Bachelor's thesis

Construction of a model action for a zero-dimensional model of the nonanalytic behaviour of the gluon propagator in Landau gauge at zero source strength

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Abstract

Possible forms of the action of a zero dimensional model free energy describing the behaviour of the gluon field in Landau gauge were studied. In [1] it was found, that the free energy of the gluon field is not analytic in a source term in this gauge. The goal of this bachelor's thesis was to find an action, which would lead to said free energy when perturbed with a source term or to a non-analyticity in the source parameter at zero. For a non-linear parameterization of the source, an action leading to the free energy in [1] was found. For a linear parameterization, an action leading to a free energy, which is not analytic in the source, was found. Both results were acquired using Mathematica. The free energies were Legendre transformed in the source parameter, to obtain the corresponding quantum effective action.

Contents

1 Introduction

An assumption frequently made in physics, is that functions are analytic in their variables. This is often assumed for the generating functional of correlation functions and its source parameter in quantum field theory [2] [3] [4]. As it turns out, this is not the case for the gluon field in minimal Landau gauge, as it was found that there is a difference between the gluon propagator calculated with perturbation theory, which relies on the analyticity of the theory, and the results of numerical methods which do not need an analytic theory [1]. The propagator $D(k, h)$ is not analytic at the origin for source strength $h = 0$ and momentum $k = 0$, because the infrared limit $k \to 0$ and the limit to the unperturbed theory $h \to 0$ do not commute: $\lim_{k\to0} \lim_{h\to0} D(k, h) \neq \lim_{h\to0} \lim_{k\to0} D(k, h)$. This would be the case, if the quantity from which the propagator can be derived, the free energy, was non-analytic in the source parameter. The free energy is defined as (14), thus if the generating functional is not analytic in the source, the free energy and the propagator usually should not be analytic either [1].

To obtain the generating functional of correlation functions, a path-integral over a field A is calculated in every possible field configuration and thus in every possible gauge (here written as an integral over Ω).
To evaluate the integral, one needs to fix a gauge, which corresponds to doing a coordinate transform and integrating over a smaller domain of field configurations (written as integral over $\omega \subseteq \Omega$) - thus a Jacobian determinant $|\det J|$ is introduced into the equation [3]:

$$\int_\omega \mathcal{D}A e^{-S(A)} = \int_\Omega \mathcal{D}A e^{-S(A)-S_{GF}(A)}$$

where the Jacobian determinant was written as a gauge fixing term in the action: $|\det J| = e^{S_{GF}(A)}$. This makes it possible, to separate the integral into an integration in one gauge, and an integration over the gauge transformations which becomes an irrelevant factor [3].

For the gluon field in minimal Landau gauge it was discovered, that [1]

$$\int_\Omega \mathcal{D}A e^{-S(A)-S_{GF}(A)} \neq \lim_{J\to0} \int_\Omega \mathcal{D}A e^{-S(A)-S_{GF}(A)+\int JA}$$

where JA is a perturbation parameterized with J, which means that the term on the right-hand side is not continuous and thus not analytic in J at $J = 0$. Thus, perturbation theory cannot be used to evaluate the left-hand side of the equation. The discrepancy was found because perturbation theory calculates the integral by taking the limit $J \to 0$, and numerical methods calculate the integral without the perturbation JA [1].

The aim of this bachelor's thesis is to take a look at a simplified problem where the field is a function of a zero dimensional space and determine, what kind of gauge fixing term in the action could introduce such an effect: For which function $S_{GF}(x)$ is $\int_{-\infty}^{\infty} e^{-S_{GF}(x)+Jx} dx$ not analytic in J at $J = 0$? A more detailed explanation of the theory can be found in section 2 [1]. Chapter 3 contains the action with non-linear parameterization of the perturbation that solved the simplified problem. In chapter 4 an action with a linearly parameterized perturbation which lead to a non-analytic free energy is discussed. In chapter 5, some thoughts about the functions that were discarded and ideas about how to find further suitable actions are expressed. Furthermore, the Legendre transform of the non-analytic free energy, that was obtained for the linear parameterization, was calculated numerically in chapter 6. Chapter 7 gives a short summary of the results.

All calculations were done with the help of Mathematica, some results that were identified as wrong or incomplete, were modified manually.

2 Theory

2.1 Analytic functions

Let $D \subseteq \mathbb{R}$ be open. A function $f : D \to \mathbb{R}$ is real analytic in D, if it is infinitely often differentiable and for each point $x_0 \in D$ the function's Taylor series converges to the function itself pointwise in a neighbourhood of x_0. A function is real analytic at a point $x_0 \in D$, if there exists a neighborhood of x_0 in D, for which the Taylor series of the function at the point x_0 is defined and convergent to the function [5] [6].

The Taylor series of a function at the point x_0 is defined as [6]

$$f(x) = \sum_{n=0}^{\infty} \frac{\partial f^n(x_0)}{\partial x^n} \frac{(x - x_0)^n}{n!}, \tag{1}$$

where $\frac{\partial f^n(x_0)}{\partial x^n}$ denotes the n-th derivative of the function f at the point x_0. A complex analytic function can be defined in the same way, though an open set has to be open in $\mathbb{C}$ [7]. Furthermore f is real analytic on a set D open in $\mathbb{R}$ if and only if there exists a complex analytic extension of f on an open set $G \subseteq \mathbb{C}$ with $D \subseteq G$ [5]. If a function is defined on an open real interval, its analytic continuation into a simply connected open set in $\mathbb{C}$ containing the set is unique [8].

For a function to be analytic in a domain $G \subseteq \mathbb{C}$ it is a necessary condition, that the function is continuous on G. Composites, sums and products of analytic functions are analytic themselves. Polynomials, the exponential function and the error function are analytic on $\mathbb{C}$. A rational function with poles $x_1 \dots x_n \in \mathbb{C}$ is analytic on $\mathbb{C} \setminus (\bigcup_{i=1}^{n} \{x_i\})$ [8] [9].

2.2 Legendre transform

The Legendre transform f^* of a function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, with $f(x) \neq \infty$ for at least one point $x \in \mathbb{R}$, is defined as [10]

$$f^*(x) = \sup_{c \in \mathbb{R}} \{cx - f(c)\} \tag{2}$$

A function is called convex on an interval $[a, b]$, if $\forall x_1, x_2 \in [a, b]$ the inequality [6]

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} \tag{3}$$

is fulfilled, meaning the secant of two points of the function is never below the function itself between the two points. A twice differentiable function is convex on an interval $I \subseteq R$ if and only if its second derivative is non-negative on I [6].

The Legendre transform f^* of a twice differentiable function f with invertible first derivative and positive second derivative can be calculated as follows [10]:

$$f^*(x) = (c \cdot x - f(c))\Big|_{c=\left(\frac{\partial f}{\partial x'}\right)^{-1}(x)}. \tag{4}$$

Here $\left(\frac{\partial f}{\partial x'}\right)^{-1}(x)$ is the inverse of the first derivative of f at the point x.
A function f is lower semicontinuous if for every converging sequence (x_n) [10]

$$f\left(\lim_{n\to\infty} (x_n)\right) \leq \liminf_{n\to\infty} f(x_n). \tag{5}$$

Every continuous function is thus also lower semicontinuous. For a function f which is lower-semicontinuous and convex, $(f^*)^*(x) = f(x)$ holds: the Legendre transform is invertible and its own inverse [10].

2.3 Path integral

2.3.1 In quantum mechanics

The quantum mechanical propagator $\langle x_N, t_N | x_1, t_1 \rangle$ describes the probability density that a particle propagates from point x_1 at time t_1 to point x_N at time t_N. It can be calculated by integrating over all possible paths the particle could take, weighted by a factor which is dependent on the classical action $S = \int_{t_1}^{t_N} \mathcal{L}(\dot{x}, x, t)dt$ of each particular path [11]:

$$\langle x_N, t_N | x_1, t_1 \rangle = \int_{x_1,t_1}^{x_N,t_N} \mathcal{D}[x(t)] \exp\left(i \int_{t_1}^{t_N} \mathcal{L}(\dot{x}, x, t)dt \right) \tag{6}$$

$\mathcal{D}[x(t)]$ denotes the summation over all possible paths. In practice this is done by discretizing time and integrating over position space at different points in time, then taking the limit to infinitely many points in time [11]:

$$\int_{x_1,t_1}^{x_N,t_N} \mathcal{D}[x(t)] = \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t} \right)^{\frac{(N-1)}{2}} \int_{\mathbb{R}^p} dx_{N-1} \cdots \int_{\mathbb{R}^p} dx_2. \tag{7}$$

x_1, x_N, t_1 and t_N are fixed, the t_n are equally spaced, $\Delta t = \frac{t_N - t_1}{N-1}$. p is the dimension of space. Using, that for $N \to \infty$ it holds that $\Delta t \to 0$, the action can be approximated to be a function of the start and end point of the integration of the Lagrangian. In this way, the propagator can be calculated [11].

2.3.2 In quantum field theory of a scalar field

In quantum field theory of a scalar field, the action S is written as a functional of a scalar field $\phi(x)$ with $x \in \mathbb{M}$, where $\mathbb{M}$ denotes Minkowski spacetime [2] [3]:

$$S[\phi(x)] = \int_{\mathbb{M}} \mathcal{L}(\phi(x), \partial_\mu \phi(x), x)dx. \tag{8}$$

$\partial_\mu \phi(x)$ is the four-derivative of the field $\phi(x)$ with respect to x and $\mathcal{L}$ is the Lagrangian density describing the system. The path integral is then calculated by summing over all different possible field configurations, similar to the path integral in quantum mechanics. Instead of time, minkowski space-time is discretized to define the path integral. The field configurations at times $t_1 \to -\infty(1 - i\epsilon)$ and $t_N \to \infty(1 - i\epsilon)$ with $\epsilon \to 0^+$ are fixed [2] [3].

To obtain the n-point correlation functions $\langle T(\phi(x_1) \dots \phi(x_n)) \rangle \coloneqq \langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle$ which are used to describe field interactions, the path integral is performed [2] [3]:

$$\langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle = \frac{\int \mathcal{D}[\phi(x)] \phi(x_1) \dots \phi(x_n) \exp\left(iS[\phi(x)]\right)}{\int \mathcal{D}[\phi(x)] \exp\left(iS[\phi(x)]\right)} \tag{9}$$

where $S[\phi(x)]$ is the action from (8), T is the time ordering operator, $|\Omega\rangle$ is the vacuum state and $\mathcal{D}[\phi(x)]$ denotes the integration over all possible field configurations of the field $\phi(x)$. $\phi(x_1) \dots \phi(x_n)$ are also fields, on the left-hand side of (9) they are operators. The term in the denominator is a normalization constant [2] [3].

2.4 Gauge theories and gauge transformations

A gauge field theory is a theory, which is described by a field whose configuration is not uniquely determined by the physical state that it is supposed to describe. A well known case in classical physics is electrodynamics, where the electromagnetic four-potential can be transformed to a different potential, that still describes the same physical electromagnetic field via a gauge transformation [3]:

$$A_\mu(x) \to A_\mu(x) + c\partial_\mu \alpha(x) \tag{10}$$

Here A_μ is the electromagnetic four-potential, $\mu \in \{0, 1, 2, 3\}$ is the index running over all spacetime dimensions, c is a constant, $\alpha(x)$ is a real valued field and ∂_μ is the four-gradient. These gauge transformations leave the Lagrangian and thus the action invariant [3].
Since the Lagrangian does not change under these transformations, there are "directions" in the path integral which are constant, thus making the integral divergent. When a gauge is fixed, this problem is solved, as the integration is not performed over these equivalent field configurations anymore. In practice, gauge fixing is achieved by adding an additional gauge fixing term to the action [3].

