Mechanics

Support material for philosophy of science seminar in SS 2018 at the KFU Graz

Axel Maas

Contents

1 Introduction **1**

2 Newtonian mechanics **3**

2.1 Kinematics 4

2.2 Newton's laws 5

2.2.1 Preliminaries 5

2.2.2 The first law 5

2.2.3 The second law 6

2.2.4 The third law 7

2.3 Gravity 8

2.4 The potential 9

Chapter 1

Introduction

(Theoretical) mechanics is, in a sense, a somewhat ancient topic. Essentially completely formulated in its modern form in the 19th century, it has matured into a mathematically consistent and closed theory. Even the advent of special relativity only required a minor modification of the underlying vector spaces in the mathematical formulation, and could therefore be technically easily accommodated. Of course, the interpretational impact has been of much more significance.

The natural question is therefore why it is necessary to discuss it then at all. The answer to this is that all modern models in physics draw from the conceptual structure of classical mechanics. General relativity introduced again modifications of the arena. Quantum physics modified the nature of coordinates themselves. But the basic formulation tools of theoretical mechanics, especially the Lagrangian formulation of chapter **??** and the Hamiltonian formulation of chapter **??**, are still mathematical cornerstones of these theories. It is not the concept which changed, merely the entities on which it is applied, at least from a mathematical point of view. Of course, again the shifts in physical interpretation had a much larger impact.

Thus, theoretical mechanics remains both mathematical and in terms of conception still a cornerstone of even the most modern areas of physics. Understanding theoretical mechanics in these formulations is therefore forming the foundation on which these are build.

Of course, not to mention its ubiquitous engineering, and other, applications. Though this is not the focus here.

However, these formulation as Lagrangian and Hamiltonian mechanics in chapters **??** and **??**, as powerful as they are, are very dissimilar from the usual concept of Newton's law, even if they embody the same physics. In fact, at first sight it is far from obvious that reformulating mechanics mathematically in this way could be anything but obscuring. It is only when making the transition to quantum physics and general relativity that their conceptual importance becomes really evident, though the stuctural simplification can be utilized in itself.

If one would embark on either topic without first going through the mathematical reformulation of mechanics would require to cope at the same time with quite different mathematical and conceptual problems. Classical mechanics, and to some extent special relativity, are therefore role models for the future.

To provide a smooth transition from the experimental view on mechanics to the theoretical formulation, the first step will be to give a more theoretical perspective on Newtonian mechanics in chapter 2, sometimes also called analytical mechanics. In this way, the theoretical approach to physics problems is best outlined, as here the subject of study, ordinary Newtonian mechanics, is already well acquainted from experimental physics, so that one can concentrated on the more abstract theoretical formulation.

Another step is then to introduce the ideas of special relativity in chapter **??**. This is somewhat orthogonal, and will therefore be relegated to a suitable spot.

The first version of reformulation of mechanics is then Lagrangian mechanics in chapter **??**. There are two versions of this, both having their own advantages. Lagrange's equations of the fist kind in section **??** are extremely useful when it comes to problems in applications of classical mechanics. However, some of the concepts will also be relevant in statistical physics, the microscopical explanation of thermodynamics. Lagrange's equation of the second kind in section **??** are the natural way to formulate problems embodying special relativity and quantum physics simultaneously, which is the arena of particle physics. Afterwards, another reformulation is given in terms of Hamilton's mechanics in chapter **??**. This reformulation is especially useful for non-relativistic quantum physics. It may appear odd at first sight that this more special case, as it does not lend itself so easily to special relativity, is treated after the more general case. But this formulation is actually easier to understand from the Lagrangian formulation as then from starting outright towards it. Finally, these methods will then be used to treat some special topics in classical mechanics in chapter **??**.

Chapter 2

Newtonian mechanics

The aim of this first chapter is not necessarily to introduce new physics or new physics concepts. After all, Newtonian physics is familiar from experimental physics. The main aim here is to provide a new perspective on it. So far, the exposure to physics was mainly by means of understanding experiments. The theoretical approach is somewhat different. The basic idea is to start from a set of fixed rules, e. g. Newton's laws. The next step is then to derive consequences of these laws, e. g. the movement of the earth around the sun. Comparison between these derived results and experiments then decides whether the basis of the derivation is actually useful to describe experiments or not.

