Electroweak Physics

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Contents

Chapter 1

Introduction

There are four forces which are observed currently in nature. One is the gravitational force, binding any form of matter together. The second one is the strong nuclear force, binding the components of the nuclei together. The third is the electromagnetic force, which is responsible for almost all of the non-gravitative effects which can be observed on earth in day-to-day life, like solid-state physics. The last force is the weak nuclear force, which is responsible for part of the nuclear decays. It is remarkable that the latter is the only one which makes a difference between left and right, in the sense that it violates parity maximally. This particular property of the weak force has profound consequences. One of them is that no fermionic particles which is charged under the weak force can have a static mass. Only dynamical effects can produce such a mass. To provide such a dynamical effect, interactions with a special particle, the Higgs particle, are invoked. In this context the usual picture of the spontaneous broken weak gauge symmetry comes about, a somewhat unprecise description, as will be seen. Furthermore, the electromagnetic and weak interaction turn out to be intimately linked, whereby the interaction becomes named the electroweak interactions. Herein, the formal and phenomenological aspects of these interactions will be described.

This will start with a short discussion of the phenomenology and ideas which lead to the formulation of the electroweak sector of the standard model. In particular, this will show that the description cannot proceed without describing also the physics of the Higgs particle, asthis is necessary to produce the characteristic properties of the particles mediating the electroweak interaction, and also for providing the mass to all particles observed so far, except the Higgs itself.

The formulation of this theory will be in terms of two gauge theories, an Abelian one and a non-Abelian one, which do mix. The profound consequences of this property will be described, both by perturbative means and in the non-perturbative domain. Finally, at the end the electroweak phase transition will be discussed briefly, showing that it may have far-reaching consequences for the evolution of the universe.

Good introductory textbooks on this topic are

- Gauge theories in particle physics by I. Aitchison and A. Hey (IOP publishing)
- Gauge theories by M. Böhm, A. Denner, and H. Joos (Teubner)
- The quantum theory of fields I & II by S. Weinberg (Cambridge).
- An introduction to quantum field theory by M. Peskin & D. Schroeder (Perseus)
- Quantum field theory by A. Das (World Scientific)
- Finite temperature field theory (1st edition) by J. Kapusta (Cambridge)

and a more modern treatement of the field theory can be found in the review article arXiv:1712.04721, which I wrote.

Chapter 2

Phenomenology of the weak interactions

In the following a motivation will be given for what electroweak physics needs to cover. As it turns out, the electroweak sector of the standard model is a formidable quagmire of around half a dozen very different mechanisms and issues, which intertwine in a highly non-trivial way. Each of them on its own has interesting and far-reaching consequences. But only in their combination they lead to the unique physics of the standard model. Nonetheless, this makes it often hard to really get an overview of what is going on. In the following sections, and chapters, it will be attempted to disentangle the various features, and ultimately recombine them. Here, this will be started with a description of the observed phenomena, which are eventually found to be all tied to the electroweak sector of the standard model.

2.1 The Fermi constant

The first weak phenomena observed was the β-decay of the nucleon. The first attempts of a field-theoretical formulation were based on a four-fermion interaction of type

$$G_F(\bar{p}\gamma_\mu n)(\bar{e}\gamma^\mu\nu), \tag{2.1}$$

where p, n, e, and ν represent the fields of the involved proton, neutron, electron, and the (anti)neutrino and are four-component spinors. The characteristic scale for the weak process was set by the Fermi constant G_F, which is of order 1.14×10^{-5} GeV^{-2} $\approx$ (296 GeV)$^{-2}$. This is not a renormalizable interaction, and as such should be only the low-energy limit of an underlying renormalizable theory.

In fact, it was suggested that there should exist, similarly to QED, an exchanged boson. However, to produce such a scale, it would have to be massive with a mass M_w. Assuming a coupling constant g' for the process, the scale would be set by expressions of the type

$$G_F \approx \frac{g'^2}{M_w^2}, \tag{2.2}$$

indicating a mass scale of about (a few) hundred GeV for the process, if $g' \approx 1$. This already sets the scale of the weak interactions. Furthermore, the appearance of a mass-scale indicates that the weak interactions will have only a limited range, of the order of $1/M_w$, and would be screened beyond this scale. Its range is therefore much shorter than that of any other force.

Over the time other weak effects were found, all characterized by the scale G_F. In particular, after some time the postulated interactions bosons have been observed directly. There are two charged ones, the $W^{\pm}$, and a neutral one Z (or sometimes Z^0), with masses about 80 and 91 GeV, respectively.

2.2 Parity violation

Another observation, which was made early on, was that the weak interactions did not respect parity. The discrete transformation parity P inverts the sign of each vector, e. g., coordinate r or momenta p

$$Pp = -p.$$

Pseudovectors or axial vectors, however, do not change sign under parity transformation. Such vectors are obtained from vectors, e. g., by forming a cross product. Thus the prime example of an axial vector are (all kind of) angular momenta

$$PL = P(r \times p) = Pr \times Pp = r \times p = L.$$

It was observed that a polarized neutron which decays will emit the electrons preferentially in one direction. Therefore, the interaction must couple spin s and momenta, and would therefore have also a contribution proportional to sp. However, the momenta of the decay products also depend on the invariant mass, p^2, and thus on a scalar contribution. Since therefore both scalars (scalar products of two vectors or two axial vectors) and pseudoscalars (products of a vector and an axial vector) appear imply that the interaction is not having a definite transformation behavior under parity, and is thus parity violating. In fact, it turned out that it is maximally parity violating.

To give this a more formal version, it is necessary to consider how couplings to fermions transform under parity. The parity transformation of a Dirac spinor is obtained by multiplying it with γ_0, the time-Dirac matrix. So, for spinors ψ the parity transformation is given by

$$P\psi = \gamma_0\psi.$$

Since furthermore $\gamma_0\gamma_\mu\gamma_0 = -\gamma_\mu$, the four-fermion coupling (2.1) will indeed transform as a scalar

$$P((\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi)) = (\bar{\psi}\gamma_0\gamma_\mu\gamma_0\psi)(\bar{\psi}\gamma_0\gamma^\mu\gamma_0\psi) = (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi).$$

To obtain a pseudoscalar coupling, one of the vectors would have to be replaced by an axial vector. This can be obtained if there would be a matrix γ_5 such that $\gamma_0\gamma_5\gamma_0 = \gamma_5$. In fact, such a matrix is given by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

This matrix anticommutes with all Dirac matrices,

$$\{\gamma_5, \gamma_\mu\} = 0.$$

As a consequence, the current

$$\bar{\psi}\gamma_5\gamma_\mu\psi,$$

is an axial vector, and can be used to obtain a pseudoscalar coupling.

However, this is not yet indicating to which extend the weak interactions should be parity violating, and this can also not be predicted on basis of the standard model to be developed. Experimentally, however, it is found that parity is maximally violated. For massless particles (e. g. for neutrinos to a very good accuracy) this would imply that only one of the helicity states would be affected by the weak interactions. The helicity state for a Dirac spinor is projected out as

$$\frac{1 \pm \gamma_5}{2}\psi.$$

The sign is determined by whether left-handed or right-handed states should be selected. Experiment finds that only left-handed states are involved, and thus a minus sign is appropriate. Furthermore, the weak interactions are found to violate also the charge conjugation symmetry maximally. Hence, the sign is not reversed for the anti-particle state. Therefore, the correct four-fermion interaction version of the weak interactions would be (appropriately normalized)

$$\frac{G_F}{\sqrt{2}}\left(\bar{\psi}\frac{1-\gamma_5}{2}\gamma_\mu\frac{1-\gamma_5}{2}\psi\right)\left(\bar{\psi}\frac{1-\gamma_5}{2}\gamma^\mu\frac{1-\gamma_5}{2}\psi\right),$$

which therefore exhibits maximal violation of C and P individually.

2.3 Flavor violation

In any strong or electromagnetic process the identity of particles is conserved. What we call an electron or quark, or in general flavor, remains the same is. E. g., the strangeness content is not changing. This is not true for weak processes. In fact, they will change the very concept of flavor, as will be discussed in section 6.3.3. For the moment, however, it is only important that the weak interactions do not respect flavor for any of the fermionic particles. In case of the leptons this effect is suppressed by the small neutrinos masses involved, but in case of quarks this is a significant effect. This happens in a particular pattern.

Consider, as an eexample, the weak decay of a neutron compared to that of a strange Λ baryon, it is found experimentally that the relative strengths can be expressed as

$$\begin{aligned} g_{\Delta S=0} &= g' \cos\theta_C \\ g_{\Delta S=1} &= g' \sin\theta_C \end{aligned}$$

where g' is a universal strength parameter for the weak interactions, its coupling constant. The angle parameterizing the decay is a swecond universal quantity, called the Cabibbo angle. A similar relation also holds in the leptonic sector for the muon quantum number

$$\begin{aligned} g_{\Delta\mu=0} &= g' \cos\theta_C^L \\ g_{\Delta\mu=1} &= g' \sin\theta_C^L, \end{aligned}$$

where, however, $\sin\theta_C^L$ is almost one, while in the quark sector $\sin\theta_C$ is about 0.22. Corresponding observations are also made for other flavors.

This result implies that the mass eigenstates of the matter particles are not at the same time also weak eigenstates, but they mix. Hence, on top of the P-violating and C-violating factors of $(1-\gamma_5)/2$, it is necessary to include something into the interaction which provides this mixing. This can be done by introducing a flavor-dependent unitary coupling matrix

$$G' = \frac{g'}{\sqrt{2}} \begin{pmatrix} 0 & \cos\theta_C^{(L)} & 0 \\ \cos\theta_C^{(L)} & 0 & \sin\theta_C^{(L)} \\ 0 & -\sin\theta_C^{(L)} & 0 \end{pmatrix}.$$

This is equivalent to just use a doublet

$$\begin{pmatrix} u & d\cos\theta_C + s\sin\theta_C \end{pmatrix},$$

e. g., in the quark sector. Such a doublet structure can be associated with (weak) charges $Q_u = 1/2$ and $Q_{ds} = -1/2$. This is called the weak isospin. When writing down the structure of the standard model version of the weak interactions, it will be found that this weak isospin will form a gauge group for the weak interactions.

Hence, the flavor (and mass) eigenstates of the fermions are effectively rotated by a unitary matrix. For two generations, this matrix, the Cabibbo matrix is uniquely given by

$$V_C = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix},$$

with again the Cabibbo angle θ_C, with a value of about $\sin\theta_C \approx 0.22$. For three generations, there exist no unique parametrization of the mixing matrix, called the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The standard form is

$$\begin{aligned} V_{CKM} &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & e^{-i\delta_{13}}s_{13} \\ -s_{12}c_{23} - e^{i\delta_{13}}c_{12}s_{23}s_{13} & c_{12}c_{23} - e^{i\delta_{13}}s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - e^{i\delta_{13}}c_{12}c_{23}s_{13} & -c_{12}s_{23} - e^{i\delta_{13}}s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}. \qquad (2.3) \\ c_{ij} &= \cos\theta_{ij} \\ s_{ij} &= \sin\theta_{ij} \end{aligned}$$

Here, to reduce the number of free parameters to 4 (θ_{12}, θ_{13}, θ_{23}, and δ_{13}) it has been used that within the standard model this matrix must be unitarity. Testing whether this matrix is indeed unitary by measuring the nine components individual is currently recognized as a rather sensitive test for physics beyond the standard model. There is a second such matrix for neutrinos, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, having the same structure. The difference is that it is not strongly diagonal-dominant, as is the quark matrix.

The presence of this matrix gives also rise to the possibility that not only C and P are violated separately, but that also the compound symmetry CP is violated (and therefore also T by virtue of the CPT-theorem). That occurs to lowest order in perturbation theory by a box-diagram exchanging the quark flavor of two quarks by the exchange of two W bosons, the gauge bosons of the weak interaction to be introduced later.

That such a process violates CP can be seen as follows. The process just described is equivalent to the oscillation of, e. g., a $d\bar{s}$ bound state into a $s\bar{d}$ bound state, i. e., a neutral kaon K^0 into its anti-particle $\bar{K}^0$. The C and P quantum numbers of both particles are P$= -1$, C$= 1$ and P$= 1$, C$= 1$, respectively, and hence CP$= -1$ and CP$= 1$. Thus, any such transition violates CP. Performing the calculation of the corresponding diagram yields that it is proportional to the quantity

$$\begin{aligned} \chi &= \sin\theta_{12}\sin\theta_{23}\sin\theta_{13}\cos\theta_{12}\cos\theta_{23}\cos^2\theta_{13}\sin\delta_{13} \\ &\quad \times(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2). \end{aligned}$$

Thus, such a process, and therefore CP violation, is possible if there is mixing at all (all θ_{ij} non-zero) with a non-trivial phase δ_{13}, and the masses of the quarks with fixed charge are all different. They may be degenerate with ones of different charge, however. Since such oscillations are experimentally observed, this already implies the existence of a non-trivial quark-mixing matrix. For the lepton case, the smallness of the neutrino masses already indicate that the effects will be much smaller. Whether there is a non-vanishing δ_{13} is at the time of writing yet experimentally not known definitely, but there are indications that δ_{13} is close to its maximal value. Which is again quite different from the quark case, where the value of δ_{13} is much smaller.

Within the standard model, there is no explanation of this mixing, and these are only parameters. However, since these mixings only correspond to a base transformation of the fields with a fixed transformation, the additional complications of mixing will be ignored throughout most of the following. In the end, it will be seen that they originate from flavor-violating interactions of the fermions with the Higgs.

2.4 Summary of phenomenology

To summarize, a theory of the (electro)weak physics must provide the following properties:

- Massive intermediate vector bosons
- Parity and charge conjugation violations
- This will require that a dynamical mechanism exists to provide masses to fermions for theoretical consistency
- A mechanism for CP violation
- A connection to electromagnetism such that the intermediate vector bosons become charged
- Universality and a suitable scale of the weak coupling

In the following the corresponding theory of the weak interaction used in the standard model, the Glashow-Salam-Weinberg theory, will be constructed. A first step is to consider the general type of theories of spin 1 vector bosons.

2.5 Intermediate vector bosons and the necessity for a gauge theory

A genuine fermion four-point coupling is not representing a (perturbatively) renormalizable interaction, and can therefore be only an effective low-energy approximation of the interaction. That it is nonetheless a rather good description is due to the size of the Fermi constant of about 250 GeV. Only at energies comparable to this scale the approximation is expected to break down.

The only known renormalizable interaction involving four fermions is that due to the exchange of an intermediate boson. In fact, the angular distribution of cross-sections due to the weak interaction indicate that at sufficiently high energies the fermion four-point-interaction is resolved into the exchange of an intermediate boson of spin one. This boson has to have an (effective) mass of the order of the Fermi constant. More detailed investigations showed that the weak interaction can change the electric charge of a particle, or can leave it unchanged. Therefore, there must exist charged and uncharged intermediate vector bosons, which are the $W^{\pm}$ bosons and the neutral Z boson. Their masses have been measured with rather good precision and are about 80 GeV for the $W^{\pm}$ boson and 91 GeV for the Z boson, in fact of the order of the Fermi constant.

Hence, in any diagram there are now a vertex of two fermions with an intermediate gauge boson G' reading at tree-level

$$i\frac{g'}{\sqrt{2}}\bar{\psi}\gamma^{\mu}\frac{1-\gamma_5}{2}G'_{\mu}\psi,$$

and two vertices are connected by the propagator for a massive spin 1 boson reading

$$D_{\mu\nu}=\left(g_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{M^2}\right)\frac{1}{k^2-M^2},$$

where its mass is M. Such a theory is superficially renormalizable, and everything seems to be working at first glance. However, this is not the case.

If the weak interactions should be an asymptotically free theory, which seems to be desirable and which seems to be consistent with experimental results so far, then at very large energies only the tree-level (or Born) contribution should be relevant for cross-sections[1]. However, this leads to a contradiction. Consider the case of Drell-Yan production of weak bosons. To be explicit, consider the creation of a $W^{\pm}$ pair by annihilation of two neutrinos

[1] It may also be an asymptotically safe theory instead, and then different arguments apply. At the moment, this seems not to be the case based on experimental results.

and the exchange of a lepton, neglecting lepton mixing for the moment[2]. The amplitude for this process is given by

$$M_{DY}^{W^\pm} = g'^2 \epsilon_\mu^{-*}(k_2, \lambda_2)\epsilon_\nu^{*\dagger}(k_1, \lambda_1)\bar{v}(p_2)\gamma^\mu(1-\gamma_5)\frac{p_{1\rho}\gamma^\rho - k_{1\rho}\gamma^\rho + m}{(p_1-k_1)^2 - m^2}\gamma^\nu(1-\gamma_5)u(p_1),$$

with the mass m of the exchanged lepton, polarization vectors ϵ for the produced W bosons with polarization state λ_i and v and u spinors of the incoming neutrinos.

Select the longitudinal components to be measured. The corresponding polarization vectors are

$$\epsilon^\mu(k,0) = \frac{1}{M}\begin{pmatrix}|\vec{k}| & 0 & 0 & k^0\end{pmatrix} = \frac{k^\mu}{M} + \frac{M}{k^0 + |\vec{k}|}\begin{pmatrix}-1 & \frac{\vec{k}}{|\vec{k}|}\end{pmatrix},$$

which tend to k_μ in the high-energy limit, which will be taken here. The mass of the lepton m is in all cases much smaller than M, and can therefore also be neglected. The production cross-section for longitudinally polarized $W^\pm$ if all energy scales are large then becomes

$$\frac{d\sigma}{d\Omega} = \frac{g'^4}{8\pi^2 M^4}(p_1 k_2)(p_2 k_2) = \frac{g'^4}{8\pi^2 M^4}E^2 \sin^2\theta,$$

where θ is the angle between the incoming neutrino momenta and E is the center-of-mass energy. Therefore, the total cross-section rises like E^2. If this is indeed the leading contribution at high energies, this will violate unitarity, i. e., probability is no longer conserved at high energies, and the theory is ill-defined. That this is the case can be seen by the fact that any total cross-section can be written as

$$\sigma = \frac{4\pi}{E^2}\sum_J (2J+1)\sin^2\delta_J,$$

on grounds of flux conservation, i. e., unitarity. However, since the phase shift δ_J are real a total cross section for each partial wave amplitude must fall as $1/E^2$ This is not the case here, for each partial wave the cross section diverges with E. Thus, unitarity is violated.

A consequence, due to the optical theorem, is that loops containing two virtual longitudinal gauge bosons will diverge stronger than that of ordinary gauge bosons. Their propagators are thus effectively equivalent to $1/k$ instead of a $1/k^2$ propagator, thus yielding a divergence just as for a four-fermion theory. Thus, with intermediate vector bosons the theory is not only not unitary, but also not renormalizable, at least not perturbatively. This could potentially change non-perturbatively, but experiement shows that a different route is taken by the weak interactions.

This result could appear somewhat strange because other theories, like QED, which also work with intermediate vector bosons, do not violate unitarity. The reason is that

[2] This has the advantage that no other process at tree-level interferes.

the process is not existing in this case, as longitudinal real photons do not exist due to gauge invariance. Their contribution is exactly canceled by time-like ones. This indicates already that one possibility to save the weak interactions as an asymptotic free theory would be to turn it into a gauge theory.

Chapter 3

Hidden gauge symmetry

3.1 The issue with the mass

The previous arguments show that a theory of the electroweak interactions requires two apparently contradictory properties: A gauge symmetry and massive vector bosons. This is contradictory since a mass term

$$M^2 A_\mu^a A_a^\mu \tag{3.1}$$

is not invariant under a gauge transformation

$$\begin{aligned} A_\mu^a &\to A_\mu^a + D_\mu^{ab}\phi^b \\ D_\mu^{ab} &= \delta^{ab}\partial_\mu + gf^{abc}A_\mu^c, \end{aligned}$$

neither in the Abelian ($f^{abc} = 0$) nor in the non-Abelian ($f^{abc} \neq 0$) case. Apparently, the only possibilities are that either the gauge symmetry is realized in a form which is different from Maxwell/Yang-Mills theory or a very different kind of theory is needed.

But it is, in fact, the former possibility, which appears to be realized in the case of the electroweak interactions. This is an empirical result. What happens is that the gauge symmetry becomes hidden: It appears that there is no gauge symmetry, and the gauge bosons are massive, but this is an emergent feature, which comes from the interactions. The symmetry is thus not visible anymore, as it is in Yang-Mills theory itself. This will happen through the interaction with another particle, the Higgs.

In the following it will be discussed how to hide a gauge symmetry. Since this becomes somewhat complicated when directly applied to the case of a non-Abelian gauge theory, this will be discussed first for simpler situations before returning to the electroweak interactions in general and to introduce the celebrated Glashow-Salam-Weinberg theory, including the Brout-Englert-Higgs mechanism. In fact, it turns out that a hidden symmetry is by far not a very exotic mechanism, and it occurs also for global symmetries or Abelian gauge symmetries. As will be seen, more familiar effects like magnetism and superconductivity originate from the same mechanism.

3.2 Hiding the symmetry

3.2.1 Global symmetries

3.2.1.1 Magnetism

The technically simpelst possibility for hiding a symmetry occurs with a global symmetry. While there are many different possibilities how to realize this effect, the simpelst one involves for the moment only a single real scalar field. As will be seen, this gives a field-theoretical model of magnetism.

A suitable, perturbatively renormalizable, potential for this field is

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \omega\phi^3 + \frac{1}{2}\frac{\mu^2}{f^2}\phi^4. \tag{3.2}$$

Without the ϕ^3 term the field enjoys a Z_2 symmetry, $\phi \to -\phi$. Having a non-zero value of ω would explicitly break this symmetry. However, here the symmetry should be hidden, and thus it needs to be there in the first place. Thus, ω will be set to zero for the moment. Note that the ϕ^2 term looks odd at first, as it seems to give the ϕ-particle a tachyonic mass. This will be a case where the effects of the interactions are so important that this tree-level behavior will be altered, and ultimately provide a normal mass to the particle. This will also provide an important lesson abou the expansion point of perturbation theory.

This potential has minima at $\phi = \pm f/\sqrt{2}$, $V(\pm f/\sqrt{2}) = -f^2\mu^2/8$. It has a maximum at $\phi = 0$. Therefore, weak quantum fluctuations will always drive the system away from the classical $\phi = 0$ situation and into either of the minima. Of course, quantum effects will also shift the minima away from f, and it cannot be excluded a-priori that these could not distort the maximum sufficiently such that $\phi = 0$ would again be a minimum. In fact, examples are known where this is the case. But for suitabley chosen values of the parameters this is not the case, and this will be assumed henceforth.

But concerning the minima, there are two equiavlent ones at $\pm f/\sqrt{2}$. This is due to the potential being Z_2 symmetric. Nonetheless, classically the groundstate has to have a unique value for the field. In classical physics, however, it is necessary to specify initial conditions to fully pose the problem. The initial conditions will then select which minimum[1].

[1]It is important to note that classical physics can be formulated as an initial value problem and has

The situation changes drastically at the quantum level. The path integral averages with a symmetric weight over all histories of the field ϕ, and thus over all initial conditions. Because the symmetry is neither broken by the Lagrangian, nor by the measure[2]. Thus, for every field configuration there is another one with the same weight, but with value $-\phi$. Hence, the (space-time averaged) expectation value $\langle\phi\rangle = 0$ always. In addition, the space-time-avergaed $\langle\phi^2\rangle > 0$ always, as the field fluctuates always. It thus also does not help.

Does this imply that at the quantum level the classical maximum is preferred? No. But is shows that $\langle\phi\rangle$ is not a suitable observable to discuss the dynamics. It is necessary to sharpen the expectation. In fact, what one implicitly assumes is that in such a situation, if one performs a measurement, one obtains a result in which the field points at every point in space-time in, essentially, the same direction. A measurement is doing nothing more than to select a fixed configuration. Thus, what the actual question is, is whether such a field configuration is typical for the system. This question can be answered in two distinct ways.

The first one is to construct an observable which tests if the average field configuration has this property. The important insight is that this is not a local property, as the two previous observables. Rather, the question is, whether there is a preferred orientation globally. This is the same as asking whether for point the neighboring field values are aligned, or not. This can be tested with

$$\left\langle\left(\int d^d x\phi(x)\right)^2\right\rangle > 0 \tag{3.3}$$

as this will only be non-vanishing if the system has long-range orders. And that is what is usually denoted as magnetism. However, this observable will not identify which minimum, because either has this feature. In fact, asking which minimum makes no sense, as both are equally likely.

The alternative is to ask, whether the system reacts to an external perturbation. For this, switch now on ω, which breaks the symmetry explicitly. Depending on the sign of ω, either of the minima will be the preferred one, as it skwes the potential in one of the directions. Now measure

$$\lim_{|\omega|\to 0}\langle\phi\rangle \neq 0 \tag{3.4}$$

then a detrministic evaluation under the equations of motion. Thus, in this theory, if somebody insists on setting the initial conditions $\phi = \partial_i\phi = 0$, the field will remain in the metastable equilibirum forever.

[2]This is not proven, but the absence of such an anomaly can be shown in this case relatively straightforward, as the Jacobian under a symmetry transformation indeed is unity.

with implicit averaging over space-time. Alternatively, and more conventional, one could also add a source-term $\jmath\phi$, which would have the same effect. Depending on whether (3.4) vanishes or not is again a statement about long-range order. If long-range order is possible, even an infinitesimal perturbation will create it. If not, then an infinutesimal perturbation has no effect. Thus, also this procedure identifies reliably the dynamics sought-for. In addition, it yields a prefereed minimum.

In most theories, (3.3) and (3.4) will yield the same answer. However, especially in gauge theories, this is not necessarily the case, as non-analyticities can develop in the limit (3.4), which yield that (3.4) indicates a breaking of symmetry, even if (3.3) does not. Thus, generically (3.4) is the better choice to detect this kind of dynamics. However, because it is a global observable, it is usually harder to calculate.

This situation is often denoted (in an abuse of language) as a spontaneous breakdown of the symmetry. But this is not quite correct. In the case of (3.3) the symmetry is always intact, while in (3.4) it is awalys explicitly broken, even in case of an infinitesimal ω (or j). Still, this name stuck, also to separate the situation at hand from the case of a finite ω or j, which is an explicit symmetry breaking. Another notion is that of a hidden symmetry for the situation (3.4), as the symmetry would actually be there if ω woud be zero, but seems to be absence in the limit.

3.2.1.2 Perturbation theory

The most important reason to do this trick is the applicability of perturbation theory. Perturbation theory is, essentially, as small-field expansion around zero field. There is no way whatsoever to obtain a non-zero expectation value for any field just in perturbation theory. This implies that hiding a symmetry, and therefore giving mass to gauge bosons, is necessarily an inherently non-perturbative process. A perturbative description, which is often and especially in the standard model surprisingly good, can still be achieved by the following trick: Rather then to expand around the vacuum, expand around a classical solution, in which one of the minima has been selected using initial conditions, e. g., the value $\phi = f/\sqrt{2}$. This yields a new perturbative expansion, but now with a vacuum expectation value $\langle\phi\rangle = f/\sqrt{2}$. If the fluctuations are small, especially if the quantum potential remains close, in a suitable meaning, to the classical one, this will provide a very good quantitative description. Given the success of perturbation theory in electroweak physics, this seems to be not the case at presently achievable energy scales. One needs to note, nonetheless, that the vacuum expectation value has been set by hand in this approach. It is therefore necessary to take care that only quantities insensitive to this choice will be reliable. Hence, the absolute value of the magnetization will come out right, but not the orientation. The consequences of such a perturbative treatment will now be discussed in detail.

However, a discrete group like Z_2 is of limited use phenomenologically. All experiments indicate that continuous groups are central in particle physics, especially for the weak interaction. Therefore, at the very least a charged scalar will be necessary, which will be discussed now. Actually, it will be necessary to upgrade it at least to a doublet later on.

The potential for such a charged scalar field is similarly given by

$$V(\phi,\phi^\dagger) = -\frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{2}\frac{\mu^2}{f^2}(\phi^\dagger\phi)^2. \tag{3.5}$$

A cubic term $\phi^2\phi^\dagger + \phi^{\dagger 2}\phi$ has been omitted directly. In this case, a global phase symmetry is present, i. e., the theory is invariant under the replacement

$$\begin{aligned}\phi &\to e^{-i\theta}\phi \approx \phi - i\theta\phi\\ \phi^\dagger &\to e^{i\theta}\phi^\dagger \approx \phi^\dagger + i\theta\phi^\dagger.\end{aligned}$$

To analyze the situation further, it is useful to rewrite the complex field in terms of its real and imaginary part

$$\phi = \sigma + i\chi,$$

and correspondingly for its hermitian conjugate. The Lagrangian then takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\chi\partial^\mu\chi) + \frac{\mu^2}{2}(\sigma^2+\chi^2) - \frac{1}{2}\frac{\mu^2}{f^2}(\sigma^2+\chi^2)^2,$$

and therefore describes two real scalar fields, which interact with each other and having the same (tachyonic) tree-level mass μ. The corresponding transformations take the (infinitesimal) form

$$\begin{aligned}\sigma &\to \sigma + \theta\chi\\ \chi &\to \chi - \theta\sigma,\end{aligned}$$

and therefore mixes the two flavors.

To find the extrema of the potential, it is necessary to inspect the derivatives of the potential

$$\begin{aligned}\frac{\partial V}{\partial\sigma} &= -\mu^2\sigma + \frac{\mu^2}{f^2}\sigma(\sigma^2+\chi^2)\\ \frac{\partial V}{\partial\chi} &= -\mu^2\chi + \frac{\mu^2}{f^2}\chi(\sigma^2+\chi^2).\end{aligned}$$

The extrema of this potential therefore occur at $\sigma = \chi = 0$ and at

$$\sigma^2 + \chi^2 = f^2 = \phi^\dagger \phi.$$

To analyze whether these extrema are maxima or minima the second derivatives of the potential are necessary, reading

$$\begin{aligned}
\frac{\partial^2 V}{\partial \sigma^2} &= -\mu^2 + \frac{\mu^2}{f^2}(3\sigma^2 + \chi^2) \\
\frac{\partial^2 V}{\partial \chi^2} &= -\mu^2 + \frac{\mu^2}{f^2}(3\chi^2 + \sigma^2) \\
\frac{\partial^2 V}{\partial \sigma \partial \chi} &= 2\frac{\mu^2}{f^2}\sigma\chi
\end{aligned}$$

Obviously, at zero field, the second derivatives are smaller or equal to zero, and therefore the potential at zero field is maximal. The situation at the second extremum is symmetric, so it is possible to make any choice to split the f^2 between σ and χ. Making now the explict choice and splitting it as $\sigma = f$ and $\chi = 0$ yields immediately

$$\begin{aligned}
\frac{\partial^2 V}{\partial \sigma^2} &= 2\mu^2 \\
\frac{\partial^2 V}{\partial \chi^2} &= 0 \\
\frac{\partial^2 V}{\partial \sigma \partial \chi} &= 0.
\end{aligned}$$

It is therefore a true minimum, and will be the ground-state of the system, provided quantum corrections are not too large. Replacing in the Lagrangian the fields now by

$$\begin{aligned}
\sigma &\to \sigma + f \\
\chi &\to \chi,
\end{aligned} \tag{3.6}$$

a new (and equally well-defined) Lagrangian is obtained with the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \chi \partial^\mu \chi) - \mu^2 \sigma^2 + \frac{\mu^2}{f}\sigma(\sigma^2 + \chi^2) + \frac{1}{2}\frac{\mu^2}{f^2}(\sigma^2 + \chi^2)^2, \tag{3.7}$$

where irrelevant constant and linear terms have been dropped. This Lagrangian, with the fluctuation field σ, describes two scalar particles, one with (normal) tree-level mass $\sqrt{2}\mu$, and one with zero mass. These interact with cubic and quartic interactions. It is noteworthy that the cubic coupling constant is not a free parameter of the theory, but it is uniquely determined by the other parameters. That is, as it should be, since by a mere field translation no new parameters should be introduced into the theory.

The fact that one of the particles is actually massless is quite significant. It is called a Goldstone particle, and the Goldstone theorem states quite generally that for any theory with a positive-definite metric in the internal space and a hidden symmetry such a particle must exist. However, it must not always be, as in the present case, an elementary particle. E. g., the corresponding Goldstone boson of chiral symmetry breaking in the chiral limit of QCD, the pion, is a composite object. Indeed, the corresponding excitations in a ferromagnet are spin-waves, and thus even collective excitations. In a gauge theory, which, even in the case of QED, has no positive metric space, the theorem is not holding, and no such particle must appear. Indeed, it will be found that the Goldstone boson actually takes on the role of the yet missing third polarization of the gauge bosons. The explicit construction of Goldstone's theorem will be given in section 3.2.2.

The shifted theory has as a new minimum of the potential at all fields being zero. Therefore, it is amiable to the usual perturbative expansion. Thus, the decisive step for being able to doing perturbation theory was the shift (3.6).

It should be noted, as an aside, that it is always possible to shift the potential such that the lowest energy state has energy zero. In this case, the potential takes the form

$$V = \frac{\mu^2}{2f^2}\left(\phi\phi^\dagger - \frac{f^2}{2}\right)^2.$$

This is the same as the potential (3.5), up to a constant term of size $\mu^2 f^2/8$, which is irrelevant.