2.5 Perturbation theory

Expression (9) can be difficult to calculate directly, which is why perturbation theory is used. The generating functional of correlation functions (also called partition function) is defined as [2]:

$$Z[J] :- \frac{\int \mathcal{D}[\phi(x)] \exp\left(i \int d^4x \left(\mathcal{L}(\phi(x), \dot{\phi}(x), x) - J(x)\phi(x)\right)\right)}{\int \mathcal{D}[\phi(x)] \exp\left(i \int d^4x \mathcal{L}(\phi(x), \dot{\phi}(x), x)\right)} \tag{11}$$

If the generating functional is analytic in the source $J(x)$, the n-point correlation functions can be retrieved from it via [3]

$$\langle T(\phi(x_1) \dots \phi(x_n)) \rangle = \frac{1}{(-i)^n} \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)}\bigg|_{J=0} \tag{12}$$

Here, $\frac{\delta}{\delta J(x_1)}$ denotes the functional derivative with respect to the field $J(x_1)$.
The propagator, or two point function, of a scalar field is thus

$$\langle T\phi(x_1)\phi(x_2) \rangle = -\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \tag{13}$$

The free energy (or generating functional of connected diagrams) is defined by [2]

$$\exp(W[J]) = Z[J] \tag{14}$$

The propagator can then also be calculated with

$$\langle T\phi(x_1)\phi(x_2) \rangle = -\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \tag{15}$$

The quantum effective action Γ is defined as the legendre transform of the free energy [12] [13]

$$\Gamma[\phi^c] = \int \phi^c(x) J(x) dx - W[\phi^c] \tag{16}$$

with

$$\phi^c = \frac{\delta W[J(x)]}{\delta J(x)} \tag{17}$$

2.6 Wick rotation

By performing the substitution $x_0 \to ix_0$, one can calculate quantities like the partition function in Euclidean space-time and without the factor of the imaginary unit in the exponent of the integrand [2] [3]. This was used in the calculation of the following results.

2.7 Result for gluon propagator in minimal Landau gauge

The gluon fields are gauge fields whose possible gauge transformations are described by a non-abelian group [12]. Gluons are bosons, so their vector fields are scalar valued. The gauge transformations for their vector potentials look similar to (10), although there appear some additional terms, because the structure constants of the transformation are not zero in a non-abelian theory [12] [3].

The gluon propagator is only defined if a gauge has been chosen. Some behaviour of gluons is impossible to describe with perturbation theory [12].

In [1] the gluon propagator in minimal Landau gauge was studied and it was found, that it is not analytic in the source term at zero source strength. The free energy per euclidean volume was defined as $w(J) = W(J)/L^d$ and parameterized as $w(h,k) \equiv W(J)/L^d$, and the source that was studied, was parameterized with $J = h\cos(x)$. L^d is the volume of the periodic box that was studied. A zero-dimensional model for the free energy per Euclidean volume was constructed, which reproduces the observed behaviour of the propagator at vanishing source strength h [1]:

$$w(k,h) = \sqrt{\hat{\gamma}^2(k) + \alpha^2 h^2 k^2} - \hat{\gamma}(k) \tag{18}$$

giving a propagator $D(k,h)$ of the form

$$D(k,h) = \frac{\partial^2 w(k,h)}{\partial h^2} = \frac{\alpha^2 k^2 \hat{\gamma}^2(k)}{(\hat{\gamma}^2(k) + \alpha^2 h^2 k^2)^{\frac{3}{2}}} \tag{19}$$

and a quantum effective action $\gamma(a)$

$$\gamma(a) = \hat{\gamma}(k)\left(1 - \left(1 - \frac{a^2}{\alpha^2 k^2}\right)\right) \tag{20}$$

Here, k is the momentum, α is a constant, a is the k-th component of the Fourier transform of the classical configuration, $\hat{\gamma}(k)$ is a function of k, which can be chosen in such a way, that $\lim_{k\to 0}\lim_{h\to 0} D(k,h) \neq \lim_{h\to 0}\lim_{k\to 0} D(k,h)$, the propagator is not analytic at $h = 0$ [1].

The model of a "zero dimensional path integral" with Wick rotation was studied, to get a glimpse of what the action describing this behaviour might look like. In zero dimensions, the Wick rotated partition function $Z[J]$ (11) is given by [4]:

$$z(j) = \int_{-\infty}^{\infty} dx \exp\left(-S(x) + jx\right), \tag{21}$$

in this case the fields j and x are just variables. The free energy is then defined by

$$\exp\left(w(j)\right) = \int_{-\infty}^{\infty} dx \exp\left(-S(x) + jx\right). \tag{22}$$

For the quantum effective action, (16) reduces to [1]

$$\gamma(a) = ja - w(a) \tag{23}$$

with

$$a = \frac{\partial w(j)}{\partial j} \tag{24}$$

in the zero dimensional case that was studied.

3 Non-linear parameterization that led to desired function for easier problem

In this bachelor's thesis a simplified version of the model free energy (18) was considered:

$$w(h) = \sqrt{b^2 + ch^2} - b \tag{25}$$

Here, h is the source parameter, b and c are real parameters.

3.1 Result without perturbation

For the unperturbed Action $S_0(x) = dx^2$ was used, with $d \in \mathbb{R}^+$. Then, the free energy for the unperturbed action $w_0(d)$ was calculated to be

$$\exp(w_0(d)) = \sqrt{\frac{\pi}{d}} \tag{26}$$

$$w_0(d) = \log\left(\sqrt{\frac{\pi}{d}}\right) \tag{27}$$

where the principal branch of the logarithm was chosen for this thesis. Note, that the normalization in (26) is different from the normalization in (11).

3.2 Result with perturbation

A source term with non-linear parameterization was added to the previously unperturbed action, to obtain the free energy (25) from the simplified model of (18) in [1]:

$$w(h,b,c) = \log\left(\int_{-\infty}^{\infty} dx \exp\left(-S_0(x) + 2\sqrt{d\log\left(\frac{\sqrt{d}\exp\left(-b+\sqrt{b^2+ch^2}\right)}{\sqrt{\pi}}\right)}x\right)\right) = -b + \sqrt{b^2+ch^2} \tag{28}$$

(18) can thus easily be obtained from (28) by replacing b with $\hat{\gamma}(k)$ and c with α^2k^2.

For a function $f(h)$ one can set the Action to $S_0(x) + 2\sqrt{a\log\left(\sqrt{\frac{a}{\pi}}f(h)\right)}x$ giving

$$\int_{-\infty}^{\infty} dx \exp\left(-S_0(x) + 2\sqrt{d\log\left(\sqrt{\frac{d}{\pi}}f(h)\right)}x\right) = f(h) \tag{29}$$

Thus, for a function $f(h) = \exp(w(h))$ where $w(h)$ is not analytic in h, one can find a perturbation to $S_0(x)$ which gives the wanted function and is of the form $g(h)x$.

4 Simpler parameterization

For a simpler parameterization of the perturbation, no action was found, that gave a free energy exactly of the form (25). However, an action, leading to a dependence on the source term, which is not analytic at vanishing source strength, was found.

4.1 Result without perturbation

The action

$$S_0'(x) = \begin{cases} cx^2 & x \geq 0 \\ b\sqrt{|x|} & x < 0 \end{cases} \tag{30}$$

with $c, b \in \mathbb{R}^+$, was used as the unperturbed term, giving a free energy per unit euclidean volume of $w_0(c,b) = \log\left(\frac{4\sqrt{c}+b^2\sqrt{\pi}}{2\sqrt{c}b^2}\right) = \log\left(\frac{2}{b^2} + \frac{\sqrt{\pi}}{2\sqrt{c}}\right)$. Again, the normalization is different from (11).

4.2 Result with perturbation

Perturbing the action S_0' with the source term hx and calculating the path integral via (21) leads to the partition function

$$z(h,c,b) = \begin{cases} \frac{2}{b^2} + \frac{\sqrt{\pi}}{2\sqrt{c}} & h = 0 \\ \frac{\sqrt{\pi}e^{\frac{h^2}{4c}}\left(\mathrm{erf}\left(\frac{h}{2\sqrt{c}}\right)+1\right)}{2\sqrt{c}} - \frac{\sqrt{\pi}be^{\frac{b^2}{4h}}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{h}}\right)}{2h^{3/2}} + \frac{1}{h} & h > 0 \\ \infty & h < 0 \end{cases} \tag{31}$$

and the free energy

$$w(h,c,b) = \begin{cases} \log\left(\frac{2}{b^2} + \frac{\sqrt{\pi}}{2\sqrt{c}}\right) & h = 0 \\ \log\left(\frac{\sqrt{\pi}e^{\frac{h^2}{4c}}\left(\mathrm{erf}\left(\frac{h}{2\sqrt{c}}\right)+1\right)}{2\sqrt{c}} - \frac{\sqrt{\pi}be^{\frac{b^2}{4h}}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{h}}\right)}{2h^{3/2}} + \frac{1}{h}\right) & h > 0 \\ \infty & h < 0 \end{cases} \tag{32}$$

The integral diverges for parameter values $h < 0$, this function is not defined in a neighbourhood of $h = 0$ and thus does not have a defined derivative at the point $h = 0$, thus it is not analytic there.
Taking the limit to zero source strength of the function describing the free energy for $h > 0$ gives

$$\lim_{h\to 0}\left(\log\left(\frac{\sqrt{\pi}e^{\frac{h^2}{4c}}\left(\mathrm{erf}\left(\frac{h}{2\sqrt{c}}\right)+1\right)}{2\sqrt{c}} - \frac{\sqrt{\pi}be^{\frac{b^2}{4h}}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{h}}\right)}{2h^{3/2}} + \frac{1}{h}\right)\right) = \log\left(\frac{2}{b^2} + \frac{\sqrt{\pi}}{2\sqrt{c}}\right) = w_0(c,b), \tag{33}$$

the function is continuous for $h \in \mathbb{R}_0^+$.

Figure 4 shows the source dependence of the free energy (32) for several parameter values. Parameter b changes a term of the function that quickly becomes insignificant away from the origin, parameter c changes the term of the function that becomes dominant for large h. For small values of the parameter b the free energy seems to have a minimum, for higher values of b, no more extrema are present. If both parameters are large, the free energy can become negative. Overall, the qualities of the function change only slowly with changing parameter b for the values that were plotted: making the value b 50 times larger, visually does not alter the function much. The parameter c seems to mostly influence, how "stretched" the function becomes horizontally. Both of the parameters influence the value of the limit, that the free energy takes on for $h \to 0$.

5 Some observations

For actions similar to (30), one could probably find similar behaviours for the free energy. In appendix B some other actions with linear parameterizataion that were studied are listed. It seems, that for this source term, the dependence of the partition function on the source takes on some of the same qualities as the dependence of the unperturbed action on the field variable. If the unperturbed term is even in the field variable, the partition function is even in the source parameter. For an unperturbed action in the form of a monomial of the absolute value of x, which can be written as symmetric hypergeometric function dependent on a monomial of $|x|$, which is convergent everywhere, the resulting h-dependence is a symmetric sum of generalized hypergeometric functions which have an argument proportional to a monomial of h and are convergent everywhere.

