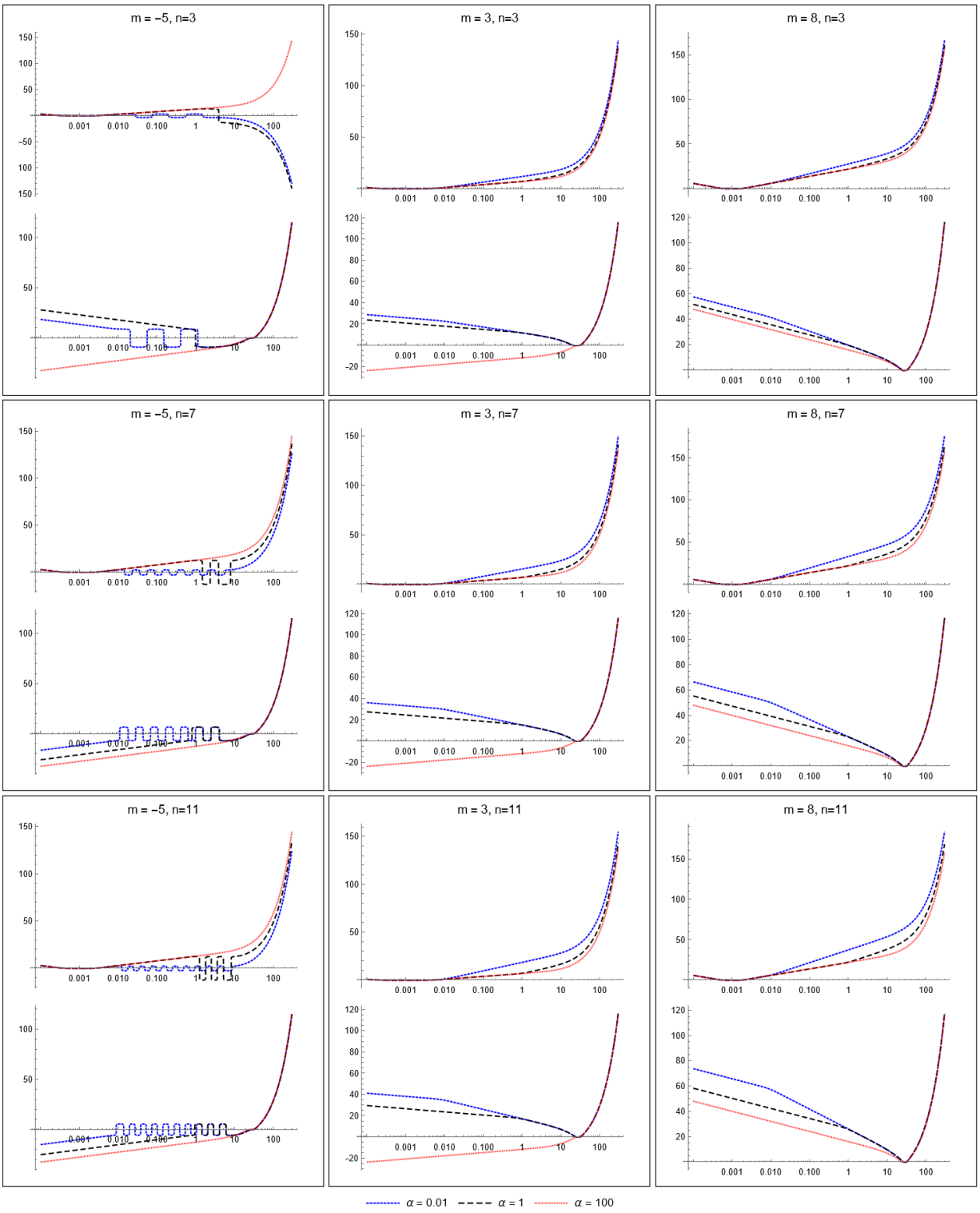

Fig. 2: Solutions to (63) for μ_1 for multiple combinations of parameters. The different colors correspond to different values of α and the individual frames to different combinations of m and n. The 2 different plots in each frame are for different initial conditions - the top one uses (65a) and aims to suppress a possible singularity at 0, while the bottom one uses (65b) and is meant to supress the unbounded long range solution.