There is a another feature of the theory with hidden symmetry. Return to the situation with the unshifted fields. The charge Q associated with the symmetry and the corresponding Noether current generates the transformation as

$$\begin{aligned}[Q,\sigma] &= i\chi\\ [Q,\chi] &= -i\sigma.\end{aligned}$$

From this follows immediately

$$\langle 0|\,[Q,\chi]\,|\,0\rangle = \langle 0|Q\chi - \chi Q|0\rangle = -i\langle 0|\sigma|0\rangle = -if.$$

That is of course only possible, if the hermitian charge Q is not annihilating the vacuum state

$$Q|0\rangle = |q\rangle \neq 0.$$

So, the vacuum state is charged, despite the Hamiltonian[3] being invariant under the symmetry transformation $[Q,H]=0$. However, it follows

$$0 = [Q,H]|0\rangle = HQ|0\rangle - E_0 Q|0\rangle = H|q\rangle - E_0|q\rangle = 0,$$

[3]The Hamiltonian is invariant when the Lagrangian is, provided boundary terms can be dropped.

and therefore it appears that the state q must be degenerate in energy with the vacuum, which has energy E_0. However, this observation is misleading. The state q is not normalizable,

$$\begin{aligned}\langle q|q\rangle &= \langle 0|QQ|0\rangle = \langle 0|\int d^3x J^0(x)Q|0\rangle = \int d^3x\langle 0|e^{iPx}J^0(0)e^{-iPx}Q|0\rangle \\ &= \int d^3x\langle 0|e^{iPx}J^0Qe^{-iPx}|0\rangle = \langle 0|J^0Q|0\rangle\int d^3x = \infty.\end{aligned}$$

where it has been used that the, by definition, coordinate-independent charge Q is commuting with the generator of translations P_μ. That Q is still time-independent follows from its commutation with the Hamiltonian, and therefore its trivial Heisenberg equation of motion. Furthermore, $P|0\rangle = 0$ has been used, under the assumption that the ground-state is isotropic, which should be realized in a particle physics theory. Therefore the state q is indeed not normalizable. Q is therefore mapping a state from the (normalized) Hilbert space outside the Hilbert space, and is hence no longer well-defined. That is also a general statement that the charge of a hidden symmetry in its naive form is no longer a well-defined operator. Therefore, no operator Q exists in this case, i. e., no well-defined operator.

3.2.1.3 The catch

While the previous construction works well, it is, in fact, not formally correct. The step where the hidden symmetry becomes an explicitly broken one is when f in the shift (3.6) was onwards treated as a constant in deriving the new Lagrangian (3.7). This is not correct beyond a perturbative expansion close to the minimum.

In fact, actually also the value f changes under a symmetry transformation, as it is part of the field σ, as

$$\begin{aligned}\sigma &\to \sigma + \theta\chi \\ \chi &\to \chi - \theta(\sigma - f) \\ f &\to f + \theta\chi,\end{aligned}$$

i. e. f has actually to be transformed. The implicit step which has been done was identifying the shift value f with the parameter f of the Lagrangian. Setting for a moment

$f = v$ to differentiate between both, yields

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\chi\partial^\mu\chi) + \frac{\mu^2 v}{\sqrt{2}}\left(\frac{v^2}{f^2} - 1\right)\sigma + \frac{\mu^2 v^2}{4}\left(\frac{v^2}{2f^2} - 1\right)\\ &\quad -\mu^2\left(\frac{3}{2}\frac{v^2}{f^2} - \frac{1}{2}\right)\sigma^2 - \mu^2\left(\frac{1}{2}\frac{v^2}{f^2} - \frac{1}{2}\right)\chi^2 + \frac{\sqrt{2}\mu^2}{f}\frac{v}{f}\sigma(\sigma^2+\chi^2) + \frac{\mu^2}{2f^2}(\sigma^2+\chi^2)^2 \quad (3.8)\\ &\overset{v=f}{=} \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\chi\partial^\mu\chi) - \mu^2\sigma^2 + \frac{\sqrt{2}\mu^2}{f}\sigma(\sigma^2+\chi^2) + \frac{1}{2}\frac{\mu^2}{f^2}(\sigma^2+\chi^2)^2 - \frac{\mu^2 f^2}{8}.\end{aligned}$$

where the explicit identification has been done in the last line. One consequence is that in the Lagrangian (3.8), i. e. before the identification $v = f$ has been made, at tree-level both fields are still massive. However, these tree-level masses are not symmetry-invariant quantities, and there is no Goldstone boson. Only in the small-field expansion where $v \approx f$, this is a negligble problem.

Of course, as long as no transformation is performed, this does not change. Hence, once the symmetry is explicitly broken by a choice of vacuum, calculations can be performed, but they will never reflect the underlying symmetry, but the explicit broken one. Keeping this in mind, this is very often sufficient. However, using suitable, and more demanding, non-perturbative methods, it is also possible to perform corresponding calculations without this choice, yielding for any invariant quantity the same results. In fact, massless states will then also appear, though they are not necessarily created by the elementary fields, and may become only accessible using composite operators. This has yet been not fully explored for global symmetries. For gauge symmetries, this is much better understood, as will be discussed in section 6.3.2.

3.2.2 The Goldstone theorem

Staying with a perturbative/fixed setup, it is possible to make much more general statments about the apperance of Goldstone bosons.

3.2.2.1 Classical Goldstone theorem

In the previous (Abelian) case there appeared one massless and one massive scalar after hiding the symmetry. The Goldstone theorem states this more generally: If a symmetry group G of size dim$G$ is hidden, then there exists as many massless modes as there are generators. If the group is only hidden partly than only as many massless modes appear as generators are hidden.

This can be shown as follows. Take as the symmetry group a (semi-)simple Lie-group G. Than the symmetry transformation of the associated real fields transforming under a real representation of the symmetry group are given by

$$\delta\phi_i = iT_a^{ij}\phi_j\theta^a \tag{3.9}$$

with arbitrary infinitesimal parameters θ^a counting from 1 to dim$G$. The Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi^i - V(\phi)$$

must be invariant under a group transformation. Since the kinetic term is trivially so, this implies for the potential

$$0 = \delta V = \frac{\partial V}{\partial\phi_i}\delta\phi^i = i\frac{\partial V}{\partial\phi_i}T^a_{ij}\phi^j\theta_a.$$

Since the parameters are arbitrary, this can only be satisfied if

$$\frac{\partial V}{\partial\phi_i}T^a_{ij}\phi^j = 0$$

holds. Differentiating this equation with respect to ϕ_k yields

$$\frac{\partial^2 V}{\partial\phi_k\phi_i}T^a_{ij}\phi^j + \frac{\partial V}{\partial\phi_i}T^a_{ik} = 0.$$

The symmetry is hidden by expanding around the minimum of the potential, and therefore the first derivatives have all to vanish. The symmetric matrix of second derivatives is positive at a minimum, i. e., has only positive or zero eigenvalues

$$\frac{\partial^2 V}{\partial\phi_k\partial\phi_i} = (M^2)^{ki}.$$

Expanding now, as before, the field around the classical minimum at $\psi_i = \phi_i - f_i$, the quadratic order of the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi_i\partial^\mu\psi^i - \frac{1}{2}(M^2)_{ki}\psi_k\psi_i + ...$$

Since the mass matrix M is semi-definite positive, all particles have at tree-level only positive or zero mass.

The conditional equation for a classical minimum reads

$$(M^2)^{ki}T^a_{ij}f^j = 0.$$

If the classical minimum is invariant under a subgroup H of G, this subgroup is called the stability group of G. As a consequence for generators t_a out of H the conditional equation reads

$$t^a_{ij}f^j = 0.$$

Therefore, the value of the mass matrix is irrelevant for this direction, and there can be dim$H$ massive modes. However, for the coset space G/H with generators τ^a, the corresponding equations

$$\tau^a_{ij} f^j \neq 0.$$

are not fulfilled, and therefore the corresponding entries of the mass-matrix have to vanish. Since these represent $\dim(G/H)$ equations, there must be $\dim(G/H)$ massless modes, the Goldstone modes.

A simple example of these concepts is given by the linear σ-model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi^i - \frac{\lambda}{4}\left(\phi_i \phi^i - \frac{\mu^2}{\lambda}\right)^2.$$

This is invariant under the group O(N) if the real field ϕ has N components. The classical minimum is given by the condition

$$\phi^i \phi_i = f^i f_i = f^2 = \frac{\mu^2}{\lambda} > 0 \tag{3.10}$$

provided μ^2/λ is greater than zero. This minimum characterizes a vector of length f^2 on the N-sphere, and is therefore invariant under the group O($N-1$), being thus the stability group of this theory. Since O(N) has $N(N-1)/2$ generators, there are $N-1$ generators no longer manifest, and thus there exists $N-1$ massless modes. Setting conventionally the direction to $(0, ..., 0, f)$ and rewriting the theory with the shifted field $\phi = (\vec{\pi}, \sigma + f)$ yields the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \mu^2 \sigma^2 + \frac{1}{2}\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - \frac{\lambda}{4}(\vec{\pi}^2 + \sigma^2)^2 - \lambda v \sigma(\vec{\pi}^2 + \sigma^2).$$

And, indeed, the $N-1$ fields $\vec{\pi}$ are massless, while the field σ is massive with mass $\sqrt{2}\mu$, in accordance with the Goldstone theorem.

It should be noted that this theory for the group O(4), being isomorphic to SU(2)×SU(2), is one of the models of chiral symmetry breaking. It will also play a central role in the electroweak sector of the standard model as the Higgs sector before introducing the weak gauge interaction.

3.2.2.2 Quantized Goldstone theorem

To determine the consequence of hiding symmetry at the quantum level it is useful to investigate the normalized partition function[4],

$$T[J_i] = \frac{Z[J_i]}{Z[0]} = \frac{1}{Z[0]} \int \mathcal{D}\phi_i \exp\left(i\int d^4x(\mathcal{L} + J_i\phi_i)\right),$$

with the same Lagrangian as before. Since the Lagrangian and the measure are invariant under a symmetry transformation[5], the variation of the partition function must vanish

$$0 = \delta Z[J_i] = \int \mathcal{D}\phi_i e^{iS+i\int d^4xJ_i\phi_i} \int d^4x \left(\frac{\partial\delta\phi_i}{\partial\phi_j} + \delta\left(iS + i\int d^4xJ_i\phi_i\right)\right).$$

The first term is the deviation of the Jacobian from unity. As the measure is invariant, it vanishes. The second is the variation of the action, which also vanishes. Only the third term can contribute. Since all variations are arbitrary, it thus follows

$$\int d^4xJ_iT^a_{ik}\frac{\delta T[J_i]}{i\delta J_k} = 0,$$

where it has been used that $Z[0]$ is a constant, and the order of functional and ordinary integration has been exchanged, and

$$\frac{\delta T[J_i]}{i\delta J_i} = \frac{1}{Z[0]} \int \mathcal{D}\phi_i\phi_i \exp\left(i\int d^4x(\mathcal{L} + J_i\phi_i)\right).$$

Furthermore, it has been used that all variations are independent, thus delivering dim$G$ independent equations.

Since

$$\delta T \equiv \delta\left(e^{T_c}\right) = e^{T_c}\delta T_c,$$

and the factor $\exp(T_c)$ is not depending on x, since it is a functional, this can be rewritten in terms of the generating functional for connected Green's functions as

$$\int d^4xJ_iT^a_{ik}\frac{\delta T_c[J_i]}{i\delta J_k} = 0.$$

[4]It is the uses of sources in this expression, which is the equiavlent of the external magnetic field in section 3.2.1.1, which is really the explicit breaking of the symmetry. The discontinuity in this source is really the origin of any mismathc with a full non-perturbative description.

[5]If the measure would not be invariant, this would lead to an anomaly. This happens, e. g., in the case of chiral symmetry. For the purpose of the electroweak physics, this is not the case, and therefore invariance is assumed here.

This can be furthermore transformed into an equation for the vertex (i. e., connected and amputated Green's functions) generating functional Γ, which is related to the connected one by a Legendre transformation[6]

$$\begin{aligned} i\Gamma[\phi] &= -i\int d^4x J_i\phi_i + T_c[J] \\ \langle\phi_i\rangle &= \frac{\delta T_c[J]}{i\delta J_i} = \langle 0|\phi_i|0\rangle\,[J_i] \\ J_i &= -\frac{\delta\Gamma[\phi]}{i\delta\phi_i}, \end{aligned} \tag{3.11}$$

by simply exchanging the derivative and the source. This yields finally

$$\int d^4x \frac{\delta\Gamma}{\delta\phi_i} T^a_{ik}\langle\phi_k\rangle = 0. \tag{3.12}$$

For the fields developing a vacuum expectation value it then holds when sending the sources to zero

$$\begin{aligned} f_i &= \langle 0|\phi_i|0\rangle = \frac{\delta T_c}{i\delta J_i}[0] \\ J_i &= -\frac{\delta\Gamma}{i\delta\phi_i}[f_i] \to 0. \end{aligned} \tag{3.13}$$

The inverse propagator of the fields ϕ_i is given by

$$\frac{i\delta^2\Gamma}{\delta\phi_i(x)\delta\phi_j(y)}[f_i] = -(D^{-1})_{ik}(x-y). \tag{3.14}$$

An expression for this object can be obtained by differentiating (3.12) with respect to the field once more yielding

$$\int d^4x \left(\frac{\delta^2\Gamma}{\delta\phi_i(x)\delta\phi_j(y)} T^a_{ik}\langle\phi_k\rangle + \frac{\delta\Gamma}{\delta\phi_i} T^a_{ii}\delta(x-y)\right).$$

The last term vanishes since the generators are Hermitian and traceless and, even if not, because $\delta\Gamma/\delta\phi_i = 0$, while the first one is just the Fourier-transform of the inverse propagator at zero momentum, yielding

$$(D^{-1})_{ij}(p=0)T^a_{ik}f_k = 0.$$

Thus, there must vanish as many inverse propagators as there are non-zero f_i. At tree-level the inverse propagator is given by[7]

$$(D^{-1})_{ij} = \delta_{ij}(p^2+m^2),$$

[6] Using the same notation for the field and its one-point Green's function.

[7] Note that without supersymmetry all symmetry generators are bosonic, and therefore the propagator is that of a bosonic particle. It is (pseudo)scalar, as vector (and higher-spin) particles would break Lorentz symmetry.

and thus this implies that the pole mass must vanish, the propagator becomes that of a massless particle, just as classically. However, in the full quantum theory, the mass becomes momentum-dependent, and the full propagator takes the form

$$D_{ij}^{-1} = Z_{ij}(p^2)(p^2 + M_{ij}(p^2)).$$

Thus, only the combination $Z(0)/M(0)$ must vanish. This is the so-called screening-mass of the particle, which therefore has to vanish. The propagator has also a pole at this point, and therefore in this case the screening mass coincides with (a) pole mass, giving the particle a genuine massless mode. However, there can also be further poles at non-zero momentum, i. e. massive, and potentially stable, excitations with the same quantum numbers as the Goldstone mode. This is the quantized version of the Goldstone theorem.

It should be noted here that this derivation only applies to a global symmetry. From an axiomatic point of view, the reason for failure is that implicitly a positive definite Hilbert space has been assumed, which is not the case for a gauge theory. Thus, it is useful to first introduce the gauged version of the theory before dwelling on the finer points of it.

3.3 An interlude: Foundations of gauge theories

Before delving into the case of local symmetries, it is useful to shortly rehearse the formulation of gauge theories[8]. Their quantization will be discussed later in more details.

3.3.1 Abelian gauge theories

The simplest possible gauge theory is the quantum-field-theoretical generalization of electrodynamics. In classical electrodynamics, it was possible to transform the gauge potential A_μ by a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \omega,$$

where ω is an arbitrary function. A defining property of such gauge transformations is the fact that they do not alter any measurable quantities. In particular, the electric and magnetic fields $\vec{E}$ and $\vec{B}$, which are obtained from the gauge potentials by

$$\begin{aligned} \vec{E}_i &= -\frac{\partial}{\partial t} A_i - \partial_i A_0 \\ \vec{B}_i &= (\vec{\nabla} \times \vec{A})_i, \end{aligned}$$

[8] A much more deatiled treatement can be found in the lecture quantum field theory II.2

are invariant under such transformations. From the vector potential, it is possible to form the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which is also invariant under gauge transformations. The Maxwell equations can then be written in the compact form

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} &= 0, \end{aligned}$$

where j^μ is the matter current. These are the equations of motions of the classical Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu.$$

Consistently quantizing this theory is actually highly non-trivial, and will be briefly rehearsed in section 5.1.1. However, for the current purpose it is mainly interesting how the Lagrangian of the quantized version of this theory looks. It turns out that the first term proportional to $F_{\mu\nu}F^{\mu\nu}$ is already the Lagrangian of the quantized electromagnetic field. It then remains to construct the electric current. If an electron is represented by a spinor ψ, this spinor is actually no longer invariant under a gauge transformation. However, as in quantum mechanics, only the phase can be affected by a gauge transformation, as the amplitude is still roughly connected to a probability (or electric) current, and thus may not be affected. Therefore, under a gauge transformation the spinors change as

$$\psi \to \exp(-ie\omega)\psi,$$

where the same function ω appears as for the vector potential, which is now representing the field of the photon, and is called gauge field. Since ω is a function, the kinetic term for an electron is no longer invariant under a gauge transformation, and has to be replaced by

$$i\bar{\psi}(\gamma^\mu(\partial_\mu + ieA_\mu))\psi.$$

This replacement

$$\partial_\mu \to \partial_\mu + ieA_\mu = D_\mu$$

is called minimal coupling, and D_μ the covariant derivative. This is now gauge invariant. Thus, the (gauge-invariant) Lagrangian of quantum electro dynamics (QED) is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

where a mass term has been added, which is trivially gauge-invariant. The second term is thus the quantum version of the $j^\mu A_\mu$ term at the classical level.

This type of gauge theories is called Abelian, as the phase factor $\exp(i\omega)$ with which the gauge transformation is performed for the fermions is an element of the U(1) group. Thus, U(1) is called the gauge group of the theory. It can furthermore be shown that the vector potential, or gauge field, is an element of the corresponding u(1) algebra.

3.3.2 Non-Abelian gauge theories: Yang-Mills theory

Although U(1) gauge theory already provides an enormous host of interesting physical effects, e. g. solid state physics, its complexity is not sufficient to describe all the phenomena encountered in the standard model, e. g. nuclear physics. A sufficiently complex theory is obtained when the Abelian gauge algebra u(1) is replaced by the non-Abelian gauge algebra[9] su(N), where N is referred to often as the number of colors. In case of the strong interactions N is 3, and for the weak interactions it is 2.

In this case, all fields carry an additional index, a, which indicates the charge with respect to this gauge group. E. g., the gluon index runs from 1 to 8, while the quark index runs from 1 to 3, because the former have charges corresponding to the adjoint representation of SU(3), and the latter to the fundamental one.

In particular, a gauge field can now be written as $A_\mu = A_\mu^a \tau_a$, with τ_a the generators of the algebra of the gauge group and A_μ^a are the component fields of the gauge field for each charge. The gauge transformation of a fermion field is thus

$$\begin{aligned} \psi &\rightarrow g\psi \\ g &= \exp(i\tau^a\omega_a), \end{aligned}$$

with ω_a arbitrary functions and a takes the same values as for the gauge fields. The corresponding covariant derivative is thus

$$D_\mu = \partial_\mu + ieA_\mu^a \tau_a,$$

with the τ_a in the fundamental representation of the gauge group. The corresponding gauge transformation for the gauge fields has then to take the inhomogeneous form

$$A_\mu \rightarrow gA_\mu g^{-1} + g\partial_\mu g^{-1}.$$

The expression for $F_{\mu\nu}$ is then also no longer gauge invariant, and has to be generalized to

$$F_{\mu\nu} = F_{\mu\nu}^a \tau_a = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu].$$

[9] Such gauge theories can be constructed for any Lie group, but the following is restricted to those relevant for the standard model.

This quantity is still not gauge invariant, and thus neither are magnetic nor electric fields. However, the expression

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

is. Hence, the Lagrangian for a non-Abelian version of QED reads

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{3.15}$$

which at first looks simple, but just because the explicit form has been defined into appropriate quantities, and $F_{\mu\nu}$ and D_μ are now matrix-valued in the group space of the gauge group.

3.4 Hiding local symmetries

The situation for hiding a local symmetry analogous, though more involved, to the case of a global symmetry. For simplicity, regard the Lagrangian of an Abelian theory, coupled to a single, complex scalar, the so-called Abelian Higgs model,

$$\begin{aligned}\mathcal{L} &= ((\partial_\mu + iqA_\mu)\phi)^\dagger(\partial^\mu + iqA^\mu)\phi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &\quad +\frac{1}{2}\mu^2\phi^\dagger\phi - \frac{1}{2}\frac{\mu^2}{f^2}(\phi^\dagger\phi)^2.\end{aligned}$$

Note that the potential terms are not modified by the presence of the gauge-field. Therefore, the extrema have still the same form and values as in the previous case, at least classically. However, it cannot be excluded that the quartic $\phi^\dagger\phi A_\mu A^\mu$ term strongly distorts the potential. Once more, this does not appear to be the case in the electroweak interaction, and it will therefore be ignored.

The general equation of motion, without fixing the gauge, for the photon is

$$\partial^2 A_\mu - \partial_\mu(\partial^\nu A_\nu) = iq(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) - 2q^2\phi^\dagger\phi A_\mu. \tag{3.16}$$

To bring this equation into a more simple form, rewrite the scalar field as[10]

$$\phi(x) = \left(\frac{f}{\sqrt{2}} + \rho(x)\right)\exp(i\alpha(x)).$$

[10]Note that if the space-time manifold is not simply connected and/or contains holes, it becomes important that α is only defined modulo 2π. For flat Minkowski (or Euclidean) space, this is of no importance. However, it can be important, e. g., in finite temperature calculations using the Matsubara formalism. It is definitely important in ordinary quantum mechanics, where, e. g., the Aharanov-Bohm effect and flux quantization depend on this, when not using a formulation employing only gauge-invariant quantities.

This is another reparametrization for the scalar field, compared to σ and χ previously. It is such that at $\rho = 0$ this field configuration will be a classical minimum of the potential for any value of the phase α. Inserting this parametrization into the equation of motion (3.16) yields

$$\partial^2 A_\mu - \partial_\mu(\partial^\nu A_\nu) = -q^2 f^2 \left(A_\mu + \frac{1}{q}\partial_\mu \alpha\right) - 2q^2 f\rho A_\mu - q^2\rho^2 A_\mu.$$

While the two last terms do describe the interactions of the fluctuating field ρ with the gauge boson, the first two terms describe only the interaction with the phase. So far, no gauge has been fixed, despite the fact that the theory has a non-trivial classical vacuum.

Now, it is possible to make the deliberate gauge choice, the so-called unitary gauge,

$$\partial_\mu A^\mu = -\frac{1}{q}\partial^2\alpha. \tag{3.17}$$

This is always possible. It is implemented by first going to Landau gauge and then perform the gauge transformation

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{q}\partial_\mu\alpha \\ \phi &\to \exp(-i\alpha)\phi. \end{aligned}$$

This gauge choice has two consequences. The first is that it makes the scalar field real everywhere. Therefore, the possibility of selecting the vacuum expectation value of ϕ to be real is a gauge choice. Any other possibilities, e. g. purely imaginary, would be equally well justified gauge choices. This also implies that the actual value of the vacuum expectation value of ϕ is a gauge-dependent quantity. The equation for the gauge-fixed field then takes the form

$$\partial^2 A_\mu - \partial_\mu(\partial^\nu A_\nu) = -q^2 f^2 A_\mu - 2q^2 f\rho A_\mu - q^2\rho^2 A_\mu, \tag{3.18}$$

i. e., the first term on the right-hand side has now exactly the form of a screening term, and yields an effective mass qf for the photon field. If the field ρ would be negligible, it would further follow, as before, that $\partial_\mu A^\mu = 0$. That this is not the case can be seen by the gauge condition (3.17). The interaction terms enforce this condition on the gauge field away from $\partial_\mu A^\mu = 0$. Furthermore, if ρ could be neglected, the equation of motion for A_μ would now be indeed

$$(\partial^2 + M^2)A_\mu = 0.$$

Together with the gauge condition for A_μ this implies that the field A_μ acts now indeed as a massive spin-1 field. At the level of the Lagrangian, this mass would appear from the two-scalars-two-gauge boson term out of the covariant kinetic term for the scalars, and the explicit appearance of the condensate,

$$(D_\mu\phi)^\dagger(D^\mu\phi) = -\phi^\dagger\partial^2\phi - iq\phi^\dagger A^\mu\partial_\mu\phi + iq\phi A^\mu\partial_\mu\phi^\dagger + q^2\phi\phi^\dagger A_\mu A^\mu.$$

The last term provides then the mass-term due to its dependence on f^2. The Goldstone theorem actually guarantees that a mass will be provided to each gauge boson associated with one of the hidden generators. Hence, a massless Goldstone boson is effectively providing a mass to a gauge boson by becoming its third component, and vanishes by this from the spectrum. It is the tri-linear couplings which provides the explicit mixing terms delivering the additional degree of freedom for the gauge boson at the level of the Lagrangian. Hence, the number of degrees of freedom is preserved in the process: In the beginning there were two scalar and two vector degrees of freedom, now there is just one scalar degree of freedom, but three vector degrees of freedom. The gauge-transformation made nothing more than to shift one of the dynamic degrees of freedom from one field to the other. This was possible due to the fact that both the scalar and the photon are transforming non-trivially under gauge-transformations. It is thus a remarkable fact that the equation of motion for the gauge field depends on the gauge.

This can, and will be, generalized below. In general it turns out that for a general covariant gauge there are indeed six degrees of freedom, four of the vector field, and two from the scalars. Only after calculating a process it will turn out that certain degrees of freedom cancel out, yielding just a system which appears like having a massive vector particle and a single scalar. It will also be this effect which will permit to show that the theory is now indeed renormalizable. But it is now somewhat to be expected: A Yang-Mills theory coupled to arbitrary fields, such that asymptotic freedom is preserved, is always renormalizable, and the infrared, non-perturbative effect of condensation of matter fields would not spoil the high-momentum behavior necessary for renormalization.

Another quite striking result from the equation (3.18) is the fact that the two interaction couplings and the effective mass are completely fixed by the two parameters of the original theory. Therefore, in this form the hidden symmetry still surfaces in the form of relations between various coupling parameters. By measuring such relations, it is in principle possible to determine whether a theory has a hidden symmetry or not.

Note that though the original scalar field ϕ was charged, the radial excitation as the remaining degree of freedom is actually no longer charged: The coupling structure appearing in (3.18) is not the one expected for a charged field. Still, the theory now seems to provide a reasonable starting point for a perturbative expansion with ρ expanded around zero field. And also in this expansion, due to the particular relations of the coupling constants, it will be found that the theory is renormalizable, and thus its degree of divergence is in fact less than its superficial degree of divergence[11].

A possibility to make this behavior more explicit, though at the cost of having unphysical degrees of freedom which only cancel at the end, are 't Hooft gauges. To explicitly demonstrate them use once more the decomposition

$$\phi = \frac{1}{\sqrt{2}}(f + \sigma + i\chi)$$

for the scalar field. Then the equation of motion of the photon becomes

$$(\partial^2 + (qf)^2)A_\mu - \partial_\mu(\partial^\nu A_\nu) = -qf\partial_\mu\chi + q(\chi\partial_\mu\sigma - \sigma\partial_\mu\chi) - q^2A_\mu(\sigma^2 + 2f\sigma + \chi^2).$$

Again, the effective mass for A_μ is explicit. Interestingly, the field χ appears linearly in the equation of motion. Therefore, the photon and this scalar, the would-be Goldstone boson, will mix. This equation is not yet meaningful, as it is not possible to obtain a photon propagator from it, since the corresponding differential operator cannot be inverted. Therefore, select the 't Hooft gauge

$$\partial_\mu A^\mu = qf\xi\chi, \tag{3.19}$$

where ξ is a free gauge parameter. Though it is not entirely trivial to find a gauge transformation ϕ such that the gauge-fixed field satisfies the 't Hooft condition (3.19), it is always possible to any order in perturbation theory. However, the scalar fields in general do not take a simpler form. In particular, in contrast to the previously employed unitary gauge, the components are still present. Nonetheless, by virtue of the gauge condition the equation of motion for A takes thus always the form

$$(\partial^2 + (qf)^2)A_\mu - \left(1 - \frac{1}{\xi}\right)\partial_\mu(\partial^\nu A_\nu) = q(\chi\partial_\mu\sigma - \sigma\partial_\mu\chi) - q^2A_\mu(\sigma^2 + 2f\sigma + \chi^2).$$

In this equation the mixing of A with χ has been removed. In particular, in the limit of σ and χ to zero, the tree-level propagator can be obtained as the Green's function of the remaining operator and takes the form

$$D^{\mathrm{tl}}_{\mu\nu} = \left(-g_{\mu\nu} + \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi M^2}\right)\frac{1}{k^2 - M^2 + i\epsilon},$$

with $M = qf$. As has been seen previously, the problems with renormalization occurred since the propagator had a term scaling as $k_\mu k_\nu/M^2$ beforehand. This term is here no

[11]Similar to a supersymmetric theory, where relations between the degrees of freedom and coupling constants imposed by supersymmetry also reduce the actual degree of divergence below the superficial degree of divergence.

longer present, since in the limit of $k^2 \to \infty$ this becomes just the propagator of a massless vector boson in general covariant gauges with gauge parameter ξ. Therefore, the theory actually renormalizes in the same way as ordinary Maxwell theory. The price to be payed, however, is that the unphysical degree of freedom χ is still present, and has to be included in any calculation. Only in the end it will cancel out, in just the same way as in unitary gauges the non-renormalizable divergences disappear.

Furthermore, this particular gauge also provides in perturbation theory a smooth limit of the propagator for a vanishing mass. In this limit, exactly the massless gauge boson propagator is reproduced.

Both properties are significantly different from the previous case. By inversion of the equation of motion (3.18) yields the propagator

$$D_{\mu\nu}^{\mathrm{tl}} = \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right) \frac{1}{k^2 - M^2 + i\epsilon}.$$

This propagator has the property to spoil superficial renormalizability, as it behaves essentially like $1/(k^2M^2)$ at large momenta. Furthermore, there is no smooth behavior for vanishing mass.

Thus, in both cases, and in general, there is no possibility to evade that something will not be manifest, either the physical spectrum or renormalizability, and only becomes restored at the end of a calculation. How to transfer this observation to non-Abelian gauge theories, and make it more formal, will be discussed in the following.

Chapter 4

The Glashow-Salam-Weinberg theory

It is now clear that a theoretical description of the weak interactions requires a gauge theory, as well as some matter field(s) to hide it and provide a mass to the weak gauge bosons. The fact that there are three distinct gauge bosons, two Ws and one Z, indicates that the gauge theory has to be more complex than just the U(1) gauge theory of QED. It will thus be a non-Abelian gauge theory. Furthermore, since two of them are charged the connection to the electromagnetic interactions will not be trivial, and there will be some kind of mixing. All of these aspects will be taken into account in the following.

4.1 Phenomenological construction the gauge group

Phenomenologically, the weak interactions provides transitions of two types. One is a charge-changing reaction, which acts between two gauge eigenstates. In case of the leptons, these charge eigenstates are almost (up to a violation of the order of the neutrino masses) exactly a doublet - e. g., the electron and its associated electron neutrino. Therefore, the gauge group of the weak interaction should provide a doublet representation. In case of the quarks this is less obvious, but also they furnish a doublet structure. Hence, an appropriate gauge group for the weak interaction will contain at least SU(2). Since only the left-handed particles are affected, this group is often denoted as $SU(2)_L$, but this index will be dropped here. Therefore, there are three doublets, generations, of leptons and quarks, respectively in the standard model. However, the mass eigenstates mix all three generations, as will be discussed in detail below.

Since the electric charge of the members of the doublets differ by one unit, the off-diagonal gauge bosons, the W, must carry a charge of one unit. Furthermore, such a gauge group has three generators. The third must therefore be uncharged, as it mediates interactions without exchanging members of a doublet: It is the Cartan of the SU(2) group.

The quantum number distinguishing the two eigenstates of a doublet is called the third component of the weak isospin2 t, and will be denoted by t_3 or I_W^3. Therefore, the gauge group of the weak interactions is called the weak isospin group

However, the weak gauge bosons are charged. Therefore, ordinary electromagnetic interactions have to be included somehow. Since ordinary electromagnetism has a one-dimensional representation, its gauge group is the Abelian U(1). The natural ansatz for the gauge group of the electroweak interactions is thus the gauge group SU(2)×U(1)[1]. With this second factor-group comes a further quantum number, which is called the hypercharge y. The ordinary electric charge is then given by

$$eQ = e\left(t_3 + \frac{y}{2}\right). \tag{4.1}$$

Thus, the ordinary electromagnetic interaction must be somehow mediated by a mixture of the neutral weak gauge boson and the gauge boson of the U(1). This is dictated by observation: It is not possible to adjust otherwise the quantum numbers of the particles such that experiments are reproduced. The hypercharge of all left-handed leptons is -1, while the one of left-handed quarks is $y = +1/3$.

