Calculation of eigenvalues of the Faddev-Popov-operator to a certain vector potential configuration – a variational approach

Maximilian Pfandner
12003925

Supervisor: Axel Maas

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Department of Physics
University of Graz
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Abstract

While discussing the Gibrov-Zwanziger scenario and fixing to landau gauge, it is indeed possible to resolve the eigenvalue equation of the Faddev-Popov-operator, by restricting to the region of first Gibrov horizon. To perform this task, the approach of variation should be considered and examined if it could provide a valid approximation for the eigenkets, respect the eigenvalues, for this problem.

Starting with a compact introduction to gauge- and Yang-Mills-theory, this thesis continues with outlining the Faddev-Popov-equation and the related variables, as well as structural properties. Following the procedure of variation in theory, a first attempt to solve the equation approximately will be performed. After obtaining this approximate eigenket and eigenvalue, the initial trail ansatz will be analysed in a manner of qualitative improvement, until a promising candidate will be found.

Also, if this thesis, could not prove the qualitative enhancement of the most auspicious trial ansatz, there are hints for a sufficient improvement compared to the initial one. Moreover, demonstrating the approximate solubility of the Faddev-Popov-equation by variational approach will conclude this bachelor thesis.

Contents

1 Introduction into gauge theories

Since the Faddev-Popov-problem sensitively relays on selecting a suitable choice of gauge condition, it is essential to introduce and classify the concept of gauge theories in the first place, so the actual problem can be formulated.

Talking about gauge theories, it is substantial to discuss different types of gauge invariances: Occurring as 'global' or 'local' theories, it is fundamental to distinguish between these two types for the following sections. To make this more illustrative a double-slit-system should be considered. As expected, an interference pattern can be observed on a screen located at the back of the slit, after an electromagnetic source is switched on in front of the slits. If someone decides to insert an optical half-wave plate into the aperture, after the double slit and covering both slits sufficiently, there will not be any change of the interference pattern. The pattern, or more physically spoken the phase difference δ_p of the emitted waves (described as ϕ_1 and ϕ_2) from the slits, stays unchanged if the plate covers both slits. This is an example of a global invariance related to the considered system. In general, this describes a change of a property being constant and therefore independent in space and time, not affecting an observable. So, everything within the system changes everywhere instantaneously in the same way.

In contrast to this, local gauge invariance is much more restrictive. For instance inserting the half-wave plate in a way, covering only one slit, the interference pattern would be changed due to a difference in δ_p. Hence this system fails to describe local phase invariance, because the observable property is altered. A local invariance describes a change of a property, depending on space and time, which does not affect the observable within a certain system.

Both kinds have far-reaching effects on classical and quantum-mechanical theories in physics, although they are not altering physical fields.[1] [2]

1.1 General Introduction with respect to electrodynamics

To introduce gauge theory, it is convenient to start with the theory of electromagnetism, since it is well understood and demonstrative. We shall start with the fundamentally important equations of electromagnetism, known as the Maxwell equations:

$$\nabla \cdot \vec{E} = \rho_{em} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \times \vec{B} = \vec{j_{em}} + \frac{\partial \vec{E}}{\partial t} \tag{4}$$

Where $\vec{E}$ and $\vec{B}$ are the observable quantities, respectively the electric and the magnetic field strength. ρ_{em} and $\vec{j_{em}}$ describe the electric charge- and the current-density. A much more common way to express $\vec{E}$ and $\vec{B}$, also referring to special relativity, is to introduce a three-dimensional vector potential $\vec{A_{EM}}$ (this shall not be mistaken with the potential introduced with the Faddev-Popov-problem) and a scalar potential V. In terms of these potentials $\vec{E}$ and $\vec{B}$ are defined by the following expressions:

$$\vec{B} = \nabla \times \vec{A_{EM}} \tag{5}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A_{EM}}}{\partial t} \tag{6}$$

One might have noticed that the physical fields remain unchanged after applying certain transformations to $\vec{A_{EM}}$ and V. These transformations are well defined and called gauge transformations. $\vec{A_{EM}}$ should transform like

$$\vec{A_{EM}} \rightarrow \vec{A_{EM}}' = \vec{A_{EM}} + \nabla\chi \tag{7}$$

and for V:

$$V \rightarrow V' = V - \frac{\partial \chi}{\partial t} \tag{8}$$

Where χ is a arbitrary function, depending on space and time. Introducing a relativistic four-dimensional vector potential $\tilde{\vec{A}}_\mu$ containing A_{EM} and V and a four dimensional partial derivation operator ∂_μ the transformations (7) and (8) could be rewritten in a more compact form:

$$\tilde{\vec{A}}_\mu = (\vec{A_{EM}}, V) \tag{9}$$

$$\partial_\mu = (\frac{\partial}{\partial t}, \nabla) \tag{10}$$

$$\tilde{\vec{A}}_\mu \rightarrow \tilde{\vec{A}}'_\mu = \tilde{\vec{A}}_\mu + \partial_\mu \chi \tag{11}$$

This gauge transformation applies to the non-quantum-mechanic, but relativistic theory of electromagnetism. Starting form this basis, it is indeed possible to continue in the waters of quantum mechanics. [1][2]