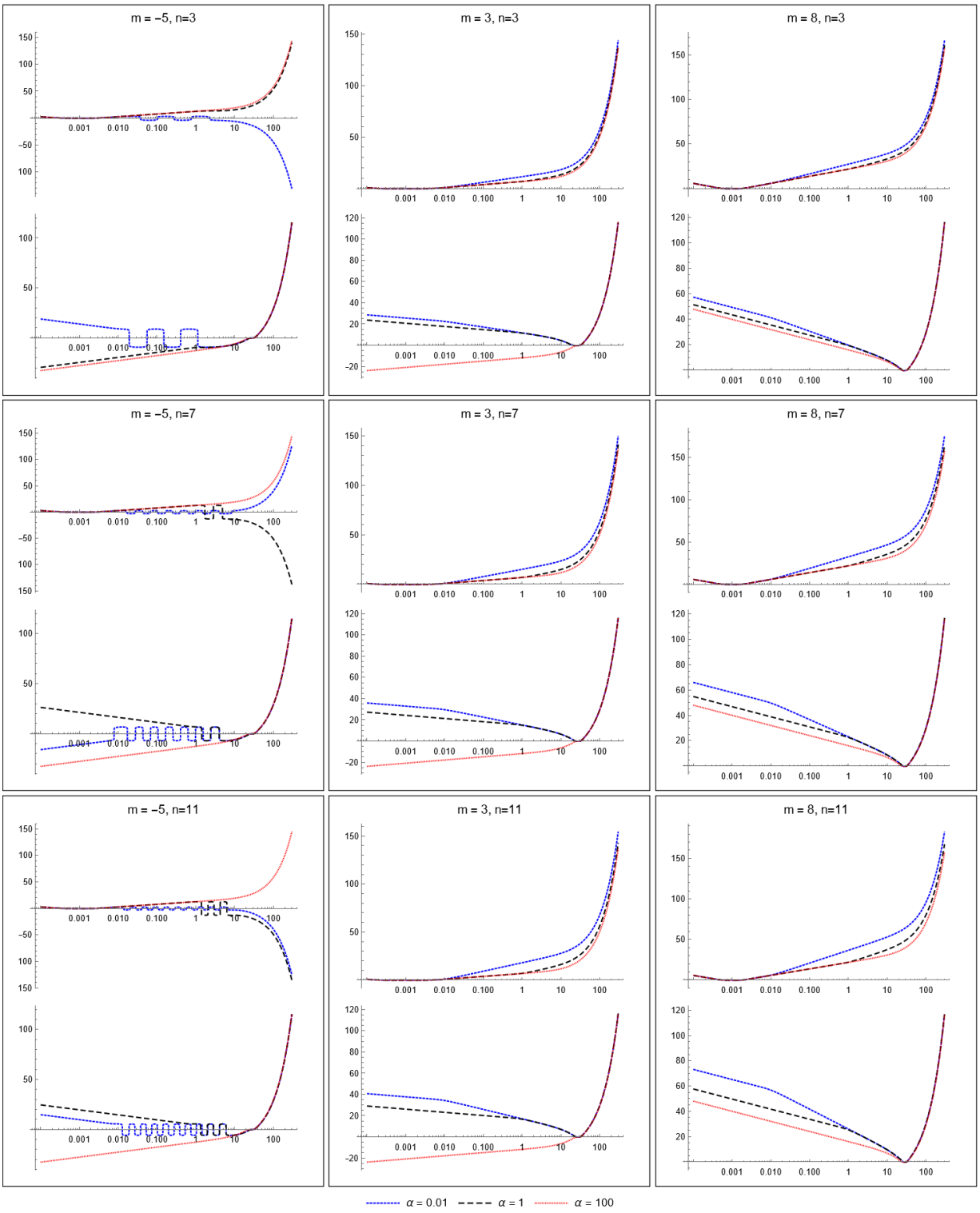

Fig. 3: Solutions to (63) for μ_2 for multiple combinations of parameters. The different colors correspond to different values of α and the individual frames to different combinations of m and n. The 2 different plots in each frame are for different initial conditions - the top one uses (65a) and aims to suppress a possible singularity at 0, while the bottom one uses (65b) and is meant to supress the unbounded long range solution.