Of course, the aim of this exercise is to end up with the minimal basis to describe all experimental results. This is often considered as the basic laws of nature. Whether they are indeed ingrained in reality in some way is a highly non-trivial questions, and not (yet?) resolved. However, to even pose this question requires to be able to connect some limited set of basic principles with experiments. This is the task of theoretical physics. This also includes, of course, to identify this basic set. The resulting basis is called a (standard) model or a theory, while the constructions still awaiting experimental tests are usual considered as hypothesis. There are fine distinctions between these names which, however, play no role in this lecture, nor actually in the day-to-day research.

Note, however, that any experiment can at most falsify a theory. It can never prove it to be correct. Thus, any theory or model can only be considered to be an adequate description for the time being. Though often theories are refuted entirely based on contradiction to experiment, there are some cases where they do contradict experiments, but are not really refuted. In this case, the theory turns out to be the limit of a more general theory in a particular case. E. g., Newtonian mechanics will be the limit of special relativity for small speeds. Only when experiments become sensitive enough they can detect this situation.

A special limit theory is by no means useless. When building a bridge, nobody will do the statics using special relativity, but ordinary Newtonian mechanics. Thus such a limit theory, more often called an effective theory, is by no means useless. Quite often the more general theory is of less practical use.

2.1 Kinematics

Before starting with physics, sometimes also called dynamics, it is useful to first consider the description of (point) particles, the central entities of mechanics. This is a pure description, and there is no answers to why a particle behaves in a certain way. To separate this from the actual reasons of their movement this is often called kinematics.

The starting point for the description of a particle is the path it follows during an interval of time. In general, this path is described by a vector-valued function $\vec{r}(t)$ in a vector space. This vector space describes the position of the particle in space, while time acts as a parameter to identify the position of the particle along its path, which is also called a trajectory.

The speed $\vec{v}$ of a particle is defined to be

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt},$$

i. e. the rate of change of the position of the particle along its path. Usually, the path of particles will be infinitely often differentiable, so this is a well-defined quantity. In mechanics, however, usually only the second differential,

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}$$

plays also an important role. The others may appear, but are not central quantities.

With these definitions, the behavior of a particle can be described (though not explained). If there are multiple particles, their paths will be indexed.

Given the acceleration, it is possible to obtain the path by twofold integration,

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) + \int\limits_{t_0}^{t} dt' \int\limits_{t_0}^{t'} dt'' a(t''),$$

where the quantities $r(t_0)$ and $v(t_0)$ are the position and speed at time t_0, the initial conditions. Similarly, if the speed or higher derivatives are given, the path can be obtained by a single or even more integrations.

2.2 Newton's laws

2.2.1 Preliminaries

While the previous section provided the tools to describe the movement of a particle, it did not give any reason how, e. g., the acceleration comes about. This is the question of dynamics.

As alluded to earlier, the first step in theoretical physics is to define a basis, i. e. a set of laws which describe the dynamics. In analogy to mathematics, these are sometimes also called axioms, but more often nowadays just as the model, given the insight that all models have their limits.

Classical mechanics is the theory which is based upon Newton's laws. They were found in a long sequence of interplay between theory and experiment. This history of physics will not be traced out here, but it should be noted that the formulation of the model is by no means a trivial exercise, and all laws of physics have been created based on a multitude of experimental insights, no matter how many spectacular things later were predicted by them.

2.2.2 The first law

To formulate Newton's laws, it is necessary to introduce a few more concepts. The first is to define a force as the origin of dynamics. I. e., without forces, the particle will not change its kinematics. To give this statement a more precise meaning requires the notion of an inertial system.

The first law of Newton is that if no forces act on a particle then there exists a coordinate system in which its acceleration and all higher derivatives of its path vanish identically, and thus the particle either remains at rest or moves at constant speed. If the constant speed is zero, this frame is called the rest frame of the particle.