It did not seem to be possible for Mathematica, to calculate the path integral for most actions that have three different terms with an x-dependence, which made it difficult to calculate (21) for actions containing the perturbation and a dependency on x that is not analytic, for example square roots, because then the integral diverges for every nonzero value of h. The only functions of this form that were found to be integrable in a closed form expression, contain as a third term $d|x|$ which makes the integral convergent for $h \in (-d, d)$. Complex plots of these results seem to suggest, that they are analytic at $h = 0$, figures 6 and 7 show this for two different actions for one set of parameter values each. Some of the results look like they could be non-analytic at $h = 0$, if d could be set to zero, which would be possible, if one could find another term making the integral finite which is not linear in x. Thus making the integral convergent with another term, could produce something, which has a similar non-analytic structure as some terms of the results have at $h = \pm d$. Restricting the integrals on one side with the unperturbed action, similar to (30), and setting the parameter $d = 0$, could lead to partition functions, that have a similar structure at $h = 0$ as the functions have at $h = \pm d$, maybe without the divergence at $h = 0$ if the action is chosen correctly.

The function (31) seems to have a structure, that makes it impossible to describe the function values for $h \geq 0$ with an entire function. This need not be the case, as another action that leads to a non-analytic partition function was found, that is also convergent for $h = 0$ and divergent for $h < 0$ but here the dependence for $h \geq 0$ can be described by an entire function: B.2

6 Legendre transform of results

6.1 Non-linear parameterization

The legendre transform of (25) with respect to the source parameter h can easily be obtained from (20) by means of substitution.

6.2 Linear parameterization

In the following, it was assumed that the source strength $h \geq 0$ for (32), since the path integral for the action (30) diverges for values of $h < 0$.

To obtain a quantum effective action for the model with linear parameterization, the result (32) was Legendre transformed numerically, as its derivative

$$\frac{\partial w(h,c,b)}{\partial h} = \frac{\sqrt{\pi}c^{3/2}be^{\frac{b^2}{4h}}\left(b^2+6h\right)\operatorname{erfc}\left(\frac{b}{2\sqrt{h}}\right)+2\sqrt{h}\left(\sqrt{c}\left(2h^3-c\left(b^2+4h\right)\right)+\sqrt{\pi}h^4e^{\frac{h^2}{4c}}\left(\operatorname{erf}\left(\frac{h}{2\sqrt{c}}\right)+1\right)\right)}{4ch^{5/2}\left(\sqrt{\pi}he^{\frac{h^2}{4c}}\left(\operatorname{erf}\left(\frac{h}{2\sqrt{c}}\right)+1\right)+2\sqrt{c}\right)-4\sqrt{\pi}c^{3/2}bh^2e^{\frac{b^2}{4h}}\operatorname{erfc}\left(\frac{b}{2\sqrt{h}}\right)}$$

could not be inverted analytically.

Visually it seems, that the function might be convex at least for some combinations of parameter values c,b, (figure 4) since it seems to be convex close to the origin, and for large h, the term proportional to the convex function h^2 dominates. An attempt was made to test this with the help of Mathematica, by checking if the second derivative is positive under the assumption that $h \geq 0$, for some combinations of c and b, which was unsuccessful. Nevertheless, the Legendre transform of the function was calculated numerically under the assumption, that (4) holds, in an interval, where the free energy looks convex for the combinations of parameter values which were used here. The code which was used can be found in appendix A.
The functions numerical evaluation is unreliable for certain parameter values, especially for small values of h. Thus, the parameter values that were studied, were chosen such that the numerical evaluation did not seem too unreliable. The interval of values of h that were transformed, $h \in [0.1, 5]$, was also chosen accordingly. Figure 5 shows plots of the Legendre transform of (32) for several parameter values.

7 Conclusion and outlook

As was shown in section 3, it is easy to find an action leading to non-analytic behaviour in the source parameter, if the parameterization of the source can be non-linear. One could probably find many actions whose free energies behave the same in the source strength if a non-linear parameterization of the source is allowed, although there might arise the problem of getting a result of the path-integral, that depends on the parameter function in a way that is not invertible analytically, thus making it difficult to get the exact result (25).

An action containing a source term with linear parameterization of the source strength can also be used to construct a free energy which is not analytic in the source strength parameter, as was shown by the example which was built in section 4: (32). Further actions that give similar results could probably be constructed, by constraining the integral for the positive real axis by the unperturbed action and the negative axis by the perturbation term, like in the given example.

One problem encountered during the search for suitable actions, was the difficulty of analytically integrating interesting functions (some examples are in B.3). It is possible, that some actions were not recognized as fitting the wanted criteria because of this.

A numerical legendre transform was calculated for (32) and plotted in figure 5, the analytical legendre transform for (25) can be found in [1] (with some substitution).

A Code

In[∘]:= w[h_, a_, b_] = Log[ ]; (*function that will be transformed*)

$$w[h\_, a\_, b\_] = \mathrm{Log}\left[\frac{2\sqrt{a}\,\sqrt{h} + e^{\frac{h^2}{4a}}\,h^{3/2}\sqrt{\pi} + e^{\frac{h^2}{4a}}\,h^{3/2}\sqrt{\pi}\,\mathrm{Erf}\left[\frac{h}{2\sqrt{a}}\right] - \sqrt{a}\,b\,e^{\frac{b^2}{4h}}\sqrt{\pi}\,\mathrm{Erfc}\left[\frac{b}{2\sqrt{h}}\right]}{2\sqrt{a}\,h^{3/2}}\right];$$

Logarithmus

```
trialf[h_, a_, b_] = w[h, a, b];
(* legendre transform
 inputs:
   trialf: function dependant on 3 variables, is legendre transformed in 1st variable "h"
   a,b: other two variable values, act as parameters
   startpt: lowest value of h for which transform is calculated
   endpt: highest value of h for which transform is calculated
   delt: stepsize between values of h, for wich transform is calculated

 outputs: List with 5 entries {α,β,γ,δ,ϵ};
  α: values of derivative for parameter values a,b dependent on what values h has;
  β: values of h for which transform is calculated;
  γ: variable of legendre transform: derivative of function at values h, for which transform is calculated;
  δ: values of legendre tranform at values of γ;
  ϵ: listplot of legendre transform;

steps:
  -calculate points where derivative of function is evaluated: evpts,
  -evaluate derivative of function at these points: derevpts,
  -calculate transform by assuming, function is convex: trafo(p=derevpts=df/dh (h))  = h*(df/dh) (h)-f(h)
*)
legendre[trialf, a_, b_, startpt_, endpt_, delt_] :=
  {df[y_] = D[trialf[y, a, b], y], evpts = Table[i, {i, startpt, endpt, delt}], derevpts = df[evpts], trafo = evpts*derevpts - trialf[evpts, a, b],
           leite ab                            Tabelle
   ListPlot[{Transpose[{derevpts, trafo}]}, TicksStyle → Directive[18], AxesLabel → {Style["p", 18], Style["γ(p,a,b)", 18]},
   listenbezo⋯ transponiere                Stil der Teilstri⋯ Anweisung    Achsenbeschrif⋯ Stil              Stil
    PlotLabel → Style[Text["(a,b) = "[a, b]], FontSize → 20]]};
    Beschriftung ⋯ Stil  Text              Schriftgröße

avalues = {0.1, 1, 5, 8}; (*values of a for which transform will be evaluated*)
bvalues = {0.1, 1, 2, 5}; (*values of b for which transform will be evaluated*)
startpth = 0.01; (*smallest value of h for which transform will be evaluated*)
endpth = 5; (*largest value of h for which transform will be evaluated*)
delth = 0.00001; (*stepsize of h values that will be evaluated*)

Do[M = legendre[trialf, a, b, startpth, endpth, delth];
iteriere
 Print[M[[5]]], {b, bvalues}, {a, avalues}] (*prints plots of transform for different parameter values*)
 gib aus

(*Do[M=legendre[trialf,a,b,startpth,endpth,delth];
 Export[StringJoin[{"a_b","_",ToString[a],"_",ToString[b],"num_leg",".png"}],M[[5]]
 ],{b,bvalues},{a,avalues}] (*saves plots for different parameter values*)*)
```

B Some other examples

In the next section, ${}_pF_q(x)$ denotes the generalized hypergeometric functions, $\Gamma(x)$ the Gamma function, $G_{k,l}^{m;n}(x)$ the (generalized) Meijer G functions, $E_n(x)$ the generalized exponential integral, Ai(x) and Ai'(x) the Airy function of the first kind and its derivative, Bi(x) and Bi'(x) the Airy function of the second kind and its derivative, $K_n(x)$ the modified Bessel function of the second kind and $D_+(x)$ the Dawson plus function. The results of the integration of the 0-dimensional path integral corresponding to the actions are written below the actions. For real unperturbed actions S_0, there is no way that for $h = 0$, $z(h)$ becomes zero or less, thus for $w(h)$ to be non-analytic at $h = 0$, $z(h)$ must be non-analytic at that point because the logarithm is analytic on the positive reals. The functions were integrated with the assumption, that $h \in \mathbb{R}$ unless otherwise noted. For brevity, only the results of the calculation (using (21)) were written down. For some h where the result is not defined, it is because the integral diverges and in some other cases the calculation of the integral was unsuccessful.

B.1 Discarded examples

The following functions were classified as analytic or probably analytic at $h = 0$ because of the functions they are composed of, because of the appearance of their complex plots around the point $h = 0$. The functions that are undefined at $h = 0$ can probably be made analytic by adding a point with the value of the integral for $h = 0$. For some of these functions, it is easy to see that they are analytic in h:
Generalized hypergeometric functions are entire, if none of their parameters are nonpositive integers and they don't have more p than q parameters, $p \leq q$ [14]. The Airy functios of the first and second kind are entire and so are their derivatives [15]. The generalized exponential integral $E_p(x)$ is anayltic in the parameter p, if $x \neq 0$ [16].
For the other appearing functions, the situation is not as obvious. It was assumed that they are analytic at $h = 0$ because of the appearence of their complex plots. Examples of this are figures 7 and 6.

The functions that diverge for $h = 0$ were also discarded because they do not fit the wanted criteria of the model.

$\boldsymbol{S_0(x) = ax^4}$ with $\boldsymbol{a > 0}$

$$\frac{h^2\Gamma\left(\frac{3}{4}\right)\,{}_0F_2\left(;\frac{5}{4},\frac{3}{2};\frac{h^4}{256a}\right)}{4a^{3/4}} + \frac{2\Gamma\left(\frac{5}{4}\right)\,{}_0F_2\left(;\frac{1}{2},\frac{3}{4};\frac{h^4}{256a}\right)}{\sqrt[4]{a}}$$

$\boldsymbol{S_0 = ax^6}$ with $\boldsymbol{a > 0}$

$$\frac{h^4\Gamma\left(\frac{5}{6}\right)\,{}_0F_4\left(;\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3};\frac{h^6}{46656a}\right) + 12\sqrt{\pi}\sqrt[3]{a}h^2\,{}_0F_4\left(;\frac{2}{3},\frac{5}{6},\frac{7}{6},\frac{4}{3};\frac{h^6}{46656a}\right)}{72a^{5/6}} + \frac{2\Gamma\left(\frac{7}{6}\right)\,{}_0F_4\left(;\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6};\frac{h^6}{46656a}\right)}{\sqrt[6]{a}}$$

$\boldsymbol{S_0(x) = a|x|^3 \; a > 0}$

$$\frac{3h^2\left({}_1F_2\left(1;\frac{4}{3},\frac{5}{3};-\frac{h^3}{27a}\right) + {}_1F_2\left(1;\frac{4}{3},\frac{5}{3};\frac{h^3}{27a}\right)\right) + 4\;3^{2/3}\pi a^{2/3}\left(\mathrm{Bi}\left(-\frac{h}{\sqrt[3]{3}\sqrt[3]{a}}\right) + \mathrm{Bi}\left(\frac{h}{\sqrt[3]{3}\sqrt[3]{a}}\right)\right)}{18a}$$