Right-handed particles are neutral under the weak interaction. In contrast to the $t = 1/2$ doublets of the left-handed particle, they belong to a singlet, $t = 0$. All in all, the following assignment of quantum numbers for charge, not mass, eigenstates will be necessary to reproduce the experimental findings:

- Left-handed neutrinos: $t = 1/2$, $t_3 = 1/2$, $y = -1$ ($Q = 0$)
- Left-handed leptons: $t = 1/2$, $t_3 = -1/2$, $y = -1$ ($Q = -1$)
- Right-handed neutrinos: $t = 0$, $t_3 = 0$, $y = 0$ ($Q = 0$)
- Right-handed leptons: $t = 0$, $t_3 = 0$, $y = -2$ ($Q = -1$)
- Left-handed up-type (u, c, t) quarks: $t = 1/2$, $t_3 = 1/2$, $y = 1/3$ ($Q = 2/3$)
- Left-handed down-type (d, s, b) quarks: $t = 1/2$, $t_3 = -1/2$, $y = 1/3$ ($Q = -1/3$)
- Right-handed up-type quarks: $t = 0$, $t_3 = 0$, $y = 4/3$ ($Q = 2/3$)
- Right-handed down-type quarks: $t = 0$, $t_3 = 0$, $y = -2/3$ ($Q = -1/3$)

[1]Actually, the correct choice is SU(2)/Z_2×U(1), though this difference is not relevant in perturbation theory, and therefore neglected here.

- W^+: $t = 1$, $t_3 = 1$, $y = 0$ ($Q = 1$)
- W^-: $t = 1$, $t_3 = -1$, $y = 0$ ($Q = -1$)
- Z: $t = 1$, $t_3 = 0$, $y = 0$ ($Q = 0$)
- γ: $t = 0$, $t_3 = 0$, $y = 0$ ($Q = 0$)
- Gluon: $t = 0$, $t_3 = 0$, $y = 0$ ($Q = 0$)
- Higgs: a complex doublet, $t = 1/2$ with weak hypercharge $y = 1$. This implies zero charge for the $t_3 = -1/2$ component, and positive charge for the $t_3 = 1/2$ component and negative charge for its complex conjugate

This concludes the list of charge assignments for the standard model particles. The Higgs case is special, and will be detailed in great length below, starting with section 4.2

Since at the present time the photon field and the Z boson are not yet readily identified, it is necessary to keep the gauge boson fields for the SU(2) and U(1) group differently, and these will be denoted by W and B respectively. The corresponding pure gauge part of the electroweak Lagrangian will therefore be

$$\begin{aligned}\mathcal{L}_g &= -\frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\\ G^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + gf^{abc}W^b_\mu W^c_\nu,\end{aligned}$$

where g is the weak isospin gauge coupling. f^{abc} are the structure constants of the weak isospin gauge group, which is just the SU(2) gauge group.

Coupling matter fields to these gauge fields proceeds using the ordinary covariant derivative, which takes the form

$$D_\mu = \partial_\mu + \frac{ig}{2}\tau_a W^a_\mu + \frac{ig'y}{2}B_\mu,$$

where g' is the hypercharge coupling constant, which is modified by the empirical factor y. Note that y is not constrained by the gauge symmetries, and its value is purely empirical. Thus, why it takes the rational values it has is an unresolved question to date. However, this fact would come about naturally, if the weak gauge group would originate from a different gauge group at higher energies, say SU(2)×U(1)⊂SU(3), which is hidden to the extent that all other fields charged under this larger gauge group are effectively so heavy that they cannot be observed with current experiments.

The matrices τ_a are determined by the representation of SU(2) in which the matter fields are in. For a doublet, these will be the Pauli-matrices. For the adjoint representation, these would be given by the structure constants, $\tau^a_{bc} = f^{abc}$, and so on. For fermions, of course, this covariant derivative is contracted with the Dirac matrices γ_μ. Precisely, to couple only to the left-handed spinors, it will be contracted with $\gamma_\mu(1-\gamma_5)/2$ for the W_μ^a term and with γ_μ for the kinetic and hypercharge term. By this, the phenomenological couplings are recovered in the low-energy limit, as the propagator of a massive gauge boson then becomes proportional to $1/M^2$, thus recovering the Fermi-coupling g^2/M^2. How this mass disappears in the case of the non-Abelian gauge group will be discussed in section 4.3.

4.2 Group structure and custodial symmetry

One aspect, which has been glossed over so far, is the necessity of two complex doublets for the Higgs. Because SU(2) is a pseudoreal group, it would essentially be sufficient to have a single complex doublet to write down a gauge-invariant theory. However, in this case the number of Goldstone bosons is too small to provide three massive gauge bosons, as demanded by experiment. Thus, the presence of two doublets is experimentally necessary, but not theoretically. However, the presence of the second doublet has more consequences than just giving all three weak gauge bosons mass.

To understand the consequences it is best to concentrate first on the pure Higgs sector, i. e. only considering the two doublets, and no other fields. In this case the theory becomes the linear σ-model

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi^2),$$

where ϕ, the Higgs field, is now a four-dimensional real vector. This theory has therefore an O(4) symmetry, i. e. it is invariant under rotations of the Higgs field.

The group O(4) is isomorphic to SU(2)×SU(2). Rewriting the 4-dimensional real vector first as a complex doublet and then this doublet as a 2×2 matrix

$$X = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} = (\phi_i^\dagger\phi_i)^{\frac{1}{2}}\varphi, \qquad (4.2)$$

where φ is an SU(2) matrix, makes this structure manifest. The corresponding Lagrangian takes the form

$$\mathcal{L} = \mathrm{tr}\partial_\mu X^\dagger\partial^\mu X - V(\det X) = \mathrm{tr}\partial_\mu X^\dagger\partial^\mu X - V(\mathrm{tr}X^\dagger X),$$

which makes the symmetries manifest. The two SU(2) groups now act once as a left-multiplication or as a right-multiplication on (4.2). Thus, these are usually called $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$.

When passing to the gauged theory, it is actually only the $SU(2)_L$ which becomes the weak gauge group. Thus, the name is doubly motivated, both from the fermions on which it acts and from the present setup. The theory retains the group $SU(2)_R$ as a global symmetry group, as long as the remainder of the standard model is not included. Thus, there is a global symmetry, called custodial symmetry. This symmetry will remain unbroken, as the Brout-Englert-Higgs effect only acts in the gauged sector, and therefore spontaneously break only $SU(2)_L$. The consequence of the custodial group is, e. g., that in absence of the hypercharge the three gauge bosons all receive the same mass.

This is best seen by considering unitary gauge. In this case, the Higgs field is rotated such that it becomes real. This condition is not invariant under custodial transformation. This gauge condition therefore locks the gauge and custodial symmetry. To maintain the invariance of the gauge condition under a custodial transformation, which must be possible as the gauge condition cannot break a global symmetry, requires then to also perform a gauge transformation to maintain a real Higgs field. This locking is the mechanism how the custodial symmetry guarantees the same masses for the gauge bosons.

When coupling the hypercharge gauge boson, or fermions, the custodial symmetry is explicitly broken. This happens by the fact that it is actually the U(1) subgroup of the $SU(2)_R$, which becomes broken. This uniquely provides the hypercharge quantum numbers of the Higgs field and the electric charges of the electroweak gauge bosons, but not of the fermions. This will yield, e. g. that the $W^{\pm}$ and the Z are no longer mass degenerate. This breaking is actually mild, as can be seen from the small mass difference of the weak gauge bosons.

Note that if the custodial symmetry would be spontaneously broken, the Goldstone theorem would require the existence of massless excitations, up to corrections due to the small explicit breaking, like the pions of QCD. Since these are not observed, this requires the custodial symmetry to be not spontaneously broken. This constrains the form of the potential, as it is possible to write down further terms, which would yield an absolute minimum breaking the custodial symmetry spontaneously.

While in the standard model the custodial symmetry only plays furthermore a minor role, it is quiet important in beyond-the-standard-model physics. The requirement of its existence and it being unbroken substantially limits possible larger theories in which the standard model can be embedded. During most of the remainder, the custodial symmetry will not play an important role, but it will become important again when considering the gauge-invariance of the observable, and thus physical, spectrum in chapter 6.3. Especially, it will also be important to justify the use of perturbation theory.

4.3 Hiding the electroweak symmetry

To have a viable theory of the electroweak sector it is necessary to hide the symmetry such that three gauge bosons become massive, and one uncharged one remains massless. Though this can be of course arranged in any gauge, it is most simple to perform this in the unitary gauge. Since three fields have to be massive, this will require three pseudo-Goldstone bosons. Also, since empirically two of them have to be charged, as the $W^{\pm}$ bosons are charged, the simplest realization is by coupling a complex doublet scalar field, the Higgs field , to the electroweak gauge theory

$$\begin{aligned} \mathcal{L}_h &= \mathcal{L}_g + (D_\mu\phi)^\dagger D^\mu\phi + V(\phi\phi^\dagger) \\ \phi &= \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \end{aligned} \tag{4.3}$$

where the field ϕ is thus in the fundamental representation of the gauge group. Because of the hypercharge being the U(1) subgroup of the custodial charge, tts hypercharge will be 1. The potential V can only depend on the gauge-invariant combination $\phi^\dagger\phi$, and thus can, to be perturbatively renormalizable, only contain a mass-term and a quartic self-interaction. The mass-term must be again of the wrong sign (imaginary mass), such that there exists a possibility for the ϕ field to acquire a (gauge-dependent) vacuum-expectation value. At that point there is not yet the familiar bosons, but three weak isospin gauge bosons W^a and a hypercharge gauge boson B.

To work in unitary gauge it is best to rewrite the Higgs field in the form

$$\phi = e^{\frac{i\tau^a\alpha_a}{2}}\begin{pmatrix}0 \\ \rho\end{pmatrix}.$$

There are now the three α^a fields and the ρ field. Performing a gauge transformation such that the phase becomes exactly canceled, and setting $\rho = f + \eta$ with f constant makes the situation similar to the one in the Abelian Higgs model. Note that by a global gauge transformation the component with non-vanishing expectation value can be selected still at will. It is convenient to use the matrix representation for the Higgs field for this purpose, which after gauge-fixing is

$$X = f1 + \eta,$$

and thus the vacuum-expectation value is propertional to the unit matrix in this gauge.

The bilinear interaction term with the gauge fields then reads

$$\frac{g^2f^2}{2}W_\mu^a W_b^\mu \mathrm{tr}\tau^a 1\tau^b 1 + \frac{g'^2f^2}{2}B_\mu B^\mu \mathrm{tr}1^2 + \text{terms with } \eta.$$

At this point, it appears that both gauge symmetries are broken, and all four gauge bosons would be massive. However, this is not the case, as a simultaneous and opposite U(1) gauge transformation in the hypercharge and one proportionally to τ^3 in the weak isospin leaves the vacuum expectation value invariant. for this remember that they act on X as left multiplication and right multiplication, and thereby cancel. If the second term would be absent, then in the employed conventions $\mathrm{tr}\tau^a\tau^b = 1/2$ this yields a mass-term with equal mass $M_W = gf/2$ for all three gauge bosons of the weak isospin. As advertised, this is due to the then exact custodial symmetry. It corresponds to performing a global SU(2) left multiplication and right multiplication, which again leaves the vacuum expectation value invariant. Thus, there is a linking between global weak isospin and custodial symmetry. This is what is meant colloquial as the breaking of the ((electro)weak) gauge symmetry. The remaining diagonal subgroup of $\mathrm{SU}(2)_L^{\mathrm{local}} \times \mathrm{SU}(2)_R^{\mathrm{global}} \to \mathrm{SU}(2)$ is often also refereed to as the custodial symmetry . It is also the U(1) subgroup of this diagonal subgroup which will become the electromagnetic gauge symmetry below.

To make the unbroken diagonal U(1) subgroup manifest a change of variables can be used. Defining

$$\begin{aligned} A_\mu &= W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \\ Z_\mu &= W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W, \end{aligned}$$

which are the fields given the name of the photon A_μ and the Z boson Z_μ. The mixing parameter θ_W is the (Glashow-)Weinberg angle θ_W, and is given entirely in terms of the coupling constants g and g' as

$$\begin{aligned} \tan\theta_W &= \frac{g'}{g} \\ \cos\theta_W &= \frac{g}{\sqrt{g^2 + g'^2}} \\ \sin\theta_W &= \frac{g'}{\sqrt{g^2 + g'^2}}. \end{aligned}$$

Using the inverse transformations

$$\begin{aligned} W_\mu^3 &= A_\mu \sin\theta_W + Z_\mu \cos\theta_W \\ B_\mu &= A_\mu \cos\theta_W - Z_\mu \sin\theta_W \end{aligned}$$

yields that no longer a mass-term for the photon appears, due to the relative minus sign. However, the mass for the Z boson increases according to

$$M_Z \cos\theta_W = M_W.$$

The actual value depends on the relative strengths of the weak isospin and hypercharge interaction. In the standard model, this yields an increase of somewhat more than 10%. This is a change between the so-called charge eigenbasis, in which the hypercharge and the weak isospin interactions are separated, to a mass eigenbasis, in which the states are eigenstates of the mass operator. While technically not necessary, it is usually the mass eigenstates, especially the photon, which are easier to prepare experimentally. However, this implies that part of what is perceived as electric interactions is actually weak isospin interaction.

Of course, this changes also the form of the coupling for matter to the neutral fields. In particular, the SU(2) symmetry is no longer manifest, and the $W^{1,2}$ cannot be treated on the same footing as the neutral bosons. E. g., the neutral part of the coupling in the covariant derivative now takes the form

$$D_\mu^{(N)} = \partial_\mu + ig\sin\theta_W A_\mu \left(t_3 + \frac{y}{2}\right) + i\frac{g}{\cos\theta_W} Z_\mu \left(t_3\cos^2\theta_W - \frac{y}{2}\sin^2\theta_W\right).$$

Using the relation (4.1), it is possible to identify the conventional electric charge as

$$e = g\sin\theta_W,$$

i. e., the observed electric charge is smaller than the hypercharge. It should be noted that this also modifies the character of the interaction. While the interaction with the photon is purely vectorial, and the one with the $W^{1,2}$ bosons remains left-handed (axial-vector), the interaction with the Z boson is now a mixture of both, and the mixing is parametrized by the Weinberg angle.

In addition, the original interaction between the W^i entails now a tree-level interaction not only of the Z with $W^{1,2}$, but also of the photon. It is in this context useful to introduce the linear combination

$$W^\pm = \frac{1}{\sqrt{2}}\left(W^1 \mp iW^2\right)$$

as the new states no interaction with the photon with a coupling strength given by $\pm e$, while the coupling to the Z is reduced to $g\cos\theta_W$. Thus the 'electric' charges of the $W^\pm$ bosons is actually entirely a weak isospin interaction, and just stems from a switch to a mass eigenbasis. These electrically charged states have still the same mass M_W. However, their coupling to other particles via a covariant derivative is purely left-handed, and derived from the weak isospin. This also allows them to couple to formally electrically neutral states, like neutrinos.

4.4 Fermion masses

So far, the construction seems to reproduce very well and elegantly the observed phenomenology of the electroweak interactions.

However, there is one serious flaw. The Dirac equation for fermions, like leptons and quarks, has the form

$$0 = (i\gamma^\mu D_\mu - m)\psi = \left(i\gamma^\mu D_\mu + \frac{1-\gamma_5}{2}m + \frac{1+\gamma_5}{2}m\right)\psi.$$

The covariant derivative is a vector under a weak isospin gauge transformation, and so is the spinor $(1-\gamma_5)/2\psi$. However, the spinor $(1+\gamma_5)/2\psi$ is a singlet under such a gauge transformation. Hence, not all terms in the Dirac equation transform covariantly, and therefore weak isospin cannot be a symmetry for massive fermions. Another way of observing this is that the mass term for fermions in Lagrangian can be written as

$$\mathcal{L}_{m\psi} = m(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L),$$

and therefore cannot transform as a gauge singlet. Thus an ordinary tree-level mass term is forbidden in the standard model.

However, massive fermions must be accommodated in the theory: Since the experimentally observed quarks and (at least almost) all leptons have a mass, it is necessary to find a different mechanism which provides the fermions with a mass without spoiling the isoweak gauge invariance.

A possibility is by utilizing the BEH-effect also for the fermions and not only for the weak gauge bosons. By adding an interaction

$$\mathcal{L}_h = g_f\phi_k\bar{\psi}\left(\alpha^k_{ij}\frac{1-\gamma_5}{2} + \beta^k_{ij}\frac{1+\gamma_5}{2}\right)\psi + \left(g_f\phi_k\bar{\psi}\left(\alpha^k_{ij}\frac{1-\gamma_5}{2} + \beta^k_{ij}\frac{1+\gamma_5}{2}\right)\psi\right)^\dagger,$$

this is possible. The constant matrices α and β have to be chosen such that the terms become gauge-invariant. Their precise form will be given later. If, in this interaction, the Higgs field acquires a vacuum expectation value, $\phi = f+$quantum fluctuations, this term becomes an effective mass term for the fermions, and it is trivially gauge-invariant. Alongside with it comes then an interaction of Yukawa-type of the fermions with the Higgs-field. However, the interaction strength is not a free parameter of the theory, since the coupling constants are uniquely related to the tree-level mass m_f of the fermions by

$$g_f = \frac{\sqrt{2}m_f}{f} = \frac{e}{\sqrt{2}\sin\theta_W}\frac{m_f}{m_W}.$$

However, the 12 coupling constants for the three generations of quarks and leptons are not further constrained by the theory, introducing a large number of additional parameters in the theory. Though phenomenologically successful, this is the reason why many feel this type of description of the electroweak sector is insufficient. However, even if it would be incorrect after all, it is an acceptable description at energies accessible so far, and thus will be discussed further.

4.5 The Glashow-Salam-Weinberg theory in general

Combining the last three sections, the Glashow-Salam-Weinberg theory is of the following type: It is based on a gauge theory with a gauge group G, which is SU(2)×U(1). There are left-handed and right-handed fermions included, which belong to certain representations L^a and R^a of the gauge group G. These have been doublets for the left-handed quarks and leptons, and singlets for the right-handed quarks and leptons. There is also the scalar Higgs field which belongs to the representation P^a, again a doublet in the standard model.

The corresponding generators of the representation fulfill the commutation relations

$$
\begin{aligned}
[L^a, L^b] &= i f^{abc} L^c \\
\left[R^a, R^b\right] &= i f^{abc} R^c \\
\left[P^a, P^b\right] &= i f^{abc} P^c,
\end{aligned}
$$

with the structure constants f^{abc} of the gauge group G.

The most general form of the Lagrangian is therefore

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \bar{\psi}_i (i\gamma_\mu D^\mu_{ij} - M_{ij})\psi_j + \frac{1}{2} d^\mu_{ij}\phi^j d^{ik}_\mu \phi_k \\
&\quad -\frac{\lambda}{4}(\phi_i\phi^i)^2 - \mu^2 \phi_i\phi^i + g_f \bar{\psi}_i \phi_r \left(X^r_{ij} \frac{1-\gamma_5}{2} + Y^r_{ij} \frac{1+\gamma_5}{2} \right) \psi_j \qquad (4.4) \\
F^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu \\
D^{ij}_\mu &= \delta^{ij}\partial_\mu - igW^a_\mu \left(L^{ij}_a \frac{1-\gamma_5}{2} + R^{ij}_a \frac{1+\gamma_5}{2} \right) \\
d^{ij}_\mu &= \delta^{ij}\partial_\mu - igW^a_\mu P^{ij}_a .
\end{aligned}
$$

Note that the fields W^a_μ contain both the photon and the weak gauge bosons. Therefore, the index a runs from 1 to 4, with indices 1 to 3 from the SU(2) part, and 4 from the U(1) part. Consequently, all structure constants in which an index is 4 vanish. The parametrization of the broken part has not yet been performed. The Yukawa coupling to generate the fermion masses must also be gauge invariant. This can be achieved if the matrices X and Y fulfill the conditions

$$\begin{aligned}[L^a, X^r] &= X^s P^a_{sr} \\ [R^a, Y^r] &= Y^s P^a_{sr}.\end{aligned}$$

Since X and Y appear linearly in these conditions their overall scale is not fixed. This permits the different fermion species, though belonging to the same representation, to acquire different masses. That this is sufficient can be seen, e. g. for the right-handed coupling term by performing an infinitesimal gauge transformation

$$\begin{aligned}\delta\psi_i &= i\theta_a R^a_{ij}\psi^j \\ \delta\phi_r &= i\theta_a P^a_{rs}\phi^s,\end{aligned}$$

with the arbitrary infinitesimal transformation functions θ^a. The Yukawa term then transforms as follows

$$\begin{aligned}&\delta\left(g_F\bar{\psi}_i\phi_r Y^r_{ij}\frac{1+\gamma_5}{2}\right)\psi_j \\ = \; & i\theta^a g_f\left(-R^a_{ki}\bar{\psi}^k\phi_r Y^r_{ij}\frac{1+\gamma_5}{2}\psi_j + \bar{\psi}_i P^a_{rs} Y^r_{ij}\phi^s\frac{1+\gamma_5}{2}\psi_j + \bar{\psi}_i\phi_r Y^r_{ij}\frac{1+\gamma_5}{2}R^a_{jk}\psi^k\right) \\ = \; & i\theta^a g_f\bar{\psi}_i\phi_r\frac{1+\gamma_5}{2}\psi_j\left(Y^r_{ik}R^a_{kj} - R^{ik}_a Y^r_{kj} + P^a_{sr}Y^s_{ij}\right) \\ = \; & i\theta^a g_f\bar{\psi}_i\phi_r\frac{1+\gamma_5}{2}\psi_j\left([Y^r, R^a]_{ij} + P^a_{sr}Y^s_{ij}\right) = 0.\end{aligned}$$

Likewise the calculation proceeds for the left-hand case. Explicit representation of X and Y then depend on the chosen gauge group G.

To hid the symmetry, a shift of the scalar field by its vacuum expectation value can be performed.

4.6 The electroweak sector of the standard model

The Lagrangian (4.4) is transformed into the electroweak sector of the standard model by choosing its parameters appropriately. First of all, the vacuum expectation value for the scalar field is chosen to be $f/\sqrt{2}$. Its direction is chosen such that it is manifestly electrically neutral. That is provided by the requirement

$$Q\phi = 0 = \left(\frac{\tau_3}{2} + \frac{y_\phi}{2}\right)\phi = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = 0.$$

The field is then split as

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(f+\eta+i\chi) \end{pmatrix},$$

with $\phi^- = (\phi^+)^\dagger$, i. e., its hermitian conjugate. The fields ϕ^+ and ϕ^- carry integer electric charge plus and minus, while the fields η and χ are neutral. Since a non-vanishing value of f leaves only a U(1) symmetry manifest, $\phi^\pm$ and χ are would-be Goldstone bosons. This leads to the properties of the Higgs fields and vector bosons as discussed previously.

The fermions appear as left-handed doublets in three generations

$$\begin{aligned} L_i^L &= \begin{pmatrix} \nu_i^L \\ l_i^L \end{pmatrix} \\ Q_i^L &= \begin{pmatrix} u_i^L \\ d_i^L \end{pmatrix}, \end{aligned}$$

where i counts the generations, l are the leptons e, μ and τ, ν the corresponding neutrinos ν_e, ν_μ, ν_τ, u the up-type quarks u, c, t, and d the down-type quarks d, s, b. Correspondingly exist the right-handed singlet fields l_i^R, ν_i^R, u_i^R, and d_i^R. Using this basis the Yukawa interaction part reads

$$\mathcal{L}_Y = -\bar{L}_i^L G_{ij}^{lr} l_j^R \phi^r + \bar{L}_i^L G_{ij}^{\nu r} \nu_j^R \phi^r + \bar{Q}_i^L G_{ij}^{ur} u_j^R \phi^r + Q_i^L G_{ij}^{dr} d_j^R \phi^r + h.c..$$

The matrices G are obtained from X and Y upon entering the multiplet structure of the fields. These create the mass matrices

$$M_{ij}^l = \frac{1}{\sqrt{2}} G_{ij}^l f \qquad M_{ij}^\nu = \frac{1}{\sqrt{2}} G_{ij}^\nu f \qquad M_{ij}^u = \frac{1}{\sqrt{2}} G_{ij}^u f \qquad M_{ij}^d = \frac{1}{\sqrt{2}} G_{ij}^d f.$$

It is then possible to transform the fermion fields[2] f into eigenstates of these mass matrices, and thus mass eigenstates, by a unitary transformation

$$\begin{aligned} f_i^{fL} &= U_{ik}^{fL} f_k^L \\ f_i^{fR} &= U_{ik}^{fR} f_k^R, \end{aligned} \tag{4.5}$$

for left-handed and right-handed fermions respectively, and f numbers the fermion species l, ν, u, and d and i the generation. The fermion masses are therefore

$$m_{fi} = \frac{1}{\sqrt{2}} \sum_{km} U_{ik}^{fL} G_{km}^f (U^{fR})_{mi}^\dagger f.$$

[2] The vacuum expectation value f of the Higgs field, carrying no indices, should not be confused with the fermion fields f_i^f, which carry various indices, f denoting the fermion class.

In this basis the fermions are no longer charge eigenstates of the weak interaction, and thus the matrices U correspond to the CKM matrix in the quark sector and to the PMNS matrix in the lepton sector. Thus, these matrices are not an independent pheomena. Rather, they encode off-diagonal flavor interactions of the Higgs with the fermions while in the charge eigenbasis of the electroweak interactions. This also implies that CP violation originates in actuality from the Higgs-Yukawa sector, and not from the weak interactions, in contrast to the separate violations of P and C symmetry. In the limit of vanishing Yukawa couplings the standard model becomes CP-invariant at tree-level.

Furthermore, neutral interactions (or also called neutral currents) are not changing flavors as always combinations of type $U^{fL}(U^{fL})^\dagger$ appear, For flavor-changing (non-neutral) currents2 the matrices

$$\begin{aligned} V^q &= U^{uL}(U^{dL})^\dagger \\ V^l &= U^{\nu L}(U^{lL})^\dagger \end{aligned} \tag{4.6}$$

remain, providing the flavor mixing. Finally, the electric charge is given by

$$e = \sqrt{4\pi\alpha} = g' \sin\theta_W = g\cos\theta_W,$$

with the standard value $\alpha \approx 1/137$.

Putting everything together, the lengthy Lagrangian for the electroweak standard model emerges:

$$\begin{aligned}
\mathcal{L} &= \bar{f}_i^{rs}(i\gamma^\mu\partial_\mu - m_f)f_i - eQ_r\bar{f}_i^{rs}\gamma^\mu f_i^{rs}A_\mu \\
&+ \frac{e}{\sin\theta_W\cos\theta_W}(I^3_{Wr}\bar{f}_i^{rL}\gamma^\mu f_i^{rL} - \sin^2\theta_W Q_r\bar{f}_i^{rs}\gamma^\mu f_i^{rs})Z_\mu \\
&+ \frac{e}{\sqrt{2}\sin\theta_W}(\bar{f}_i^{rL}\gamma^\mu V^r_{ij}f_j^{rL}W_\mu^+ + \bar{f}_i^{rL}\gamma^\mu(V^r)^+_{ij}f_j^{rL}W_\mu^-) \\
&- \frac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W_\mu^-W_\nu^+ - W_\nu^-W_\mu^+)|^2 \\
&- \frac{1}{4}\left|\partial_\mu Z_\nu - \partial_\nu Z_\mu + ie\frac{\cos\theta_W}{\sin\theta_W}(W_\mu^-W_\nu^+ - W_\nu^-W_\mu^+)\right|^2 \\
&- \frac{1}{2}\left|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ - ie(W_\mu^+A_\nu - W_\nu^+A_\mu) + ie\frac{\cos\theta_W}{\sin\theta_W}(W_\mu^+Z_\nu - W_\nu^+Z_\mu)\right|^2 \\
&+ \frac{1}{2}\left|\partial_\mu(\eta + i\chi) - i\frac{e}{\sin\theta_W}W_\mu^-\phi^+ + iM_Z Z_\mu + \frac{ie}{2\cos\theta_W\sin\theta_W}Z_\mu(\eta + i\chi)\right|^2 \\
&+ |\partial_\mu\phi^+ + ieA_\mu\phi^+ - ie\frac{\cos^2\theta_W - \sin^2\theta_W}{2\cos\theta_W\sin\theta_W}Z_\mu\phi^+ - iM_W W_\mu^+ - \frac{ie}{2\sin\theta_W}W_\mu^+(\eta + i\chi)|^2 \\
&- f^2\eta^2 - \frac{ef^2}{\sin\theta_W M_W}\eta(\phi^-\phi^+ + \frac{1}{2}|\eta + i\chi|^2)
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&- \frac{e^2 f^2}{4\sin^2\theta_W M_W^2}(\phi^-\phi^+ + \frac{1}{2}|\eta + i\chi|^2)^2 \\
&- \frac{e m_{ri}}{2\sin\theta_W M_W}(\bar{f}_i^{rs} f_i^{rs}\eta - 2I_{Wr}^3 i\bar{f}_i^{rs}\gamma_5 f_i^{rs}\chi) \\
&+ \frac{e}{\sqrt{2}\sin\theta_W}\frac{m_{ri}}{M_W}(\bar{f}_i^{rR} V_{ij}^r f_j^{rL}\phi^+ + \bar{f}_i^{rL}(V^r)^+_{ij} f_j^{rR}\phi^-) \\
&+ \frac{e}{\sqrt{2}\sin\theta_W}\frac{m_{ri}}{M_W}(\bar{f}_i^{rL} V_{ij}^r f_j^{rR}\phi^+ + \bar{f}_i^{rR}(V^r)^+_{ij} f_j^{rL}\phi^-),
\end{aligned}
$$

where a sum over fermion species r is understood and I_W^3 is the corresponding weak isospin quantum number.

Note that this Lagrangian is invariant under the infinitesimal gauge transformation

$$
\begin{aligned}
A_\mu &\to A_\mu + \partial_\mu\theta^A + ie(W_\mu^+\theta^- - W_\mu^-\theta^+) \qquad (4.8)\\
Z_\mu &\to Z_\mu + \partial_\mu\theta^Z - ie\frac{\cos\theta_W}{\sin\theta_W}(W_\mu^+\theta^- - W_\mu^-\theta^+) \\
W_\mu^\pm &\to W_\mu^\pm + \partial_\mu\theta^\pm \mp i\frac{e}{\sin\theta_W}(W_\mu^\pm(\sin\theta_W\theta^A - \cos\theta_W\theta^Z) - (\sin\theta_W u - \cos\theta_W Z_\mu)\theta^\pm) \\
\eta &\to \eta + \frac{e}{2\sin\theta_W\cos\theta_W}\chi\theta^Z + i\frac{e}{2\sin\theta_W}(\phi^+\theta^- - \phi^-\theta^+) \\
\chi &\to \chi - \frac{e}{2\sin\theta_W\cos\theta_W}(v+\eta)\theta^Z + \frac{e}{2\sin\theta_W}(\phi^+\theta^- + \phi^-\theta^+) \\
\phi^\pm &\to \phi^\pm \mp ie\phi^\pm\left(\theta^A + \frac{\sin^2\theta_W - \cos^2\theta_W}{2\cos\theta_W\sin\theta_W}\theta^Z\right) \pm \frac{ie}{2\sin\theta_W}(f + \eta \pm i\chi)\theta^\pm \\
f_i^{f\pm L} &\to f_i^{f\pm L} - ie\left(Q_i^\pm\theta^A + \frac{\sin\theta_W}{\cos\theta_W}\left(Q_i^f \mp \frac{1}{2\sin^2\theta_W}\right)\theta^Z\right)f_i^{\pm L} + \frac{ie}{\sqrt{2}\sin\theta_W}\theta^\pm V_{ij}^\pm f_j^{\pm L} \\
f_i^{fR} &\to -ieQ_i^f\left(\theta^A + \frac{\sin\theta_W}{\cos\theta_W}\theta^Z\right)f_i^{fR}. \qquad (4.9)
\end{aligned}
$$

The $\pm$ index for the left-handed fermion fields counts the isospin directions. The infinitesimal gauge functions θ^α are determined from the underlying weak isospin θ^i and hypercharge θ^Y gauge transformations by

$$
\begin{aligned}
\theta^\pm &= \frac{1}{g'}(\theta^1 \mp i\theta^2) \\
\theta^A &= \frac{1}{g}\cos\theta_W\theta^Y - \frac{1}{g'}\sin\theta_W\theta^3 \\
\theta^Z &= \frac{1}{g'}\cos\theta_W\theta^3 + \frac{1}{g}\sin\theta_W\theta^Y.
\end{aligned}
$$

It is now straightforward to upgrade the Lagrangian of the electroweak sector of the standard model (4.7) to the full Lagrangian of the standard model by adding the one for the strong interactions, QCD,

$$\begin{aligned}\mathcal{L}_s &= -\frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu} - g\bar{f}_i^{rs}\gamma^\mu\omega^a f_i^{rs}G^a_\mu \\ G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g'' h^{abc}G^b_\mu G^c_\nu,\end{aligned}$$

where the generators ω^a and the structure constants h^{abc} belong to the gauge group of the strong interactions is the so-called color group SU(3)[3], and g'' is the corresponding coupling constant. G^a_μ are the gauge fields of the gluons, and the fermions now have also an (implicit) vector structure in the strong-space, making them three-dimensional color vectors.