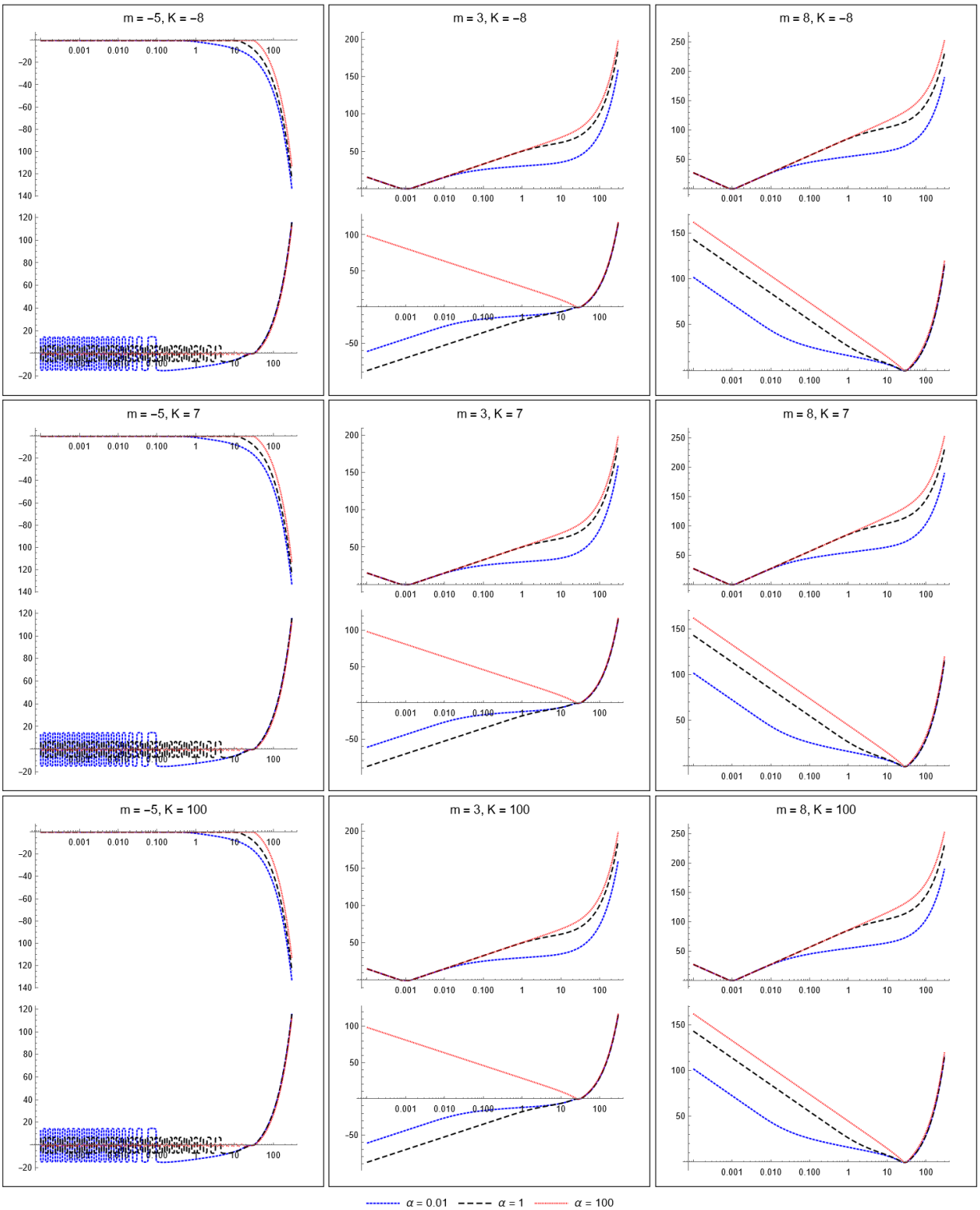

Fig. 4: Solutions to (63) for μ_3 for multiple combinations of parameters. The different colors correspond to different values of α and the individual frames to different combinations of m and K. The 2 different plots in each frame are for different initial conditions - the top one uses (65a) and aims to suppress a possible singularity at 0, while the bottom one uses (65b) and is meant to supress the unbounded long range solution.

Looking at Figures 2 to 4 one can immediately make some observations:
Firstly, all solutions without exception grow incredibly fast for large arguments and the maximum value achieved is always lower for the solution at the bottom of the frames, which was meant to suppress such unbounded long range solutions. On the other hand, the same solutions also show a steep increase in the values towards lower arguments. Specifically, solutions grow linearly in this pseudeo double logarithmic representation, which correspons to a behavior of the type $d_m(z) \xrightarrow[\rho\to 0]{} Ae^{-bz}$ in accordance with (59c).
So one can conclude that the initial conditions that suppress the unbounded long range solution, enhances the at 0 divergent solutions and vice versa. The convergent short range asymptote does not connect to the bounded long range asymptote and therefore no solution fulfilling the boundary conditions exists.