$\boldsymbol{S_0(x) = a|x|^5 \; a > 0}$

$$\Big[24a^{3/5}h\Gamma\left(\frac{2}{5}\right)\left({}_0F_3\left(;\frac{3}{5},\frac{4}{5},\frac{6}{5};\frac{h^5}{3125a}\right)-{}_0F_3\left(;\frac{3}{5},\frac{4}{5},\frac{6}{5};-\frac{h^5}{3125a}\right)\right)+120a^{4/5}\Gamma\left(\frac{6}{5}\right)\left({}_0F_3\left(;\frac{2}{5},\frac{3}{5},\frac{4}{5};-\frac{h^5}{3125a}\right)+\right.$$
$$\left.+{}_0F_3\left(;\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{h^5}{3125a}\right)\right)+12a^{2/5}h^2\Gamma\left(\frac{3}{5}\right)\left({}_0F_3\left(;\frac{4}{5},\frac{6}{5},\frac{7}{5};-\frac{h^5}{3125a}\right)+{}_0F_3\left(;\frac{4}{5},\frac{6}{5},\frac{7}{5};\frac{h^5}{3125a}\right)\right)+$$
$$+h^4\,{}_1F_4\left(1;\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};-\frac{h^5}{3125a}\right)+h^4\,{}_1F_4\left(1;\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};\frac{h^5}{3125a}\right)+4\sqrt[5]{a}h^3\Gamma\left(\frac{4}{5}\right)\left({}_0F_3\left(;\frac{6}{5},\frac{7}{5},\frac{8}{5};\frac{h^5}{3125a}\right)+\right.$$
$$\left.-{}_0F_3\left(;\frac{6}{5},\frac{7}{5},\frac{8}{5};-\frac{h^5}{3125a}\right)\right)\Big]\frac{1}{120a}$$

$\boldsymbol{S_0(x)=a|x|^{\frac{5}{3}}}$ with $\boldsymbol{a>0}$

$$\Big[\pi h^4\,{}_3F_4\left(1,\frac{4}{3},\frac{5}{3};\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};-\frac{27h^5}{3125a^3}\right)+\pi h^4\,{}_3F_4\left(1,\frac{4}{3},\frac{5}{3};\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};\frac{27h^5}{3125a^3}\right)+4\;3^{7/10}a^{9/5}h\Gamma\left(-\frac{4}{15}\right)\cdot$$
$$\cdot\Gamma\left(\frac{16}{15}\right)\Gamma\left(\frac{7}{5}\right)\left({}_2F_3\left(\frac{11}{15},\frac{16}{15};\frac{3}{5},\frac{4}{5},\frac{6}{5};-\frac{27h^5}{3125a^3}\right)-{}_2F_3\left(\frac{11}{15},\frac{16}{15};\frac{3}{5},\frac{4}{5},\frac{6}{5};\frac{27h^5}{3125a^3}\right)\right)+6\sqrt[10]{3}a^{12/5}\Gamma\left(\frac{1}{5}\right)\cdot$$
$$\cdot\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{13}{15}\right)\left({}_2F_3\left(\frac{8}{15},\frac{13}{15};\frac{2}{5},\frac{3}{5},\frac{4}{5};-\frac{27h^5}{3125a^3}\right)+{}_2F_3\left(\frac{8}{15},\frac{13}{15};\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{27h^5}{3125a^3}\right)\right)+3\;3^{9/10}a^{3/5}h^3\cdot$$
$$\cdot\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{17}{15}\right)\Gamma\left(\frac{22}{15}\right)\left({}_2F_3\left(\frac{17}{15},\frac{22}{15};\frac{6}{5},\frac{7}{5},\frac{8}{5};\frac{27h^5}{3125a^3}\right)-{}_2F_3\left(\frac{17}{15},\frac{22}{15};\frac{6}{5},\frac{7}{5},\frac{8}{5};-\frac{27h^5}{3125a^3}\right)\right)+9\;3^{3/10}\cdot$$
$$\cdot a^{6/5}h^2\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{14}{15}\right)\Gamma\left(\frac{19}{15}\right)\left({}_2F_3\left(\frac{14}{15},\frac{19}{15};\frac{4}{5},\frac{6}{5},\frac{7}{5};-\frac{27h^5}{3125a^3}\right)+{}_2F_3\left(\frac{14}{15},\frac{19}{15};\frac{4}{5},\frac{6}{5},\frac{7}{5};\frac{27h^5}{3125a^3}\right)\right)\Big]\frac{1}{20\pi a^3}$$

$\boldsymbol{S_0(x)=a|x|^{\frac{3}{2}}}$ with $\boldsymbol{a>0}$

$$\frac{1}{27a^2}\Bigg(9h^2\left({}_2F_2\left(1,\frac{3}{2};\frac{4}{3},\frac{5}{3};-\frac{4h^3}{27a^2}\right)+{}_2F_2\left(1,\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{4h^3}{27a^2}\right)\right)+8\sqrt[3]{3}\pi a^{2/3}\Bigg(\sqrt[3]{3}h\sinh\left(\frac{2h^3}{27a^2}\right)\cdot$$
$$\cdot\operatorname{Bi}\left(\frac{h^2}{3\sqrt[3]{3}a^{4/3}}\right)+3a^{2/3}\cosh\left(\frac{2h^3}{27a^2}\right)\operatorname{Bi}'\left(\frac{h^2}{3\sqrt[3]{3}a^{4/3}}\right)\Bigg)\Bigg)\text{ if }h\in\mathbb{R}$$

$\boldsymbol{S_0(x)=d|x|+a|x|^{-\frac{1}{2}}}$ with $\boldsymbol{d,a>0}$

$$\frac{1}{\sqrt{\pi}(d-h)(d+h)}\left(dG^{3,0}_{0,3}\left(\frac{1}{4}a^2(d-h)\middle|\;0,\tfrac{1}{2},1\right)+hG^{3,0}_{0,3}\left(\frac{1}{4}a^2(d-h)\middle|\;0,\tfrac{1}{2},1\right)+\right. \tag{34}$$
$$\left.+dG^{3,0}_{0,3}\left(\frac{1}{4}a^2(d+h)\middle|\;0,\tfrac{1}{2},1\right)-hG^{3,0}_{0,3}\left(\frac{1}{4}a^2(d+h)\middle|\;0,\tfrac{1}{2},1\right)\right)\text{ if }h\in(-d,d)$$

$\boldsymbol{S_0(x)=d*|x|+a|x|^{-\frac{1}{3}}}$

$$\frac{\sqrt{3}}{2\pi(d-h)(d+h)}\left[dG^{4,0}_{0,4}\left(\frac{1}{27}a^3(d-h)\middle|\;0,\tfrac{1}{3},\tfrac{2}{3},1\right)+hG^{4,0}_{0,4}\left(\frac{1}{27}a^3(d-h)\middle|\;0,\tfrac{1}{3},\tfrac{2}{3},1\right)+\right.$$
$$\left.+dG^{4,0}_{0,4}\left(\frac{1}{27}a^3(d+h)\middle|\;0,\tfrac{1}{3},\tfrac{2}{3},1\right)-hG^{4,0}_{0,4}\left(\frac{1}{27}a^3(d+h)\middle|\;0,\tfrac{1}{3},\tfrac{2}{3},1\right)\right]\text{ if }h\in(-d,d)$$

$S_0(x) = d|x| + a|x|^{\frac{1}{3}}$ with $d, a > 0$

$$\Big[9h\sqrt[3]{d-h}\sqrt[3]{d+h}\,{}_1F_2\left(1;\frac{1}{3},\frac{2}{3};\frac{a^3}{27(h-d)}\right)+9d\sqrt[3]{d-h}\sqrt[3]{d+h}\,{}_1F_2\left(1;\frac{1}{3},\frac{2}{3};\frac{a^3}{27(h-d)}\right)-9h\sqrt[3]{d-h}\cdot$$
$$\cdot\sqrt[3]{d+h}\,{}_1F_2\left(1;\frac{1}{3},\frac{2}{3};-\frac{a^3}{27(d+h)}\right)+9d\sqrt[3]{d-h}\sqrt[3]{d+h}\,{}_1F_2\left(1;\frac{1}{3},\frac{2}{3};-\frac{a^3}{27(d+h)}\right)-2\ 3^{2/3}\pi ad\sqrt[3]{d+h}\cdot$$
$$\cdot\,\mathrm{Bi}\left(-\frac{a}{\sqrt[3]{3}\sqrt[3]{d-h}}\right)-2\ 3^{2/3}\pi ah\sqrt[3]{d+h}\mathrm{Bi}\left(-\frac{a}{\sqrt[3]{3}\sqrt[3]{d-h}}\right)+2\ 3^{2/3}\pi ah\sqrt[3]{d-h}\mathrm{Bi}\left(-\frac{a}{\sqrt[3]{3}\sqrt[3]{d+h}}\right)-2\ \cdot$$
$$\cdot\,3^{2/3}\pi ad\sqrt[3]{d-h}\mathrm{Bi}\left(-\frac{a}{\sqrt[3]{3}\sqrt[3]{d+h}}\right)\Big]\frac{1}{9(d-h)^{4/3}(d+h)^{4/3}}\ \text{ if } h\in(-d,d)$$

$S_0(x) = d|x| + a\sqrt{|x|}$ with $d, a > 0$

$$\frac{1}{2(d-h)^{3/2}(d+h)^{3/2}}\left(\sqrt{\pi}(-a)d\sqrt{d+h}e^{\frac{a^2}{4d-4h}}\mathrm{erfc}\left(\frac{a}{\sqrt{4d-4h}}\right)-\sqrt{\pi}ad\sqrt{d-h}e^{\frac{a^2}{4(d+h)}}\mathrm{erfc}\left(\frac{a}{2\sqrt{d+h}}\right)+\right.$$
$$\left.-\sqrt{\pi}ah\sqrt{d+h}e^{\frac{a^2}{4d-4h}}\mathrm{erfc}\left(\frac{a}{\sqrt{4d-4h}}\right)+\sqrt{\pi}ah\sqrt{d-h}e^{\frac{a^2}{4(d+h)}}\mathrm{erfc}\left(\frac{a}{2\sqrt{d+h}}\right)+4d\sqrt{d-h}\sqrt{d+h}\right)$$
if $h\in(-d,d)$

$S_0(x) = d|x| + \frac{a}{|x|}$ with $d, a > 0$

$$\frac{2\left(\sqrt{a(d+h)}K_1\left(2\sqrt{\frac{d-h}{a}}\right)+\sqrt{a(d-h)}K_1\left(2\sqrt{\frac{d+h}{a}}\right)\right)}{\sqrt{a(d-h)}\sqrt{a(d+h)}}\ \text{ if } h\in(-d,d) \tag{35}$$

$S_0(x) = d|x| + \frac{a}{|x|^2}$ with $d, a > 0$

$$\left[dG_{0,3}^{3,0}\left(\frac{1}{4}a(d-h)^2\middle|\ 0,\tfrac{1}{2},1\right)+hG_{0,3}^{3,0}\left(\frac{1}{4}a(d-h)^2\middle|\ 0,\tfrac{1}{2},1\right)+dG_{0,3}^{3,0}\left(\frac{1}{4}a(d+h)^2\middle|\ 0,\tfrac{1}{2},1\right)+\right.$$
$$\left.-hG_{0,3}^{3,0}\left(\frac{1}{4}a(d+h)^2\middle|\ 0,\tfrac{1}{2},1\right)\right]\frac{1}{\sqrt{\pi}(d-h)(d+h)}\ \text{ if } h\in(-d,d)$$

$S_0(x) = b\sqrt{|x|}$ when $h\in\mathbb{C}$, $b>0$

$$\begin{cases}\frac{4}{b^2} & h=0\\ \frac{b\left(\sqrt{\pi}e^{\frac{b^2}{4h}}\mathrm{erf}\left(\frac{b}{2\sqrt{h}}\right)+2D_+\left(\frac{b}{2\sqrt{h}}\right)\right)}{2h^{3/2}} & \Re(h)=0\ \&\ h\neq 0\end{cases}$$

$S_0(x) = a * \exp(|x|)$ with $a > 0$

$$E_{1-h}(a)+E_{h+1}(a)$$

$S_0(x) = ax^2 + d\log\left(|x|^{\frac{1}{2}}\right)$ with $a, d > 0$

$$a^{\frac{d-2}{4}}\Gamma\left(\frac{2-d}{4}\right)\,{}_1F_1\left(\frac{1}{2}-\frac{d}{4};\frac{1}{2};\frac{h^2}{4a}\right)\ \text{ if } d<2$$

$$S_0(x)=\begin{cases}cx^2 & x\geq 0\\ 0 & x<0\end{cases}$$

$$\begin{cases}\infty & h\leq 0\\ \frac{\sqrt{\pi}e^{\frac{h^2}{4a}}\left(\mathrm{erf}\left(\frac{h}{2\sqrt{a}}\right)+1\right)}{2\sqrt{a}}+\frac{1}{h} & h>0\end{cases}$$

$\boldsymbol{S_0(x) = b \exp(ax)}$ with $\boldsymbol{a, b > 0}$

$$\begin{cases} \infty & h \leq 0 \\ \frac{b^{-\frac{h}{a}}\Gamma\left(\frac{h}{a}\right)}{a} & h > 0 \end{cases}$$

The last two results were discarded, because they diverge for $h = 0$.