1.2 Quantum mechanical approach with respect to electrodynamics

Since electromagnetism enters quantum mechanics, it is obviously a well suited guinea pig for examining the effects of gauge transformations regarding to quantum mechanics. Continuing with

the expressions of the last chapter the Schrödinger equation with the solution function $\phi(x,t)$ of a charged particle (with q representing the charge) in terms of $\vec{A_{EM}}$ and V is given by:

$$\left(\frac{1}{2m}(-i\nabla - q\vec{A_{EM}})^2 + qV\right)\phi(x,t) = i\frac{\partial\phi(x,t)}{\partial t} \tag{12}$$

Applying the transformations from the previous section respect to $\vec{A_{EM}}$ and V, also the equation changes due to the gauge properties. These transformations also effects the solution function $\phi(x,t)$, which is now called $\phi(x,t)'$ with respect to the transformed potentials.

$$\left(\frac{1}{2m}(-i\nabla - q\vec{A_{EM}}')^2 + qV'\right)\phi'(x,t) = i\frac{\partial\phi'(x,t)}{\partial t} \tag{13}$$

Solving the gauge transformed equation $\phi'(x,t)$, the initial solution times a space and time dependent phase factor, solves the altered equation (e.g. for local-gauge-invariance).

$$\phi'(x,t) = e^{iq\chi(x,t)}\phi(x,t) \tag{14}$$

The function $\chi(x,t)$, appearing within the face factor, is the same expression occurring at the gauge transformations, applied to the initial Schrodinger equation. Therefore, performing a gauge transformation like (7) and (8) also alters the solution function of the Schrodinger equation, so a third gauge transformation shall be added:

$$\phi(x,t) \rightarrow \phi'(x,t) = e^{iq\chi(x,t)}\phi(x,t) \tag{15}$$

Inserting this result into the initial equation, the Schrodinger equation does not change, by performing the mentioned gauge transformations, at all. This phenomena is called gauge-covariance (comparison between (12) and (13).

Like in the previous section it is common to introduce a four dimensional vector potential $\tilde{\vec{A}}_\mu$ and a four dimensional derivation operator ∂_μ. For the quantum mechanical chase a further operator D_μ will be created. This operator is gauge invariant by definition (with respect to the transformations (7) and (8)).

$$D_\mu = \partial_\mu + iq\tilde{\vec{A}}_\mu \tag{16}$$

This expression is fundamentally important for Abelian gauge theory like quantum electrodynamics and has a an analogy in non-Abelian gauge theory like the strong and weak interaction theory. [1][2]

2 Brief introduction into SU(2) Yang Mills Theory

After examining gauge theories, one related gauge concept of particular importance, not only for the Faddev-Poppov-problem, but also for the entire field of particle physics, respectively for the standard model of particle physics itself, will be addressed below.

2.1 Introducing Yang-Mills Theory

First introduced by Chen Ning Yang and Robert Laurence Mills in the mid 1950s, Yang-Mills-gauge-theory is considered as *the* keystone of the standard model of particle physics. It is fundamentally important to describe strong and electroweak interactions, depending on the chosen gauge algebra (see also [3] and [2]). The key finding originally postulated by Yang and Mills was to assume local gauge invariances within non-Abelian gauge theories (e. g. SU(2)- or SU(3)-gauge algebra). Therefore the concept shall be briefly illustrated on the basis of SU(2)-Yang-Mills-theory. [2][3]

2.2 Special Conditions of SU(2)

Starting with the global invariance of a single wave function ϕ,

$$\phi' = e^{i\chi}\phi \tag{17}$$

where this shall not be mistaken with the local invariance of the phase factor in (15). By considering multiple states this leads to a non-Abelian theory:

The suspicion of proton and neutron being not entirely independent states, but energy-degenerated states with the same energy, neglecting electromagnetic interactions, gave the decisive impulse to assume that each state could be constructed by linearly combining the state of the proton (ϕ_p) and the state of the neutron (ϕ_n) with some complex coefficients ($\alpha, \beta, \gamma, \delta$).

$$\phi_p \to \phi_p' = \alpha\phi_p + \beta\phi_n \tag{18}$$

$$\phi_n \to \phi_n' = \gamma\phi_p + \delta\phi_n \tag{19}$$

These new states satisfy both corresponding Schrödinger equations by construction. Therefore it is indeed possible to consider the neutron and the proton state as components of a state-vector. (Commonly introduced as $\vec{\phi}_{\frac{1}{2}}$, caused by the fact of being a $\frac{1}{2}$-spin-system)

$$\vec{\phi}_{\frac{1}{2}} = \begin{pmatrix}\phi_p \\ \phi_n\end{pmatrix} \to \begin{pmatrix}\phi_p' \\ \phi_n'\end{pmatrix} = \begin{pmatrix}\alpha\phi_p + \beta\phi_n \\ \gamma\phi_p + \delta\phi_n\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\vec{\phi}_{\frac{1}{2}} = \mathbf{V}\vec{\phi}_{\frac{1}{2}} \tag{20}$$

Since the normalisation of $\vec{\phi}_{\frac{1}{2}}$ has to be preserved when transforming it to $\vec{\phi}_{\frac{1}{2}}{}'$, unitarity is required for $\mathbf{V}$. In order to rewrite (20) analog to (17) additionally $Det[\mathbf{V}] = 1$ is required:

$$\vec{\phi}_{\frac{1}{2}}{}' = \begin{pmatrix} e^{i\chi}\phi_p \\ e^{i\chi}\phi_n \end{pmatrix} = e^{i\chi}\mathbf{1}\begin{pmatrix} \phi_p \\ \phi_n \end{pmatrix} = \mathbf{V}\begin{pmatrix} \phi_p \\ \phi_n \end{pmatrix} \tag{21}$$

Consecutively $\mathbf{V} = e^{i\chi}\mathbf{1}$, where $\mathbf{1}$ is the 2x2 identity matrix. In conclusion $\mathbf{V}$ is required to be unitary ('U'), with the special property $Det[\mathbf{V}] = 1$ ('S') and finally being a 2x2-matrix ('2'). Therefore it is quit useful to combine these characteristic within a new class of matrix, namely 'SU(2)', respectively with adding the simple matrix multiplication as combination law, to a 'SU(2)'-algebra.