The importance of the concept of an inertial system follows from the following idea (also called gedankenexperiment). A particle can be observed. If now the observer moves with respect to the particle, it will appear to be moving, even if it is the observer, which moves. Thus, the kinematics depend on the relative motion of observer and particle. However, if there are forces, i. e. something changing the kinematics, then there is a source of change not identical to just a change of coordinates, and therefore, no matter how, there is no coordinate system in which the particle behaves as expected.

There is one loophole to be fixed. If the observer would be accelerated, it may look like the kinematics change. Therefore, inertial systems are restricted to such systems which may at most move relatively with a constant speed with respect to the coordinate system of the particle. If this is not the case, so-called pseudo forces, like the Coriolis force, emerge. This case will be discussed in section **??**.

The abbreviated, sloppy, version of this law is: If there are no forces, the particle moves at constant speed (which includes zero speed).

2.2.3 The second law

Newton's first law identifies what happens in the absence of forces. Newton's second law describes what happens in the presence of forces. However, this requires first to define forces a little bit more. Note that this is indeed a definition of forces, not an explanation. This is one of the essences of theoretical physics, at least currently: It cannot explain everything in the sense of requiring nothing external. It can at best explain a multitude of experimental observations with a very limited number of external inputs. In this sense, also forces are something motivated and defined by experiment, but without derivation.

The definition is that a force is a vector, which is possibly time-dependent, $\vec{F}(t)$. This also implies that forces add like vectors, and the total force is the sum of all the individual forces. This is sometimes also called the fourth law, since this is also not something derivable.

In addition, every particle is assigned an inertial mass m, which is a property of said particle. There is no explanation of the origin of this mass in mechanics, and for any given problem the value of this mass has to be determined by experiment. This mass is an additive property, that is the mass of two particle M is just the sum of the individual particles' masses

$$M = m_1 + m_2,$$

and so on for more particles. This mass can have any arbitrary, positive value, and it is not quantized (comes in portions). It is an intrinsic property of a particle. In classical mechanics, furthermore, this property is immutable in time, $m(t) = m$.

Having these two concepts, the next step is, for the sake of convenience, to define a new quantity, the momentum of a particle,

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}(t)}{dt}.$$

This will make the analysis especially of mass distributions, i. e. large numbers of particle with small masses, simpler, as well as the generalization beyond classical mechanics.

With this, it is possible to formulate the second law as

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}(t) \tag{2.1}$$

i. e. the force changes the momentum and equals the acceleration up to a factor of the mass. As said, this is a definition, so there is no reason for it, except that it fits well with experiment.

Though mass is an immutable concept of a particle, it may still be useful to think of a time-dependent mass, e. g. if thinking of an ensemble of particles. When pouring particles on a scale, e. g., the mass on the scale changes. If this is the case, the second law takes the form

$$\vec{F} = \frac{d\vec{p}}{dt} = m(t)\vec{a}(t) + \vec{v}(t)\frac{dm(t)}{dt}, \tag{2.2}$$

i. e. the definition in terms of the momentum is the basic one, not the one in terms of the acceleration or the speed. This also elevates the momentum to be the central kinematical quantity in classical mechanics, and actually far beyond. This equation is also called the (Newtonian) equation of motion.

The equation is often also referred to as Newton's law or as the dynamical equation. While the first law and the third law describe only general features of systems with forces, it is this equation which actually describes the impact of forces on particles.

Though not explicitly noted, the force is in general not a constant, but it may (and in general will) depend on the position of the particles, as well as derivatives of the position, like the speed, the acceleration, or, in principle, even higher derivatives. In practice, the majority of cases involve only forces which are position-dependent, and sometimes dependent on the speed. Forces involving the speed are also often called frictional forces, as they usually appear in the context of friction phenomena.

The force may, in addition, depend also explicitly on the time, not only implicitly through the position of the particle as a function of time. This happens especially often if there is an external source of the force.