B.2 Not analytic but result/action was complex

$\boldsymbol{S_0(x) = ax^{\frac{3}{2}}}$ for $\boldsymbol{a > 0}$

$$\frac{1}{27}\pi e^{\frac{2h^3}{243}}\left(4\ 3^{2/3}\mathrm{Bi}'\left(\frac{h^2}{9\ 3^{2/3}}\right) + 2(-3)^{2/3}\mathrm{Bi}'\left(-\frac{\sqrt[3]{-1}h^2}{9\ 3^{2/3}}\right) + \sqrt[3]{3}h\left(\mathrm{Bi}\left(\frac{h^2}{9\ 3^{2/3}}\right) - i\mathrm{Ai}\left(\frac{h^2}{9\ 3^{2/3}}\right)\right)\right) \text{ if } h \geq 0$$

The integral does not converge for $h < 0$, thus the result is not analytic for the same reasons as (31), but contrary to (31), the function describing this result on $h \geq 0$ is entire.

B.3 Some examples, that could not be integrated

$S_0(x) = ax^2 + bx^{\frac{1}{2}}$
$S_0(x) = ax^2 + b|x|^{\frac{1}{2}}$
$S_0(x) = ax^2 + b|x|^{\frac{1}{3}}$
$S_0(x) = a|x|^3 + b|x|^{\frac{1}{3}}$
$S_0(x) = a|x|^3 + bx^2$
$S_0(x) = ax^4 + bx^2$

References

[1] Axel Maas and Daniel Zwanziger. “Analytic and numerical study of the free energy in gauge theory”. In: *Phys. Rev. D* 89 (3 Feb. 2014), p. 034011. DOI: 10.1103/PhysRevD.89.034011. URL: https://link.aps.org/doi/10.1103/PhysRevD.89.034011.

[2] Michael E Peskin and Daniel V. Schroeder. *An Introduction To Quantum Field Theory*. Addison-Wesley, 1995. ISBN: 9780201503975.

[3] L. H. Ryder. *Quantum Field Theory*. Cambridge University Press, June 1996. ISBN: 978-0-521-47814-4. DOI: 10.1017/CBO9780511813900.

[4] A. Zee. *Quantum field theory in a nutshell*. Princeton University Press, 2003. ISBN: 978-0-691-14034-6.

[5] Steven G. Krantz and Harold R. Parks. *A primer of real analytic functions*. Birkhäuser, 2002. ISBN: 9780817642648.

[6] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier, 2007. ISBN: 978-0-12-373637-6.

[7] Walter Rudin. *Real and complex analysis*. McGraw Hill, 1987. ISBN: 978-0-07-054234-1.

[8] Ruel V. Churchill, James W. Brown, and Roger F. Verhey. *Complex Variables and Applications*. McGraw Hill, 1976. ISBN: 978-0073383170.

[9] Eric W. Weisstein. *Erf*. URL: https://mathworld.wolfram.com/Erf.html (visited on 29/9/2023). From MathWorld–A Wolfram Web Resource.

[10] Helmut Eschrig. *The Fundamentals of Density Functional Theory*. Teubner Texte zur Physik. Vieweg+Teubner Verlag Wiesbaden, 2012. ISBN: 978-3-322-97620-8. DOI: https://doi.org/10.1007/978-3-322-97620-8.

[11] J. J. Sakurai and San Fu Tuan. *Modern Quantum Mechanics Revised Edition*. Reading, Mass.: Addison-Wesley, 1994. ISBN: 0-201-53929-2.

[12] Axel Maas. “Gauge bosons at zero and finite temperature”. In: *Physics Reports* 524.4 (2013), pp. 203–300. ISSN: 0370-1573. DOI: https://doi.org/10.1016/j.physrep.2012.11.002. URL: https://www.sciencedirect.com/science/article/pii/S0370157312004097.

[13] Reinhard Alkofer and Lorenz von Smekal. “The infrared behaviour of QCD Green’s functions: Confinement, dynamical symmetry breaking, and hadrons as relativistic bound states”. In: *Physics Reports* 353 (2001). ISSN: 0370-1573. DOI: https://doi.org/10.1016/S0370-1573(01)00010-2. URL: https://www.sciencedirect.com/science/article/pii/S0370157301000102.

[14] *Generalized hypergeometric function*. URL: https://en.wikipedia.org/wiki/Generalized_hypergeometric_function (visited on 12/10/2023).

[15] Eric W. Weisstein. *Airy Functions*. URL: https://mathworld.wolfram.com/AiryFunctions.html (visited on 15/10/2023). From MathWorld–A Wolfram Web Resource.

[16] *Generalized Exponential Integral*. URL: https://dlmf.nist.gov/8.19#viii (visited on 19/10/2023).

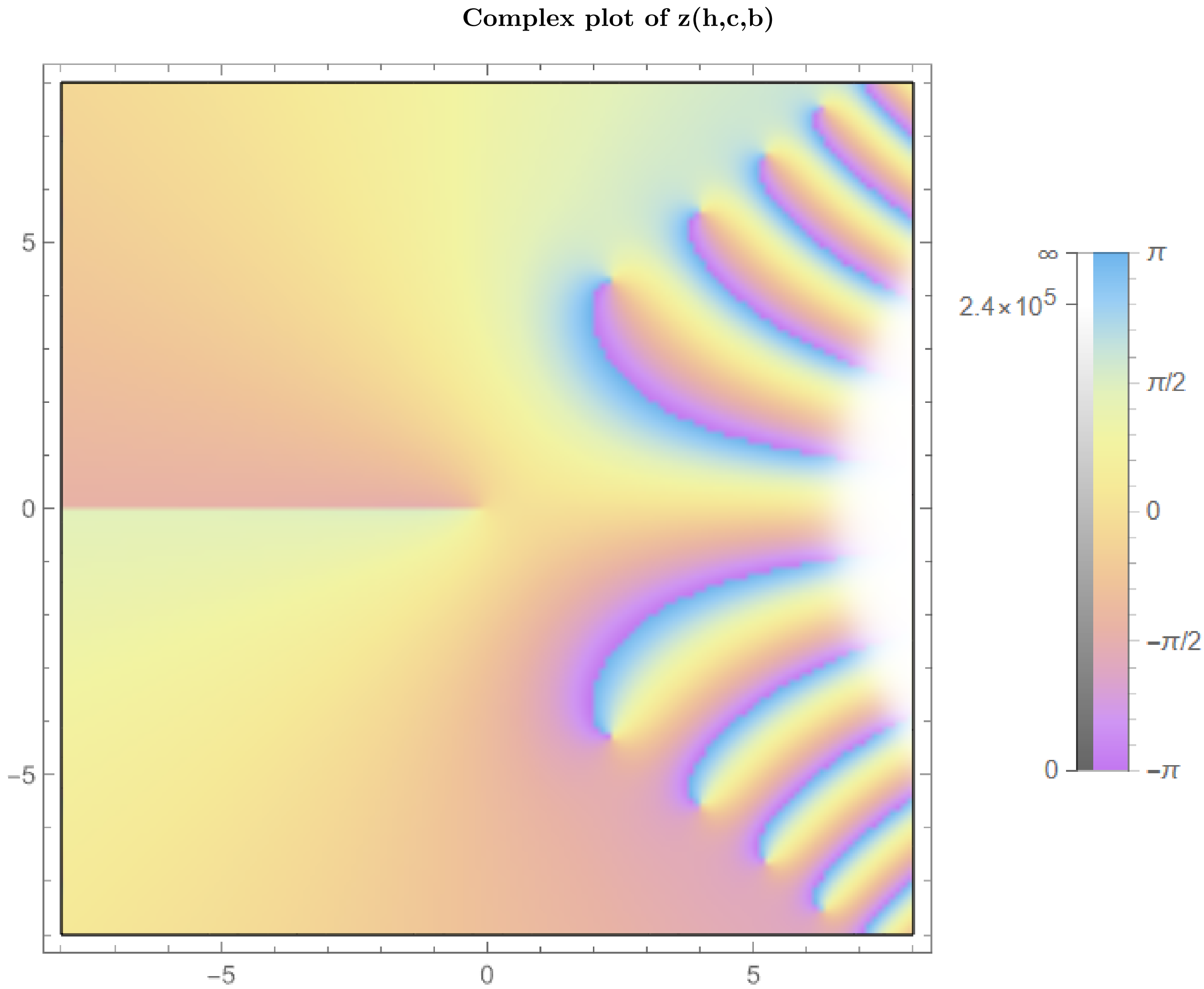


Figure 1: A complex plot of the generating functional (31) for parameter values $(c, b) = (1, 1)$ in dependence of the source strength h. The colour indicates the argument of the function value, the brightness the absolute value. There is a branch cut for $h \in (-\infty, 0]$, otherwise it is analytic, because the functions that make up $z(h, c, b)$ are all analytic except at $h \in (-\infty, 0]$. Thus, it seems that this function will always have a branch cut, where the branch cut of the square root was put.