This algebraic structure can be reformulated by deriving the infinitesimal transformation of $\mathbf{V}$ to:

$$V = e^{i\vec{\alpha}\frac{\boldsymbol{\tau}}{2}} \tag{22}$$

where the components of $\vec{\alpha}$ are constant and real, by analogy to the phase-factor χ in the one-dimensional case, while $\boldsymbol{\tau}$ represent the so called pauli-matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

2.3 Arrival at local non-Abelian gauge theory in SU(2)

Indeed (23) forms a non-Abelian gauge algebra, but still a global one. In order to introduce a local gauge theory,(22) shall be space (represented by x) and time (represented by t) independent.

$$\mathbf{V} = e^{i\vec{\alpha}(x,t)\frac{\boldsymbol{\tau}}{2}} \tag{24}$$

Therefore conveying the transformation for $\vec{\phi}_{\frac{1}{2}}$ like:

$$\vec{\phi}_{\frac{1}{2}} \rightarrow \vec{\phi}_{\frac{1}{2}}{}' = e^{i\vec{\alpha}(x,t)\frac{\boldsymbol{\tau}}{2}}\vec{\phi}_{\frac{1}{2}} \tag{25}$$

Finally, this describes Yang-Mills-Theory in SU(2). [1],[2] Although this breve introduction follows a more technical path, postulating (25), leads to far-reaching effects in particle physics originally illustrated (in [4]) by Chen Ning Yang and Robert Laurence Mills with the following example:

The differentiation between a neutron and a proton is then a purely arbitrary process. As usually conceived, however, this arbitrariness is subject to the following limitation: once one chooses what to call a proton, what a neutron, at one space-time point, one is then not free to make any choices at other space-time points.

3 Description of the Faddev-Popov-operator, appropriate gauge fixing and calculation related space transformations

3.1 Transformation to Euclidean 4D-space-time

The Faddev-Popov-operator initially acts within the so called Minkowski space. This concept describes a four-dimensional space time with real space (x_1, x_2, x_3) and time tc (c is the speed of light in the vacuum) axis with a certain kind of norm:

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 - (ct)^2} \tag{26}$$

This leads clearly not to a space with Euclidean metric. To avoid complications at the further calculations, a so called Wick rotation is performed. While every space component x_i stays untouched, the rotation $t \to -it$ is performed to rotate the imaginary axis of the complex plain into the real axis. As consequence the time axis is now imaginary and the Euclidean metric:

$$d = |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (ct_x - ct_y)^2} \tag{27}$$

can be applied, describing the same physics after a back transformation after finishing the demanded calculations, which will be performed within a four dimensional Euclidean space. Here the index of t denotes the dependence to the space-time-point (x_1, x_2, x_3, t_x) or (y_1, y_2, y_3, t_y) [5]

3.2 Description of the terms and variables of the Faddev-Popov-operator, regarding to its structure and the concrete problem

To gain further insights to the underling theory, it shall be referred to [3] and [6].

Starting with the so called Gibrov-Zwanziger scenario, a local gauge condition will be introduced, more precisely the Landau gauge (introduced within the next subchapter). This resolves the Gibrov-Singer ambiguity by selection of one representative of the gauge orbit, which describes the history in the four-dimensional space of every component of $\vec{A}_\mu$. As following step, the considered region should be restricted to the first Gibrov horizon (which also includes perturbation theory). Within this boundary the eigenvalues of the Faddev-Poppov-operator (displayed below) are strictly positive semi-definite and approaching zero at the horizon. By crossing it, the eigenvalues become negative.

$$M^{ab} = -\partial_\mu(\partial_\mu \delta^{ab} + g f^{abc} A_\mu^c) \tag{28}$$

This expression can be inserted to its eigenvalue equation, with the eigenvalue ω^2 and its eigenstate of every component Φ (obeying Einstein‘s sum convention a, b and c reaching form one to three):

$$M^{ab}\Phi^b = -(\partial_\mu^2\delta^{ab} + \partial_\mu g f^{abc} A_\mu^c)\Phi^b = \omega^2\Phi^a \tag{29}$$

At first ∂_μ is taken into account. Since the entire problem underlies the Euclidean four-dimensional space time, the derivation operator is introduced after selecting a coordinate system.

Due to avoid unnecessary complexity within the calculation a set of two polar coordinates (instead of four-dimensional hyperspherical coordinates) is introduced:

$$r_\mu = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \rho\cos\eta \\ \rho\sin\eta \end{pmatrix} \tag{30}$$

Consecutively the partial derivative can be illustrated as four-dimensional nabla-operator. Using this information, the four-dimensional Laplacian can be derived to:

$$\partial_\mu^2 = \frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{\rho}\partial_\rho\rho\partial_\rho + \frac{1}{\rho^2}\partial_\eta^2 \tag{31}$$

The remaining constants are g describing the gauge coupling and f^{abc} is the structure constant of the gauge group of SU(2) in equation (29). The remaining $\delta_{a,b}$ is a common Kronecker-δ.