This is also demonstrated by looking at the solutions for $\mu = 0$, which are represented by the modified Bessel functions:

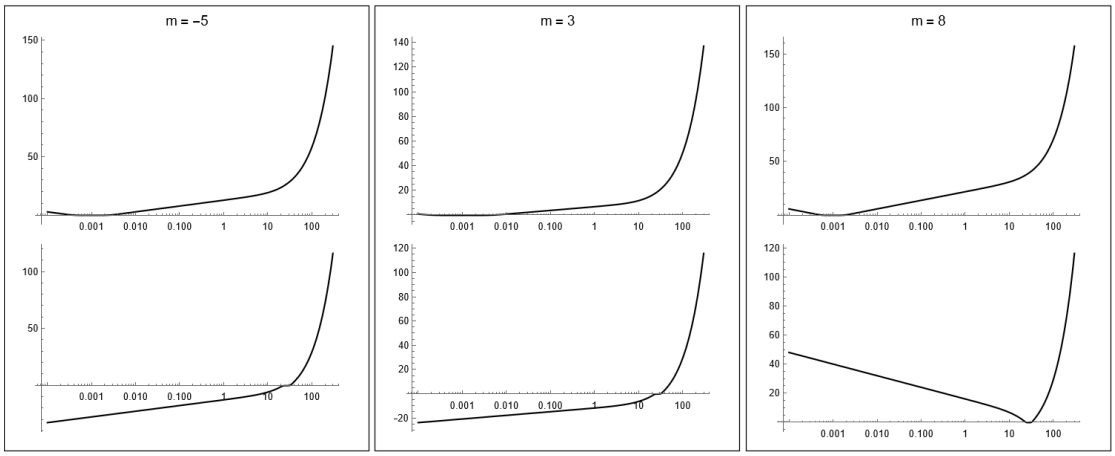

Fig. 5: Solutions to (63) for $\mu = 0$ for multiple values of m. The 2 different plots in each frame are for different initial conditions - the top one uses (65a) and aims to suppress the around 0 unbounded modified Bessel function of second kind $K_m(z)$, while the bottom one uses (65b) and is meant to supress the modified Besself function of first kind $I_m(z)$, which grows exponential like for large arguments.

Again one makes the same observations as in Figures 2 to 4 but without the appearance of any oscillating solutions.

4 Summary and Outlook

In an attempt to find eigenfunctions to negative eigenvalues of the Faddeev-Popov operator in $SU(2)$, an oriented center vortex field configuration ansatz (41) was used and the system of equations was reduced to a single ordinary differential equation (55) in an almost identical fashion to [2].
The asymptotic behavior of this equation was investigated for both small and large arguments. The results already show at this point that, while individually there are small and long range asymptotes fulfilling the necessary boundary conditions, only ever 1 of the 2 linearly independent solutions does so. This already provides a hint that it might require very specific conditions to for either the small range asymptote or the long range asymptote to be bounded, which is the bare minimum required of an admissable solution. But to then find an admissable solution, the bounded short range asymptote would have to connect to the bounded long range asymptote and hence 2 specific sets of conditions would have to hold simultaneously. Something, however, that cannot be ruled out at this stage.
To further examine the equation, 3 explicit profile functions (64) were chosen and for different initial conditions and parameter combinations solved numerically. The results shown in Figures 2 to 4 strongly suggest that no bounded solutions exist for the examined cases. Both linearly indpentent solutions seem to always diverge either for small or large arguments.
This leads to the hypothesis that (55) does not have admissable solutions in general, irrespective of the profile function chosen. A claim further supported by the observation that solutions seem to be very robust under modifications of the profile function and the fact that the asymptotic equations (56) and (60) do not depend on it.
But to truly dismiss the existence of valid solutions to (64), a more rigorous mathematical anlysis would be necessary. And even then only the ansatz (44) could be dismissed.
Therefore ony may conclude that, while the existence of admissable solutions under the employed oriented center vortex field ansatz cannot be disproven based on the shown results, they do indicate that searching for solutions with negative eigenvalues under these asumptions does not appear particularly promising.

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