4.7 Majorana neutrinos

So far the leptons and neutrinos have been treated in the same way as the quarks. This consistent with all known results. However, there is a pecularity for neutrinos. Right-handed neutrinos are not charged under any of the gauge interactions, and only interact with the standard model by Yukawa interactions with the Higgs. As the size of the Yukawa couplings for the neutrinos is exceedingly small, there has not been any observation of right-handed neutrinos, in contrast for all other standard model particles. Hence, they are strictly speaking not necessary.

This leaves the question of how to then achieve the neutrino masses, which have been observed. The solution to this are so-called Majorana neutrinos. More appropriately would be to call them Weyl fermions, but in four dimensions this does not matter. In such a case only a two-dimensional Weyl spinor is necessary, i. e. only the left-handed component of the usual Dirac spinor is used. The original Dirac, or actually Majorana, spinor can be reconstructed as

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ -\psi^{2*} \\ \psi^{1*} \end{pmatrix}.$$

The physical content is made manifest by performing a charge conjugation

$$C\Psi = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ -i\sigma_2\psi^* \end{pmatrix}^* = \begin{pmatrix} \psi \\ -i\sigma_2\psi^* \end{pmatrix} = \Psi. \qquad (4.10)$$

Thus, such a particle is its own anti-particle. It is in this context important that the weak isospin group, SU(2), is pseudo-real. Hence, particles and anti-particles are actually

[3] Actually SU(3)/Z_3, similar to the case of the weak isospin group.

related. A similar construction would not be possible for a quark. A scalar mass-term can be build from such a Weyl fermion as

$$m\psi^T(i\sigma_2)\psi$$

where σ_2 is the second Pauli matrix, and which is gauge-invariant. Note that in this case no Yukawa interaction of the neutrino with the Higgs is possible. The mass of such a Majorana neutrino has therefore a distinct other origin as those for Dirac neutrinos.

Because of the property (4.10), a propagating left-handed neutrino can convert into an anti-neutrino. This gives rise to new phenomena. An especially important signature is the so-called neutrinoless double β-decay. In this process, the anti-neutrino emitted during a β-decay of a nucleus converts to an neutrino, and allows then for second β-decay, in which the neutrino is absorbed by a neutron to emit a second electron and become a proton. Hence the name for the process. Similar phenomena can occur also in high-eenrgetic processes like the deacy of $W^\pm$ bosons, and thus yield also distinct signatures like same-sign lepton pairs in collider experiemnts. However, due to the feeble interaction of neutrinos it is kinematically complicated to show beyond doubt that not just a low-energy neutrino has escaped undetected.

Current experimental results remain consistent with both the existence and absence of a right-handed neutrino. It is currently not clear how long experiments will need to decide the matter. In the following the existence of a right-handed neutrino is assumed, for simplicity. However, everything could also be done with Majorana neutrinos, usually with few changes.

Chapter 5

Quantization

Before entering the realm of actual calculations in the electroweak theory, it is necessary to discuss the quantization, and thus also renormalization, of theories with hidden symmetries. This will be done here. After this, the quantized version of the Glashow-Salam-Weinberg theory is at our disposal, and can be used for calculations of phenomenology.

5.1 Gauge-fixing

5.1.1 Case with manifest symmetry

To understand the differences in case of a Brout-Englert-Higgs effect, it is useful to rehearse briefly the case without.

The naive quantization for a non-Abelian Yang-Mills theory with classical Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\end{aligned}$$

proceeds by writing down the generating functional

$$\begin{aligned}Z[J^a_\mu] &= \int \mathcal{D}A^a_\mu \exp\left(iS[A^a_\mu] + i\int d^4x J^a_\mu A^\mu_a\right)\\ S[A^a_\mu] &= \int d^4x\mathcal{L}.\end{aligned}$$

In the case of an Abelian field with $f^{abc} = 0$, this integral reduces to a Gaussian one. Hence, it should be possible to integrate it. It takes the form

$$Z[J^a_\mu] = \int \mathcal{D}A^a_\mu \exp\left(i\int d^4x \left(\frac{1}{2}A^\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + J^\mu A_\mu\right)\right).$$

However, it is not possible to perform this integral, since this would require the matrix $g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ to be invertible, which is not the case. This can be seen directly by the fact that its momentum-space version $\delta_{\mu\nu}k^2 - k_\mu k_\nu$ is a projection operator which vanishes when contracted with k_μ.

An alternative way to see this is to note that any gauge transformation

$$A_\mu^a \to A_\mu^{ag} = A_\mu^a + D_\mu^{ab}\theta^b \tag{5.1}$$

with θ^b arbitrary leaves S invariant. Thus, there are flat directions of the integral, namely along the so generated so-called gauge orbit, and thus the integral diverges. There are only two possibilities to escape. The one is to perform the quantization on a discrete space-time grid in a finite volume, as is done in lattice gauge theory, determine observables and only after this take the continuum limit. Making thereby everything finite permits also to make the flat directions finite, and hence tractable.

The other is by performing gauge-fixing, i. e., cutting off the flat directions of the integral. Here, as the aim is to work directly in the continuum, the latter possibility will be the only one[1].

In the following only the procedure for the perturbative case will be discussed. The extension to the non-perturbative case is far from obvious for non-Abelian gauge theories, due to the presence of the Gribov-Singer ambiguity. In case of Yang-Mills theory, some possibilities exist to resolve this issue, though a formal proof is yet lacking. How this can be done in the case without manifest gauge symmetry is yet unknown, but empirically it turns out to be a less severe problem, and can therefore be more comfortably ignored for know. A few more words on this will be discussed in section 6.2.

Select thus a gauge condition $C^a(A_\mu^a, x) = 0$ which perturbatively selects exactly one gauge copy. I. e., for a set of gauge-fields related by gauge transformations (5.1) there is one, and only one, which satisfies the condition C^a. An example of such a condition is, e. g., the Landau gauge $C^a = \partial^\mu A_\mu^a$, but so are the 't Hooft gauge condition (3.19) or the unitary gauge (3.17) condition.

To make the path integral well-defined, it is necessary to factor off the irrelevant number of field configurations equivalent under the gauge transformation (5.1), and just remain with one representative for each physically inequivalent field configuration. An alternative, given below by covariant gauges, is to average over all copies with a uniquely defined weight for each gauge copy.

[1]Note that there are, in principle, other possibilities also in the continuum. However, none of them is so far useful for practical calculations in the non-Abelian case.

To do this consider the functional generalization of the Dirac-δ function. The expression

$$\Delta^{-1}[A_\mu^a] = \int \mathcal{D}g\delta(C^a[A_\mu^{ag}])$$

contains an integration over all gauge-transformations g for a fixed physical field configuration A_μ^a, but by the δ-function only the weight of the one configuration satisfying the gauge condition is selected. The measure is taken to be the invariant Haar-measure. Hence, when performing the change of variables $g \to gg'$ with some group element g' it remains unchanged by definition. As a consequence, Δ is actually gauge-invariant. Evaluating it at the gauge-transformed configuration $A_\mu^{ag'}$ yields

$$\begin{aligned}\Delta[A_\mu^{ag'}]^{-1} &= \int \mathcal{D}g\delta(C^a[A_\mu^{agg'}]) = \int \mathcal{D}(gg'^{-1})\delta(C^a[A_\mu^{ag}]) \\ &= \int \mathcal{D}g\delta(C^a[A_\mu^{ag}]) = \Delta[A_\mu^a]^{-1}.\end{aligned}$$

In perturbation theory it is always possible to invert Δ. Thus, the relation

$$1 = \Delta[A_\mu^a] \int \mathcal{D}g\delta(C^a[A_\mu^{ag}])$$

is meaningful. Beyond perturbation theory this is in general not the case, and a more subtle construction is required.

Inserting this into the functional integral yields

$$\begin{aligned}Z &= \int \mathcal{D}A_\mu^a \Delta[A_\mu^a] \int \mathcal{D}g\delta(C^a[A_\mu^{ag}]) \exp(iS[A_\mu^a]) \\ &= \int \mathcal{D}g \int \mathcal{D}A_\mu^{ag'} \Delta[A_\mu^{ag'}]\delta(C^a[A_\mu^{agg'}]) \exp(iS[A_\mu^{ag'}]) \\ &= \int \mathcal{D}g \int \mathcal{D}A_\mu^a \Delta[A_\mu^a]\delta(C^a[A_\mu^a]) \exp(iS[A_\mu^a])\end{aligned}$$

where in the last line the inner variables of integration have been changed from $A_\mu^{ag'}$ to $A_\mu^{ag'^{-1}g^{-1}}$ and it has been used that all expressions except the δ-function are invariant. Hence, the group measure integral is not influencing anymore the remaining integral, and contributes only a factor, which can be removed by appropriate normalization of the functional integral. It would have been possible to also replace the action by any gauge-invariant functional, in particular expressions involving some observable f in the form $f[A_\mu^a] \exp(iS[A_\mu^a])$. Thus, gauge-fixing is not affecting the value of gauge-invariant observables. Due to the δ-function, on the other hand, now only one gauge-equivalent field configuration contributes, making the functional integral well-defined.

It remains to clarify the role of the functional Δ. Select an infinitesimal gauge transformation $g = 1 + i\theta^a\tau^a$ with τ^a the generators of the Lie algebra. Since only perturbative calculations are done, the measure in the defining integral for Δ can be replaced by an integration over θ^a

$$\Delta[A_\mu^a]^{-1} = \int \mathcal{D}\theta^a \delta(C^a[A_\mu^{a\theta}]).$$

In perturbation theory it is always possible to resolve the condition $C^a[A_\mu^{a\theta}] = 0$ to obtain θ^a as a function of C^a. Hence, by exchanging C and θ as variables of integration yields

$$\Delta[A_\mu^a]^{-1} = \int \mathcal{D}C^a \left(\det \frac{\delta C^b}{\delta \theta^c}\right) \delta(C^a) = \left(\det \frac{\delta C^b}{\delta \theta^c}\right)_{C^a=0},$$

where it has been used that when resolving θ for C inside an expression for C, like in the δ-function, any modification will be of higher order in C and can therefore be dropped in perturbation theory. The appearing determinant is just the corresponding Jacobian. Thus, the function Δ is given by

$$\Delta[A_\mu^a]^{-1} = \left(\det \frac{\delta C^a(x)}{\delta \theta^b(y)}\right)_{C^a=0} = \det M^{ab}(x,y). \tag{5.2}$$

The Jacobian has the name Faddeev-Popov operator, the determinant goes by the name of Faddeev-Popov determinant2.

To get a more explicit expression it is useful to use the chain rule

$$\begin{aligned} M^{ab}(x,y) &= \frac{\delta C^a(x)}{\delta \theta^b(y)} = \int d^4z \frac{\delta C^a(x)}{\delta A_\mu^c(z)} \frac{\delta A_\mu^c(z)}{\delta \theta^b(y)} \\ &= \int d^4z \frac{\delta C^a(x)}{\delta A_\mu^c(z)} D_\mu^{cb} \delta(y-z) = \frac{\delta C^a(x)}{\delta A_\mu^c(y)} D_\mu^{cb}(y). \end{aligned}$$

If the condition C^a depends on more than one field, it is necessary to include this explicitly. In this case, the chain rule has a term for all fields on which C^a depends, multiplied by their variation with a gauge transformation,

$$M^{ab}(x,y) = \frac{\delta C^a(x)}{\delta \theta^b(y)} = \int d^4z \sum_{ij} \frac{\delta C^a(x)}{\delta \omega_j^i(z)} \frac{\delta \omega_j^i(z)}{\delta \theta^b(y)}.$$

In this case i counts the field-type, while j is a multi-index, encompassing gauge indices, Lorentz indices etc..

To proceed further, a choice of C^a is necessary. Choosing, e. g., the Landau gauge $C^a = \partial^\mu A_\mu^a = 0$ yields

$$M^{ab}(x,y) = \partial_\mu D_{ab}^\mu \delta(x-y).$$

Due to the presence of the δ-function the functional Δ can then be replaced by $\det M^{ab}$ in the path integral.

To obtain the standard class of covariant gauges, select the condition $C^a = \partial_\mu A_\mu^a + \Lambda^a$ for some arbitrary function Λ^a. In general, this will make the Lorentz symmetry not manifest. This can be recovered by integrating the path integral over all possible values of Λ with some arbitrary weight function. Since the path integral will not depend on Λ, as this is a gauge choice, the integration is only an arbitrary normalization. The path integral then takes the form

$$\begin{aligned} Z &= \int \mathcal{D}\Lambda^a \mathcal{D}A_\mu^a \exp\left(-\frac{i}{2\xi}\int d^4x \Lambda^a \Lambda_a\right) \det M \delta(C) \exp(iS) \\ &= \int \mathcal{D}A_\mu^a \det M \exp\left(iS - \frac{i}{2\xi}\int d^4x C^a C_a\right), \end{aligned} \tag{5.3}$$

where the δ-function has been used in the second step. Note that for any local expression in the fields C^a can be replaced by $C^a - \Lambda^a$ freely, as this corresponds only to terms linear in A_μ^a or constants, and thus can always be dropped.

It is furthermore possible to recast the determinant also in the form of an exponential. Using the rules of Grassmann numbers it follows immediately that

$$\det M \sim \int \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\left(-i\int d^4x d^4y \bar{c}^a(x) M^{ab}(x,y) c^b(y)\right),$$

where the Faddeev-Popov ghost fields c and $\bar{c}$ are Grassmann-valued scalar fields. Since these are just auxiliary fields, this is not at odds with the spin-statistics theorem. The fields are in general gauges not related, but may be so in particular gauges. This is, e. g., the case in Landau gauge where there exists an associated symmetry. If the condition C^a is local in the fields, the Faddeev-Popov operator will be proportional to $\delta(x-y)$, and this ghost term will become local.

It is furthermore often useful to introduce an additional auxiliary field, the Nakanishi-Lautrup field b^a. This is obtained by rewriting

$$\exp\left(-\frac{i}{2\xi}\int d^4x C^a C_a\right) = \int \mathcal{D}b^a \exp\left(-i\int d^4x \left(\frac{\xi}{2} b^a b_a + b_a C^a\right)\right).$$

The total expression reads

$$Z = \int \mathcal{D}A_\mu^a \mathcal{D}b^a \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\left(iS - i\int d^4x \left(\frac{\xi}{2} b^a b_a + b_a C^a\right) - \int d^4x d^4y \bar{c}^a(x) M^{ab}(x,y) c^b(y)\right).$$

Choosing the gauge $C^a = \partial^\mu A_\mu^a = 0$, this takes the form

$$Z = \int \mathcal{D}A_\mu^a \mathcal{D}b^a \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\left(iS - i\int d^4x \left(\frac{\xi}{2} b^a b_a + b^a \partial_\mu A_a^\mu\right) - i\int d^4x \bar{c}^a \partial^\mu D_\mu^{ab} c^b\right).$$

Furthermore, the ever-so popular Landau gauge corresponds to the limit $\xi \to 0$, as this is corresponding to the case where all of the weight of the weight-function is concentrated only on the gauge copy satisfying $\partial^\mu A_\mu^a = 0$. However, in principle this limit may only be taken at the end of the calculation.

It is furthermore important that if the gauge group is Abelian, in particular U(1), then the gauge transformation of the gauge fields reduces as

$$D_\mu^{ab}\phi^b \to \partial_\mu \phi,$$

and thus the ghost term takes the form

$$-i\int d^4x \bar{c}^a \partial^2 c^a.$$

Hence, the ghosts decouple, and will not take part in any dynamical calculations. However, their contribution can still be important, e. g., in thermodynamics. The decoupling of the ghost is not a universal statement. Choosing a condition which is not linear in the gauge fields will also in an Abelian theory introduce interactions with ghosts. Furthermore, from the sign of this term it is also visible that the kinetic term of the ghosts has the wrong sign compared to ordinary scalars, a sign of their unphysical spin-statistic relation.

Note that the program remains unchanged in the presence of matter fields, as long as the matter fields are not appearing in the gauge condition. Without hidden symmetries, there is usually little advantage to do so, but this changes fundamentally once a symmetry should be hidden.

5.1.2 Abelian case with hidden symmetry

In quantized calculations using perturbation theory the 't Hooft gauge (3.19) is most convenient, as renormalizability is manifest order by order of perturbation theory, and also beyond perturbation theory. The simplest example is the case of an (unmixed) Abelian gauge theory.

Splitting at the classical level the real Higgs doublet as $(\phi, f+\eta)$, where the fields ϕ and η fluctuate and f is the expectation value of the vacuum, the Lagrangian reads

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_\mu F^\mu + \frac{(gf)^2}{2}A_\mu A^\mu + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \mu^2\eta^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - gfA^\mu\partial_\mu\phi \\ &\quad +2igA^\mu(\phi\partial_\mu\phi + \eta\partial_\mu\eta) - 2g(gf)\eta A_\mu A^\mu - g^2(\phi^2+\eta^2)A_\mu A^\mu - \frac{1}{2}\frac{\mu^2}{f^2}(\phi^2+\eta^2)^2.\end{aligned}$$

The 't Hooft gauge is exactly the gauge in which the bilinear mixing part is removed from the Lagrangian. In the Abelian case this is achieved by

$$\begin{aligned} C[A_\mu,\phi] &= \partial^\mu A_\mu + \xi(gf)\phi \\ \mathcal{L}_f &= -\frac{1}{2\xi}(\partial^\mu A_\mu + \xi(gf)\phi)^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (gf)A^\mu\partial_\mu\phi - \frac{\xi}{2}(gf)^2\phi^2, \end{aligned}$$

where a partial integration has been performed. It should be noted that the gauge-parameter ξ now enters twice. Once by parameterizing the width of the averaging functional and once in the gauge condition itself. In principle it would be permitted to choose a different parameter in both cases, but then the mixing terms would not be canceled. In this gauge the Goldstone boson becomes in general massive, exhibiting clearly that the Goldstone theorem is not applying to the case of a gauge theory.

Furthermore, the limit $\xi \to 0$ corresponds to the case of ordinary Landau gauge with the mixing term and also the mass-shift for the ϕ-field due to the gauge-fixing removed from the Lagrangian. Thus, in the Landau gauge in contrast to other covariant gauges the Goldstone boson remains massless at tree-level[2].

This can already be read off from the tree-level propagators

$$\begin{aligned} D_{\eta\eta} &= \frac{i}{k^2-\mu^2+i\epsilon} \\ D_{\phi\phi} &= \frac{i}{k^2-\xi(gf)^2+i\epsilon} \\ D_{\mu\nu} &= i\left(g_{\mu\nu} + \frac{(1-\xi)k_\mu k_\nu}{k^2-\xi(gf)^2}\right)\frac{1}{k^2-(gf)^2+i\epsilon}. \end{aligned}$$

Thus, in the Landau gauge the Goldstone field ϕ becomes massless while the gauge boson field remains massive. However, the corresponding propagator reduces to the one of a massless particle in the high-energy limit in contrast to the naive one. Hence renormalizability remains manifest.

The ghost contribution now reads[3]

$$\mathcal{L}_g = -\bar{c}(\partial^2 + \xi(gf)^2 + \xi g(gf)\eta)c.$$

Thus the ghosts are massive with the same mass as for the Goldstone field, visible from the propagator

$$D_{\bar{c}c} = -\frac{i}{k^2-\xi(gf)^2+i\epsilon}$$

[2]Some subtleties appear here when not constructing the Landau gauge as a limit $\xi \to 0$. This will be discussed later.

[3]Note that a phase transformation mixes real and imaginary parts of a complex field.

It is this precise relation which guarantees the cancellation of both unphysical degrees of freedom in any process. Note that even though the group is Abelian the ghosts no longer decouple in this gauge. Only in the Landau gauge the then massless ghost will decouple.

Thus, all propagators behave as $1/k^2$ at large k, and hence the theory in this gauge is, indeed, superficially renormalizable.

5.1.3 Non-Abelian case with hidden symmetry

The non-Abelian case proceeds in much the same way as the Abelian case. Again, there appears a mixing term

$$-ig(\partial_\mu\phi^i)A_a^\mu(T_{ij}^a f_j),$$

which depends on the representation of the scalar field as represented by the coupling matrices T^a and the condensate f_i, which can be turned in any arbitrary direction in the internal gauge space by a global gauge transformation. Similar to the Abelian case the gauge fixing Lagrangian can be written as

$$\mathcal{L}_f = -\frac{1}{2\xi}(\partial^\mu A_\mu^a + i\xi g\phi_i T_{ij}^a f_j)^2,$$

which removes the mixing at tree-level directly. In contrast to the Abelian case the masses for the Goldstone bosons are now determined by a mass matrix coming from the part quadratic in the Goldstone field of the gauge-fixing term

$$\frac{1}{2}M_{ij}^2\phi^i\phi^j = -\frac{1}{2}g^2(T_{ik}^a f^k f^l T_{alj})\phi_i\phi_j.$$

As a consequence only the Goldstone bosons belonging to a hidden 'direction' acquire a mass. As in the previous case the masses appearing equal the one for the gauge bosons up to a factor of $\sqrt{\xi}$.

In essentially the same way the ghost part of the Lagrangian is

$$\mathcal{L}_g = -\bar{c}^a(\partial_\mu D_{ab}^\mu - \xi g^2 f^i T_a^{ij} T_b^{jk}(f_k + \eta_k))c^b.$$

Similarly to the previous case the ghosts pick up a mass of the same size as for the Goldstone bosons. Furthermore, again an additional direct interaction with the remaining (Higgs) boson is present. As in the Abelian case, in the Landau gauge limit both effects cease and the ghost part takes the same form as in the theory with manifest symmetry.

The appearance of the same mass, up to factors of ξ, for the longitudinal gauge bosons, the ghosts, and the would-be Goldstone bosons is of central importance for the cancellations of unphysical poles in the S-matrix, and is seen to happen order-by-order in perturbation theory.

Another interesting feature of the 't Hooft gauge is that in the limit $\xi \to \infty$ the unitary gauge condition (3.17) is recovered. This gauge is therefore a limiting case in which the perturbative physical spectrum becomes manifest while renormalizability at finite orders of perturbation theory is lost. The first effect can be directly seen from the fact that the tree-level masses of ghosts and Goldstone bosons then diverge and thus these degrees of freedom decouple, as a consequence of the Appelquist-Carrazone theorem.

It should be noted that when gauge-fixing the electroweak standard model two gauge-fixing prescriptions are necessary. One fixes the manifest U(1)-degree of freedom, corresponding to QED. In that case, a standard gauge-fixing prescription like Feynman gauge with decoupling ghosts is very convenient. The other one fixes the non-Abelian SU(2) part of the weak interactions.

5.1.4 Gauge-fixing the electroweak standard model

The situation in the electroweak standard model is a bit more involved due to presence of mixing. To implement the 't Hooft gauge the gauge-fixing conditions

$$\begin{aligned} C^{\pm} &= \partial^\mu W_\mu^\pm \mp iM_W\xi_W\phi^\pm = 0 \\ C^Z &= \partial^\mu Z_\mu - M_Z\xi_Z\chi = 0 \\ C^A &= \partial_\mu A_\mu = 0 \end{aligned} \tag{5.4}$$

are chosen. Therefore, there are three independent gauge-fixing parameters ξ_Z, ξ_W, and ξ_A. It is not necessary to split the ξ_W parameter further, as particle-antiparticle symmetry permits the cancellation of all mixing terms immediately by the gauge-fixing Lagrangian

$$\mathcal{L}_f = -\frac{1}{2\xi_A}C^{A2} - \frac{1}{2\xi_Z}C^{Z2} - \frac{1}{\xi_W}C^+C^-.$$

As a consequence, all mixed contributions vanish.

The corresponding ghost contribution has to determined a little bit carefully. Since the gauge transformation (4.9) mixes the field, this must be taken into account. Also, the gauge-fixing condition involve the scalar fields, which also therefore have to be included. Denoting a general gauge boson as V_μ^a with $a = A$, Z, and $\pm$ yields

$$\mathcal{L}_g = -\int d^4z d^4y \bar{u}^a(x) \left(\frac{\delta C^a(x)}{\delta V_\nu^c(z)} \frac{\delta V_\nu^c}{\delta\theta^b(y)} + \frac{\delta C^a(x)}{\delta\phi^c(z)} \frac{\delta\phi^c(z)}{\delta\theta^b(y)} \right) u^b(y).$$

The corresponding ghost Lagrangian for the electroweak standard model then takes the lengthy form

$$
\begin{aligned}
\mathcal{L}_g = & -\bar{u}^+(\partial^2+\xi_W M_W^2)u^+ + ie(\partial_\mu\bar{u}^+)\left(A_\mu - \frac{\cos\theta_W}{\sin\theta_W}Z_\mu\right)u^+ \\
& -ie(\partial^\mu\bar{u}^+)W_\mu^+\left(u^A - \frac{\cos\theta_W}{\sin\theta_W}u^Z\right) + \bar{u}^-(\partial^2+\xi_W M_W^2)u^- \\
& -eM_W\xi_W\bar{u}^+\left(\frac{\eta+i\chi}{2\sin\theta_W}u^+ - \phi^+\left(u^A - \frac{\cos^2\theta_W-\sin^2\theta_W}{2\cos\theta_W\sin\theta_W}u^Z\right)\right) \\
& +ie(\partial_\mu\bar{u}^-)\left(A_\mu - \frac{\cos\theta_W}{\sin\theta_W}Z_\mu\right)u^- - ie(\partial^\mu\bar{u}^-)W_\mu^-\left(u^A - \frac{\cos\theta_W}{\sin\theta_W}u^Z\right) \\
& +eM_W\xi_W\bar{u}^-\left(\frac{\eta-i\chi}{2\sin\theta_W}u^- - \phi^-\left(u^A - \frac{\cos^2\theta_W-\sin^2\theta_W}{2\cos\theta_W\sin\theta_W}u^Z\right)\right) \\
& +\bar{u}^Z(\partial^2+\xi_Z M_Z^2)u^Z - ie\frac{\cos\theta_W}{\sin\theta_W}(\partial^\mu\bar{u}^Z)(W_\mu^+u^- - W_\mu^-u^+) - \bar{u}^A\partial^2u^A \\
& -eM_Z\xi_Z\bar{u}^Z\left(\frac{\eta u^Z}{2\cos\theta_W\sin\theta_W} - \frac{\phi^+u^- + \phi^-u^+}{2\sin\theta_W}\right) + ie(\partial^\mu\bar{u}^A)(W_\mu^+u^- - W_\mu^-u^+).
\end{aligned} \tag{5.5}
$$

It should be remarked that neither the Abelian ghost u^A nor the photon A_μ decouples from this dynamics, as a consequence of mixing. Furthermore, the masses and couplings are proportional to the gauge parameters, and will therefore vanish in the case of Landau gauge. Another particular useful gauge is the 't Hooft-Feynman gauge in which all $\xi_a = 1$. In this gauge, the vector boson propagator becomes particularly simple

$$
D^a_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2 - M_a + i\epsilon}.
$$

It is therefore in widespread use in perturbative calculations, though other properties may be more simple in Landau gauge or other gauges.

5.2 Feynman rules

After gauge-fixing, the Feynman rules can be obtained in the usual way. In particular, they read:

- Draw for the process in question all topological inequivalent diagrams, up to a given number of each type of vertex, which are both connected (no graph may consists of separate subgraphs without connection) and 1-particle irreducible (no graph may be separated in unconnected subgraphs by cutting at most one line).

- Each diagram can be translated into a mathematical expression by the following rules

- Follow an external line. For each vertex encountered, write the corresponding mathematical form of the vertex, with indices connected to the corresponding lines attached to the vertex. For each line connecting two vertices, write a propagator between the two vertices. Its momentum is determined by momentum conservation at the vertices. Its indices are joined to the vertex its emerges from to the vertex where it ends.
- Any loop in the graph will produce by this one momentum which is not external. Integrate over this (and divide by appropriate factors of 2π, depending on convention, here $(2\pi)^4$ in four dimensions).
- Multiply for each closed loop of fermions with -1.
- Divide by a symmetry factor which counts all equivalent ways of drawing the diagram.

• Add all the diagrams belonging to a process, and evaluate the expression.

There are two mathematical quantities which are needed for the calculations. These are the tree-level propagators and the vertices. Given a Lagrangian, these can be calculated simply by functional derivatives. For the propagator D of fields A, B, including their indices, this is obtained by

$$(\Gamma^{AB}+i\epsilon)\delta(p-q) = (D^{AB}(p)^{-1}+i\epsilon)\delta(p-q) = -i\int dydze^{ipy-iqz}\left(\frac{\delta^2}{\delta A(y)\delta B(z)}i\int d^4x\mathcal{L}\right)_{\alpha=0},$$

where α denotes the set of all fields. The calculation of the vertices is somewhat simpler as no inversion is included. Thus for three-point vertices the formula is

$$\begin{aligned}&\Gamma^{ABC}(p,q,k=-p-q)\delta(p+q+k)\\&= -i\left(\int dydzdwe^{ipy+iqz+ikw}\left(\frac{\delta^3}{\delta A(y)\delta B(z)\delta C(w)}i\int d^4x\mathcal{L}\right)_{\alpha=0}\right),\end{aligned}$$

and correspondingly for four-point vertices

$$\begin{aligned}&\Gamma^{ABCD}(p,q,k,l=-p-q-k)\delta(p+q+k+l)\\&= -i\left(\int dydzdwdve^{ipy+iqz+ikw+ivl}\left(\frac{\delta^4}{\delta A(y)\delta B(z)\delta C(w)\delta D(v)}i\int d^4x\mathcal{L}\right)_{\alpha=0}\right).\end{aligned}$$

By explicit calculation, this permits to determine the Feynman rules for the electroweak standard model. For the explicit versions to be used here, a covariant gauge is chosen for the QED part, and a 't Hooft gauge for the weak interaction part.

5.2.1 Propagators

There are four different ones. The first is the gauge boson V propagator

$$D_{\mu\nu}^{VV} = -\frac{-ig_{\mu\nu}}{k^2 - M_V^2 + i\epsilon} + \frac{i(1-\xi_V)k_\mu k_\nu}{(k^2 - M_V^2)(k^2 - \xi_V M_V^2)},$$

where M_A for the photon is zero, and there are different ξ values for the Abelian and non-Abelian part. Also, the ξ are different for the $W^\pm$ and Z bosons due to mixing, effectively giving rise to three ξ values, ξ_W, ξ_Z, and ξ_A. The corresponding masses appearing here and below can be taken form chapter 4. Note that the color indices are absorbed into the V index.

The ghost propagator is given by

$$D^{\bar{G}G} = \frac{i}{k^2 - M_G^2 + i\epsilon}.$$

Note that the mass is given uniquely in terms of the gauge boson mass. This also applies to the scalar propagator of the Higgs and the would-be Goldstone bosons S

$$D^{SS} = \frac{i\delta^{ss}}{k^2 - M_S^2 + i\epsilon},$$

where of course the mass of the Higgs is independent of the gauge boson masses.

The fermion propagator is given as a Dirac-matrix by

$$D^{ffij} = \frac{i(\gamma_\mu p^\mu + m_F)\delta^{ij}}{p^2 - m_F^2 + i\epsilon},$$

where the tree-level masses are determined by the Higgs coupling, and the propagator is diagonal in flavor and charge space.

5.2.2 Vertices

5.2.2.1 Three-point vertices

In the following the short form $+$ and $-$ are used as indices for $W^\pm$ and for the would-be Goldstone bosons $\phi^\pm$.

The three-gauge-boson vertex is given by

$$\Gamma_{\mu\nu\rho}^{VVV}(p,q,k) = ieC^{VVV}(g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-k)_\mu + g_{\rho\mu}(k-p)_\nu)$$

The coupling matrix C here and henceforth absorbs the color indices. It is in this case totally antisymmetric with non-zero values

$$C^{A+-} = 1$$
$$C^{Z+-} = -\frac{\cos\theta_W}{\sin\theta_W}.$$

Here, and below, always only non-zero entries for the coupling matrices C will be given. The symmetry of these coupling constants is determined by the nature of the particles. For two external bosonic legs a vertex must be totally symmetric when exchanging all indices of these two legs, and totally antisymmetric for two fermionic legs.

The three-scalar vertex reads

$$\begin{aligned}
\Gamma^{SSS}(p,q,k) &= ieC^{SSS} \\
C^{\eta\eta\eta} &= -\frac{2}{3\sin\theta_W}\frac{M_H^2}{M_W} \\
C^{\eta\chi\chi} &= C^{\eta+-} = -\frac{1}{2\sin\theta_W}\frac{M_H^2}{M_W}.
\end{aligned}$$

The matrix C is symmetric. Its mass dimensions arises since the three-scalar couplings originate from the four-scalar couplings by replacing one of the scalar fields by its vacuum expectation value, which is of course only non-zero for an η Higgs field.