The eigenstate of every component represents the wave function of the corresponding color component, each a scalar-value function, coincided as:

$$\vec{\Phi} = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \end{pmatrix} \tag{32}$$

Like the eigenstate, also the potential $\vec{A}$ has three components, each for every color. However, every component of the potential is vector-valued by itself, being a topological field in the four-dimensional space (denoted with the subscript μ).

$$\vec{A} = \begin{pmatrix} A_\mu^1 \\ A_\mu^2 \\ A_\mu^3 \end{pmatrix} \tag{33}$$

3.3 Landau gauge condition

Since the considered problem includes, inter alia, a perturbational approach and relays on gauge fixing, it is important to select an appropriate gauge condition (the so called gauge fixing process), though it shall be a local one, considering this case. The most suitable candidate is the Landau Gauge condition:

$$\partial_\mu \vec{A}^a_\mu = 0 \tag{34}$$

Even though other covariant gauge constraints could provide a still working condition, Landau gauge provides some additional advantages, which should be mentioned, but due to their complexity, no discussed in further detail. At first, the so called renormalisation appears in its most simple form, compared to other gauge theories. Further Landau gauge is also less complicated, when it comes to problems related to manifest-covariance. Finally, the so called transverse and longitudinal tensors occur less coupled within this gauge constraint. Additional and more profound information about the benefits of landau gauge can be found within the source of this chapter [6], [3].

3.4 Formulating the aim of this thesis

After performing a wick transformation and defining the Faddev-Popov-operator in the corresponding space (the Euclidean space time and the so called color space regarding to the vector potential) and choosing the appropriate gauge condition, the actual purpose for this project can be formulated: The aim of this thesis is to find an approximation for eigenkets and eigenvalues of the Faddev-Popov-operator, passing the first Gibrov-horizon. Furthermore this approximation shall be further improved and examined, if the variational approach (introduced in the following chapter) provides a sufficiently valid approximation method for the Faddev-Popov-eigenvalue-equation.

Moreover, most of the more sophisticated analytical tasks of this project and also figure (1) were performed with the use of the 'Mathematica'-software from wolfram.

4 Introducing the variational method

As sophisticated quantumechanical problems easily approach the horizon of exact analytical solutions, different approximation techniques become more important. Among them, the variational approach, relaying on educated guessing and the experience of the user, is a common way to handle such difficulties. It exploits the fact that a basis $|i\rangle$ of a part of the actual Hamiltonian H describes the entire Hilbertspace in the same manner the actual basis $|E_i\rangle$ would provide a valid description of the same Hilbertspace. (In fact, it is not necessarily demanded to be a part of the initial Hamilton-operator, if its basis still describes the entire Hilbertspace) Since this property is fulfilled for every element of the basis $|i\rangle$, it is possible to rewrite every element by using the projector in terms of $|E_i\rangle$.

$$|i\rangle = \sum_i \langle E_i||E_i\rangle|i\rangle \tag{35}$$

Starting with the first ket of the partial basis $|0\rangle$, acting on the full Hamiltonian, a first approximation for a upper energy threshold e_0 for the groundsate energy of the full Hamiltonian is found.

$$e_0 = \frac{\langle 0|H|0\rangle}{\langle 0||0\rangle} \geq E_0 \tag{36}$$

This provides a first approximation with the ket $|0\rangle$ and the approximate eigenvalue e_0. To improve this approximation further, since $|i\rangle$ only contains parts of the actual $|E_i\rangle$, as long it is not the exact solution, the basis elements $|i\rangle$ have to depend on further variational parameters. So $|i\rangle$ becomes $|i(\vec{\lambda_i})\rangle$. Therefore every 'trial solution' ket can be further adapted to solve the actual problem, by minimization respect to every $\vec{\lambda_i}$:

$$\partial_{\vec{\lambda_i}} = 0 \tag{37}$$

Depending on the number and structure of these parameters occurring in $|i(\vec{\lambda_i})\rangle$ higher derivatives might be necessary to determine whether the extremum is a minimum or not. After a certain level of approximation has been reached, the next element of the basis $|j(\vec{\lambda_j})\rangle$ can be applied to the projection in terms of $|E-i\rangle$. Because every used $|i(\vec{\lambda_i})\rangle$ is only an approximation for the real $|E_i\rangle$ and there are still parts of $|E_i\rangle$ among the other $|E-i\rangle$, it is indeed possible to find kets $|j(\vec{\lambda_j})\rangle$ with $j > 0$ leading to a lower energy threshold:

$$e_j > e_0 \tag{38}$$

Therefore $|j(\vec{\lambda_j})\rangle$ becomes the new $|0(\vec{\lambda_0})\rangle$, likewise e_j becomes the new e_0

To make statements about the quality of a certain trial ket, it is a common procedure to apply this ket and its eigenvalue to the actual eigenvalue equation. As following step the left side of the equation will be subtracted from the right side. After squaring this expression and integrating it (if necessary, over the entire space), a cost function, depending on the variational parameters, remains.

Arriving at the actual problem of this thesis the variational approach seems to be a suited approximation method for the eigenvalue equation of the Faddev-Poppv-operator, since finding bounded states of the operator is the main interest. In terms of variation, this breaks down to finding a trial solution with a negative energy threshold, because the actual groundstate will be even lower than the approximation (by construction). As mentioned before, a trial ket has to be chosen. Therefore, a trial ansatz for a ket shall be found, fulfilling the condition of a bounded sate.