Because the force can then be seen as a function of other vectors, and has usually a well-defined value for every point in space and time, the force is often also considered a force-field, though the name force is still used for brevity. Only if the force does not depend on the position, but at most explicitly on time, it is strictly speaking not a force field.

2.2.4 The third law

The third law is of a substantially different nature than the two first laws. The two first laws describe how particles are affected by the presence or absence of forces, but do not make any statement about the origin of the forces. This is changed by the third law. Its statement is that if the force on a particle emanates from another particle, then the target particle acts always with the same but opposite force on the source particle,

$$\vec{F}_s = -\vec{F}_t,$$

i. e. any action induces a, equal in magnitude but opposite in sign, reaction. If there are more than two particles involved, this statement applies pairwise to each possible pairing of particles. Note that still no statement is made about how any of the involved particles creates the forces, but it requires for any possibility that this balance of action and reaction is satisfied. Again, there is no reason for this at the present time, since it is an axiom.

An important approximation is very often that it is assumed that a force is external. I. e., though the back reaction occurs, it is so weak that, for all practical purposes, the origin of the force is not changed, and therefore the force on the particle does not change. An example for this is the movement of the earth around the sun. There, the impact of the third law can be taken exactly into account, but it can also be shown how it becomes irrelevant for the sun being much heavier than the earth.

2.3 Gravity

As noted before, Newton's laws do not explain the origin of the forces, just how they act on particles. They are therefore sometimes called a dynamical principle, but require still the force $\vec{F}$ to actually describe motion. It is the realm of fundamental physics to deduce these forces from experiment, and investigate, which of these forces can be deduced from other forces. The most basic ingredients known today are general relativity and the standard model of elementary particles. Though not completely covering all experimental observations, all known forces can, in principle, be deduced from them. However, the actual derivation of, say, the forces involved in standing on a floor from these elementary theories is practically far too involved and too complicated. Therefore, rather than using them, it is much better to use effective forces, which neglect all those aspects which play no role, i. e. are too weak to make any practically measurable difference, rather than the full forces of these theories. This is then considered as an effective theory, rather than a, more or less, fundamental one. Especially, in the realm of classical mechanics only such effective forces play a practical role.

The probably best known of these effective forces is the gravitational force. Given the distance between two bodies, $|\vec{r}_1 - \vec{r}_2|$, the gravitational force between them is

$$\vec{F} = \frac{G m_1^g m_2^g (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}, \tag{2.3}$$

where $G \approx 6.6741(4)$ $m^3/(kgs)$ is Newton's constant, a number obtained from measurement. The quantities m_i^g are the so-called heavy or gravitational mass. Just like the inertial mass, they are a property of particles, and must be deduced by measurement. They act as a so-called charge, i. e. they are features of particles which determine the influence of a force on this particle. It is found experimentally that the gravitational masses are always positive.

It is a remarkable, and highly non-trivial, experimental finding that for a particle of inertial mass m and gravitational mass m^g

$$m = m^g$$

holds, i. e. gravitational and inertial mass are the same. This feature, also known as the equivalence principle, is the basis for general relativity, and thus one of the most basic foundations of modern physics. There is no explanation of it, though, it is again an axiom. However, since experimentally extremely well supported, in the following no more distinction between the gravitational and the inertial mass will be made.

There is an interesting special version of the gravitational force (2.3). Set one of the particles fixed at the coordinate origin, $\vec{r}_2 = \vec{0}$. Then

$$\vec{F} = m_1 G m_2 \frac{\vec{e}_r}{r^2} = m_1 \vec{g}$$

where the so defined vector $\vec{g}$ is called the gravitational acceleration due to the body 2. E. g., on the surface of the earth, $\vec{g}$ always points to the center of the earth and has a value, depending on latitude[1], of about 9.8 m/s. Since the radius of the earth is large, $\vec{g}$ changes only very slowly when moving just a little bit vertically, and therefore can be taken to be roughly constant for the couple of kilometers, from the deepest ocean trench to the traveling altitudes of planes, where human activities usually take place.