The tree-level decay vertex of a gauge boson into two scalars is given by

$$\begin{aligned}
\Gamma_\mu^{VSS}(p,q,k) &= ieC^{VSS}(p-q)_\mu \\
C^{Z\chi\eta} &= -\frac{i}{2\cos\theta_W\sin\theta_W} \\
C^{A+-} &= -1 \\
C^{Z+-} &= \frac{\cos^2\theta_W-\sin^2\theta_W}{2\cos\theta_W\sin\theta_W} \\
C^{\pm\mp\eta} &= \mp\frac{1}{2\sin\theta_W} \\
C^{\pm\mp\chi} &= -\frac{i}{2\sin\theta_W},
\end{aligned}$$

where C is antisymmetric in the two scalar legs. The corresponding tree-level decay of a scalar into two gauge bosons is given by

$$\begin{aligned}
\Gamma_{\mu\nu}^{SVV}(p,q,k) &= ieg_{\mu\nu}C^{SVV} \\
C^{\eta ZZ} &= \frac{M_W}{\cos^2\theta_W\sin\theta_W} \\
C^{\eta+-} &= \frac{M_W}{\sin\theta_W} \\
C^{\pm\mp A} &= -M_W \\
C^{\pm\mp Z} &= -\frac{M_W\sin\theta_W}{\cos\theta_W},
\end{aligned}$$

with C symmetric in the two gauge bosons. The mass dimensions appears as this vertex comes from the two-gauge-boson-two-scalar vertex upon replacing one external line of η by its vacuum expectation value.

The fermion-gauge-boson vertex is given by

$$\begin{aligned}
\Gamma^{V\bar{F}F} &= ie\gamma_\mu\left(C_L^{V\bar{F}F}\frac{1-\gamma_5}{2}+C_R^{V\bar{F}F}\frac{1-\gamma_5}{2}\right)\\
C_L^{A\bar{f}_if_j} &= C_R^{A\bar{f}_if_j}=-Q_f\delta_{ij}\\
C_L^{Z\bar{f}_if_j} &= -\frac{\sin\theta_W}{\cos\theta_W}Q_f\delta_{ij}\\
C_R^{Z\bar{f}_if_j} &= \frac{I^3_{Wf}-\sin^2\theta_W Q_f}{\sin\theta_W\cos\theta_W}\delta_{ij}\\
C_L^{+\bar{f}^r_{ui}f^r_{dj}} &= \frac{1}{\sqrt{2}\sin\theta_W}V^r_{ij}\\
C_L^{-\bar{f}^r_{di}f^r_{uj}} &= \frac{1}{\sqrt{2}\sin\theta_W}V^{r\dagger}_{ij}\\
C_R^{+\bar{f}^r_if^r_j} &= 0,
\end{aligned}$$

which is antisymmetric in the two fermion indices. f_u and f_d denote here and below up-type fermions (up-quark-like and neutrinos) and down-type fermions (down-quark-like and electron-like), respectively. There is also no summations in i and j implied.

The scalar-two-fermion vertex is given by

$$\begin{aligned}
\Gamma^{S\bar{F}F} &= ie\left(C_L^{S\bar{F}F}\frac{1-\gamma_5}{2}+C_R^{S\bar{F}F}\frac{1-\gamma_5}{2}\right)\\
C_L^{\eta\bar{f}_if_j} &= C_R^{\eta\bar{f}_if_j}=-\frac{1}{2\sin\theta_W}\frac{m_{fi}}{M_W}\delta_{ij}\\
C_L^{\chi\bar{f}_if_j} &= -C_R^{\chi\bar{f}_if_j}=-\frac{i}{\sin\theta_W}I^3_{Wf}\frac{m_{fi}}{M_W}\delta_{ij}\\
C_L^{+\bar{f}^r_{ui}f^r_{dj}} &= -\frac{m_{dj}}{m_{ui}}C_R^{+\bar{f}^r_{ui}f^r_{dj}}=\frac{1}{\sqrt{2}\sin\theta_W}\frac{m_{ui}}{M_W}V^r_{ij}\\
C_L^{-\bar{f}^r_{di}f^r_{uj}} &= -\frac{m_{uj}}{m_{di}}C_R^{-\bar{f}^r_{di}f^r_{uj}}=-\frac{1}{\sqrt{2}\sin\theta_W}\frac{m_{di}}{M_W}V^{r\dagger}_{ij},
\end{aligned}$$

with again antisymmetric fermionic indices.

In addition to these, at tree-level, gauge-independent vertices there are the gauge-dependent vertices with ghost. There are two in case of the 't Hooft gauge. The first is the gauge boson ghost vertex

$$\begin{aligned}
\Gamma_\mu^{V\bar{G}G}(p,q,k) &= ieq_\mu C^{V\bar{G}G}\\
C^{A\bar{c}^\pm c^\pm} &= C^{\pm\bar{c}^Ac^\mp}=C^{\mp\bar{c}^\mp c^A}=\pm 1\\
C^{Z\bar{c}^\pm c^\pm} &= C^{\pm\bar{c}^Zc^\mp}=C^{\mp\bar{c}^\mp c^Z}=\mp\frac{\cos\theta_W}{\sin\theta_W},
\end{aligned}$$

where $\pm$, Z and A correspond to the ghosts used to fix the respective gauge degree of freedom. Note that there is a ghost associated with the photon field, which is coupling due to the mixing with the Z. This also includes the non-interacting Abelian ghost. After mixing, it can no longer be dismissed. C is again antisymmetric in the fields, but the momentum-dependence is not trivial, since $\bar{c}$ and c are independent fields.

In the 't Hooft gauge there is also a coupling of the scalars to the ghosts given by

$$\begin{aligned}
\Gamma^{S\bar{G}G} &= ieC^{S\bar{G}G}\xi_{\bar{G}} \\
C^{\eta\bar{c}^Z c^Z} &= -\frac{M_W}{2\cos^2\theta_W \sin\theta_W} \\
C^{\eta\bar{c}^\pm c^\pm} &= -\frac{M_W}{2\sin\theta_W} \\
C^{\chi\bar{c}^\pm c^\pm} &= \mp\frac{iM_W}{2\sin\theta_W} \\
C^{\pm\bar{c}^\pm c^A} &= M_W \\
C^{\pm\bar{c}^\pm c^Z} &= -\frac{\cos^2\theta_W - \sin^2\theta_W}{2\cos\theta_W \sin\theta_W} M_W \\
C^{\pm\bar{c}^Z c^\mp} &= \frac{M_W}{2\cos\theta_W \sin\theta_W}.
\end{aligned}$$

The ξ has to be chosen which is associated with the outgoing anti-ghost. Again, C is antisymmetric in the ghost fields. The appearance of M_W is in this case not by reduction of a four-point vertex but due to the 't Hooft gauge condition (3.19).

5.2.2.2 Four-point vertices

The four-gauge-boson vertex is given by

$$\Gamma^{VVVV}_{\mu\nu\rho\sigma}(p,q,k,l) = ie^2 C^{VVVV}(2g_{\mu\nu}g_{\sigma\rho} - g_{\nu\rho}g_{\mu\sigma} - g_{\rho\mu}g_{\nu\sigma}).$$

C is of mixed symmetry in V-space and the only non-zero values are

$$\begin{aligned}
C^{+-ZZ} &= -\frac{\cos^2\theta_W}{\sin^2\theta_W} \\
C^{+-AZ} &= \frac{\cos\theta_W}{\sin\theta_W} \\
C^{+-AA} &= -1 \\
C^{++--} &= \frac{1}{\sin^2\theta_W}.
\end{aligned}$$

It is noteworthy that also the photon, due to the mixing of the B_μ and the W^3_μ undergoes this type of interaction. C is again antisymmetric.

The four scalar vertex is simply given by

$$\begin{aligned}
\Gamma^{SSSS}(p,q,k,l) &= ie^2 C^{SSSS} \\
C^{\eta\eta\eta\eta} &= C^{\chi\chi\chi\chi} = -\frac{3}{4\sin^2\theta_W}\frac{M_H^2}{M_W^2} \\
C^{\eta\eta\chi\chi} &= C^{\eta\eta+-} = C^{\chi\chi+-} = -\frac{1}{4\sin^2\theta_W}\frac{M_H^2}{M_W^2} \\
C^{+-+-} &= -\frac{1}{2\sin^2\theta_W}\frac{M_H^2}{M_W^2}.
\end{aligned}$$

The matrix C is totally symmetric.

The mixed two-gauge-boson-two-scalar vertex is given by

$$\begin{aligned}
\Gamma^{VVSS}_{\mu\nu}(p,q,k,l) &= ie^2 g_{\mu\nu} C^{VVSS} \\
C^{ZZ\eta\eta} &= C^{ZZ\chi\chi} = \frac{1}{2\cos^2\theta_W \sin^2\theta_W} \\
C^{+-\eta\eta} &= C^{+-\chi\chi} = C^{+-+-} = \frac{1}{2\sin^2\theta_W} \\
C^{AA+-} &= 2 \\
C^{ZA+-} &= -\frac{\cos^2\theta_W - \sin^2\theta_W}{\cos\theta_W \sin\theta_W} \\
C^{ZZ+-} &= \frac{(\cos\theta_W^2 - \sin^2\theta_W)^2}{2\cos^2\theta_W \sin^2\theta_W} \\
C^{\pm A\mp\eta} &= -\frac{1}{2\sin\theta_W} \\
C^{\pm A\mp\chi} &= \mp\frac{i}{2\sin\theta_W} \\
C^{\pm Z\mp\eta} &= -\frac{1}{2\cos\theta_W} \\
C^{\pm Z\mp\chi} &= \mp\frac{i}{2\cos\theta_W},
\end{aligned}$$

where C is symmetric in both pairs of indices.

5.3 Renormalization

5.3.1 Global case

The renormalization of a theory with hidden symmetry can be formulated in terms of the same theory but with manifest symmetry, i. e., with a potential having its minimum at zero field value. That this is the case is not surprising from a physical point of view:

The breaking of the symmetry occurs at a scale of the order of the vacuum expectation values f. When renormalizing at sufficiently large scales, this scale can be neglected. This implies that the theory is still renormalizable if the theory with manifest symmetry is renormalizable. However, for objects at the scale of f, i. e., masses and the vacuum expectation value, some changes can be expected.

It is much simpler to first investigate this for the case of a global symmetry before the more complicated case of a gauge symmetry. In the following it will be assumed that the theory is perturbatively renormalizable. Then, in the absence of anomalous UV-IR mixing it is sufficient to show renormalizability perturbatively. Including non-perturbative effects will then only provide at most a finite shift of the renormalization constants. It will be later on described what possible pitfalls there are.

The obvious possibility appears to just renormalize for an arbitrary value of the mass μ^2 and then perform an analytical continuation from positive mass to negative mass. However, it turns out that the renormalization process is non-analytic at $\mu^2 = 0$. Thus, this possibility is not available. An alternative is to provide a path from the positive to the negative mass domain using an auxiliary coupling in form of a linear shift of the relevant fields at the level of the Lagrangian

$$\mathcal{L}^c = \mathcal{L} + c_i\phi_i.$$

This breaks the symmetry explicitly, if the c_i are invariant under the symmetry transformation. Setting the c_i to zero recovers the original theory. Furthermore, it is useful to introduce the shifted variables which have all zero vacuum expectation value as

$$\psi_i = \phi_i - f_i.$$

The corresponding generating functional is then given by

$$\begin{aligned} T^c[J_i] &= e^{T_c^c[J]} = \frac{1}{Z[0]}\int \mathcal{D}\psi \exp\left(i\int d^4x(\mathcal{L}(\psi_i + f_i) + (J_i + c_i)\psi_i)\right) \\ &= \frac{1}{Z[0]}\int \mathcal{D}\phi \exp\left(i\int d^4x(\mathcal{L}(\phi) + (J_i + c_i)(\phi_i - f_i))\right) \\ &= \exp\left(T_c[J_i + c_i] - i\int d^4x c_i f_i - i\int d^4x J_i f_i\right), \end{aligned}$$

where use has been made of the fact that the measure is invariant under a translation of the fields. This generating functional is connected to the original one

$$T_c^c[J_i] = T_c[J_i + c_i] - i\int d^4x c_i f_i - i\int d^4x J_i f_i.$$

Note that the second term is equal to $T_c[c_i]$, since the value of the field for this source, f_i, would make the Lagrangian vanish: The kinetic term since it is constant, and by definition it is a minimum at zero of the potential. Due to (3.13), it follows that the first derivatives of T_c^c with respect to the sources vanish,

$$\frac{\delta T_c^c}{i\delta J_i}[0] = \frac{\delta T_c}{i\delta J_i}[0 + c_i] - 0 - i\int d^4x\delta(x-y)f_i = 0.$$

Performing a Legendre transformation yields

$$\begin{aligned} i\Gamma^c[\psi] &= T_c^c[J_i] - i\int d^4xJ_i\psi_i = T_c[J_i + c_i] - T_c[c_i] - i\int d^4xJ_i(\psi_i + f_i) \\ &= i\Gamma[\psi_i + f_i] - i\Gamma[f_i] + i\int d^4xc_i\psi_i \\ \psi_i &= \frac{\delta T_c^c[J_i]}{i\delta J_i} = \frac{\delta T_c[J_i + c_i]}{i\delta J_i} - f_i. \end{aligned} \tag{5.6}$$

where the relations (3.11) of the original theory has been used,

$$c_i = -\frac{\delta\Gamma}{\delta\phi_i}[f_i].$$

For the perturbative expansion it follows that

$$\begin{aligned} i\Gamma^c[0] &= 0 \\ \Gamma_i^c(x) &= \frac{\delta\Gamma^c}{\delta\psi_i(x)}[0] = -c_i + 0 + c_i = 0. \end{aligned}$$

For the higher orders, it is useful to expand around ψ_i. Since on the right-hand side the argument is $(\psi_i + f_i)$, the expansion around ψ_i is of the formal type

$$\Gamma[\psi + f] = \sum_n(\psi + f)^n\Gamma_n = \sum_n \psi^n \sum_{m=0}^{\infty}\frac{1}{m!}f^m\Gamma_{n+m},$$

while the expansion of Γ^c in ψ_i is of the ordinary type. Comparing then the same orders of ψ_i yields for $n > 1$

$$\Gamma^c_{i_1...i_n}(x_1, ..., x_n) = \sum_{m=0}^{\infty}\int \Pi_{k=1}^m \left(d^4y_kf_{j_k}\right)\Gamma_{i_1,...,i_n,j_1,...,j_m}(x_1, ..., x_n, y_1, ..., y_m).$$

Thus, the Green's functions of the hidden case are completely determined by the ones with manifest symmetry. However, this connection requires knowledge of all Green's functions of the manifest case, contracted with the condensate on superfluous external legs, to determine only one of the hidden case. Thus, this connection is outside of perturbation theory only of limited practical use. Note that it is principle possible to invert this relation. For the purpose of demonstrating renormalizability this is, however, very useful. It also permits, e. g., the proof that the Goldstone theorem in fact holds in each order of perturbation theory, which is not trivial.

For the purpose of renormalization it is now useful to return to the relation (5.6). The renormalization of the manifest theory implies that the renormalized vertex functional is related to the bare one as

$$\Gamma_R(\phi_i, -\mu^2, \lambda) = \Gamma(\phi_{i0}, -\mu_0^2, \lambda_0) = \Gamma(\sqrt{Z}\phi_i, -\mu^2 - \delta\mu^2, Z_\lambda\lambda).$$

Since if the manifest case is renormalizable, so must be the hidden case by virtue of relation (5.6). Therefore in the latter case it holds that

$$\Gamma_R^c(\psi_i, -\mu^2, \lambda, c_i, f_i) = \Gamma^c(\psi_{i0}, -\mu_0^2, \lambda_{i0}, c_{i0}, f_{i0}) = \Gamma^c\left(Z^{\frac{1}{2}}\psi_i, -\mu^2 - \delta\mu^2, Z_\lambda\lambda, Z^{-\frac{1}{2}}c_i, Z^{\frac{1}{2}}f_i\right).$$

The renormalization of c_i follows from the fact that the action must not change under renormalization. The condensate is just an arbitrary split-off of the field, and must therefore renormalize in exactly the same way as the field. This formula turns out to be sufficient to perform the analytical continuation by the sequence

$$(\mu^2 < 0, c_i = 0) \to (\mu^2 < 0, c_i \neq 0) \to (\mu^2 > 0, c_i \neq 0) \to (\mu^2 > 0, c_i = 0).$$

Thus, the theory with hidden symmetry renormalizes in exactly the same way as the one with manifest symmetry, and only the vacuum expectation value has to be renormalized additionally.

The quantitative value of the renormalization constants may differ in principle. However, it can be shown that that only the mass counter-term actually differs. Of course, it is always allowed to modify nonetheless the renormalization constants by finite parts. They can be the same as in the manifest version of the theory, but they need not to be so. This can be of great practical relevance. Also, renormalization schemes can be found in which also the infinite parts are different. Still, the renormalization of the theory with hidden symmetry is always possible, if it is possible for the theory with manifest symmetry.

5.3.2 Local case

The procedure to renormalize a gauge theory with hidden symmetry is a direct extension of the global case. In case of the theory with manifest symmetry the renormalization is performed as

$$\begin{aligned}\Gamma_R(\phi, A^\mu, g, -\mu^2, \lambda, ...) &= \Gamma(\phi_0, A_0^\mu, g_0, -\mu_0^2, \lambda_0, ...)\\ &= \Gamma\left(Z^{\frac{1}{2}}\phi, Z_A^{\frac{1}{2}}A^\mu, Z_g g, -\mu^2 - \delta\mu^2, Z_\lambda\lambda, ...\right),\end{aligned}$$

where the dots denote additional parameters which depend on the gauge, like ghost wave-function renormalization or renormalization of the gauge parameters. These are renormalized for linear gauges, as are used throughout here, such that the gauge-fixing term is not changing under renormalization, i. e., that the gauge condition is not changed under renormalization. This actually is a particular choice of renormalization scheme.

As in the global case it is possible to introduce an additional parameters c_i which break the gauge symmetry explicitly. The resulting theory is of course no longer renormalizable, but non-renormalizable, requiring an infinite number of counter terms. However, it can still be renormalized formally. As previously the parameters c_i can be used to analytically move through the coupling plane from the theory with manifest symmetry to the one without, yielding

$$\begin{aligned}\Gamma^c_R(\phi, A^\mu, g, -\mu^2, \lambda, c^i, f^i, ...) &= \Gamma^c(\phi_0, A^\mu_0, g_0, -\mu_0^2, \lambda_0, c_0^i, f_0^i, ...)\\ &= \Gamma^c\left(Z^{\frac{1}{2}}\phi, Z_A^{\frac{1}{2}}A^\mu, Z_g g, -\mu^2-\delta\mu^2, Z_\lambda\lambda, Z^{-\frac{1}{2}}c^i, Z^{\frac{1}{2}}f_i, ...\right).\end{aligned}$$

Thus, once more the theory without manifest symmetry can be renormalized in the same way as the one with manifest symmetry. Finite renormalizations can then be performed afterwards to obtain more useful results, e. g., by shifting the value of f_i to the true minimum of the renormalized potential, etc.. This is particularly useful as it eliminates tadpoles from the calculations.

Chapter 6

The physical spectrum

Though unitary gauge appears to provide insight into the spectrum by reducing the number of degrees of freedom by a particular gauge choice, this is not sufficient due to the lack of manifest renormalizability in this gauge. A more general construction of the state space is required. This will be done in the following first at the perturbative level, yielding the (perturbative) physical spectrum. The extension beyond perturbation theory will require the perturbative case, and will thus be done afterwards.

A possibility to establish the physical state space is by use of the BRST (Becchi-Rouet-Stora-Tyutin) symmetry, which is a residual symmetry after gauge-fixing. Perturbatively, it permits to separate physical from unphysical fields. Again, it is useful to first rehearse the BRST symmetry and its role in case of a manifest gauge symmetry before extending it to the case with hidden symmetry.

6.1 Physical states, gauge invariance, and perturbation theory

6.1.1 BRST symmetry for manifest gauge symmetries

The starting point for the rehearsal is the gauge-fixed Lagrangian with Nakanishi-Lautrup fields included

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \frac{\xi}{2}b^a b_a + b^a C_a - \int d^4z \bar{u}^a(x)\frac{\delta C^a}{\delta A^c_\nu}D^{cb}_\nu u^b(z).$$

Herein the gauge condition is encoded in the condition $C^a = 0$. Furthermore, matter fields are ignored, and therefore the ghost contribution has a rather simple structure compared to the case in the electroweak theory. These contributions will be reinstantiated later when discussing the full electroweak theory.

This Lagrangian is invariant under the BRST transformation

$$\begin{aligned}
\delta_B A_\mu^a &= \lambda D_\mu^{ab} u^b = & \lambda s A_\mu^a \\
\delta_B u^a &= -\lambda \tfrac{g}{2} f^{abc} u^b u^c = & \lambda s u^a \\
\delta_B \bar{u}^a &= \lambda b^a = & \lambda s \bar{u}^a \\
\delta_B b^a &= 0 = & \lambda s b^a.
\end{aligned}$$

Herein, λ is an infinitesimal Grassmann number, i. e., it anticommutes with the ghost fields. As a consequence, the BRST transformation s obeys a generalized Leibnitz rule

$$s(FG) = (sF)G + (-1)^{\text{Grassmann parity of } F} F s G.$$

The Grassmann parity of an object is 1 if it is Grassmann, i. e. contains an odd number of Grassmann numbers, and 0 otherwise. An amazing property of the BRST symmetry is that it is nil-potent, i. e., $s^2 = 0$, which follows by explicit calculation.

Showing the invariance is straightforward for the classical Lagrangian, as the transformation for the gauge boson is just an ordinary gauge transformation with gauge parameter λu^a, which is an ordinary real function. The remainder part is more involved, and requires to exploit the algebra structure extensively. This will be skipped here.

There is also a more straightforward proof. It holds that the gauge-fixing part of the Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_f &= s\left(\bar{u}^a \left(\frac{\xi}{2} b^a + C^a\right)\right) \\
&= \frac{\xi}{2} b^a b_a + b^a C_a + \bar{u}^a \int d^4 y \frac{\delta C^a}{\delta A_\mu^b(y)} D_\mu^{bc} u^b(y).
\end{aligned}$$

Hence, the gauge-fixing part of the Lagrangian is BRST-invariant, since $s^2 = 0$. This can be generalized to other gauge conditions by adding arbitrary BRST-exact terms $s(\bar{u}^a F_a)$ to the Lagrangian, leading to the so-called anti-field formalism for gauge-fixing.

The BRST transformation for matter fields also takes the form of a gauge transformation with the parameter λu^a, and is also nilpotent. Therefore, all matter Lagrangian contributions automatically satisfy invariance under a BRST transformation. For a matter field ϕ in representation τ^a it takes the form

$$\delta_B \phi^i = \lambda i g u^a \tau_{ij}^a \phi^j. \tag{6.1}$$

The BRST symmetry can be understood as being a residual gauge transformation, with on top the possibility of exploiting ambiguities in the introduction of auxiliary fields. This is relatively evident in local covariant gauges. Since the gauge fixing contains an average over different gauge copies, the BRST transformation switches between different copies over which it averages. The changes of the auxiliary fields make sure to reweight the contribution of the Faddeev-Popov determinant, i. e. the Jacobian.

6.1.2 Constructing the physical state space for a manifest symmetry

The following discussion shows how to explicitly construct the state space using BRST symmetry. It extends thereby the Gupta-Bleuler construction of QED, and it can be directly extended to include also matter fields.

The first concept in constructing the physical state space is the presence of states which have not a positive norm. Choose, e. g., Feynman gauge. The corresponding tree-level propagator is then given by

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle = \delta^{ab}g_{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+i\epsilon} = -\delta^{ab}g_{\mu\nu}\int\frac{d^3p}{2(2\pi)^3|\vec{p}|}e^{-ip_i(x-y)_i}.$$

The norm of a state

$$\Psi(x) = \int d^4x f(x)A_0(x)|0\rangle = \int\frac{d^4xd^4p}{(2\pi)^4}e^{ip_0x_0-p_ix_i}f(p)A_0(x)|0\rangle$$

created from the vacuum by the operator A_μ with some arbitrary weight function $f(x)$ then reads

$$|\Psi|^2 = \int d^4x\int d^4y\langle A_0(x)A_0(y)\rangle f^\dagger(x)f(x) = -\int\frac{d^3p}{2|\vec{p}|}f^\dagger(p)f(p) < 0.$$

Hence, there are negative (and zero) norm states present in the state space. This applies actually already for QED, as the non-Abelian nature was not needed in the argument.

These cannot contribute to the physical state space, or otherwise the probability interpretation of the theory will be lost. Or at least, it must be shown that the time evolution is only connecting physical, i. e. with positive definite norm, initial states into physical final states.[1]

[1]The precise characterization of what is a final state beyond perturbation theory is open. One possibility is a non-perturbative extension of the construction to follow. Another one characterizes all physical states by the necessary condition to be invariant under renormalization - after all, physics should be independent of the scale at which it is measured. However, whether this condition is sufficient, in particular beyond perturbation theory, is also not clear. Bound states with non-zero ghost number, e. g., may also possess this property, though may not be a viable physical state.

That they indeed do not contribute can be shown using the BRST symmetry. In fact, it can be shown that

$$\begin{aligned} Q_B|\psi\rangle_{\text{phys}} &= 0, \\ [Q_B,\psi]_\pm &= s\psi. \end{aligned} \tag{6.2}$$

will be sufficient to define the physical state space. The $\pm$ indicates commutator or anticommutator, depending on whether ψ is bosonic (commutator) or fermionic (anticommutator). The BRST charge Q_B can be defined from the Noether current and is given by

$$Q_B = \int d^3x \left(b^a D_0^{ab} u^b - u^a \partial_0 b^a + \frac{1}{2} g f^{abc} u^b u^c \partial_0 \bar{u}^a \right).$$

It is fermionic. Since $s^2 = 0$ it directly follows that $Q_B^2 = 0$ as well.

The BRST charge has evidently a ghost number of 1, i. e., the total number of ghost fields minus the one of anti-ghosts is 1. This ghost number, similarly to fermion number, is actually a conserved quantum number of the theory. It is due to the invariance of the Lagrangian under the scale transformation

$$\begin{aligned} u^a &\to e^\alpha u^a \\ \bar{u}^a &\to e^{-\alpha} \bar{u}^a, \end{aligned}$$

with real parameter α. Note that such a scale transformation is possible since u^a and $\bar{u}^a$ are independent fields. Furthermore for a hermitian Lagrangian the relations

$$\begin{aligned} u^\dagger &= u \\ \bar{u}^\dagger &= -\bar{u} \end{aligned}$$

hold. As a consequence, also the BRST transformation and charge have ghost number 1 and is Hermitian.

Since the Lagrangian is invariant under BRST transformation, so is the Hamiltonian, and therefore also the time evolution and the S-matrix,

$$\begin{aligned} [Q_B, H] &= 0 \\ [Q_B, S] &= 0. \end{aligned}$$

Hence, if in fact the BRST symmetry is manifest[2], and the condition (6.2) defines the physical subspace, that is already sufficient to show that physical states will only evolve into physical states. It remains to see what kind of states satisfy (6.2).

[2]The consequences of a not manifest BRST are far from trivial, and the non-perturbative status of BRST symmetry is still under discussion, though there is quite some evidence that if it can be defined it is well defined. But how to define it is not finally settled.

Because the BRST charge is nilpotent the state space can be separated in three subspaces:

- States which are not annihilated by Q_B, $V_2 = \{|\psi\rangle | Q_B|\psi\rangle \neq 0\}$.
- States which are obtained by Q_B from V_2, $V_0 = \{|\phi\rangle || \phi\rangle = Q_B|\psi\rangle, |\psi\rangle \epsilon V_2\}$. As a consequence $Q_B V_0 = 0$.
- States which are annihilated by Q_B but do not belong to V_0, $V_1 = \{|\chi\rangle | Q_B|\chi\rangle = 0, |\chi\rangle \neq Q_B|\psi\rangle, \forall |\psi\rangle \epsilon V_2\}$.

The states in V_2 do not satisfy (6.2), and therefore would not be physical. The union of the two other states form the physical subspace.

$$V_p = V_0 \cup V_1.$$

It is this subspace which is invariant under time evolution. It is not trivial to show that all states in this space have positive semi-definite norm, but this is possible. This will be skipped here. However, all states in V_0 have zero norm, and have no overlap with the states in V_1,

$$\begin{aligned} \langle\phi|\phi\rangle &= \langle\phi|Q_B|\psi\rangle = 0 \\ \langle\phi|\chi\rangle &= \langle\psi|Q_B|\chi\rangle = 0. \end{aligned}$$

Since matrix elements are formed in this way the states in V_0 do not contribute, and every state in V_p is thus represented by an equivalence class of states characterized by a distinct state from V_1 to which an arbitrary state from V_0 can be added, and thus a ray of states. Therefore, the physical Hilbert space H_p can be defined as the quotient space

$$H_p = V_p / V_0.$$

Therefore, all states in H_p have positive norm, provided that the states in V_1 have. Such a structure is also known as the cohomology, in this case of the BRST operator.

To define the theory in the vacuum, use can be made of asymptotic states, in perturbation theory usually known as in and out states. The corresponding physical asymptotic states ψ_p^a must therefore obey

$$s\psi_p^a = 0.$$

In perturbation theory in the class of theories relevant here the asymptotic states are just the non-interacting ones. Consequently, the BRST transformation of the asymptotic states are the BRST transformation of the free fields,

$$sA_\mu^{aa} = \partial_\mu u^a \tag{6.3}$$

$$su^{aa} = 0 \tag{6.4}$$

$$s\bar{u}^{aa} = b^{aa} \tag{6.5}$$

$$sb^{aa} = 0, \tag{6.6}$$

Beyond perturbation theory, this is no longer possible, as cluster decomposition in general no longer holds in gauge theories. How to proceed beyond perturbation theory is therefore not completely understood.

From this follows that the longitudinal component of A_μ, since ∂_μ gives a direction parallel to the momentum, is not annihilated by s, nor is the anti-ghost annihilated by the BRST transformation. They belong therefore to V_2. The ghost and the Nakanishi-Lautrup field are both generated as the results from BRST transformations, and therefore belong to V_0. Since they are generated from states in V_2 it is said they form a quartet with parent states being the longitudinal gluon and the anti-ghost and the daughter states being the ghost and the Nakanishi-Lautrup field. Therefore, these fields not belonging to the physical spectrum are said to be removed from the spectrum by the quartet mechanism. Note that the equation of motion for the field b^a makes it equivalent to the divergence of the gluon field, which can be taken to be a constraint for the time-like gluon. Therefore, the absence of the Nakanishi-Lautrup field from the physical spectrum implies the absence of the time-like gluon. Finally, the transverse gluon fields are annihilated by the BRST transformation but do not appear as daughter states, they are therefore physical. In general gauges, the second unphysical degree of freedom will be the one constrained by the gauge-fixing condition to which b^a is tied, while the two remaining polarization directions, whichever they are, will be belonging to V_1.

Of course, the gauge bosons cannot be physical, since they are not gauge-invariant. Therefore, their removal from the spectrum must proceed by another mechanism, which is therefore necessarily beyond perturbation theory. This will be discussed in section 6.2.

In the unbroken case, the introduction of fermion (or other matter) fields ψ follows along the same lines. It turns out that all of the components belong to V_1 since $s\psi^{aa} = 0$, without ψ appearing on any left-hand side, and therefore all matter degrees of freedom are perturbatively physically.

Similar as for the gauge boson, this cannot be completely correct, and has to change non-perturbatively. Already a one-loop calculation shows the presence of positivity violating contributions in the spectral density of the propagators of all these particles considered physical here. In perturbation theory, it is argued that this does not matter, as asymptotically the interactions cease, and the propagators are just free ones. As will be seen, this is not quite correct.