However, there are also other variable properties within the equation, like the vector potential $\vec{A}$. It can be constructed arbitrary, while obeying the gauge constraint of landau gauge.[7]

5 Ansatz for trail eigenstate

For complexity reasons, the vector potential only has one non-vanishing color component (more information at chapter 6). To prevent too technically complex structures, the ansatz for the trial solution $\vec{\Phi}_{start}$ itself contains only a view none vanishing contributions. However, the combination

of the ansatz and the vector potential must lead to reasonable output for the eigenvalue problem. Obeying this guidelines, only the last component of the trial solution equals zero.

$$\vec{\Phi}_{start} = \begin{pmatrix} (r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{39}$$

In order to normalize the ansatz a four-dimensional integral N has to be performed along every degree of freedom.

$$N = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \vec{\Phi}_{start} \cdot \vec{\Phi}^*_{start} r\rho \, dr d\rho d\eta d\theta = \frac{\pi^2 b^8}{16} \tag{40}$$

Finally, the normalisation yields to the following expression:

$$\vec{\Phi} = \frac{\vec{\Phi}_{start}}{\sqrt{N}} = \begin{pmatrix} \frac{4}{\pi b^4}(r\rho)^2 e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -\frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{41}$$

6 Ansatz for a trial vector-potential

In order to avoid unnecessary complex behaviour within the following calculations, the vector potential is chosen to have only components along one color component. Every component of $\vec{A}$ has four dimensions itself. Taking the previous statement into account the vector potential is chosen by:

$$\vec{A} = \frac{B}{g} \cdot \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{e_\theta} \cdot e^{-\frac{r^2}{b^2}} \end{pmatrix} \tag{42}$$

7 Prove of none-vanishing action

Although the vector potential has non-vanishing terms, one must examine if this particular configuration has an actual action. This ensures $\vec{A}$ is not just a gauge-transformed version of the vacuum. Therefore, the action integral S non-Abelian version of the field-tensor, linked with the potential, is performed. Consecutively A will be inserted in the following non-Abelian expression to construct the field-tensor.

$$F^a_{\nu\mu} = \partial^2_\mu A^a_\nu - \partial^2_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{43}$$

Due to the structure of the vector potential, only the first two terms contribute to the tensor:

$$F^3_{\nu\mu} = \partial^2_\mu A^3_\nu - \partial^2_\nu A^3_\mu \tag{44}$$

Plugging in the corresponding components of the potential and performing the derivatives $F^3_{\nu\mu}$ results in:

$$F^3_{\mu\nu} = \delta_{\mu\theta}\delta_{\nu r} 2\frac{2Br}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} - \delta_{\nu\theta}\delta_{\mu r}\frac{2Br}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} + \delta_{\mu\theta}\delta_{\nu\rho} 2\frac{2B\rho}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} - \delta_{\nu\theta}\delta_{\mu\rho}\frac{2B\rho}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} \tag{45}$$

Finally, the field-tensor (containing three components by itself) in the color space appears to be:

$$\vec{F_{\mu\nu}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \delta_{\mu\theta}\delta_{\nu r} 2\frac{2Br}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} - \delta_{\nu\theta}\delta_{\mu r}\frac{2Br}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} + \\ \delta_{\mu\theta}\delta_{\nu\rho} 2\frac{2B\rho}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} - \delta_{\nu\theta}\delta_{\mu\rho}\frac{2B\rho}{b^2 g} e^{-\frac{r^2+\rho^2}{b^2}} \end{pmatrix} \tag{46}$$

Where $\mathbf{0}$ describes a $\mu \times \nu$ matrix, filled with zeros. Performing the integration along the entire four-dimensional space yields to:

$$S = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \vec{F_{\mu\nu}} \vec{F^{\mu\nu}} r\rho dr d\rho d\eta d\theta = \frac{B^2\pi^2(b^2+\pi)}{g^2} \tag{47}$$

Since the amplitude B and the constant term g, as well as the factor b, appearing in the exponential term of the trial solution, are non-vanishing, S is neither. Therefore, the case of a potential, being a gauge transformed version of the vacuum, can be excluded from the further calculations.

8 Determination of an upper energy limit

Starting from the actual eigenvalue equation with the eigenket $\vec{\Phi}_{solution}$;

$$M^{ab}\Phi^b_{solution} = \omega^2 \Phi^a_{solution} \tag{48}$$

the actual solution $\vec{\Phi}_{solution}$ will be replaced by the trial ansatz to:

$$\vec{\Phi}_{solution} \to \vec{\Phi} = \begin{pmatrix} \frac{4}{\pi b^4}(r\rho)^2 e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -\frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{49}$$

Applying this replacement to the eigenvalue equation and multiplying the equation by ϕ^a from the left side yields to:

$$\Phi^a M^{ab} \Phi^b = \Phi^a E_{border} \Phi^a \tag{50}$$

Where ω^2 is replaced by the upper energy threshold E_{border}, since $\vec{\Phi}$ represents the first trial ansatz (in analogy to $|0\rangle$ in chapter 4). Since (41) has been normalized, it is possible to find a expression for E_{border}:

$$E_{border} = \Phi^a M^{ab} \Phi^b \tag{51}$$

Formulating this in a less compact way, (51) yields to the following integration:

$$E_{border} = \frac{B}{N} \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \begin{pmatrix} (r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \partial_\mu^2((r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta)) \\ -\partial_\mu^2((r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta)) \\ 0 \end{pmatrix} +$$
$$\begin{pmatrix} (r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix}^T \cdot \left(\begin{pmatrix} \vec{0} \\ \vec{0} \\ (\vec{e_\theta} \cdot e^{-\frac{r^2}{b^2}}) \end{pmatrix} \times \begin{pmatrix} \nabla_\mu((r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta)) \\ -\nabla_\mu((r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta)) \\ \nabla_\mu(0) \end{pmatrix} \right) r\rho dr d\rho d\eta d\theta \tag{52}$$