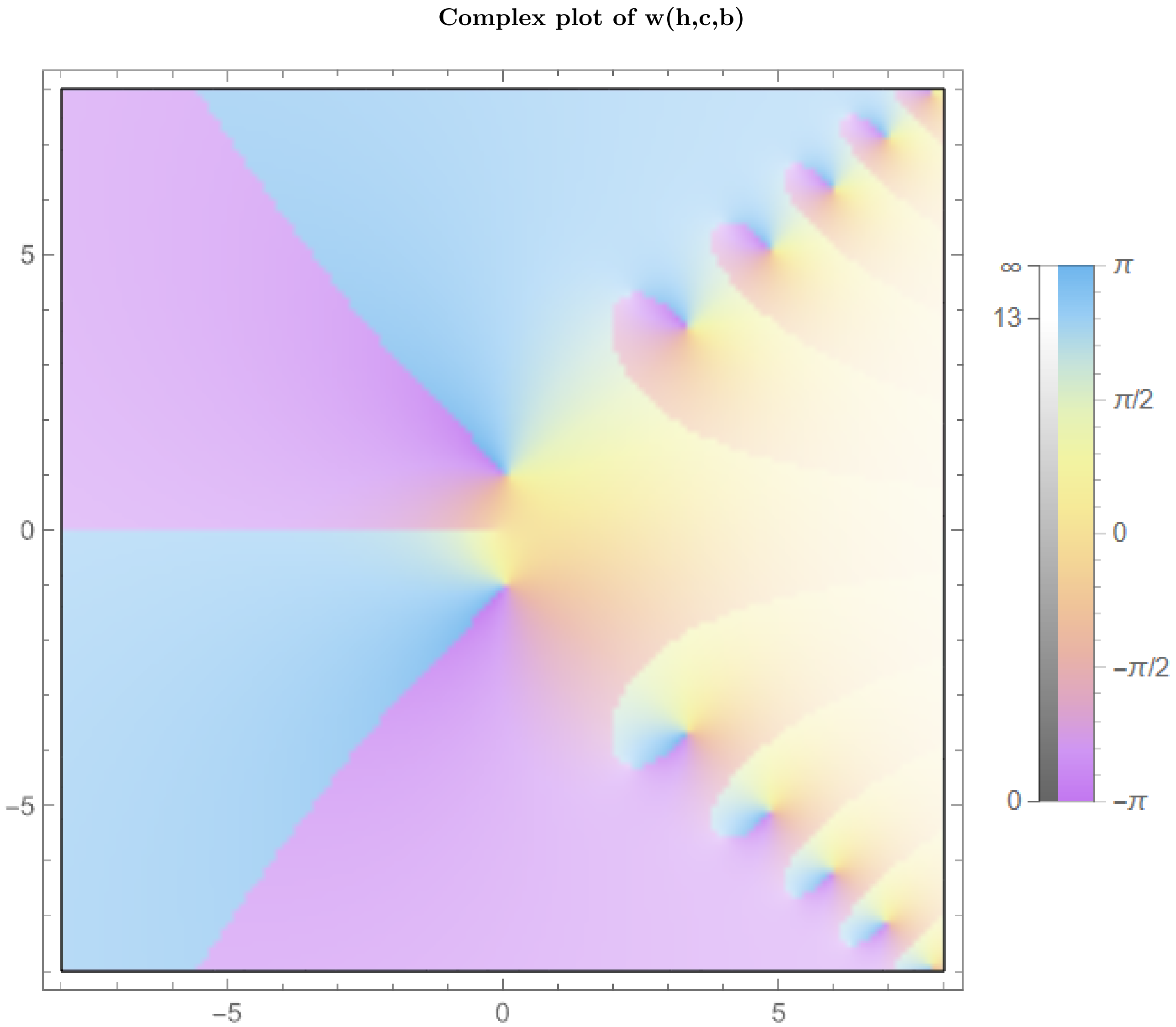


Figure 2: A complex plot of the free energy (32) for parameter values $(c, b) = (1, 1)$, some of the discontinuities here seem to stem from the discontinuity of the logarithm rather than the partition function.

In[18]:= $$\texttt{FullSimplify}\left[\texttt{Limit}\left[\frac{1}{h}+\frac{e^{\frac{h^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{h}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{b\,e^{\frac{b^2}{4h}}\sqrt{\pi}\,\texttt{Erfc}\left[\frac{b}{2\sqrt{h}}\right]}{2h^{3/2}},\ h\to p,\ \texttt{Direction}\to \texttt{I},\ \texttt{Assumptions}\to p<0\right]\right]$$

$$\texttt{Limit}\left[\frac{1}{h}+\frac{e^{\frac{h^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{h}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{b\,e^{\frac{b^2}{4h}}\sqrt{\pi}\,\texttt{Erfc}\left[\frac{b}{2\sqrt{h}}\right]}{2h^{3/2}},\ h\to p,\ \texttt{Direction}\to -\texttt{I},\ \texttt{Assumptions}\to p<0\right]$$

Out[18]= $$\frac{1}{p}+\frac{e^{\frac{p^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{p}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}+\frac{i\,b\,e^{\frac{b^2}{4p}}\sqrt{\pi}\left(i+\texttt{Erfi}\left[\frac{b}{2\sqrt{-p}}\right]\right)}{2p^{3/2}}$$

Out[19]= $$\frac{1}{p}+\frac{e^{\frac{p^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{p}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{i\,b\,e^{\frac{b^2}{4p}}\sqrt{\pi}\left(-i+\texttt{Erfi}\left[\frac{b}{2\sqrt{-p}}\right]\right)}{2p^{3/2}}$$

In[24]:= $$\texttt{Limit}\left[\texttt{Log}\left[\frac{1}{h}+\frac{e^{\frac{h^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{h}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{b\,e^{\frac{b^2}{4h}}\sqrt{\pi}\,\texttt{Erfc}\left[\frac{b}{2\sqrt{h}}\right]}{2h^{3/2}}\right],\ h\to p,\ \texttt{Direction}\to \texttt{I},\ \texttt{Assumptions}\to p<0\right]$$

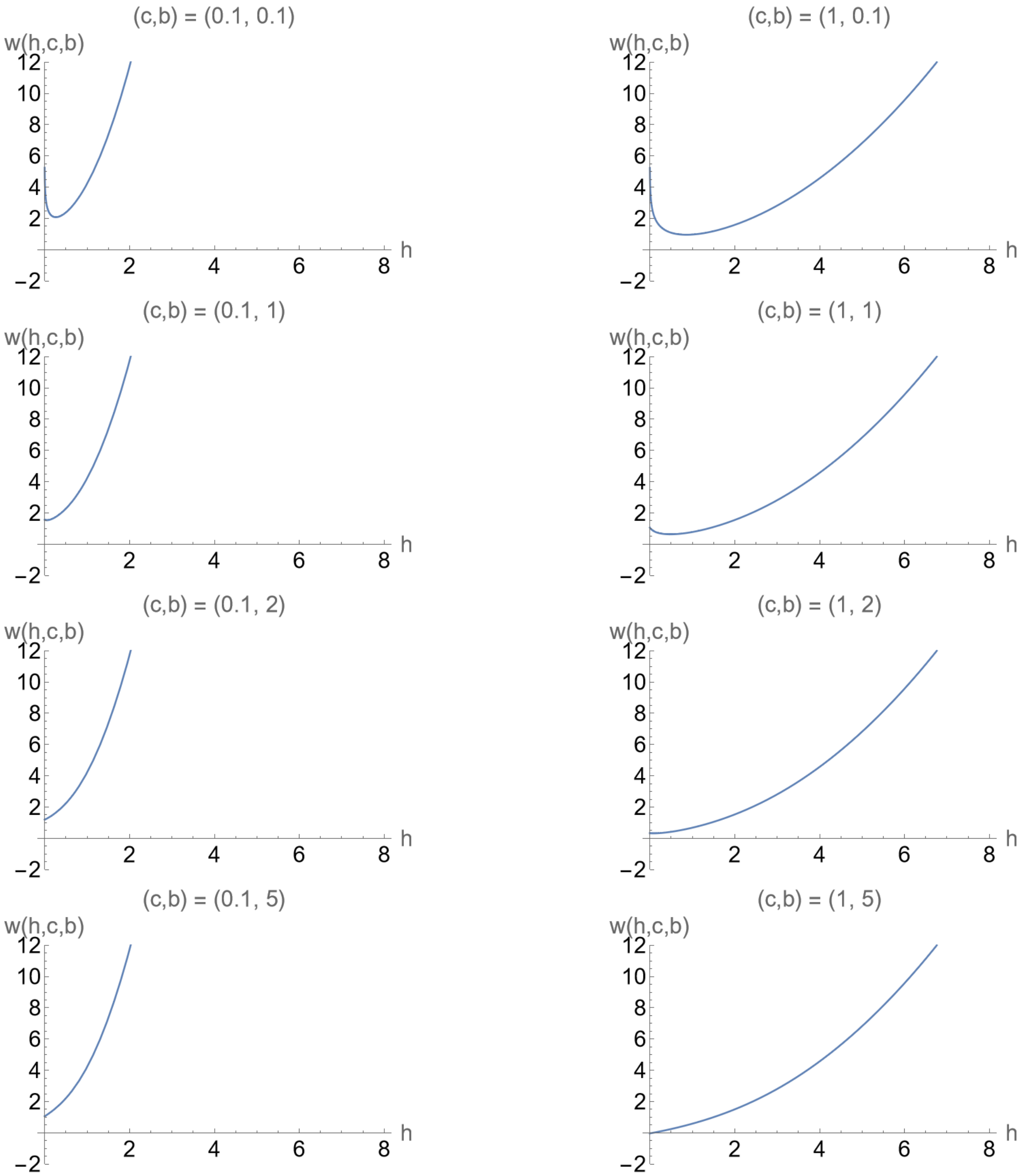
$$\texttt{Limit}\left[\texttt{Log}\left[\frac{1}{h}+\frac{e^{\frac{h^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{h}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{b\,e^{\frac{b^2}{4h}}\sqrt{\pi}\,\texttt{Erfc}\left[\frac{b}{2\sqrt{h}}\right]}{2h^{3/2}}\right],\ h\to p,\ \texttt{Direction}\to -\texttt{I},\ \texttt{Assumptions}\to p<0\right]$$

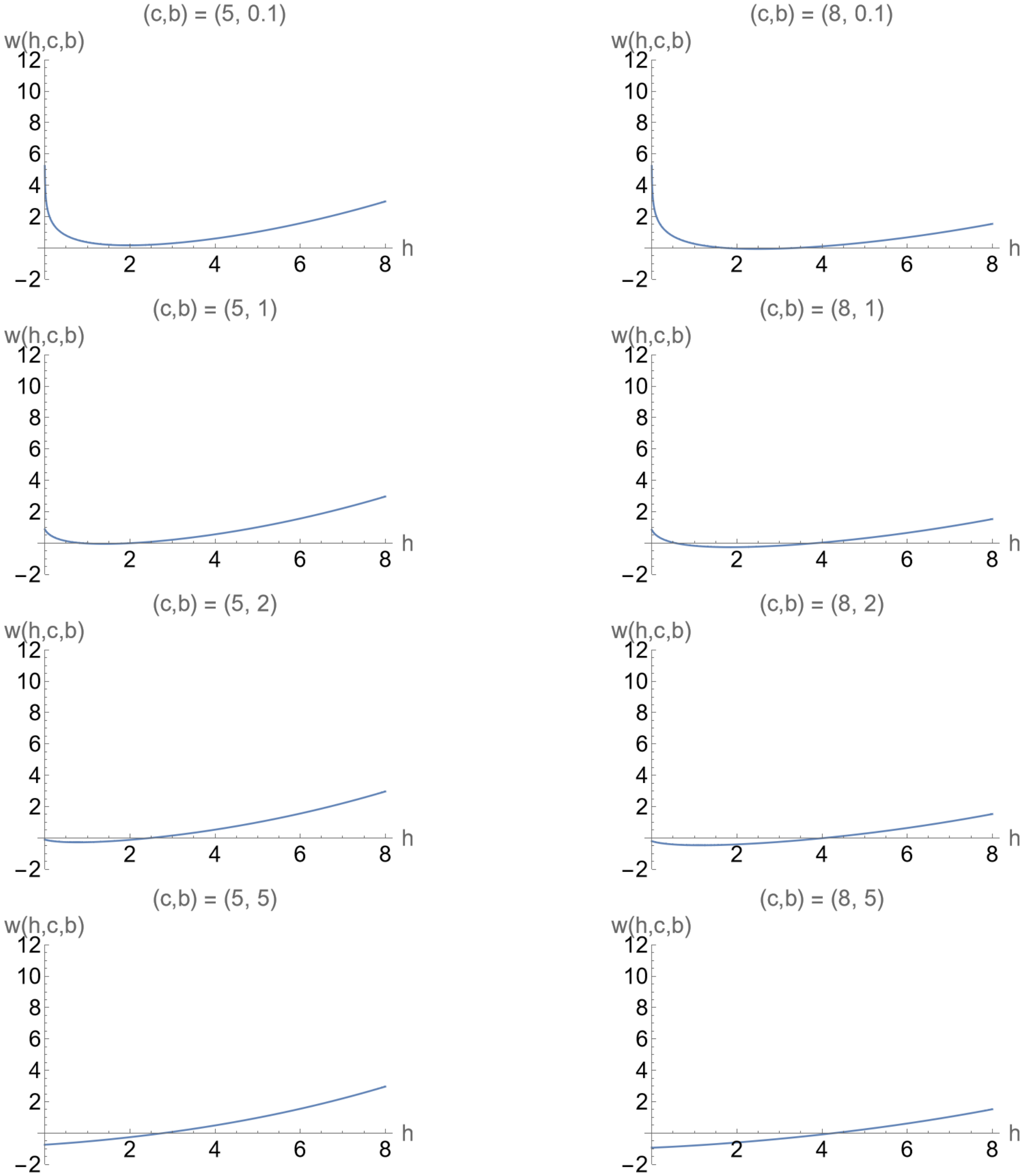
Out[24]= $$\texttt{Log}\left[\frac{1}{p}+\frac{e^{\frac{p^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{p}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}+\frac{1}{2}\,i\,b\,e^{\frac{b^2}{4p}}\sqrt{\pi}\,\texttt{Conjugate}\left[\frac{1}{p^{3/2}}\right]\left(i+\texttt{Erfi}\left[\frac{b}{2\sqrt{-p}}\right]\right)\right]$$