6.1.3 The physical spectrum with hidden symmetry

The BRST transformation for the electroweak standard model can be read off from the gauge transformations (4.8) and the gauge-fixing conditions (5.4). They read for the fields, and thus independent of the chosen (sub-)type of gauge,

$$\begin{aligned}
sA_\mu &= \partial_\mu u^A + ie(W_\mu^+ u^- - W_\mu^- u^+) \to \partial_\mu u^A \\
sZ_\mu &= \partial_\mu u^Z - ie\frac{\cos\theta_W}{\sin\theta_W}(W_\mu^+ u^- - W_\mu^- u^+) \to \partial_\mu u^Z \\
sW_\mu^\pm &= \partial_\mu u^\pm \mp ie\left(W_\mu^\pm\left(u^A - \frac{\cos\theta_W}{\sin\theta_W}u^Z\right) - \left(A_\mu - \frac{\cos\theta_W}{\sin\theta_W}Z_\mu\right)u^\pm\right) \to \partial_\mu u^\pm \\
s\eta &= \frac{e}{2\sin\theta_W\cos\theta_W}\chi u^Z + \frac{ie}{2\sin\theta_W}(\phi^+u^- - \phi^-u^+) \to 0 \\
s\chi &= -\frac{e}{2\sin\theta_W\cos\theta_W}(f+\eta)u^Z + \frac{e}{2\sin\theta_W}(\phi^+u^- + \phi^-u^+) \to -M_Z u^Z \\
s\phi^\pm &= \mp ie\phi^\pm\left(u^A + \frac{\sin^2\theta_W - \cos^2\theta_W}{2\cos\theta_W\sin\theta_W}u^Z\right) \pm \frac{ie}{2\sin\theta_W}(f+\eta\pm i\chi)u^\pm \to \pm iM_W u^\pm \\
sf_{i\pm}^L &= -ie\left(Q_{i\pm}u^A + \frac{\sin\theta_W}{\cos\theta_W}\left(Q_{i\pm} \mp \frac{1}{2\sin^2\theta_W}\right)u^Z\right)f_{i\pm}^L \to 0 \\
sf_{i\pm}^R &= -ieQ_{i\pm}\left(u^A + \frac{\sin\theta_W}{\cos\theta_W}u^Z\right)f_{i\pm}^R \to 0.
\end{aligned}$$

The precise form for the BRST transformations of the auxiliary fields depend on the chosen gauge, which will be here the ’t Hooft gauge. They read

$$\begin{aligned}
su^\pm &= \pm\frac{ie}{\sin\theta_W}u^\pm(u^A\sin\theta_W - u^Z\cos\theta_W) \to 0 \\
su^Z &= -ie\frac{\cos\theta_W}{\sin\theta_W}u^-u^+ \to 0 \\
su^A &= ieu^-u^+ \to 0 \\
s\bar{u}^a &= b^a \to b^a \\
sb^a &= 0 \to 0,
\end{aligned}$$

where the index a on the anti-ghost and the Nakanishi-Lautrup field runs over A, Z, and $\pm$.

The second step gives the asymptotic version of the BRST transformation, which can be shown, similar to the case with manifest symmetry, to be just the free-field version. It is then directly possible to read off the state-space structure.

First of all, the fermions and the Higgs field η are annihilated by s asymptotically, but do not appear as daughter states. Thus they belong to the physical subspace V_1. The same applies to all transverse degrees of freedom of Z_μ, A_μ, $W_\mu^\pm$. For the photon, the same structure emerges as previously, making the fields u^A, $\bar{u}^A$, b^A and the longitudinal component of A_μ a quartet, and thus there are only two transverse degrees of freedom for the photon.

The situation is a bit different for the would-be Goldstone bosons χ and $\phi^\pm$ and the fields Z_μ and $W_\mu^\pm$. For massive gauge fields, the component along ∂_μ, or momentum, is not the longitudinal component as in the case of a massless field. It yields the scalar component, i. e., the one defined as $k_\mu k_\nu B_\nu$. The transverse ones are given by the two transverse projectors, and the longitudinal one is the remaining degree of freedom. Hence, in the asymptotic BRST transformation the scalar component appears, while the longitudinal one is annihilated, and thus also belongs to V_1. The scalar component is not annihilated, and therefore belongs to V_2. Of course, by the equation of motion it is connected to b^a. The latter field forms with the would-be Goldstone bosons and the ghost and anti-ghost once more a quartet, and all these four fields are therefore not physical.

All in all, the physical degrees of freedom are the fermions, the Higgs, the massive gauge bosons Z_μ, $W_\mu^\pm$ with three degrees of freedom, the massless photon with two degrees of freedom, and the Higgs field. Therefore, as in unitary gauge, though less obvious, all three would-be Goldstone bosons do not belong to the physical subspace, as do not the remaining degrees of freedom of the vector fields. This therefore establishes the physical spectrum of perturbation theory.

Once more, the situation is different beyond perturbation theory, as none of these objects are gauge-invariant. None of them can belong to the physical spectrum, and only their gauge-invariant combinations can. E. g., an electron is (likely) in fact an electron-Higgs bound state. This is easily seen by calculating the perturbative production cross-section for the generation of a pair of gauge bosons, which is non-zero, though these objects can only be observed indirectly.

6.2 Beyond perturbation theory

As has now been repeatedly emphasized, the asymptotic state space can, in principle, not contain any elementary particles. The reason is that the asymptotic fields cannot be free fields, since otherwise the state space has changed from a space of gauge-dependent objects to one of gauge-singlets, and thus a local symmetry would become a global symmetry. These two spaces are not unitarily equivalent, and therefore this is strictly speaking not possible beyond perturbation theory where all results are by construction smooth in the gauge coupling.

This point can be formalized in the context of axiomatic field theory, and is known as Haag's theorem: The state spaces of an interacting theory and a non-interacting theory are not unitarily equivalent, no matter how weak the coupling. Hence, strictly speaking perturbation theory expands around the wrong vector space. However, this theorem does not make any statements about the quantitative size of the non-analytic contributions. It is thus well possible that they are a negligible effect, thus perturbation theory implicitly assumes that this is the case, and the dominant contribution comes actually from the analytic part. At least for the electroweak case, this seems to be true, as perturbation theory describes exceedingly well observations. But this does not need to be true.

Hence, in the following a correct construction will be provided, and in the end shown why, and under which conditions, perturbation theory can still give the dominant part of the answer.

To establish the answer, it is useful to neglect for the moment all non-essential parts. The remainder is just the weak gauge fields, now yielding degenerate masses for the $W^\pm$ and Z because of the absence of QED, or more precisely the hypercharge, and the two Higgs doublets.

6.2.1 Scalar QCD

The first step is to address the situation without the BEH effect. Then the theory is essentially scalar QCD, i. e. QCD with the fermionic quarks replaced by scalars, and only two colors. Such a theory is naively expected to behave like QCD, and this is indeed the case. Thus, then confinement occurs, and the only degrees of freedom observable are bound states, i. e. the analogue of hadrons.

6.2.2 Elitzur's theorem

To approach the electroweak sector, the first step is to realize that the electroweak symmetry breaking is, as emphasized, only a hiding of the symmetry by a gauge choice, and the actual gauge symmetry can never be broken. This is known as Elitzur's theorem.

The argument[3] proceeds as follows, and is best seen by first considering a simpler example. Take as a theory a theory of two-space-time dimension in cylindrical coordinates

[3] In the original argument of Elitzur is a loophole, as some types of non-analyticities are not considered. The following is a more modern view which comes to the same conclusion. It is also necessary to generally do not use in the derivations any sources which are not invariant under the symmetry transformation, as these would explicit break the symmetry, and therefore potentially make a smooth and analytic approach

r and θ with the (Euclidean) action $S = r^2$.Then the partition function is given by

$$Z = \int_{-\pi}^{\pi} d\cos\theta \int r dr e^{-r^2}$$

This partition function has a global symmetry for any rotation $\theta \to \theta + \delta$. This invariance manifests in the fact that the expectation value of every even function of θ vanishes.

To break this symmetry would require that there is a particular direction singled out, i. e. some angle θ_0 should be special, as this would break this symmetry. I. e., that for some vector-valued quantity, which has a direction,

$$\int_{-\pi}^{\pi} d\cos\theta \int r dr (a(\vec{r}^2)\vec{e}_r + b(\vec{r}^2)\vec{e}_\theta) e^{-r^2} \neq \vec{0}$$

where the coefficient functions must be only depending on $\vec{r}^2$, as otherwise the observable itself would break the symmetry explicitly. However, the whole integral has no possibility to single out such a preferred direction, as both the integral and the integral measure are invariant.

Thus, the only possibility would be to modify either the action or the integral measure. The former would be done by an explicit symmetry breaking term, the other e. g. by a gauge condition, which singles out a subrange of θ values.

The situation in gauge theories is similar. The vectors $\vec{e}_\theta$ and $\vec{e}_r$ correspond to the gauge fields, and the dependence only on $\vec{r}^2$ is equivalent to being a gauge-invariant object. Gauge orbits are then given by the variation in θ at fixed r, and gauge transformations move around this orbit. Gauge-fixing then is the same as restricting the integral on the angle θ, and therefore making non-vanishing integrals of functions of the unit vectors, and thus gauge fields, possible.

6.2.3 The Osterwalder-Seiler-Fradkin-Shenker argument

If now the symmetry cannot be broken, the question is what is about the apparent symmetry breaking by the vacuum expectation value of the Higgs field. The answer is that it was actually a gauge condition which gave the Higgs a vacuum expectation value. The 't Hooft gauge condition (3.19) singles out a particular direction by explicitly introducing a choice of direction for the vacuum expectation value of the Higgs field. However, this choice is part of the gauge choice, and any choice of direction would yield an equally valid, though possibly more cumbersome, result.

of the source to zero impossible.

Now, rather than fixing a direction once and for all, it is equally possible, just as in the construction of general linear covariant gauges, to average over all possible such choices. Then, the result would be that the vacuum expectation values would be the average over all possible direction, but this is zero, as all directions are equally preferred. Actually, without fixing this global degree of freedom the same result would be ensuing.

This seems to have drastic consequences, as without vacuum expectation value the whole construction breaks down, and especially there is no tree-level mass for the gauge bosons. This is in fact correct, and actually it can be shown that in such a gauge tthe gauge bosons remain massless to all orders in perturbation theory. But this is not a consequence of picking somehow a 'wrong' gauge: All gauge choices, which can be satisfied by all orbits[4] are equally acceptable. Thus, this cannot be a conceptual problem. In fact, in such gauges the fluctuations of the Higgs field are no longer small enough to justify perturbation theory, and hence the applicability of perturbation theory rests on the choice of a suitable gauge. In a more simple diction, this is just the statement that only in suitable coordinates perturbation theory makes sense. How to deal with the situation without using perturbation theory is detailed in section 6.3.2.

In this section, the main question is different: Since the non-vanishing of the Higgs expectation value is apparently only due to the choice of a particular gauge, how it is still possible to identify the Brout-Englert-Higgs effect?

This question has two layers, which extend the discussion of section 3.2.1 to the case of gauge symmetries.

The first is how to construct a quantity, which is still identifying the Higgs effect, even if the direction of the Higgs condensate is not fixed by the gauge choice. If keeping the analogy of a magnet, then on any single field configuration in the path integral, the Higgs field will still be aligned. Thus, the relative orientation of the Higgs field would not be influenced, especially as the different possibilities of direction in the 't Hooft gauge condition are connected by a global gauge transformation. Thus, an observable like

$$\langle v_2 \rangle = \left\langle \left| \int d^d x \phi(x) \right|^2 \right\rangle \tag{6.7}$$

would have the desired property. Note that a quantity like

$$\langle v^2 \rangle = \left\langle \int d^d x \, |\phi(x)|^2 \right\rangle ,$$

would not work. Though it is non-zero for non-vanishing relative local alignment, it will actually never vanish, expect when the Higgs field is only in a measure-zero region of

[4]Or actually by all orbits up to a measure zero set for gauge-invariant observables.

space non-zero, and vanishes otherwise. However, especially in a scalar-QCD-like phase, this can hardly be expected, and thus this observable cannot distinguish between a QCD-like behavior and a BEH-like behavior.

However, in a gauge theory this is not enough. To show that this really distinguishes between the BEH case and any alternatives, the observable must also be gauge-invariant under local gauge transformations, and (6.7) is not.

Thus, the question is, whether there is any gauge-invariant possibility to detect the BEH effect. The answer to this appears to be that it is not the case. However, the reasoning, the so-called Osterwalder-Seiler-Fradkin-Shenker argument, is not entirely trivial, and there is at least one loophole.

The problem is that to answer this question it is necessary to go beyond perturbation theory, as it was already seen that perturbation theory provides not even for the restricted case of only differing global gauge choices the correct answer. But calculations beyond perturbation theory are always more involved, and often require assumptions and/or approximations.

The probably strongest statement about the situation in the present theory can be obtained using a so-called lattice discretization, i. e. an approximation where rather than to consider the ordinary space-time, the situation is considered on a discrete and finite lattice of space-time points. The original theory is then obtained in the limit of infinite volume and zero spacing between them. For asymptotically free theories, it can be shown that there is some neighborhood around infinite volume and zero discretization where the approach becomes smooth, and thus this is a valid approach to deal with them[5]. But for not asymptotically free theories, like it is the case for the present one as will be discussed in more detail in section 7.2.2, no such statement exists[6].

Thus, for the following it is necessary to make the assumption that either the limit exists and is smooth, or if not, this has no direct implication for the result. The latter is not too high a hope: Since this only states that it should be valid up to at least some maximum discretization, which corresponds to some maximum energy, this is the statement that the results should be true in the sense of a low-energy effective theory.

The steps for the construction will only be outlined, as the technical details are too involved to present them here, and would require a thorough discussion of a discrete formulation of the theory.

The first restriction is to work at fixed Higgs length $\phi^\dagger\phi = 1$. This is actually only a

[5]Though in practice it is usually impossible to make reliable statements on how large this neighborhood is.

[6]Actually, as will be discussed in section 7.2.2, this can be an indication that the theory just does not exist as an interacting theory without an explicit cutoff, no matter the method.

technical simplification, and can be dropped. This situation is obtained when sending the Higgs-self-coupling to infinity.

The next step is to switch to unitary gauge. This is always possible, since unitary gauge does not require the BEH effect to be active to be well-defined, in contrast to 't Hooft gauges[7]. Since the length of the Higgs field is fixed, there are no Higgs degrees of freedom left in the action, and the action is classically minimized by a vanishing gauge field. It is for this fact important that there is a Higgs field and that the Higgs field fully breaks the gauge symmetry. Otherwise other configurations (instantons and monopoles, to be discussed later) would minimize the action.

Consider now any gauge-invariant operator[8]. Since the only gauge-invariant operators possible are compositions of the terms in the Lagrangian, any such operator can also be written as composition of such gauge-invariant operators. Thus, the full expectation value must be equivalent to a path integral over such gauge-invariant operators.

In the next step, expand the exponential in a series in these operators around vanishing fields, and thus vanishing field-strength tensors. On a finite, discrete lattice, this will always result in a convergent series.

The series can be merged with the expression for the gauge-invariant operator. Thus, the result is some series in gauge-invariant operators. Each term of the series is analytic. On a finite lattice, it can then be shown that this series is, for any gauge-invariant operator, bounded from above by a geometric series parametrized by the parameters. This is again only possible because of the additional potential term induced by the Higgs effect, and thus the presence of one additional parameter. The series is therefore uniformly bounded, and since every term is analytic, a general mathematical theorem guarantees then that the whole expression is an analytic function.

Th whole argument fails only if any parameter of the theory either vanishes or diverges. Thus, on the boundaries of the phase diagram it is still possible to have a phase transition,

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[7]Fixing a gauge is permitted, as only gauge-invariant statements are made, and no approximations are performed which would break gauge invariance. Thus, the final result is gauge-invariant even though a gauge has been fixed in an intermediate step. Note that this is only possible if the number of degrees of freedom of the Higgs is identical, up to the frozen radial mode, to the number of gauge fields. There can be an additional freedom left in the gauge-fixing, the so-called Gribov-Singer ambiguity. This Gribov-Singer ambiguity could, in principle, require further gauge-fixing conditions, but they will not affect gauge-invariant operators, and especially does not modify the minimum, what is the important condition here. Still, an explicit resolution, though complicated, would be helpful, but is not yet available. In addition, gauge-fixing defects may occur for points with vanishing Higgs fields. Their role is not entirely settled.

[8]The aforementioned Gribov-Singer ambiguity is one of the reasons why this proof does not pertain to gauge-dependent quantities, and they may, and do, change non-analytically in the phase diagram, providing the perturbative picture of the BEH-QCD separation.

but there can be no phase transition cutting the phase diagram in separate disconnected pieces. Thus, the phase diagram is connected, though may have phase transitions with end-points, and, of course, cross-overs.

It is visible that being on the lattice is important in the argument. It was also important that all Higgs degrees of freedom could be removed by either freezing or using the unitary gauge in an intermediate step. If the number of Higgs degrees of freedom is such that this is not possible, the argument does not hold. Thus, if the gauge symmetry is only partly broken by the Higgs field, a separation may still exist. Also, if there are surplus Higgs fields, the minimum structure may be more complicated, and the argument may not apply. Finally, when adding the remainder of the standard model, the situation is more involved, especially due to the presence of the fermion fields, and there is no similar simple argument. Thus, the phase diagram of more complicated theories has not yet been classified with the same level of rigor.

6.3 Reducing phenomeonlogy to perturbation theory

In the previous subsection the problem arose that the Higgs and W/Z fields are actually not really gauge-invariant, and in fact the whole BEH mechanism is not. The question thus arose what is actually measured when seeing peaks associated with electroweak particles in cross sections, why these peaks have (essentially) the same properties as the elementary particles, and why the elementary particles drop out of the spectrum, despite the BRST construction in section 6.1.3.

6.3.1 The failure of the perturbative state space construction

The first step is to understand why the perturbative physical space construction is failing, thus requiring to find a better one. This can be essentially traced back to two problems. One is the assumption that the limit of coupling constants to zero, underlying perturbation theory, is incorrect. This is known as Haag's theorem. Intuitively, this can be understood as that when the gauge coupling would vanish, the symmetries of the Higgs become global. But already the vector space for a theory with local or global symmetry are different. Thus, the asymptotic states space of perturbation theory cannot be unitarily equivalent to the interacting one. This also invalidates the physical state space construction, as the identifications (6.3-6.6) do not hold. As a consequence, the dependency on the (gauge) coupling constants cannot be analytic.2

The second is that this non-analyticity has consequences. A tacit statement was that gauge conditions like the 't Hooft gauge condition can be satisfied by one and only one gauge copy. In perturbation theory, this can be proven to be the case, using analyticity in the coupling constant. Thus, the proof fails non-perturbatively. The consequence is the so-called Gribov-Singer ambiguity, which implies that there are multiple solutions. Especially, it can be shown that the Faddeev-Popov determinant (5.2), which is positive definite in perturbation theory, can have now either sign. This allows for cancellations, explaing how obejcts being part of the perturbative state space can no longer be in there beyond perturbation theory.

Finally, of course, without gauge-fixing the same mechanism as discussed in section 3.2.1.1 takes place: All gauge-dependent observables necessarily vanish without a full gauge fixing. And with gauge fixing the results for gauge-dependent quantities depend on the gauge, and are thus unphysical.

At the moment, there is no full non-perturbative construction available, which could take over the role of the quartet mechanism and BRST symmetry, at least not for the case of hidden symmetry. This is still under development. However, all of this together is enough to show that the physical state space can at most contain gauge-invariant states[9].

6.3.2 The Fröhlich-Morchio-Strocchi mechanism

As before, it is simpler to first discuss only the case with the Higgs and the gauge bosons and afterwards continuing to include the remainder of the standard model, which in this case is actually possible.

The first realization necessary is that to describe physical objects requires operators which are manifestly gauge-invariant. For a non-Abelian gauge theory, like the one under discussion, this is only possible in case of composite operators, i. e. operators involving more than a single field, since any single-field operators are gauge-dependent.

Such gauge-invariant operators can then only be classified in terms of global quantum numbers, i. e. in the present case spin and parity as well as the custodial structure. Any open gauge index would yield that the quantity in question would change under a gauge transformation.

The simplest example of such an operator would be

$$\mathcal{O}_{0^+}(x) = \phi_i^\dagger(x)\phi_i(x),$$

created from the Higgs field ϕ and being a scalar and a singlet under the custodial symmetry, as well as a gauge-singlet. This operator creates a Higgs and an anti-Higgs at the

[9]They also need to be renormalization-group and renormalization-scheme invariant. However, these statements are not particular to the standard model, nor to gauge theories, and will therefore not be detailed here.

same space-time point, and therefore corresponds to a bound state of two Higgs particles, just like a meson in QCD. It is a well-defined physical state, and therefore observable.

So far, this is formally all correct. However, the immediate question appearing is that the description of the observed Higgs agrees very well with the one obtained in perturbation theory, and thus the elementary Higgs. But such a bound state, as is seen in QCD, can have widely different properties from its constituents. Thus, the two views seem to be at odds with each other.

However, there is a resolution for this apparent paradox, the so-called Fröhlich-Morcchio-Strocchi (FMS) mechanism. The mechanism itself will actually only be part of the explanation, as it is only a description of how to determine perturbatively the mass of this state.

To do this, consider the propagator of the composite state of a singlet scalar particle,

$$\langle \mathcal{O}_{0^+}(y)^\dagger \mathcal{O}_{0^+}(x)\rangle = \langle \phi_j^\dagger(y)\phi_j(y)\phi_i^\dagger(x)\phi_i(x)\rangle.$$

As usual, the poles of this correlation function will give the mass of the particle.

As the next step, select a gauge, like the 't Hooft gauge, in which the vacuum expectation value v of the Higgs field does not vanish, and rewrite $\phi_i(x) = vn_i + \eta_i(x)$. Then perform a formal expansion in the quantum fluctuation field η, yielding to leading order

$$\langle \phi_j^\dagger(y)\phi_j(y)\phi_i^\dagger(x)\phi_i(x)\rangle = v^4 + v^2\langle n_j^\dagger \eta_j^\dagger(y) n_i\eta_i(x)\rangle + \mathcal{O}\left(\left(\frac{\eta_i}{v}\right)^3\right). \quad (6.8)$$

Neglecting the higher order contributions, the only pole on the right-hand side is the one of the propagator of the fluctuation field. Thus, to this order, the masses coincide[10], and the bound state has the same mass as the elementary particle, showing why the perturbative result provides the correct mass for the observable state. Thus, this justifies why it is correct to use perturbation theory, and the perturbative spectrum, to obtain the mass of the Higgs[11].

In the same way, it is possible to construct a non-perturbative partner state for the gauge bosons, using the operator (4.2), to construct

$$\mathcal{O}^a_{1^-\mu}(x) = \mathrm{tr}\tau^a X^\dagger D_\mu X, \quad (6.9)$$

[10]Beyond leading order in the weak coupling constant the mass of the Higgs becomes scheme-dependent. It is then necessary to do this comparison in the pole scheme. It is then found that this holds to all orders in perturbation theory.

[11]The validity of the expansion, and whether for a given set of parameters, the expansion is actually valid is a dynamical question, and requires to determine both sides non-perturbatively, or the left-hand-side by experiment. It works for the ones in the standard model, but by far not for all possible parameter sets of the theory. E. g, in the QCD-like region, where v vanishes, it is not even defined.2

which is a custodial triplet vector, and a gives the corresponding index. Using that the vacuum expectation value is constant, this yields

$$\langle \mathcal{O}^{a\dagger}_{1^-\mu}(y)\mathcal{O}^{a}_{1^-\mu}(x)\rangle \sim v^4 g^2 \langle W^i_\mu(y) W_i^\mu(x)\rangle + \mathcal{O}\left(\frac{\eta}{v}\right) \tag{6.10}$$

and thus the mass of the W and Z are obtained, as well as the correct number of states, trading a custodial triplet for a gauge triplet. Note that because the masses of the gauge bosons are both scheme-invariant and gauge-parameter-invariant in perturbation theory in 't- Hooft-type gauges, this is actually an even stronger statement than for the Higgs itself.

It is possible to construct also operators for other quantum numbers, but only these two channels have a leading non-zero contribution given by one of the elementary fields. This also implies that in this expansion there are no other bound states than just these two[12]. The only states found in these channels are so-called scattering states. These are collections of other particles, whos quantum numbers and relative momenta add up such that they produce the corresponding quantum numbers of a channel. They also appear in (6.8) and (6.10) at higher orders.

This shows why the perturbative predictions provide the correct results. In fact, also scattering processes are dominated by the higher-order perturbative corrections, if the ratio η/v is sufficiently small, and in suitable kinematical regimes. Hence, to a very good approximation a perturbative description of this theory can be sufficient. Given the good accuracy of the perturbative description of the most recent experimental results, the non-perturbative corrections for the investigated processes are at most at the percent level, at least at currently accessible energies.

6.3.3 Adding the rest of the standard model

Adding the remainder of the standard model is possible, but requires a careful distinction of the various cases.

6.3.3.1 QED and mixing

Adding in hypercharge, and thus ultimately QED, makes the situation somewhat more involved. One of the reasons is that hypercharge, as an Abelian gauge theory, yields an observable charge. The observable charged states are obtained by dressing the gauge-dependent states with a Dirac phase. This structure is recovered for the bound states by the FMS mechanism.

[12]Whether this is true beyond leading order is still an open question. Since no formal proof exists, this requires to perform actual non-perturbative calculations, which is quite non-trivial.

This works because of the direct product structure of the weak and hypercharge interaction, as discussed in section 4.2. The Higgs field transforms like $\phi \to g\phi = \exp(i\alpha(y))\phi$ under a hypercharge gauge transformation $g(y)$. However, X does not transform linearly under a hypercharge transformation, but by $X \exp(i\alpha(y)\tau^3)$. This will be necessary in the following. Only because an Abelian gauge symmetry allows for the existence of an (almost) global U(1) group, the electric charge remains to classify states.

Consider the observable particles. Neither for the physical custodial triplet vector nor the scalar singlet anything changes, as they are either just linear combinations or unchanged compared to the previous case. Their operators are always mapped to the elementary $W^\pm$ and Higgs. In fact, despite their name, the $W^\pm$ bosons are not even charged under hypercharge, and thus need no Dirac phase. Just as in the perturbative treatment their apparent electric charge comes from the interaction with the W^3 component. This arises from the mixing of W^3 with the hypercharge boson to yield the Z and the photon A, as discussed in section 4.3.

For this mixing, the situation is more involved, but follows the sames rules as before. Gauge-invariant equivalent states can be constructed as

$$\mathcal{O}_Z = \sin\theta_W d\mathcal{O}^3_{1^-} + \cos\theta_W DB \approx \sin\theta_W W^3 + \cos\theta_W DB + \mathcal{O}(v^{-1}) = Z + \mathcal{O}(v^{-1}) \quad (6.11)$$

$$\mathcal{O}_A = \sin\theta_W d\mathcal{O}^3_{1^-} - \cos\theta_W DB \approx \sin\theta_W W^3 - \cos\theta_W DB + \mathcal{O}(v^{-1}) = A + \mathcal{O}(v^{-1}) \quad (6.12)$$

where D is the necessary Dirac projector, and a factor v^2 has been absorbed in the relative normalization of W^3 and B. It should be noted that both have the same quantum numbers, i. e. gauge-invariant vectors with no conserved quantum numbers, because $\mathcal{O}^3_{1^-}$ was the custodial charge multiplet member with third component zero. The FMS mechanism maps these operators to the usual elementary ones, the elementary Z boson and the photon γ. Thus, the corresponding propagators also have the correct pole structure. The basis (6.11-6.12) represents the mass eigenbasis of this channel.

Moreover, consider the three-point function for the charged W bosons and Photons. Expanding yields

$$\begin{aligned}\left\langle \left(\mathcal{O}^+_{1^-_3}\right)_\mu \left(\mathcal{O}^-_{1^-_3}\right)_\nu (\mathcal{O}_\Gamma)_\rho \right\rangle &\approx v^4 \left\langle w^+_\mu w^-_\nu \gamma_\rho \right\rangle + \mathcal{O}(v^3) \qquad (6.13)\\ &= v^4 \left(\sin\theta_W \left\langle w^+_\mu w^-_\nu w^3_\rho\right\rangle + \cos\theta_W \left\langle w^+_\mu w^-_\nu b_\rho\right\rangle\right) + \mathcal{O}(v^3)\\ &\overset{\text{tree-level}}{\approx} v^4 \sin\theta_W \left\langle w^+_\mu w^-_\nu w^3_\rho\right\rangle_{\text{tl}}.\end{aligned}$$

The left-hand-side is a combination of three separately gauge-invariant quantities. Each of them has been expanded separately. This used that the operator (6.9) in the standard choice $X \sim v1$ expands like

$$\text{tr}\tau^a x^\dagger D_\mu x = v^2 w^b_\mu \text{tr}\tau^a\tau^b + \mathcal{O}(v) = w^a_\mu + \mathcal{O}(v).$$

Hence, to leading order, the physicalvertex is given by the perturbative W^+W^-S one. As in perturbation theory, this vertex comes at tree-level actually from the SU(2) vertex due to the $W^+W^-W^3$ interaction only, combined with a factor of $\sin\theta_W$, thus reproducing the usual electric charge as coupling constant[13]. This explicitly shows that the would-be electric charges of the physical $W^\pm$ are only a mapping of custodial indices to gauge interactions.

In the same manner also the other vertices arise. Thus, the electromagnetic interaction is, up to the level of Higgs fluctuations, not modified when using the observable particles instead of the elementary degrees of freedom. Thus, in the same sense as at the perturbative level, the $W^\pm$ are electrically charged, and the two operators (6.11) and (6.12) are electrically uncharged vector singlets. The spectrum and interactions observed in experiment is recovered.

The usual singlet scalar, that is the physical Higgs, remains a singlet with respect to electric charge. It would now be possible to construct a non-singlet, gauge-invariant scalar operator, e. g. by a combination of a Z and a $W^\pm$. However, these states expand to scattering states, and thus yield no additional states in the physical spectrum. In particular, they do not expand to, e. g., charged Goldstone bosons, as the contribution of Goldstones vanish in the expansion because they are BRST non-singlets.

6.3.3.2 Flavor symmetries

The situation becomes substantially more involved when introducing fermions. For simplicity, ignore for now the Yukawa couplings as well as hypercharge. Also, consider only a single generation of left-handed leptons. The rest will be added later.

In the standard model, these leptons are doublets under the weak interactions. Therefore, these fermions can again not be observable particles, as they depend on the gauge. Gauge-invariant states can be constructed for the fermion spinor ψ describing these leptons by a dressing with a Higgs field,

$$\mathcal{O}_\Psi(y) = (X\epsilon)^\dagger(y)\psi(y), \tag{6.14}$$

where ϵ is the antisymmetric tensor of rank two[14]. This is by construction gauge-invariant, and remains a (left-handed) fermion. It is important to note that every component of the

[13] The prefactor v^4 seems to deny this possibility at first. However, at leading order in v always the same combinations of v will appear, and thus a common factor can be absorbed in the renormalization process. This will only start to play a role when subleading terms in v are included, and relative weights become important.

[14] Alternatively either the assignment of the hypercharge values of the fermions can be reversed.

operator $\mathcal{O}_\Psi$ is a left-handed spinor. Because $X^\dagger$ transforms by a left-multiplication under the custodial symmetry, the state is a custodial doublet. Thus, the weak gauge charge, which is perturbatively associated with the up/down or lepton/neutrino distinction is replaced by a custodial charge for the physical states. In fact, the same charge which distinguishes the physical $W^\pm$ and Z now distinguishes also the components of a physical fermion. These two custodial charge states are the physically observable fermions, e. g. the electron and the electron-neutrino.

To identify their connection to the elementary fermions, consider the electron-neutrino case. Denoting the two fundamental weak states ψ^1 and ψ^2 by e and ν, the expansion yields

$$\mathcal{O}_{NE} = \begin{pmatrix} N \\ E \end{pmatrix} = (X\epsilon)^\dagger \begin{pmatrix} \nu \\ e \end{pmatrix} = \begin{pmatrix} \phi_2\nu - \phi_1 e \\ \phi_1^*\nu + \phi_2^* e \end{pmatrix} \approx v \begin{pmatrix} \nu \\ e \end{pmatrix} + \mathcal{O}(v^0) \tag{6.15}$$

Thus, the usual electron and neutrino appear to leading order as the custodial states. Likewise, forming the propagator from $\mathcal{O}_{NE}$ this implies that the two custodial states expand to the tree-level propagators of the electron and the neutrino, and therefore have the same mass as the perturbative electron and neutrino. For this identification it is absolutely crucial that the Higgs is a scalar particle, as in any other case this would alter the spin-parity quantum numbers of the fermions. Thus, the perceived flavor symmetry of left-handed fermions needs to be identified with the custodial symmetry $\mathrm{SU}(2)_c$. In addition, there is a global $\mathrm{U}(1)_l$ fermion number symmetry, which only affects the fermions. Thus, at this point there is a global $\mathrm{U}(2)_L \sim\mathrm{SU}(2)_c\times\mathrm{U}(1)_l$ symmetry.

An extension to three generations is straightforward, as a generation index would be just pushed through to the bound state, and pushed back by the expansion. This gives an $\mathrm{SU}(3)_g$ additional global symmetry. Note that the total global symmetry is then $\mathrm{SU}(3)_g\times\mathrm{U}(2)_L$.

Return to the one-generation case. So far, there is no way to generate a tree-level mass for the fermions. This requires the Yukawa interaction. Introduce two right-handed additional fermions, a right-handed electron and a right-handed neutrino[15], not interacting weakly. Without any further distinguishing feature, this implies an additional $\mathrm{U}(2)_R \sim\mathrm{SU}(2)_{\text{flavor}}\times\mathrm{U}(1)_{\text{number}}$ flavor and right-handed counting symmetry. Thus at this point the global symmetry of the theory is $\mathrm{SU}(2)_c\times\mathrm{U}(1)_l\times\mathrm{U}(2)_R=\mathrm{U}(2)_L\times\mathrm{U}(2)_R$.