Referring to the previous expression the gradient only acts on the expressions on its left hand sides, while the Laplacian effects only the terms in its following brackets.

$$E_{border} = \frac{5}{b^2} - \frac{8B\sqrt{\frac{\pi}{3}}|b|^5}{81b^6} \tag{53}$$

Leading to a variational constraint $E_{border} < 0$ for bounded states:

$$E_{border} = \frac{5}{b^2} - \frac{8B\sqrt{\frac{\pi}{3}}|b|^5}{81b^6} < 0 \tag{54}$$

or expressed in terms of B:

$$\frac{405b^4}{8|b|^5}\sqrt{\frac{3}{\pi}} < B \tag{55}$$

9 Minimization of the variational parameters

For simplicity reasons b is set to $b = 1$. To achieve bounded sates $E_{border} < 0$ must be satisfied, leading to the following constraint for B:

$$\frac{405}{8}\sqrt{\frac{3}{\pi}} < B \tag{56}$$

In order to determine the quality of the trial ansatz in terms of solving the actual eigenvalue problem, the quadratic difference of every component on the left and the right side of the equal sign has to be considered.

$$D^a = M^{ab}\Phi^b - \omega^2\Phi^a \tag{57}$$

For clarity reasons the component notation is used, wherein Φ^a and Φ^b denote the components of Φ. Consecutively the integration of the sum of the quadratic differences is performed over the entire 4d-space. This results in a function $F(\omega, B)$, only depending on ω and B. $F(\omega, B)$ could be interpreted as cost or deviation function, depicting the suitability of the ansatz for the particular problem, compared to other trial solutions.

$$\int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} ((D^1)^2 + (D^2)^2 + (D^3)^2) r\rho dr d\rho d\eta d\theta = F(\omega, B) \tag{58}$$

Thus following expression can be found for $F(\omega, B)$:

$$F(\omega, B) = 46 + \frac{B^2}{32} + (-10 + \omega)\omega + \frac{16}{243}B\sqrt{\frac{\pi}{3}}(-14 + 3\omega) \tag{59}$$

The minimization of the deviation function $F(\omega, B)$ respect to ω results in the following expression:

$$\omega_{min} = \frac{1}{243}(1215 - 8B\sqrt{3\pi}) \tag{60}$$

After checking the second derivative at this point in ω space, ω_{min} can be identified as a minimum. Including the results of equation (54) B can always be chosen to reach bounded sates. Summarising the first approximate solution of the eigenvalue problem, $\vec{\Phi}$ and its approximation for the corresponding eigenvalue has been found for the particular potential configuration $\vec{A}$.

$$\vec{\Phi} = \frac{\vec{\Phi}_{start}}{\sqrt{N}} = \begin{pmatrix} \frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -\frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{61}$$

$$\omega_{min} = \frac{1}{243}(1215 - 8B\sqrt{3\pi}) \tag{62}$$

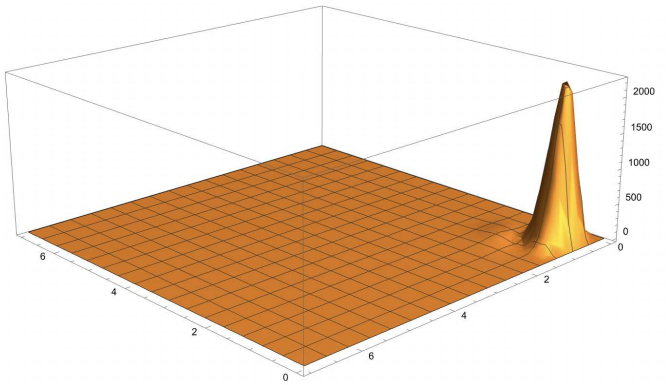

Figure 1: Displays the deviation $D^a D^a$ of the initial ansatz $\vec{\Phi}$, within the interval $r \in [0, 7]$ and $\rho \in [0, 7]$

10 Further improvements of the trial eigenstate based on the previous calculations

Taking the structure of $\vec{\Phi}$ (61) into account, the non-vanishing components of the ansatz only differ by the angular part, respectively by $\sin(\theta)$ or $\cos(\theta)$. Since these parts only appear linearly in D^1 and D^2 the angular dependencies terminate by consideration of $((D^1)^2 + (D^2)^2 + (D^3)^2)$ due to the identical radial parts (for r and ρ) within every component.

$$\Phi_{r,\rho} = (r\rho)^2 e^{-\frac{r^2+\rho^2}{b^2}} \tag{63}$$

Therefore an improvement regarding to the radial dependency of (61) seems to be intuitive.