Out[25]= $$\texttt{Log}\left[\frac{1}{p}+\frac{e^{\frac{p^2}{4c}}\sqrt{\pi}\left(1+\texttt{Erf}\left[\frac{p}{2\sqrt{c}}\right]\right)}{2\sqrt{c}}-\frac{i\,b\,e^{\frac{b^2}{4p}}\sqrt{\pi}\left(-i+\texttt{Erfi}\left[\frac{b}{2\sqrt{-p}}\right]\right)}{2p^{3/2}}\right]$$

Figure 3: Limits for the partition function (31) and free energy (32) for different directions, both of these functions are discontinuous for $h < 0$.

Free energy w(h,c,b) for action with simpler parameterizarion




(c,b) = (5, 0.1)
(c,b) = (8, 0.1)
(c,b) = (5, 1)
(c,b) = (8, 1)
(c,b) = (5, 2)
(c,b) = (8, 2)
(c,b) = (5, 5)
(c,b) = (8, 5)
w(h,c,b)
h

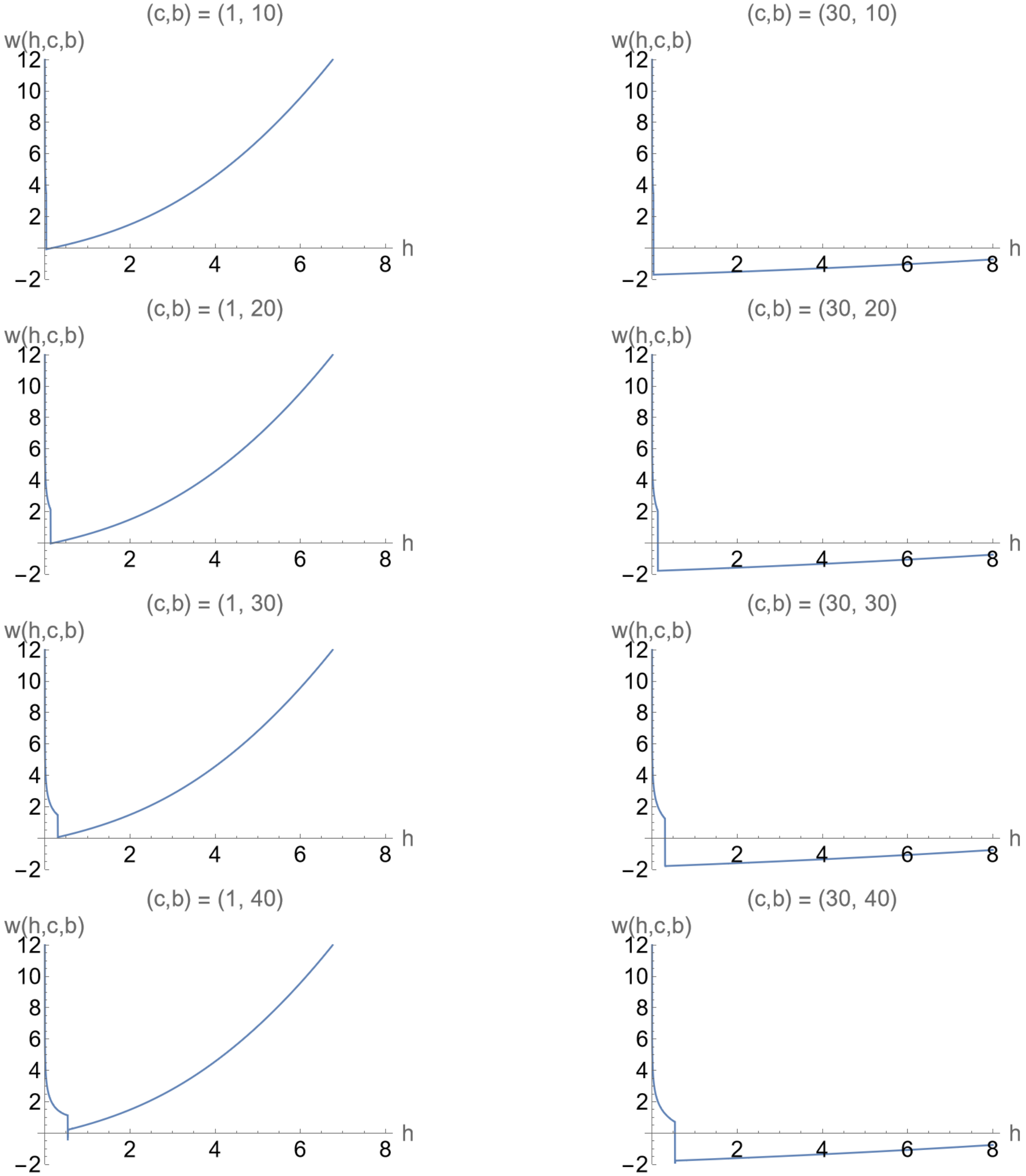
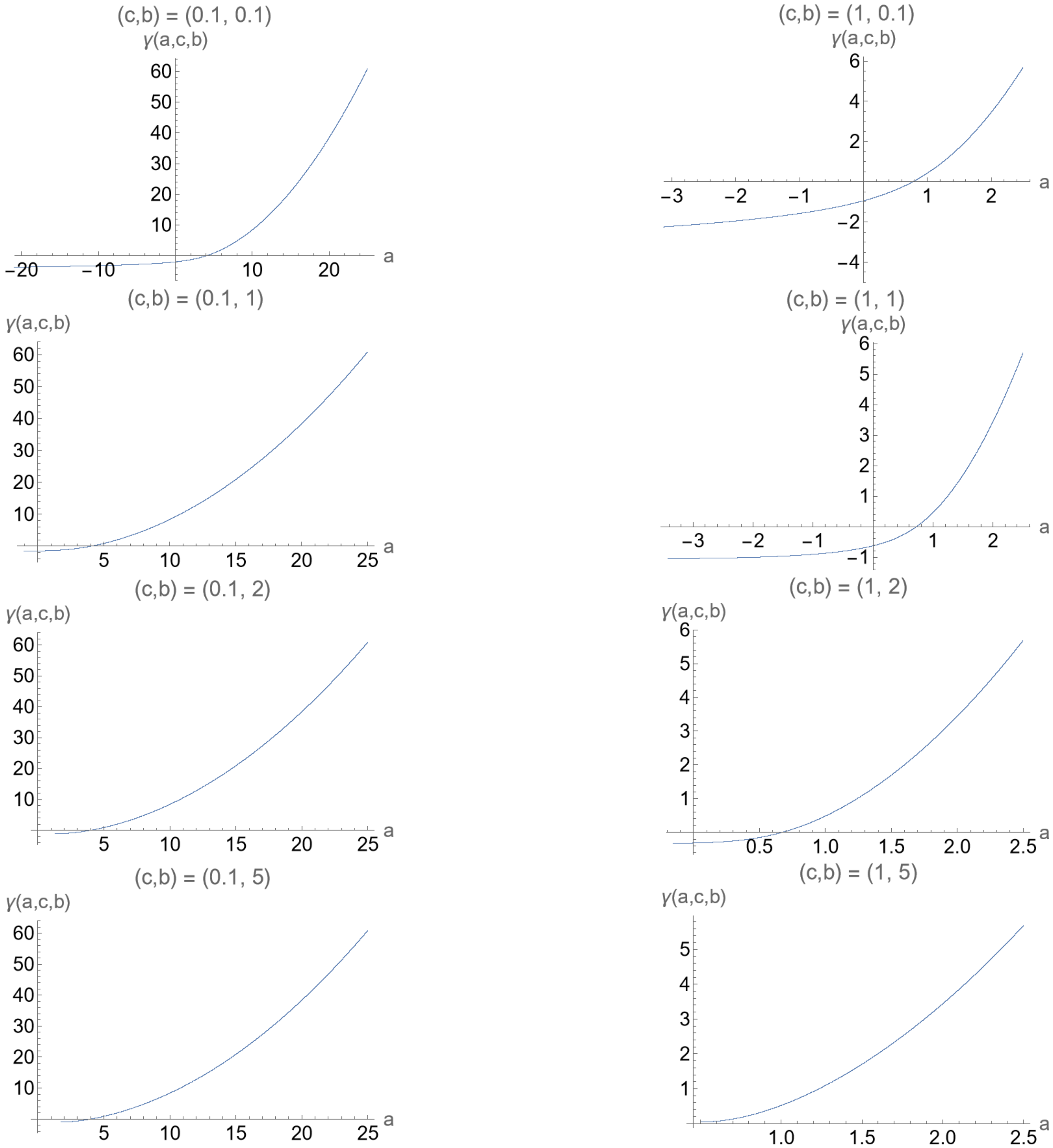


Figure 4: This figure shows the free energy that was calculated for the action (30) and its dependence on the source strength h for several parameter values. The function appears to be convex. For large values of parameter b the numerical evaluation of the function becomes very unreliable for small values of h.