Combining the two right-handed leptons in a flavor spinor Ψ_R, a possibility is a Yukawa interaction

$$\mathcal{L}_Y = g_Y\left(\bar{\mathcal{O}}_{NE}\Psi_R - \bar{\Psi}_R\mathcal{O}_{NE}\right),$$

with g_Y the Yukawa coupling. Because this locks the symmetries, this breaks the custodial

[15]If the neutrino sector is different, this would require corresponding amendments to the construction.

and flavor symmetries down to a diagonal subgroup $U(2)_d$. This diagonal subgroup can be identified with the conventional, physical flavor group. This group can be further broken down to $U(1)_E \times U(1)_N$, separate counting symmetries of both electrons and neutrinos, by introducing a matrix Ω, which is not invariant under an SU(2) transformation, as

$$\mathcal{L}_Y = g_Y \left(\bar{\mathcal{O}}_{NE} \Omega \Psi_R - \bar{\Psi}_R \Omega^\dagger \mathcal{O}_{NE} \right). \tag{6.16}$$

If the matrix Ω is diagonal diag $= (g_\nu/g_Y, g_e/g_Y)$, this gives two independent interaction terms

$$\mathcal{L}_Y = g_\nu(\phi_2 \bar{\nu}_L - \phi_1 \bar{e}_L) N_R + g_e(\phi_1^* \bar{\nu}_L + \phi_2^* e_L) E_R + \text{h.c.},$$

which for $X \approx v1$ are nothing but the usual mass terms for the neutrino and the electron of the standard model. Then, the tree-level propagators obtain their usual mass, and by virtue of the FMS mechanism so does the propagator of the bound state (6.14). If Ω is not diagonal, a change of the basis in the fermion fields can make it so, at the expense of introducing a matrix in the weak interactions, in the same way as the CKM/PMNS matrices introduce such an effect at the level of generations. Furthermore, in the presence of multiple generations, intergeneration mixing can be absorbed as usual also in the CKM/PMNS matrices.

Adding further generations proceeds almost as before. However, so far this upgrades the right-handed flavor symmetry to an (S)U(6) symmetry, which is then explicitly broken by the Yukawa terms by selecting individual flavors for the interactions with the three left-handed generations, in contrast to the generation structure before. The global generation structure for left-handed fermions and right-handed fermions can therefore still differ. The same global generation structure for the left-handed and the right-handed symmetry will be enforced by the hypercharge.

Before proceeding, return to the case of a single left-handed doublet. In section 4.2 it was pointed out that the hypercharge is actually gauging the custodial symmetry. Thus the composite state (6.14) would apparently already transform under a hypercharge transformation. Now, in addition the fermions become coupled to the hypercharge, in principle with an arbitrary value. But actually, this has to happen with the same hypercharge for both members of the elementary fermion doublet, as it would otherwise break the weak gauge symmetry. Thus, ψ transforms as $\psi \to g\psi$. Because the hypercharge gauge transformation acts like diag$(g, g^\dagger)$ on X this yields a total transformation like diag$(1, g^2)$ for the state (6.14). With the usual hypercharge assignment for the elementary fields, g^2 is equal to a transformation with twice the hypercharge value, as it is an Abelian group. This shows explicitly that the physical Neutrino state is uncharged under the hypercharge, while the physical Electron is charged, with twice the hypercharge of the elementary electron.

The actual electric charge is then, as in (6.13), obtained by considering the expansion of $\left\langle \bar{\mathcal{O}}_\Psi \mathcal{O}_\Psi \mathcal{O}_\Gamma \right\rangle$. This expands for the various combinations of custodial indices to the usual tree-level interactions, yielding the correct assignment of the electromagnetic charges to the various states.

The only thing left is the assignment of hypercharge to the right-handed fermions. To make (6.16) gauge-invariant enforces the standard assignment of section 4.1, thereby breaking the right-handed flavor symmetry already down to $\mathrm{U}(1)^3$, or $\mathrm{SU}(3)\times\mathrm{U}(1)^2$ for three generations, as the right-handed neutrino and electron cannot have the same hypercharge value. Therefore the breaking pattern for left-handed and right-handed flavor due to the presence of hypercharge is different, as in one case the custodial symmetry is actually gauged, while in the other case the assignment of the hypercharges breaks the generation symmetry explicitly. Still, it leads to the same remaining global symmetry in the left-handed and the right-handed sector. Note that differing hypercharge assignments to the differing generations would be compatible with all results. They would merely introduce a different breaking pattern of the generations, though would prevent the introduction of intergeneration transitions except for very special relative values of the hypercharges, just as in perturbation theory.

The quark sector follows exactly the same structure. From the point of view of the electroweak and Higgs interaction, it is just a copy of the lepton sector, and operates, but for different values of the couplings, just like three more generations. But it is distinguished from it by color. Color has no bearing on the FMS mechanism or any of the present results. However, reversely the change of the meaning of flavor and the global symmetry structure of the theory have implications for QCD, as will be discussed in the next section 6.3.3.3.

Before this, it should be noted that also the anomaly cancellation to be discussed in chapter 9 is not altered at all. After all, the anomaly arises from the path integral measure which involves the elementary fields only. The observable, physical states, and the FMS expansion acting on it, are merely expectation values. They therefore never interfere with the mechanism for the intrageneration anomaly cancellation.

6.3.3.3 QCD and hadrons

As noted before, with respect to the electroweak interactions the same applies to quarks as to leptons. Considering only low-energy QCD, it is possible, because of (6.15), to first apply the FMS mechanism, to obtain the standard quarks, and then reduce to QCD alone. This leads to the usual QCD, and nothing changes. Thus, QCD in isolation emerges as a low-energy effective theory in the same way as in a purely conventional treatment of the standard model. This is therefore not particularly interesting in the present context.

However, this will only be correct as long as it is good to keep just the leading term of (6.15). As discussed repeatedly, this may only be good for energies below which Higgs production is not appreciable and/or virtual Higgs corrections are negligible. Since modern experiments probe well into this region, it is worthwhile to have a closer look at how QCD operates within the context of the full standard model.

Still, even in the full standard model confinement is operative. In the present subsection, confinement will be used in the operative sense that quarks and gluons are necessarily bound inside hadrons in such a way that the hadron is gauge-invariant with respect to color. There are thus three type of gauge-invariant operators describing hadrons, which need to be considered in the following.

One are those involving only gluons, the glueballs. Since no other charges are involved, they work in precisely the same way as in pure QCD. Most importantly, they are always 'flavor' singlets, and do not carry electric charge.

The second type are operators involving quarks, but with the same quantum numbers as glueballs. Therefore, they do not carry flavor and/or electric charge. With respect to the electroweak sector they are therefore uncharged, and thus completely gauge-invariant. Also for these nothing changes. An example of this is the σ meson, which carries the quantum numbers of the vacuum,

$$\mathcal{O}_\sigma = \bar{u}_L \sigma u_L + \bar{d}_L \sigma d_L + \bar{u}_R \sigma u_R + \bar{d}_R \sigma d_R \tag{6.17}$$

where the matrix σ transforms the Weyl spinors from left-handed to right-handed, and thus acts only in Dirac space to create Lorentz scalars. Thus, this operator is a singlet for all global symmetries. Another example is the ω meson, which is a vector singlet. It should be noted that such operators mix with all singlet operators, like the (physical) Higgs. Fortunately, this mixing seems to be small experimentally, so this subtlety can be ignored here.

A change happens for the third type of operators, which is any operator not belonging to either of the two previous classes. Thus, these are all flavor-non-singlet operators from the perspective of pure QCD. Consider for the moment the one-generation case only, and ignore both hypercharge and the Yukawa interactions. Then an eight-component spinor can be constructed as

$$\Psi = \begin{pmatrix} (X\epsilon)^\dagger \psi_L \\ u_R \\ d_R \end{pmatrix}, \tag{6.18}$$

where the Weyl spinors are not detailed further.

QCD constructed from theses spinors is invariant under a global symmetry of SU(2) $\times$SU(2)$\times$U(1), dropping another U(1) due the axial anomaly. The first SU(2) acts only on the first two components, the custodial symmetry, and the second one only on the right-handed flavor symmetry. Since the custodial and right-handed flavor symmetry are not explicitly broken, the left-handed and right-handed part do not mix under the symmetry transformation. Thus, this symmetry replaces the usual chiral and flavor symmetry of QCD. By a change of basis it is possible to reestablish this symmetry, but this has also impact on the weak gauge boson physical particles, which are charged under the custodial symmetry.

Consider now a state like a pion, which has the structure

$$\pi^a = \bar{\Psi}\tau^a\gamma_5\Psi,$$

where the Dirac representation is chosen for the γ matrices. This operator mixes the various states, resulting in

$$\pi^+ = \bar{d}_R((X\epsilon)^\dagger\psi_L)^u + \overline{((X\epsilon)^\dagger\psi_L)}^d u_R \quad (6.19)$$

$$\pi^- = \bar{u}_R((X\epsilon)^\dagger\psi_L)^d + \overline{((X\epsilon)^\dagger\psi_L)}^u d_R \quad (6.20)$$

$$\pi^0 = \bar{d}_R((X\epsilon)^\dagger\psi_L)^d + \overline{((X\epsilon)^\dagger\psi_L)}^d u_R - \bar{u}_R((X\epsilon)^\dagger\psi_L)^u - \overline{((X\epsilon)^\dagger\psi_L)}^u u_R, \quad (6.21)$$

where the indices u and d identify the components of the spinors $(Y\epsilon)^\dagger\psi_L$. The first remarkable insight is that, in contrast to the σ meson (6.17), such operators always involve three valence particles, the two quarks and a higgs in form of X. This is necessary as otherwise the mixing of left-handed particles and right-handed particles to create a pseudo-scalar would not be possible gauge-invariantly.

Of course, applying the FMS mechanism will reduce (6.19-6.21) immediately to the usual operators of QCD to leading-order in v, reestablishing the pions as quark-anti-quark bound states, but multiplied with the Higgs vacuum expectation value. But this is a statement in a fixed gauge. The genuine gauge-invariant expression inside the standard model has a valence Higgs besides the valence quarks. This is remarkable for multiple reasons. One is that the mass defect is actually large. But this also implies that to excite the Higgs substructure will require a substantial amount of energy, and the corrections at the sub-leading order are generically small.

Concerning global symmetries, the pions therefore mix already the symmetries, and the operators (6.19-6.21) are neither custodial nor right-handed flavor eigenstates. If both would be separately conserved, any pion current would be needed to be decomposed into the two component currents. But this becomes irrelevant as soon as the Yukawa couplings are turned on, as both symmetries are then explicitly broken, and can mix freely. This explicit breaking also translates into the usual explicit breaking of chiral and flavor symmetry in the stand-alone QCD and/or the FMS expansion to QCD.

The introduction of hypercharge emphasizes this even more. Because the left-handed quarks and the right-handed quarks have a different hypercharge just building a Dirac spinor from them is not possible in a gauge-covariant way. However, by the combination with the Higgs field in (6.18) having also hypercharge, (6.18) transforms in a gauge-covariant fashion, and in particular allows the operators (6.19-6.21) to have a well-defined electric charge. Note that any other relative assignments of hypercharges to the higgs and the quarks would not create suitably electrically charged hadrons. That this involves also the Higgs and not just the quarks in stand-alone QCD makes the particular assignments of hypercharges to the elementary particles of the standard model even more mysterious than it already is.

All of this becomes much more dramatic for baryons. Because any baryon has an odd number of quarks, it is impossible to couple left-handed fermions together such that they form at the same time a gauge-invariant state under the strong and the weak interaction. Therefore, any baryon involving a left-handed component necessarily involves a valence Higgs. Moreover, as hadrons are essentially parity eigenstates, any baryon contains such a left-handed component.

Consider for a moment just a system of left-handed fermions. Then

$$N_L = \epsilon^{IJK} c_{ijkl} q_i^I q_j^J q_k^K (X\epsilon)^\dagger_{\tilde{i}l} \tag{6.22}$$

$$\Delta_L = \epsilon^{IJK} q_i^I q_j^J q_k^K (X\epsilon)^\dagger_{\tilde{i}i} (X\epsilon)^\dagger_{\tilde{j}j} (X\epsilon)^\dagger_{\tilde{k}k} \tag{6.23}$$

form two possible three-quark operators, which are gauge-invariant under both the strong interactions, denoted by capital indices, and the weak interaction. The operator (6.22) corresponds to cases which have one open custodial index, like the nucleons, while the operator (6.23) corresponds to cases with three open custodial indices, like the Δ^{++} and the Δ^-. The matrix c depends on the other properties of the baryon in question. E. g. for a proton it becomes

$$c^p_{ijkl} = a_1 \epsilon_{ij} \delta_{kl} + a_2 \epsilon_{ik} \delta_{jl} + a_3 \epsilon_{jk} \delta_{il}$$

while for the Δ^{++} it suffices to set in (6.23) $\tilde{i} = \tilde{j} = \tilde{k} = 1$.

Returning to the full theory with both left-handed and right-handed quarks, a nucleon operator is, e. g., constructed as

$$N^{rst} = \epsilon^{IJK} F^{rst}_{uvw} \Psi_{Iu} \left(\Psi^T_{Jv} C \gamma_5 \Psi_{Kw} \right) \tag{6.24}$$

where F acts in custodial/right-handed flavor space and the charge-conjugation matrix is $C = i\gamma_2\gamma_0$. The expression in parentheses is a scalar diquark, and as such mixes left-handed and right-handed components. The fermionic nature is carried entirely by the first quark field. To obtain definite parity states requires projection with a parity operator, essentially $(1 \pm \gamma_0)/2$. Since parity transforms left-handed to right-handed, a parity eigenstate necessarily contains both left-handed and right-handed components. Therefore, in the full standard model there is also a Higgs valence contribution in the nucleon. Actually, because the scalar diquark combines products of left-handed and right-handed components of the Ψ field, the proton, being a positive parity state, is a mixture of a state with one and three valence Higgs fields.

It is, of course, possible to write down operators with definite valence Higgs contributions. However, these are no longer mass eigenstates, and therefore are only of limited use when considering the low-energy limit of stand-alone QCD. And again, only because the Higgs is a scalar particle its presence does not alter the quantum numbers of the hadrons. If it would be, e. g., a pseudoscalar, stand-alone QCD as the low-energy limit of the standard model would not describe hadron physics correctly.

6.3.4 Describing interactions

So far only static properties have been considered. Most of our knowledge stems, however, from dynamic processes, especially scattering experiments. This will be discussed in great detail in chapter 7. However, it is necessary to have a physical picture for it. It is therefore necessary to develop a description of scattering processes based on gauge-invariance, which at the same time also explains quantitatively why the conventional perturbation theory to be used in chapter 7 works so well for the standard model.

Before doing so, it is worthwhile to discuss both how initial and final states operate. The gauge-invariant operators themselves are just bound-state operators. Thus, scattering processes which involve these states as initial and final states can be described by the usual LSZ construction[16] for bound states in terms of the corresponding matrix elements, having the bound states as in and out states. These can be calculated using the FMS mechanism, as will be shown in the following.

The more interesting question regards the elementary states, and especially, why they are not suitable in states and out states. Technically, this is not an issue, as the example of QCD shows, where quarks and gluons are never considered as in and out states. But there this is linked to confinement. Thus, this leaves only the question of why this should not occur, or even why the composite states should not decay into the elementary states. This latter question is probably the best one to discuss the problem.

[16]See the lecture on quantum field theory.

Hence, consider a matrix element

$$\left\langle \mathcal{O}(x)^\dagger \varphi(y)\varphi(z)\right\rangle, \tag{6.25}$$

where $\mathcal{O}$ is a gauge-invariant composite state and the fields φ are some of the elementary, gauge-dependent states. To define suitable asymptotic states, it will be necessary to take eventually the limit that x, y, and z are sufficiently far separated.

Of course, the final state is not gauge-invariant. But the usual perturbative argument is that at asymptotically large distances the out states become non-interacting, and thus only global transformations apply anymore. This violates Haag's theorem, and the interacting states cannot be made gauge-invariant, as was discussed in section 6.3.1. Thus, the final state is not gauge-invariant, and is unphysical. This implies especially that cluster decomposition fails for the elementary states non-perturbatively.

This does not mean that (6.25) is an irrelevant expression. Consider The situation, where the state described by $\mathcal{O}$ can decay into gauge-invariant composite states $\mathcal{O}' = \phi\varphi$, which in leading order in the FMS expansion behave as $\mathcal{O}' \sim v\varphi$. Then

$$\langle \mathcal{O}(x)\mathcal{O}'(y)\mathcal{O}'(z)\rangle = v^2 \langle \mathcal{O}(x)\varphi(y)\varphi(z)\rangle + \mathcal{O}(v).$$

Thus, to leading order in v, the decay can be approximated by (6.25). If the residual, asymptotic gauge-dependence is sufficiently small compared to the matrix element itself, this will be a very good approximation. This is the case in the standard model.

It is, of course, possible to expand even (6.25) further, by also expanding $\mathcal{O}$ according to the FMS mechanism. If such a full expansion is performed, this gives an expression to treat decay processes in this framework. This is the strategy to actually calculate cross sections and decays using a gauge-invariant starting point and the FMS mechanism. This will be exemplified in section 6.3.5 . However, the gauge-invariant matrix element is then approximated by a matrix element entirely from gauge-dependent fields. Such a gauge-dependent matrix element needs not to follow the usual rules of physical matrix element. Interestingly, however, in first applications of such calculations it appears that the unphysical contributions cancel order-by-order in a perturbative expansion. It remains to be seen, whether this is generally true.

6.3.5 Scattering processes

The simplest example for a scattering process in a gauge-invariant setup will be a lepton collider, avoiding the complications due to the QCD. The archetypical process is $e^+e^- \to \bar{f}f$, which is at the same time also experimentally reasonably easy to access with high sensitivity.

Given the discussion of section 6.3.3.2, the incoming Electron and Positron need to be considered as higgs-electron/positron bound states. A hand-waving argument why no effect has been seen so far is that the very massive valence Higgs acts entirely as a spectator particle, like the non-participating quarks and gluons in a hadron collision. Then the following picture emerges at leading order. The valence Higgs simple does not participate in the interaction, and only dresses the initial, the intermediate, and the finial state. The actual interaction therefore does not differ to this order from the conventional perturbative picture, and thus should be described to leading order just by ordinary perturbation theory. Only if the Higgs does no longer act as a spectator this will change.

Since off-shell contributions are suppressed at leading order, the requirement will be to get the Higgs on-shell. Though due to the renormalization-scheme dependence this is a non-trivial statement, this will essentially always be some kind of typical electroweak scale, and in the pole scheme the singlet scalar mass. Since the Higgs-electron-Yukawa coupling is tiny in comparison to the electroweak coupling or the three-Higgs coupling the most relevant process will require an interaction involving both Higgs from both bound states to participate[17]. This suggests a scale of $\sim 2m_h$. Hence, even at LEP2 this process will be strongly suppressed. The Applequist-Carrazone theorem suggests a suppression of at least $s/(4m_h^2)$, which is at the working point of LEP2, the Z mass, at least a suppression by a factor ten. This seems at first sight not a sufficiently strong suppression to avoid detection there.

Is is therefore necessary to investigate the process further to see why it is actually even more suppressed. To do so, it is instructive to follow the principles of the FMS mechanism. The relevant matrix element is then given by

$$\mathcal{M} = \left\langle \mathcal{O}_2^{NE}(p_1)\bar{\mathcal{O}}_2^{NE}(p_2)\mathcal{O}_i^f(q_1)\bar{\mathcal{O}}_i^f(q_2)\right\rangle \tag{6.26}$$

where only the lower custodial component, i. e. the electron component in the sense of (6.15), is considered in the initial state. The final-state is defined to be of two fermions of the same type, an exclusive measurement, and therefore the custodial index is not summed over.

At leading order in v this yields

$$\mathcal{M} \sim v^4 \left\langle e^+e^-\bar{f}f\right\rangle + \mathcal{O}(v^2) \tag{6.27}$$

and thus corroborates the deduction above. In this order the matrix element is just the ordinary, full matrix element. Expanding this matrix element in perturbation theory

[17] Or an experiment sensitive on the level of the Higgs-electron-Yukawa coupling.

further reproduces the conventional result to all orders in the couplings. Thus, there is no difference at leading order in v.

Including further terms yields

$$\mathcal{M} \approx v^4 \left\langle e^+e^-\bar{f}f\right\rangle + v^2 \left\langle \eta^\dagger\eta e^+e^-\bar{f}f\right\rangle + \left\langle \eta^\dagger\eta\eta^\dagger\eta e^+e^-\bar{f}f\right\rangle + \text{rest}. \tag{6.28}$$

Terms with an odd number of fields will have additional particles in the initial or final state, and will therefore not contribute if an exclusive measurement is done. While, for the sake of brevity, not all terms are written down and arguments are suppressed, all relevant structures appear. It is now seen that the further terms are suppressed by two or four powers in v. Thus, besides the relative suppression of matrix elements from the Appelquist-Carrazone theorem a suppression of order s/v^2 arises, which is at the Z mass of order ten as well. Both are multiplied, given a total factor 100 of suppression in the matrix element alone on dimensional grounds. This is now much more in-line with the order of sensitivity with which the process has been probed.

However, investigating the present terms further yields even more suppression. The second term in (6.28) adds a pair of Higgs fluctuation fields. Depending on the arguments, they can form an interaction in the initial or final state, or can add a propagating Higgs. The last term includes all possibilities where the Higgs can contribute in both, initial and final state. However, it is even more suppressed, and will therefore be neglected.

To understand what the processes of the second term in (6.28) involve, it is useful to expand the second term further, keeping only leading interactions

$$\left\langle \eta^\dagger\eta e^+e^-\bar{f}f\right\rangle \approx \left\langle \eta^\dagger\eta\right\rangle \left\langle e^+e^-\bar{f}f\right\rangle + \left\langle e^+e^-\right\rangle \left\langle \eta^\dagger\eta\bar{f}f\right\rangle + \left\langle \bar{f}f\right\rangle \left\langle e^+e^-\eta^\dagger\eta\right\rangle. \tag{6.29}$$

Thus, there appear three types of corrections to the leading process of (6.27). The first term adds a correction to the perturbative leading term, which is suppressed by a factor $1/v^2$ and a non-interacting spectator higgs. Such a contribution will therefore only affect the process like the presence of a spectator in a hadron collision, and thus is not relevant, except for the formation of the initial and final state.

The second and third term in (6.29) correspond to an interaction of the initial state or final state higgs with the corresponding fermions, and the other fermions acting as spectators. The latter two processes therefore correspond to a reaction of the second constituent of the fermionic bound-state in the initial or final state. At leading order perturbative corrections can then be calculated just by expanding all appearing correlation functions to the corresponding order in perturbation theory. Since double-Higgs production has not been observed at LEP2, the last matrix element will not be relevant, as it is already negligible as a leading effect. So only the one with a Higgs in the initial state could contribute.

However, at leading-order this matrix-element is in the s-channel proportional to the product of the three-Higgs coupling and the Higgs-fermion Yukawa coupling and in t and u channel to the Higgs-fermion Yukawa coupling squared. Thus, it is negligible compared to the leading term for anything but the the top. Hence, it will again be irrelevant at LEP2.

These considerations show why no contribution at LEP2 of the bound-state structure should be expected. While this was a particular investigation, similar considerations will apply to any other process. And most other processes have not been measured as well at these energies. Thus, this answers why so far nothing has been seen. All presently discussed future machines will work at, or substantially above, $\sqrt{s} = 2m_h$. Therefore, no off-shell suppression will arise. Also, double Higgs and/or double top production is reachable with reasonable sensitivity. Hence, they are the perfect machines to look for these processes or at least give a reasonable bound on them. To have a theoretical estimate, one possibility is to evaluate expressions like (6.29), or even (6.28), perturbatively. This will give a perturbative estimate of the size of the effect. While straightforward, this has not yet been completed given that the number of involved diagrams, especially when it comes to six-point functions like in (6.28), is not small even at tree-level. Not to mention loop-level. This approach misses, of course, part of the bound-state effects in the initial and final states. Whether these are quantitatively relevant is unknown.

Chapter 7

Phenomenology at tree-level

In this chapter some sample applications of the electroweak standard model will be discussed. For that, some processes will be investigated which will provide insight into the salient features of electroweak processes. For this purpose perturbation theory will be employed, in the sense of neglecting higeher orders in v in the FMS expansion, and thus keeping only the term yielding conventional perturbation theory, see ection 6.3.5. That will permit also some technical simplifications, since certain processes are trivially suppressed without any non-perturbative self-amplification effect. In particular, all neutrino masses will be set to zero, making the mixing matrix V^l a unit matrix. This is at least for the energies that are accessible at Tevatron and was accessible at LEP(2) justified, as the excellent agreement between ((N)NLO) perturbation theory and the experimental results demonstrate. Even for the LHC up to and including (at least) run 2, this holds to very good accuracy.

7.1 Drell-Yan production of W-bosons

The first process is of course the one which caused problems in the case with an intermediate vector boson which was not a gauge boson. This will show how the gauge symmetry, even when not manifest, takes care of the problem, and gives a result which is consistent with unitarity.

In the present case, it is simpler to investigate the production of $W^{\pm}$ from electron-positron scattering as then all intermediate states are massless neutrinos. This is not making a big difference compared to the case of production by neutrino annihilation otherwise. Furthermore, since the center-of-mass energy for real $W^{\pm}$ production has to be at least twice the W mass, the electron mass can be safely neglected. This process is also actually the one used for the determination of the W mass M_W experimentally at the LEP experiments.

To fix the notation, the incoming $e^\pm$ momenta and helicities will be denoted by $p_\pm$ and $\sigma_\pm$, those of the outgoing $W^\pm$ pair by $k_\pm$ and $\lambda_\pm$, respectively. It is useful to go into the center-of-mass frame in which the momenta read

$$\begin{aligned} p_\pm^\mu &= (E, 0, 0, \pm E)^T \\ k_\pm^\mu &= (E, \pm E\beta \sin\theta, 0, \pm E\beta\cos\theta)^T \\ \beta &= \sqrt{1 - \frac{M_W^2}{E^2}}, \end{aligned}$$

where the momenta of the $W^\pm$ are determined purely by momentum conservation, and the angle is arbitrarily the one between the positron and the W^+.

Very useful in this calculation, and other similar ones, are the Mandelstam variables

$$\begin{aligned} s &= (p_+ + p_-)^2 = (k_+ + k_-)^2 = && 4E^2 \\ t &= (p_+ - k_+)^2 = (p_- - k_-)^2 = && M_W^2 - 2E^2 + 2E^2\beta\cos\theta \\ u &= (p_+ - k_-)^2 = (p_- - k_+)^2 = && M_W^2 - 2E^2 - 2E^2\beta\cos\theta. \end{aligned}$$

It is furthermore necessary to give the polarization vectors for the $W^\pm$ bosons, i. e., the solutions for the non-interacting equations of motion for the $W^\pm$ bosons. These are given in the center of mass system by

$$\begin{aligned} \epsilon_\pm^{+\mu}(k_\pm, -1) &= \frac{1}{\sqrt{2}}(0, \pm\cos\theta, -i, \mp\sin\theta) \\ \epsilon_\pm^{+\mu}(k_\pm, -1) &= \frac{1}{\sqrt{2}}(0, \pm\cos\theta, i, \mp\sin\theta) \\ \epsilon_\pm^{+\mu}(k_\pm, 0) &= \frac{E}{M_W}(\beta, \pm\sin\theta, 0, \pm\cos\theta) = \frac{k_\pm^\mu}{M_W} - \frac{M_W}{E(1+\beta)}(1, -\pm\sin\theta, 0, \mp\cos\theta). \end{aligned}$$

With these two caveats are attached. First of all, since the equations of motion are gauge-dependent, so are these polarization vectors, valid only in the chosen ('t Hooft) gauge. Secondly, the helicity for a massive particle is not an independent quantum number, and therefore the assignment of -1, 1 and 0 is only valid within a given frame. In particular, the concept of, say, production of helicity + W-bosons is only valid in a fixed frame. The same process in another frame would in general yield a mixture of all helicities.

In the calculation at hand, the electron is assumed massless, and therefore all lepton helicities are conserved. Therefore, the helicities of electron and positron, labeled as $\sigma_\pm$, must obey $\sigma_- = -\sigma_+$. The $-$-sign denotes left-handed, as is conventional.

In the process as such three subprocesses can interfere. One is ordinary Drell-Yan production of $W^\pm$ bosons with an intermediate neutrino. The two other possibilities, possible due to the non-Abelian nature of the weak interaction and the photon, is that the electron-positron pair first annihilates into a virtual Z-boson or a photon, which then decays afterwards into the $W^\pm$ pair. At tree-level, these are the only processes which contribute. In the first purely weak case parity is maximally violated, and therefore only left-handed electrons contribute. In the second case, due to Z-photon mixing, both helicities can contribute.

To keep the notation simple, it is useful to use the abbreviations

$$\begin{aligned} M_t^\pm(\lambda_\pm) &= \bar{v}(p_+)\gamma_\mu\epsilon_+^\mu\gamma_\nu(k_+^\nu - p_+^\nu)\gamma_\rho\epsilon_-^\rho\frac{1\pm\gamma_5}{2}u(p_-) \\ M_s^\pm(\lambda_\pm) &= \bar{v}(p_+)(2\gamma_\mu\epsilon_+^\mu\epsilon_-^\nu k_{\nu+} - 2\gamma_\mu\epsilon_-^\mu\epsilon_+^\nu k_{\nu-} - \gamma_\mu(k_+^\mu - k_-^\mu)\epsilon_+^\nu\epsilon_{\nu-})\frac{1\pm\gamma_5}{2}u(p_-), \end{aligned}$$

and the spinors v and u denote the positron and electron spinor, respectively.

Using these abbreviations, the tree-level (Born) amplitude for the process is given for an electron of helicity σ, W helicities $\lambda_\pm$ and as a function of the Mandelstam variables by

$$\begin{aligned} M_B(\sigma,\lambda_\pm,s,t) &= \frac{e^2}{2t\sin^2\theta_W}M_t^\sigma\delta_{\sigma-} + \frac{e^2}{s}M_s^\sigma - \frac{\cos\theta_W}{\sin\theta_W}g_e^\sigma\frac{1}{s-M_Z^2}M_s^\sigma \\ &= \frac{e^2}{2\sin^2\theta_W}\left(\frac{1}{t}M_t^\sigma + \frac{1}{s-M_Z M_s^\sigma}\right)\delta_{\sigma-} - e^2\frac{M_Z^2}{s(s-M_Z^2)}M_s^\sigma \\ g_e^+ &= -\frac{\sin\theta_W}{\cos\theta_W}Q_e \\ g_e^- &= \frac{I_{W,e}^3 - \sin^2\theta_W Q_e}{\sin\theta_W\cos\theta_W} \end{aligned} \tag{7.1}$$

In the first line, the first term is due to the process with an intermediate (massless) neutrino, and there the helicity is constrained to be negative. The second is due to the virtual photon exchange, while the third is due to the virtual Z-boson exchange. In the next line the contributions from the SU(2) and the U(1) subgroup of the electroweak interactions have been split, essentially by rearranging the propagators accordingly. This is not obvious. Nonetheless, the weak contribution is only contributing if the electron is left-handed. Furthermore, because there is no direct coupling between the SU(2) group and the U(1) group, except for the mixing, both contributions are separately (perturbatively) gauge-invariant.

The relation of a matrix element with a corresponding (unpolarized) differential cross-section for such a two-to-two process is given by

$$\frac{d\sigma_B}{d\Omega} = \frac{\beta}{64\pi^2 s}\frac{1}{4}\sum_{\sigma,\lambda=\pm}|M_B(\sigma,\lambda_\pm,s,t)|^2.$$

Explicitly inserting all expressions and performing the integration over the solid angle yields for the cross section

$$\begin{aligned}\sigma_B &= \frac{\pi\frac{g^4}{(4\pi)^2}}{2\sin^4\theta_W}\frac{\beta}{s}\left(\left(1+\frac{2M_W^2}{s}+\frac{2M_W^4}{s^2}\right)\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}-\frac{5}{4}\right.\\ &+\frac{M_Z^2(1-2\sin^2\theta_W)}{s-M_Z^2}\left(2\frac{M_W^2}{s}\left(2+\frac{M_W^2}{s}\right)\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}-\frac{s}{12M_W^2}-\frac{5}{3}-\frac{M_W^2}{s}\right)\\ &\left.+\frac{M_Z^4(1-4\sin^2\theta_W+8\sin^4\theta_W)\beta^2}{48(s-M_Z^2)^2}\left(\frac{s^2}{M_W^4}+\frac{20s}{M_W^2}+12\right)\right).\end{aligned}$$

This fairly lengthy expressions exhibits nicely how quickly messed-up even simple calculations in electroweak physics become. Nonetheless, it is directly obvious that the full cross-section now decreases as $1/s$, and thus $1/E^2$ for large s. Since the helicity sum is a sum of squares this behavior therefore bounds all partial cross-sections, including the originally offending one for two longitudinal W bosons in the final state. Therefore, despite the appearance of apparently massive vector bosons, the cross-sections are now in accordance once more with unitarity, due to the underlying hidden gauge theory.