10.1 Investigations respect to the simple-polynomial behaviour of the first trial-ansatz $\Phi_{r,\rho}$

In further consideration $\Phi_{r,\rho}$ consists of an exponential term $e^{-\frac{r^2+\rho^2}{b^2}}$ and a quadratic power term $(r\rho)^2$. Hence $\Phi_{r,\rho}$ has a polynomial like behaviour at small radii r and ρ. (See also figure 1). Therefore, a polynomial behaviour like

$$\sum_{i=2}^{n} (r\rho)^i \tag{64}$$

might lead to an overall improvement. A decreasing value of the deviation function $F(\omega_{min}, B_{var})$ for every added element with corresponding index n should be expected. To examine this the amplitude B is on the one hand set to be constant (B_{const}) and on the other hand chosen to be variable (B_{var}), more precisely, set to a 10% higher amplitude then the minimal required value to achieve bounded states for a particular index (considered up to index $n = 5$).

Therefore a decrease of $F(\omega_{min}, B_{var})$ at constant $B = B_{const}$ (visual in table 2 and 3) can be observed with rising indices n and correspondingly more elements of the series (64). However, this

Table 1: Estimation of the total deviation error $F(\omega_{min}, B_{var})_n$, at variable B_{var}

n	B_{var}	ω_{min}	$F(\omega_{min}, B_{var})$
2	50	-0.5	90
3	85.9779	-0.459963	96.3167
4	154.55	-0.42689	117.63
5	340.241	-0.408875	162.461

Table 2: Estimation of the total deviation error$F(\omega_{min}, B_{var})_n$, at constant B_{const}

n	B_{const}	ω_{min}	$F(\omega_{min}, B_{var})_n$
2	350	-30.4	2620
3	350	-16.0	1210
4	350	-6.37	498
5	350	-0.538	170

Table 3: Estimation of the total deviation error $F(\omega_{min}, B_{var})_n$, at constant B_{const}

n	B_{const}	ω_{min}	$F(\omega_{min}, B_{var})_n$
2	750	-70.8	11900
3	750	-39.5	5400
4	750	-18.5	2150
5	750	-5.83	676

only appears to be true at large absolute values of B, due to the increase of the lowest affordable threshold for B to reach bounded states for every element (displayed in table 1). To conclude an ansatz in the manner of (64) can be an improvement leading to a decrease of the deviation function $F(\omega_{min}, B_{var})$ at its minimum, although B has to be large enough to obtain bounded states for the a certain maximal index n.

10.2 Introduction of a lower-order polynomial with variable coefficients

Since most of the deviating behaviour takes place at small r and ρ, a lower polynomial ansatz with variable coefficients might be better suited than the previous trial solution. Starting with the lowest order polynomial higher than the initial ansatz (61) this yields to

$$\Phi_{r,\rho} = (c \cdot (r\rho)^2 + d \cdot (r\rho)^3) e^{-\frac{r^2+\rho^2}{b^2}} \tag{65}$$

for the radial part of $\vec{\phi}$. c and d are variable and determined by minimisation process respect to the deviation function $F(\omega, B, c, d)$. Therefore, the total number of variational parameters increases by two. Starting with (65) this leads to the following expression for E_{border}.

$$E_{border} = -\frac{1024\sqrt{\frac{\pi}{3}}B\left(6c^2 + 10cd + 5d^2\right)}{243\left(256c^2 + 225\pi cd + 576d^2\right)} + \frac{768\left(c^2 + d^2\right)}{5\left(256c^2 + 225\pi cd + 576d^2\right)} + \frac{22}{5} \tag{66}$$

Leading to

$$\frac{243\sqrt{\frac{3}{\pi}}\left(640c^2+495\pi cd+1344d^2\right)}{512\left(6c^2+10cd+5d^2\right)}<B \tag{67}$$

for B to achieve a negative maximal energy threshold for bounded states. As assumed earlier at chapter 10.1, a value for B with 1.1-times of this threshold (left expression in (67)) is used for the following calculations. Furthermore the expression for the deviation function $F(\omega,B,c,d)$ appears as

$$\begin{aligned} F(\omega,B,c,d) &= \frac{384c^2\left(729B^2+512\sqrt{3\pi}B(3\omega-14)+23328((\omega-10)\omega+46)\right)}{34992\left(256c^2+225\pi cd+576d^2\right)} \\ &+ \frac{48d^2\left(2187B^2+2048\sqrt{3\pi}B(5\omega-17)+139968(\omega(3\omega-28)+122)\right)}{34992\left(256c^2+225\pi cd+576d^2\right)} \\ &+ \frac{cd\left(19683\pi\left(5B^2+80\omega(5\omega-44)+13824\right)+16384\sqrt{3\pi}B(60\omega-197)\right)}{34992\left(256c^2+225\pi cd+576d^2\right)} \end{aligned} \tag{68}$$

In order to minimize the function, the derivatives respect to ω, c and d have to equal zero. Hence a system of equations with three variables has to be solved.

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}\omega} &= \frac{2\left(768c^2\left(8\sqrt{3\pi}B+243(\omega-5)\right)+64d^2\left(80\sqrt{3\pi}B+2187(3\omega-14)\right)\right)}{729\left(256c^2+225\pi cd+576d^2\right)} \\ &+ \frac{5cd\left(2048\sqrt{3\pi}B+6561\pi(5\omega-22)\right)}{729\left(256c^2+225\pi cd+576d^2\right)} = 0 \end{aligned} \tag{69}$$