2.4 The potential

An interesting concept can be found when considering the movement of a particle under an arbitrary, but only position-dependent, force. It is best to start with a one-dimensional example first. The equation of motion then reads

$$m d_t^2 x = F(x).$$

Multiplying this equation by $d_t x$, this yields

$$m(d_t x)(d_t^2 x) = F(x) d_t x. \tag{2.4}$$

[1]Because the earth is not exactly spherical.

It can now be recognized that both sides can be rewritten as derivatives

$$d_t\left(\frac{m}{2}(d_tx)^2\right) = d_t\int F(x')dx' = -d_tV(x), \tag{2.5}$$

where the quantity $V(x)$, being the primitive of $F(x)$, has been introduced. Note that any integration constant in $V(x)$ does not play a role, as the time derivative removes it immediately. This primitive is called the potential, which generates the force. The appearing minus sign is a matter of convention.

It will happen very often that not the force, but the potential is known. The force can then be obtained by

$$F(x) = -d_xV(x), \tag{2.6}$$

i. e. the derivative of the potential is the force. This again shows that any constant terms in $V(x)$ do not play a role, and can be chosen at will. Such a kind of arbitrariness seems to be at first quite astonishing, since nature should be somehow uniquely determined. There is, however, a deeper reason behind this arbitrariness, which will become evident later. However, it is best to postpone a deeper discussion of it until more conceptual progress has been made.

Since on both sides of (2.5) total derivatives appear, the equation can be integrated to yield

$$\frac{m}{2}(d_tx)^2 = E - V(x), \tag{2.7}$$

where E is an integration constant. Note that this integration constant, which emerges from a time and not a spatial integration, cannot be dismissed. Its relevance will be discussed in more detail below. Before doing this, it is possible to solve this equation by separation of variables, yielding an implicit result

$$t - t_0 = \int\limits_{x(0)}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}}, \tag{2.8}$$

which after evaluation of the integral can then be solved for $x(t)$ to get the final solution. This solution is parametrized by the constant E and the initial condition $x(0)$. Thus, these are taking the role of the two initial conditions of position and speed at time zero used before.

While so far only the indefinite integral has been used, it is also possible to use the definite integral

$$W_{12} = \int\limits_{x_1}^{x_2} dxF(x) = V(x_1) - V(x_2).$$

This quantity is the necessary integrated force to move something between the two points x_1 and x_2. It is therefore called the work which has (to) be(en) done. It corresponds to the difference in potential. The potential therefore describes the amount of work which can be done, and actually doing some work is reducing the potential. Conversely, investing work raises the potential. Thus, the name potential. The amount of stored work for a particle at a position x with respect to zero potential is called the potential energy.

If in equation (2.7) the potential is set to zero, the constant E is entirely giving by a quantity obtained from the motion of the particle. On the other hand, solving for E of (2.7) yields

$$E = \frac{m}{2}(d_t x)^2 + V(x). \tag{2.9}$$

For a particle at rest, E is then the potential energy. Thus, in general, E is called the total energy, combining the potential energy $V(x)$ and a contribution from the motion of the particle $m(d_t x)^2/2 = p^2/(2m)$ which is called the kinetic energy, often abbreviated as T, and where p is again the momentum.

Because of the general solution (2.8), the sum $E - V$ of a particle must be positive at every point in space, since otherwise there is no solution. Thus, a particle needs to have a positive or zero kinetic energy. Furthermore, when $E = V(x)$ for some point x, the kinetic energy has to vanish, as it is a positive quantity. The second derivative may not vanish, and therefore the particle does not necessarily stops there. If the potential further increases to one side, it will actually be deflected there, a so-called inflection point of the movement. Note that the prediction of this behavior did not need the solution of the equation of motion. In fact, the energy will become a very convenient tool to simplify calculations later.

One of the probably most fundamental statements in mechanics is that the energy is x-independent, which follows from (2.7), but also time-independent for conservative, i. e. t-independent, potentials. This follows from (2.4)

$$d_t E = d_t \left(\frac{m}{2}(d_t x)^2 + V(x) \right) = m(d_t x)(d_t^2 x) - F(x) d_t x = 0. \tag{2.10}$$