Legendre transform



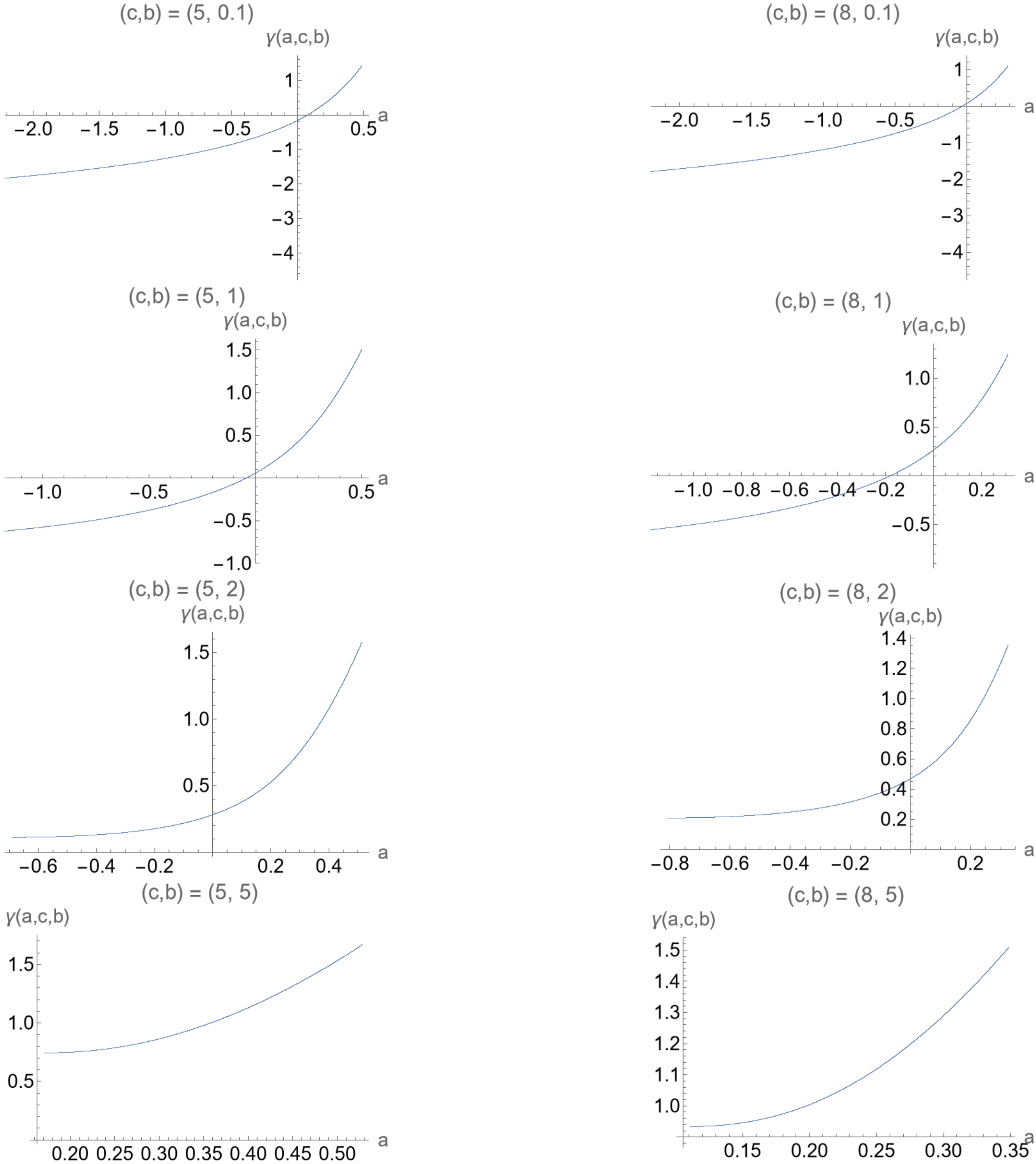


Figure 5: This figure shows the Legendre transform of the free energy (32) in the interval $h \in [0.1, 5]$ for several parameter values. It was calculated numerically, using the code in the appendix under the assumption, that the function to be transformed is convex. The abscissa marks the value of the transformed variable, the ordinate the value of the Legendre transform.

z(h) for $S_0(x) = d|x| + a|x|^{-\frac{1}{2}}$

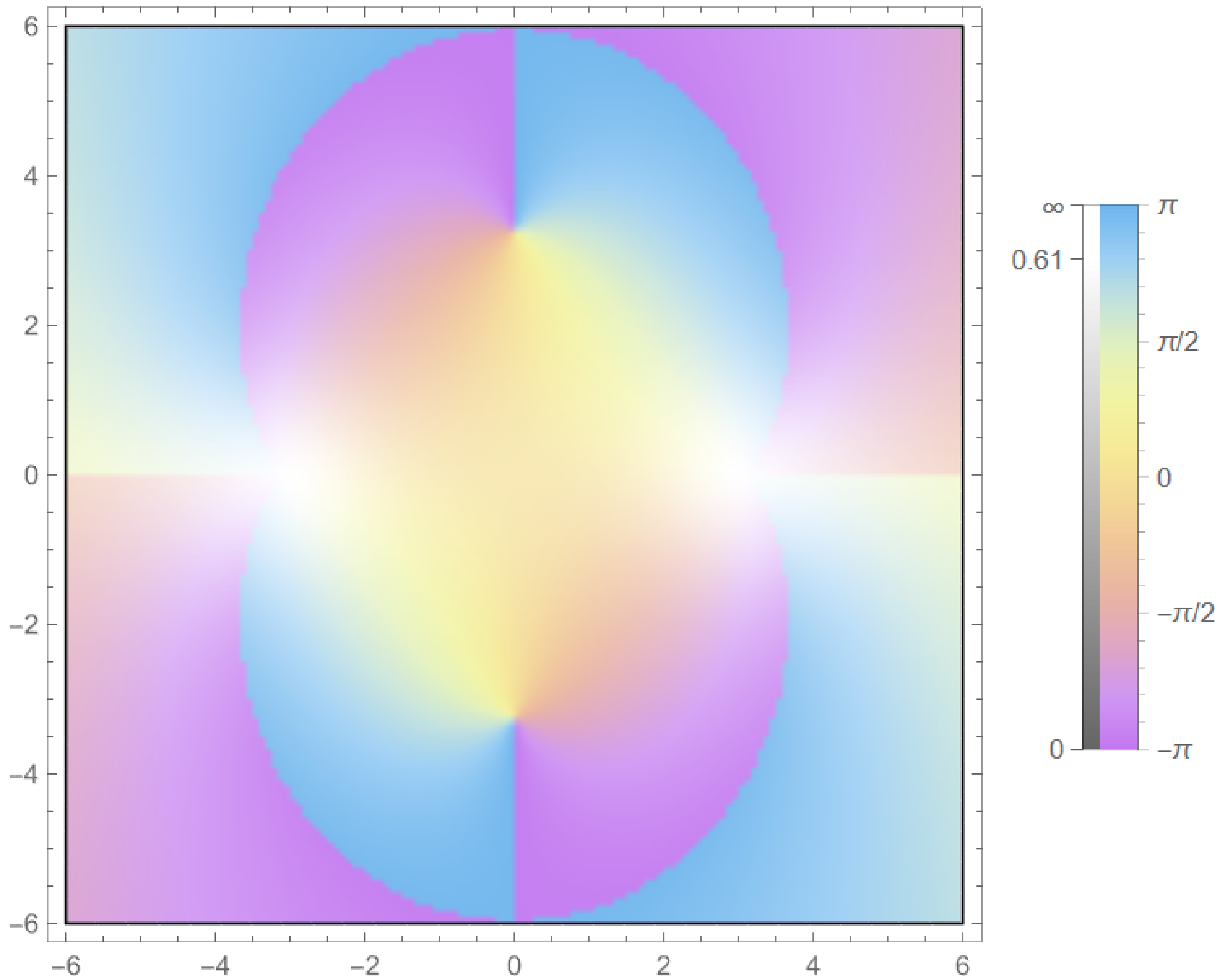


Figure 6: The function (34) describing the generating functional for $h \in (-d, d)$ of the action $S_0(x) = d|x| + a|x|^{-\frac{1}{2}}$ for $(a, d) = (1, 3)$. It looks like it could be analytic in the region where it describes $z(h)$. The parameter d restricts the region where the integral is convergent, and also the region between the two branch cuts of the solution.

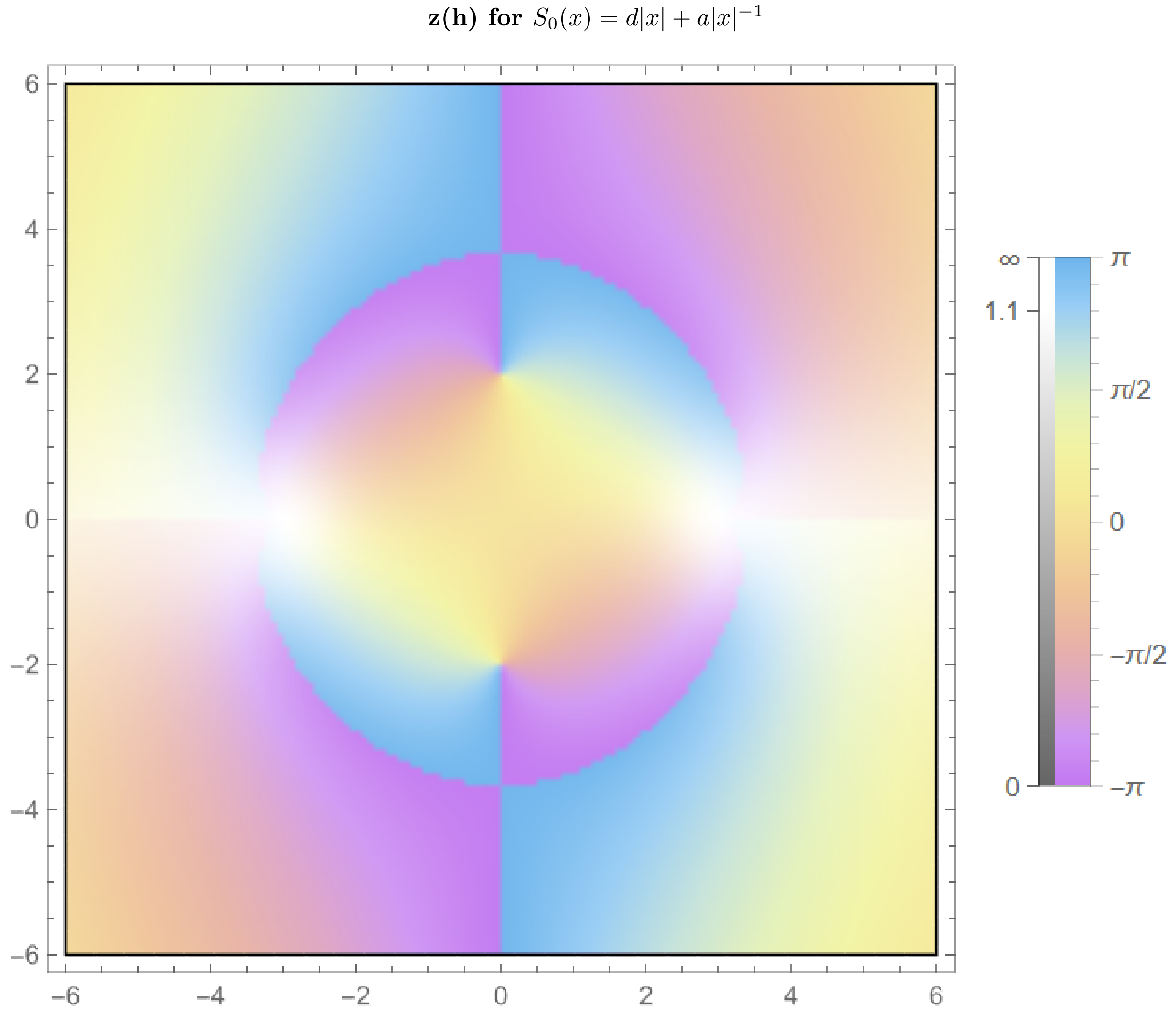


Figure 7: The function (35) describing the generating functional for $h \in (-d, d)$ of the action $S_0(x) = d|x| + a|x|^{-1}$ for $(a, d) = (1, 3)$. It also looks like it could be analytic in the radius around $h = 0$ where it describes the result of the integration.