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}c} &= d\frac{\left(3359232cd\left(5B^2-128(\omega-8)\right)+1382400\pi^{3/2}\sqrt{3}B\left(c^2(6\omega-28)+d^2(17-5\omega)\right)\right.}{2187\left(256c^2+225\pi cd+576d^2\right)^2} \\ &+ d\frac{65536\sqrt{3\pi}B\left(c^2(788-240\omega)+24cd(17\omega-92)+9d^2(60\omega-197)\right)}{2187\left(256c^2+225\pi cd+576d^2\right)^2} \end{aligned} \tag{70}$$

$$- \quad d\frac{19683\pi\left(8c^2\left(15B^2-960\omega+9152\right)+3d^2\left(35B^2+2560\omega-29312\right)\right)}{2187\left(256c^2+225\pi cd+576d^2\right)^2} \quad = \quad 0$$

$$\frac{\mathrm{d}F}{\mathrm{d}d} = d\frac{\left(-3359232cd\left(5B^2-128(\omega-8)\right)-1382400\sqrt{3}\pi^{3/2}B\left(c^2(6\omega-28)+d^2(17-5\omega)\right)\right.}{2187\left(256c^2+225\pi cd+576d^2\right)^2} \tag{71}$$

$$+ \quad d\frac{65536\sqrt{3\pi}B\left(4c^2(60\omega-197)+24cd(92-17\omega)+9d^2(197-60\omega)\right)}{2187\left(256c^2+225\pi cd+576d^2\right)^2}$$

$$- \quad d\frac{19683\pi\left(8c^2\left(15B^2-960\omega+9152\right)+3d^2\left(35B^2+2560\omega-29312\right)\right)}{2187\left(256c^2+225\pi cd+576d^2\right)^2} \quad = \quad 0$$

Starting with the first equation an expression for ω_{min} can be found depending on c and d. Inserting ω_{min} into the two remaining equations solving a system of two equations with two variables is performed. In this case the system appears to be under-constrained. Nevertheless, there still is a possibility to find a certain region for c or d in order to locate a minimum for $F(\omega, B, c, d)$. If the solution point appears to be a minimum in the $c-d-\omega$-Space, the determinate of the Hessian matrix $Det[\mathbf{H}(F(\omega_{min}, B, c_{min}, d_{min}))]$ has to be positive. The Hessian matrix (at constant B) should be defined as:

$$H(F(\omega, B, c, d))_{i,j,k} = \frac{\partial}{\partial i}\frac{\partial}{\partial j}\frac{\partial}{\partial k}F(\omega, B, c, d) \tag{72}$$

where i, j and k can be ω, c and d. At first, one of the equations is solved respect to c for example. As the following step the solution $c_{min}(d)$ will be inserted into
$Det[\mathbf{H}(F(\omega_{min}(c_{min}(d), d), B, c_{min}(d), d))]$. In order to reach a minimum, positive definiteness at this point, has to be satisfied:

$$Det[\mathbf{H}(F(\omega_{min}(c_{min}(d), d), B, c_{min}(d), d))] > 0 \tag{73}$$

Therefore, it might be possible to choose a certain interval for d to satisfy (73). This seems to be theoretically possible. However, to solve ((70) or (71)) for c respectively d the solution of a higher order polynomial will be performed. Unfortunately, the computational effort for this task is beyond the available resources for this project.

11 Results, conclusion and outlook

Applying the variational method to the eigenvalue equation of the Faddev-Popov-operator can provide an approximate solution, while restricting to the region beyond the first Gibrov-horizon and obeying the landau gauge condition.

As first solution attempt, the approximate eigenket $\vec{\Phi}$ appears to be:

$$\vec{\Phi} = \frac{\vec{\Phi}_{start}}{\sqrt{N}} = \begin{pmatrix} \frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -\frac{4}{\pi b^4}(r\rho)^2 \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{74}$$

with the corresponding approximation for the eigenvalue ω_{min}:

$$\omega_{min} = \frac{1}{243}(1215 - 8B\sqrt{3\pi}) \tag{75}$$

Here, the vector potential $\vec{A}$ was chosen by:

$$\vec{A} = \frac{B}{g} \cdot \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{e_\theta} \cdot e^{-\frac{r^2}{b^2}} \end{pmatrix} \tag{76}$$

While taking (74) into account, hints for a promising improvement have been found, leading to a serial approach, namely with a polynomial series, with variable coefficients, concerning the radial parts of $\vec{\Phi}$. Also if the actual second order trial ansatz (65) could not be performed (due to limited computational resources) in the scope of the full minimization process, it might be possible to enhance the quality of the trial eigenket regarding to the eigenvalue equation. Concluding this insights a ansatz for $\vec{\Phi_p}$ like:

$$\vec{\Phi_p} = \begin{pmatrix} \frac{4}{\pi b^4}\sum_{i=2}^{n} c_i (r\rho)^i \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \cos(\theta) \\ -\frac{4}{\pi b^4}\sum_{i=2}^{n} c_i (r\rho)^i \cdot e^{-\frac{r^2+\rho^2}{b^2}} \cdot \sin(\theta) \\ 0 \end{pmatrix} \tag{77}$$

Where c_i indicates the variable coefficients, that shall be minimized due to a a cost function like (59) depending on $n+2$ parameters including B and ω. The parameter n determines the highest considered order of the series. One might notice that, increasing n is not necessarily useful as leading to an extraordinary growth in computational effort.

Generally spoken, a polynomial structure like (77) with respect to the radii components of $\vec{\Phi_p}$ (also including the initial ansatz), could provide a good candidate for being a sufficient approximation for the Faddev-Popov-operator eigenvalue equation, since a broad spectrum of functions could be approximated by a Taylor-approximation, reflecting a similar polynomial behaviour in r and ρ like (77).

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