It is interesting to investigate how this mechanism works in detail. At energies much larger than all the masses, the tensor structures are

$$\begin{aligned}M_t^\pm &= -\frac{t}{M_W^2}\bar{v}(p_+)\gamma_\mu k_+^\mu\frac{1\pm\gamma_5}{2}u(p_-)\\ M_s^\pm &= \frac{s}{M_W^2}\bar{v}(p_+)\gamma_\mu k_+^\mu\frac{1\pm\gamma_5}{2}u(p_-)=-\frac{s}{t}M_t^\pm.\end{aligned}$$

The appearing absolute squares can then be evaluated as

$$\begin{aligned}&\left|\bar{v}(p_+)\gamma_\mu k_+^\mu\frac{1\pm\gamma_5}{2}u(p_-)\right|^2\\ &= \operatorname{tr}\left(\gamma_\mu p_+^\mu\gamma_\nu k_+^\nu\frac{1\pm\gamma_5}{2}\gamma_\rho p_-^\rho\gamma_\sigma k_+^\sigma\frac{1\pm\gamma_5}{2}\right)\\ &= \frac{s^2}{4}\beta^2\sin^2\theta.\end{aligned}$$

Here, it has been used that spinors solving the massless Dirac equations

$$\gamma_\mu p^\mu u(p)=0,$$

are given simply by $u(p)\sim\sqrt{\gamma_\mu p^\mu}$, since $p^2=0$ for massless particles. Then, by hermitian conjugation, there appears a $v\bar{v}$ and by cyclicity of the trace $u\bar{u}$, yielding the given expression. As a consequence,

$$\bar{v}(p_+)\gamma_\mu k_+^\mu\frac{1\pm\gamma_5}{2}u(p_-)=\frac{s}{2}\beta e^{i\phi}\sin\theta.$$

The phase depends on the conventions for the γ matrices and spinors. Cross-sections are not affected by the choice, and therefore it will be set to zero. As a consequence

$$M_t^\sigma = -\frac{st}{2M_W^2}\sin\theta,$$

since β now approaches 1. The cross-section just of the direct production diagram is therefore in the high-energy limit

$$\frac{d\sigma_B^W}{d\Omega} = \frac{e^2}{256\pi^2}\frac{s}{16\sin^4\theta_W M_W^4}\sin^2\theta,$$

and thus it alone would once more grow with s, and thus E^2, just as discussed for the intermediate vector boson version of the theory, violating unitarity. Only the interference with the contributions with virtual Z and photon exchange modify this, yielding the well-behaved

$$\frac{d\sigma_B}{d\Omega} = \frac{e^2}{256\pi^2}\frac{1+\sin^4\theta_W}{16s\sin^4\theta_W\cos^4\theta_W}\sin^2\theta,$$

in agreement with unitarity. Of course, in this argument there is some cheating: The direct production process is no longer gauge invariant. However, it should illustrate the point how the interplay of the gauge degrees of freedom makes it possible that unitarity is possible despite the appearance of massive spin-one bosons.

For a practical application of this calculation it would be necessary to take into account that the W bosons are not stable, their width is of the order of GeV. Therefore, they will decay and are not directly measurable[1]. Usually, in experiment they will be tagged by their decay into four fermion final states. Therefore, for comparison to experiment, it would be necessary to determine the cross section for a two-to-four fermion process, including all other possible intermediate states, which is already to lowest order a formidable task.

7.2 Properties of the Higgs

The most interesting, arguably mysterious, and most annoying, particle in the electroweak sector of the standard model is the Higgs particle. Interesting, as it is elusive, though wielding great influence in everything by generating the mass. Mysterious, since it is not yet clear how it wields this influence, though we seem to be rather able to describe it. And annoying because it is so elusive, and because there is absolutely no reason why it should

[1]In fact, gauge symmetry requires these gauge-dependent objects to be unobservable anyway, even if they would be stable, as discussed in section 6.2. However, their decay occurs so quickly that this subtlety can be usually neglected.

have the properties it seems to have, yielding the so-called hierarchy problem: Quantum corrections would naively drive the Higgs mass to be very large, of the order of the validity of the standard model, since no mechanism exists protecting it from not being so large. That is one of the reasons why so much work is invested in finding an alternative to the Higgs particle.

To further understand these problems, and someday their eventual resolution, here a number of properties of the Higgs particle are investigated.

7.2.1 Limits of perturbative considerations

In the following a number of issues using perturbation theory will be discussed[2]. For this discussion to apply, it is necessary to consider to which extent perturbation theory can be applied. In particular, most important is the question from which Higgs mass on a perturbative description will eventually break down.

The most primitive limit is given by the condition that a perturbative expansion in the four-Higgs coupling is still possible. As seen frequently, this coupling can be written in terms of the vacuum expectation value and the (now not fixed) mass of the Higgs,

$$g^{H^4} = \frac{3e^2 M_H^2}{4M_W^2 \sin^2\theta_W} = \frac{3M_H^2}{2f^2}.$$

Solving for the Higgs mass yields for the naive constraint $g^{H^4}/(4\pi) < 1$

$$M_H \lesssim \sqrt{\frac{8\pi}{3}} f \approx 700\text{GeV},$$

as a first shot at the upper limit for the applicability of perturbation theory. Similarly, a leading-order renormalization group analysis yields the running of the quartic self-coupling defined as $\lambda = 2M_H^2/f^2$

$$\frac{d\lambda}{d\log\mu^2} = \frac{3}{8\pi^2}\left(\lambda^2 + \lambda\frac{2m_t^2}{f^2} - \frac{4m_t^4}{f^4}\right) + \mathcal{O}(e^2) + \mathcal{O}(g^2_{f\neq t}).$$

Solving this equation yields a running

$$\lambda(\mu^2) = \frac{1}{2}\left(\sqrt{5}\frac{2m_t^2}{f^2}\tanh\frac{8\sqrt{5}\pi^2\frac{2m_t^2}{f^2}\Lambda_{EW} - 3\sqrt{5}\frac{2m_t^2}{f^2}\mu^2 - \frac{2m_t^2}{f^2}}{16\pi^2}\right).$$

At fixed m_t, f, and a typical electroweak scale Λ_{EW}, this equation has, due to the boundedness of the tanh no solution if the mass of the Higgs exceeds a certain limit. Therefore,

[2]These arise on top of those issues from section **??**, and exist already within perturbation theory.

a Landau pole arises, when the equation is resolved for the Higgs mass, and perturbation theory breaks down as well. This limit turns out to be of the same order as the previous one, i. e., roughly of order 1 TeV. There are other processes which have been investigated, all coming to similar results, including non-perturbative approximate lattice studies. Therefore, perturbation theory will only be useful if the mass of the Higgs remains below roughly 1 TeV. The actual value of 125 GeV is actually comfortably below this, and therefore perturbation theory is quite well applicable.

An entirely different problem is actually the lowest mass possible for the Higgs. Though naively at tree-level (in unitary or 't Hooft gauges) every mass of the Higgs is possible while retaining perturbative values for all couplings, this must not remain the case beyond tree-level. Quantum corrections can add a strictly positive term to the Higgs mass, with a value depending on the remainder of the theory. Thus, there can be an actual lower bound to the Higgs mass, depending on the remainder of the theory.

This is what actually happens in the standard model. The actual quantitative value is not entirely trivial to determine, and is strongly influenced by the ultraviolet part of the theory. Especially, the scale at which new physics processes are assumed to play a role enters these calculations. The consequence is that a quantitative precise lower bound is hard to set. However, most results end up with values rather close to the actual mass of the Higgs. Trying to lower this limit in the standard model would only be possible at either the expense of making the agreement with other observables worse, or by forcing the theory into an unstable regime, or by lowering the scale of new physics substantially below the Planck scale, even close to the electroweak scale. Especially the last point makes this problem, also known as the vacuum stability problem, a topic of very intense discussion.

This problem is not an artifact of perturbation theory, and seems to remain even in fully non-perturbative calculations. However, at the non-perturbative level, the vacuum remains stable, and only a lowering of the Higgs mass is not possible.

In addition, the Higgs mass turns out beyond leading order to be very sensitive to the parameters of the theory. Thus, setting its mass requires much more fine-tuning than for any other parameter of the standard model. This is due to it scalar nature. There is no motivation for this in the standard model, and therefore this is known as the naturalness problem. As the sensitivity is measured by the ratio of the scale of new physics to the electroweak scale, this is also known as the hierarchy problem.

7.2.2 The triviality problem

As it turns out, there is one more problem with the Higgs sector, the so-called triviality problem.

A theory is considered to be trivial, if the quantum theory is non-interacting while its classical counter-part is. There are several reasons for this to occur, but it mainly surfaces in the form that the only consistent way to remove ultraviolet divergences is by setting all interactions to zero, and at most permit for a shift of the mass and a non-trivial wave-function renormalization. Thus, the quantum theory would be non-interacting. Testing for triviality is complicated, as it requires to also include non-perturbative corrections. However, trivial theories usual show a Landau pole in perturbative calculations. However, the reverse is not true, as QCD explicitly demonstrates.

Though there is no full proof of triviality in any theory, there are several theories where varying amount of evidence exist that they are trivial.

The theory for which the evidence is most compelling is ϕ^4-theory, i. e. the ungauged Higgs sector of the standard model. Here, little doubt remains that there is no interacting quantum theory, if non-perturbative corrections are taken into account, although except for the Landau pole perturbation theory is working rather well. The Landau pole plays for perturbation theory the role of an upper limit of the theory. This also shows another feature of trivial theories: As long as an explicit cutoff remains, which then becomes a parameter of the theory, the theory remains still valid as a low-energy effective theory.

But also for the gauged version of the theory some evidence exist that it may still be trivial. It certainly has, as noted above, a Landau pole. This then shows the possibility that the standard model is also trivial. This is amplified by indications that QED, as a stand-alone theory, is under the suspicion to be trivial as well. Hence, the possibility that the full standard model is trivial is a very real possibility. However, since it is anyhow clear, at the very least by the absence of gravity, that the standard model is at most a low-energy effective theory, this is not a fundamental problem, but rather an inconvenience.

One unsettling common feature of these theories is that they all have one, or more, ultraviolet Landau poles, while no case is known where a theory without such a feature exhibits triviality. Conversely, theories without ultraviolet Landau poles seem not to be trivial. There have been two major avenues how to deal with this.

One is to require theories to show no ultraviolet Landau poles perturbatively to be non-trivial. This argument assumes that there is a one-to-one relation between both features. Then, only theories which show no Landau poles, including especially, but not only, asymptotically free theories, would appear as possible ultraviolet extensions of the standard model. This also includes the possibility that the embedding of a trivial theory in some other theory can make it once more non-trivial.

The alternative is that this is not a one-to-one relation, and theories showing perturbatively Landau poles can still be well-defined theories, so-called asymptotically safe theories.

However, this implies that a sickness of perturbation theory is cured non-perturbatively, and the theory would thus only non-perturbatively be well-defined[3]. Such theories are called asymptotically safe theories. Note that this does not prevent perturbation theory to capture the quantitatively most relevant part far away from the Landau pole, like this is also the case in the standard model. Whether the gauged Higgs theory, or the full standard model, are asymptotically safe is currently under debate, and not yet settled.

7.2.3 Higgs production

That said, and perturbation theory assumed to be further valid, the question is how to effectively produce Higgs bosons to study them. The coupling of the Higgs boson is essentially proportional to the mass of the particle to which it couples. The most efficient process would therefore be by top-antitop annihilation and the production of a Higgs boson. However, the availability of top quarks makes this rather inefficient at tree-level[4]. Another useful possibility is the Higgs-strahlung or Bjorken process. In this case, a fermion-antifermion pair annihilates into a Z boson, which then emits a Higgs boson before itself decays into other particles.

Ignoring this decay and using an experimentally readily available electron-positron pair for the parents, this process can be evaluated at tree-level. Neglecting the electron mass, there is no interference with other possibilities, like a Z emission after the annihilation of the fermions into a Higgs. The process is therefore

$$\begin{aligned} e^+(p_+,\sigma_+)+e^-(p_-,\sigma_-) &\to Z(k_Z,\lambda)+H(k_H) \\ p^\mu_\pm &= (E,0,0,\pm E)^T \\ k_{Z,H} &= E\left(1\pm\frac{M_Z^2-M_H^2}{s},\pm\beta\sin\theta,0,\pm\cos\theta\right) \\ \beta &= \frac{1}{s}\sqrt{(s-(M_Z+M_H)^2)(s-(M_H-M_Z)^2)}, \end{aligned}$$

using essentially the same conventions as in the Drell-Yan production of W-bosons, but adapted to the different masses. The angle θ is the one between the positron and the Z boson. Because the masses of the Z and the Higgs are different, the polarization vectors are different then for the center-of-mass system for the W boson, since there is now an

[3]Strictly speaking, this is anyhow true for all theories.

[4]At loop level in a hadronic environment two gluons can fuse to a top-antitop-pair, which then fuse into a Higgs. This gluon fusion is actually at the LHC the dominant process, but still not a tree-level process, as the tops form necessarily a closed loop in terms of Feynman diagrams.

asymmetry. They are given by

$$\epsilon_{\pm}^{\mu*} = \frac{1}{\sqrt{2}}(0,\cos\theta,\mp i,-\sin\theta)$$
$$\epsilon_{0}^{\mu*} = \frac{E}{M_Z}\left(\beta,\frac{s+M_Z^2-M_H^2}{s}\sin\theta,0,\frac{s+M_Z^2-M_H^2}{s}\cos\theta\right).$$

It is then straightforward to assemble the single tree-level amplitude as

$$M_B(\sigma=\pm,\lambda) = eg_e^{\pm}\frac{M_Z}{\sin\theta_W\cos\theta_W}\frac{1}{s-M_Z^2}\bar{v}(p_+)\gamma_\mu\epsilon_\lambda^{\mu*}\frac{1\pm\gamma_5}{2}u(p_-).$$

Squaring this amplitude, summing, and multiplying appropriate phase-space factors yield the unpolarized partial cross section

$$\frac{d\sigma_B}{d\Omega} = \frac{e^2}{128\pi^2}\frac{\beta M_Z^2}{(s-M_Z^2)^2}\frac{g_e^{+2}+g_e^{-2}}{\sin^2\theta_W\cos^2\theta_W}\left(1+\cos^2\theta+\left(\frac{s\beta^2}{4M_Z^2}+1\right)\sin\theta\right).$$

The first two terms come from the transverse polarized Z bosons, while the last term is due to the longitudinal Z boson, and the contributions due to $g_e^{\pm}$ are due to the corresponding lepton helicities. A factor of one-half was applied to account for the averaging over the lepton helicities. The pole in the cross section due to the Z resonance is directly visible, though it is of course not a singularity anymore as soon as the width of the Z is taken duly into account. Furthermore, the cross section for the production of transverse Zs drops like $1/s^2$, while the one for longitudinally polarized ones as $1/s$ for large energies. The corresponding total cross section reads

$$\sigma = \frac{e^2}{16\pi}\frac{\beta M_Z^2}{(s-M_Z^2)^2}\frac{g_e^{+2}+g_e^{-2}}{\sin^2\theta_W\cos^2\theta_W}\left(2+\frac{s\beta^2}{6M_Z^2}\right).$$

Reexpressing the parameters in terms of the Higgs mass, it turns out that the cross-section drops with increasing Higgs mass, i. e., in this channel a Higgs is less and less detectable the heavier it becomes. At large energies, it becomes dominated by the process in which the initial lepton pair transforms under the emission of a neutrino pair into two W which then fuse into a Higgs. Because the Z is less strongly coupled than the W, the contribution of Z fusion in this channel is almost negligible at these energies.

7.2.4 Higgs decays

The simplest decays are just given by the tree-level diagrams. Calculating the corresponding cross-section for a decay into a fermion and an antifermion, necessarily of the same type at tree-level, with total mass below the Higgs mass yields

$$\Gamma(H\to\bar{f}f) = \frac{e^2}{128\pi^2\sin^2\theta_W}M_H N_C^f\frac{m_f^2}{M_W^2}\left(1-\frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}},$$

where $N_c^f = 3$ for quarks and 1 for leptons. If the Higgs mass becomes sufficiently large, it can decay into two W and Z bosons, one of them due to the kinematics virtual and therefore further decaying into other particles. The respective widths are given by

$$\Gamma(H \to WW) = \frac{e^2}{256\pi^2 \sin^2\theta_W}\frac{M_H^3}{M_W^2}\left(1-\frac{4M_W^2}{M_H^2}\right)^{\frac{1}{2}}\left(1-\frac{4M_W^2}{M_H^2}+12\frac{M_W^4}{M_H^4}\right)$$
$$\Gamma(H \to ZZ) = \frac{e^2}{512\pi^2 \sin^2\theta_W}\frac{M_H^3}{M_W^2}\left(1-\frac{4M_Z^2}{M_H^2}\right)^{\frac{1}{2}}\left(1-\frac{4M_Z^2}{M_H^2}+12\frac{M_Z^4}{M_H^4}\right).$$

These contributions are larger, due to the correction factors, than the one for the fermions, dominating even the decay into $\bar{t}t$ at larger energies. At intermediate energies, loop processes for the decay of the Higgs into photons and gluons are also contributing. However, at energies significantly above the electroweak scale the decay is dominated by the one into W and Z and is approximately given by

$$\Gamma(H \to WW/ZZ) = \frac{3e^2}{512\sin^2\theta_W}\frac{M_H^3}{M_W^2}.$$

Resolving for the Higgs mass yields that its tree-level width in these two channels becomes larger than its mass for a Higgs above 1.4 TeV. However, at such a mass the use of tree-level perturbation theory is anyhow inadequate, and therefore this is of minor concern. For a Higgs mass between 100 and 200 GeV the decay width is not larger than a few GeV, and the Higgs is as well defined as a particle as are the W and Z bosons. In fact, for the actual Higgs mass, the total width of the Higgs is still below the experimental resolution, and thus not yet experimentally confirmed. In fact, it is theoretically predicted to be of the order of about 5-6 MeV.

7.3 Flavor physics

A particular important branch of modern electroweak physics has become flavor physics. It is centered around the flavor-violating decays of quarks. For leptons they are too rare to be (yet) experimentally accessible, except in the form of neutrino oscillations. For all but the top quark this occurs due to confinement in terms of a flavor-violating decay of a hadron, in experiment usually a meson. Thus, there are two intrinsic possibilities for flavor changes: A flavor-violating decay of a particle or an oscillation. Both will be treated in turn.

7.3.1 Decays

7.3.2 Oscillation

Chapter 8

Radiative corrections

Beyond tree-level the processes which have been discussed so far receive radiative corrections, due to loop-diagrams. It is then necessary to consistently renormalize the theory. A consequence of radiative corrections is also the hierarchy problem.

To be able to determine radiative corrections, it is necessary to perform the renormalization first. This will be done in the first step before turning to actually calculate radiative corrections.

8.1 Renormalization

8.1.1 Renormalization scheme

An assumption which pertains all of the following is that the theory can be renormalized multiplicatively. This has been proven to all orders of perturbation theory for the 't Hooft gauge, though a non-perturbative proof is still lacking. Of course, there are gauges which are not renormalizable at all, like the unitary gauge, or just not multiplicatively renormalizable, as Coulomb gauge. In the following, all calculations will be performed in the 't Hooft gauge, supplemented by a covariant gauge for the photon.

The basic setup is the introduction of the renormalization conditions. First of all, it will be required that all additive mass renormalizations just shift the masses to their measured values,

$$\begin{aligned}
M_{W0}^2 &= M_W^2 + \delta M_W^2 \\
W_{Z0}^2 &= M_Z^2 + \delta M_Z^2 \\
M_{H0}^2 &= M_H^2 + \delta M_H^2 \\
m_{fi0} &= m_{fi} + \delta m_{fi}.
\end{aligned}$$

Of course, the masses of the bosons are in fact determined by the Higgs condensate and the coupling constants. Therefore, these are not independent counter terms in general. Here, however, the renormalization of the weak gauge coupling and the Higgs four-point coupling has been traded in for renormalization of the weak gauge boson masses. The Weinberg angle is, in principle, not renormalized in this scheme, as it is fixed by definition as

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}. \tag{8.1}$$

However, it is often convenient to also formally renormalize the Weinberg angle. These renormalization constants are then, due to the relation (8.1) given in terms of the renormalization constants for the masses.

In particular, the Higgs vacuum expectation value will be renormalized such that the Higgs tadpoles t obtain a prescribed value by the condition

$$t_0 = t + \delta t.$$

If and how these quantities are related will be discussed whenever this is actually necessary.

Furthermore, of course, the entries of the CKM matrices and the coupling constant are renormalized

$$\begin{aligned} V_{ij0} &= V_{ij} + \delta V_{ij} \\ e &= e + \delta e = Z_e e = (1+\delta Z_e)e, \end{aligned}$$

where in the last line the generic way of defining the associated renormalization constants Z_i is shown. The renormalization of the second coupling constant is effectively performed by the renormalization of the Z mass. The same applies again to the Weinberg angle, which can be fixed by renormalizing the W mass.

The wave-function renormalizations are defined as

$$\begin{aligned} W_0^\pm &= Z_W^{\frac{1}{2}} W^\pm = \left(1+\frac{1}{2}\delta Z_W\right) W^\pm \\ \eta_0 &= Z_\eta^{\frac{1}{2}}\eta = \left(1+\frac{1}{2}\delta Z_\eta\right)\eta \\ \chi_0 &= Z_\chi^{\frac{1}{2}}\chi = \left(1+\frac{1}{2}\delta Z_\chi\right)\chi \\ \phi_0^\pm &= Z_\phi^{\frac{1}{2}}\phi^\pm = \left(1+\frac{1}{2}\delta Z_\phi\right)\phi^\pm \\ u_0^\pm &= \tilde{Z}_\pm u^\pm = (1+\delta\tilde{Z}_\pm)u^\pm, \end{aligned}$$

in which also the unphysical fields have been included. The situation is a bit more complicated for the photon and the Z boson fields (and thus their ghosts) as well as for the fermions, as these are mixed already at tree-level. Hence, the corresponding renormalization constants are matrix-valued

$$\begin{aligned}
\begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} &= \begin{pmatrix} Z_{ZZ}^{\frac{1}{2}} & Z_{ZA}^{\frac{1}{2}} \\ Z_{AZ}^{\frac{1}{2}} & Z_{AA}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}\delta Z_{ZZ} & \frac{1}{2}\delta Z_{ZA} \\ \frac{1}{2}\delta Z_{AZ} & 1+\frac{1}{2}\delta Z_{AA} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} \\
\begin{pmatrix} u_0^Z \\ u_0^A \end{pmatrix} &= \begin{pmatrix} \tilde{Z}_{ZZ} & \tilde{Z}_{ZA} \\ \tilde{Z}_{AZ} & \tilde{Z}_{AA} \end{pmatrix} \begin{pmatrix} u^Z \\ u^A \end{pmatrix} = \begin{pmatrix} 1+\delta\tilde{Z}_{ZZ} & \delta\tilde{Z}_{ZA} \\ \delta Z_{AZ} & 1+\delta\tilde{Z}_{AA} \end{pmatrix} \begin{pmatrix} u^Z \\ u^A \end{pmatrix} \\
f_{i0}^L &= \sqrt{Z_{ij}^{fL}} f_j^L = \left(\delta_{ij} + \frac{1}{2}\delta Z_{ij}^{fL}\right) f_j^L \\
f_{i0}^R &= \sqrt{Z_{ij}^{fR}} f_j^R = \left(\delta_{ij} + \frac{1}{2}\delta Z_{ij}^{fR}\right) f_j^R.
\end{aligned}$$

Of course, the number of independent renormalization constants is not larger than for the unmixed fields, and the off-diagonal renormalization constants will be given in terms of the independent renormalization constants.

Furthermore, the gauge-fixing parameters renormalize. Though the parameters inside the gauge choice (3.19) and the parameter with which to average over different gauge choices in (5.3) are chosen to be the same, this will in general not continue to be so beyond tree-level. This can be counter-effected by making the renormalization choice that their renormalized values are still coinciding, but this requires independent renormalization constants for both parameters. Signifying the one in (3.19) by ξ and the one in (5.3) by ξ' this leads to

$$\begin{aligned}
\xi_{0W}^{(\prime)} &= Z_{\xi_W^{(\prime)}} \xi_W^{(\prime)} \\
\xi_{0Z}^{(\prime)} &= Z_{\xi_Z^{(\prime)}} \xi_Z^{(\prime)} \\
\xi_{0A} &= Z_{\xi_A} \xi_A,
\end{aligned}$$

and which is not applying to the photon gauge choice, of course. A favorable consequence of making this renormalization choice is that the diagonality in field-space of the propagators of the would-be Goldstone bosons and the gauge bosons is retained, as well as that their tree-level perturbative propagator poles continue to coincide. This makes many calculations much simpler.

Note that the renormalization constants for further three- or four-point vertices are not independent. By virtue of the Slavnov-Taylor identities they can be tied to the ones of the coupling constants. However, they will not be needed in the following, and therefore their explicit expressions will not be given.

Inserting these renormalized quantities, the Lagrangian $\mathcal{L}_0$ can be split into the contributions containing the counter terms δZ, the counter-term Lagrangian $\mathcal{L}_c$, and those containing the renormalized fields $\mathcal{L}$,

$$\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_c,$$

making the counter-term structure explicit. In the following only one-loop corrections will be investigated, permitting to drop all contributions of order $\mathcal{O}(\delta Z^2)$, and higher. In multi-loop calculations, this is no longer possible, leading to a rather involved structure of $\mathcal{L}_c$, even longer than the one of $\mathcal{L}$.

8.1.2 Renormalization conditions

After fixing this renormalization scheme, it remains to fix the renormalization conditions. The formulation of the renormalization conditions is most conveniently done by expressions involving the renormalized correlation functions Γ_R. To make contact to the counter-terms, it is necessary to determine how these enter the renormalized correlation functions.

The renormalization conditions can be split into three classes. The renormalization of the physical parameters, like masses and coupling constants, directly enter the determination of matrix elements. Their calculation can be simplified by choosing the second class, the wave-function renormalization constants, appropriately. Finally, the choice for the renormalization constants of the unphysical fields is important for checking the consistency of the calculations by usage of the Ward-Takahashi/Slavnov-Taylor identities, or using them to further simplify the calculations.

A first convenient choice is to select δt such that the renormalized one-point Green's function of the Higgs field vanishes, i. e., imposing

$$\Gamma^\eta_R = \langle\eta\rangle_R = \Gamma^\eta + \delta t = 0.$$

As a consequence, all such tadpole graphs vanish, and need not to be taken into account. If this choice would not be made, in the following discussion additional terms would appear whenever the expectation value of the Higgs field would be relevant.

Conditions for the wave-function renormalization and mass renormalization can be obtained from the two-point functions. A sufficient condition is to required that the pole of the propagators, the inverse two-point function, occurs at the physical mass, and that the residuum at the pole is 1. These are statements on the real part of the propagator. Taking the real part is however complicated by the fact that the mixing by the CKM matrices is unitary, and therefore not real. As the CKM matrix itself is renormalized, the prescription has to be to take the real part of the self-energy, keeping the unitary structure of the CKM matrix intact, as otherwise no hermitian counter-term Lagrangian is obtained. This is denoted by $\tilde{\Re}$. The same applies for the PMNS matrix and the leptons.

The Higgs is renormalized as

$$\begin{aligned}\tilde{\Re}\Gamma_R^{\eta\eta}(M_H^2) &= 0\\ \tilde{\Re}\frac{\partial\Gamma_R^{\eta\eta}}{\partial k^2}(M_H^2) &= 1,\end{aligned}$$

since the propagator of the Higgs has the generic structure for a scalar particle,

$$\Gamma_R^{\eta\eta} = (k^2 - M_H^2) + \Sigma^\eta + (k^2 - M_H^2)\delta Z_\eta - \delta M_H^2.$$

From this follows immediately how to determine the values of the renormalization constants as

$$\begin{aligned}\delta M_H^2 &= \tilde{\Re}\Sigma^\eta(M_H^2)\\ \delta Z_\eta &= -\tilde{\Re}\frac{\partial\Sigma^\eta}{\partial k^2}(M_H^2).\end{aligned} \quad (8.2)$$

Similarly, it is always possible to solve the renormalization conditions for the counter terms. Therefore, this will not be repeated for other renormalization conditions.

The two-point correlation functions, or inverse propagators, will have contributions from the the wave-function renormalization in their kinetic parts and from the mass renormalization in their self-energy contributions, generically denoted by Σ. Thus the gauge boson propagator read

$$\Gamma_{RT}^{VV'+} = -\delta_{VV'}(k^2 - M_V^2) - \Sigma_T^{VV'+} - \left(\frac{\delta Z_{VV'}}{2}(k^2 - M_V^2) + \frac{1}{2}(k^2 - M_{V'}^2)\delta Z_{V'V} - \delta_{VV'}\delta M_V^2\right).$$

Herein V and V' take values in $\{A, Z, W^+, W^-\}$. Only the transverse part (T) of the correlation functions is determined in this way. As a consequence of the gauge condition, and thus also of the Slavnov-Taylor identities, the longitudinal part is renormalized completely by the renormalization of the would-be Goldstone bosons and the gauge parameters.

For the gauge bosons the renormalization conditions then read

$$\begin{aligned}\tilde{\Re}\Gamma_{RT}^{VV'+}(M_V^2) &= 0\\ \tilde{\Re}\frac{\partial\Gamma_{RT}^{VV+}}{\partial k^2}(M_V^2) &= -1,\end{aligned} \quad (8.3)$$

where V and V' take the values $W^\pm$, Z, and A and T indicates the transverse part of the correlation function. However, since the photon is massless due to the manifest gauge symmetry, the first condition for $V = V' = A$ on the photon propagator is actually already guaranteed by the gauge condition, and is therefore not giving an independent constraint. Also, except for $V = A, Z$ and $V' = Z, A$ the first condition is not yielding independent

information, as any other combination is trivially zero due to the manifest electromagnetic gauge symmetry. Note that the masses of the gauge bosons are not actually renormalized here, but the propagators and parameters of the Higgs potential. The masses are actually renormalization-group invariant in this class of gauges, as will be discussed below.
For the fermions, it is convenient to investigate the four possible tensor structures, given by

$$\Gamma^{f\bar{f}}_{Rij}(p) = \gamma_\mu p^\mu \frac{1-\gamma_5}{2}\Gamma^{f\bar{f}L}_{Rij}(p^2) + p_\mu\gamma^\mu\frac{1+\gamma_5}{2}\Gamma^{f\bar{f}R}_{Rij}(p^2) + \frac{1-\gamma_5}{2}\Gamma^{f\bar{f}l}_{Rij} + \frac{1+\gamma_5}{2}\Gamma^{f\bar{f}r}_{Rij}(p^2). \quad (8.4)$$

The four tensor structures for the renormalized propagator are then given by

$$\begin{aligned}
\Gamma^{f\bar{f}L}_{Rij} &= \delta_{ij} + \Sigma^{fL}_{ij} + \frac{1}{2}\left(\delta Z^{fL}_{ij} + \delta Z^{fL+}_{ij}\right)\\
\Gamma^{f\bar{f}R}_{Rij} &= \delta_{ij} + \Sigma^{fR}_{ij} + \frac{1}{2}\left(\delta Z^{fR}_{ij} + \delta Z^{fR+}_{ij}\right)\\
\Gamma^{f\bar{f}l}_{Rij} &= -m_{fi}\delta_{ij} + \Sigma^{fl}_{ij} - \frac{1}{2}\left(m_{fi}\delta Z^{fL}_{ij} + m_{fj}\delta Z^{fR+}_{ij}\right) - \delta_{ij}\delta m_{fi}\\
\Gamma^{f\bar{f}r}_{Rij} &= -m_{fi}\delta_{ij} + \Sigma^{fr}_{ij} - \frac{1}{2}\left(m_{fi}\delta Z^{fR}_{ij} + m_{fj}\delta Z^{fL+}_{ij}\right) - \delta_{ij}\delta m_{fi},
\end{aligned}$$

where the self-energy has been split in the same way.

To impose the renormalization conditions for the fermions, it is useful to again use (8.4), making left-handed and right-handed contributions explicit. The corresponding renormalization conditions then read

$$\begin{aligned}
m_{fj}\tilde{\Re}\Gamma^{f\bar{f}R}_{Rij}(m^2_{fj}) + \tilde{\Re}\Gamma^{f\bar{f}r}_{Rij}(m^2_{fj}) &= 0 \quad (8.5)\\
m_{fj}\tilde{\Re}\Gamma^{f\bar{f}L}_{Rij}(m^2_{fj}) + \tilde{\Re}\Gamma^{f\bar{f}l}_{Rij}(m^2_{fj}) &= 0 \quad (8.6)\\
\tilde{\Re}\left(\Gamma^{f\bar{f}R}_{Rii} + \Gamma^{f\bar{f}L}_{Rii} + 2\frac{\partial\left(m^2_{fi}\left(\Gamma^{f\bar{f}R}_{Rii} + \Gamma^{f\bar{f}L}_{Rii}\right) + m_{fi}\left(\Gamma^{f\bar{f}r}_{Rii} + \Gamma^{f\bar{f}l}_{Rii}\right)\right)}{\partial p^2}\right)(m^2_{fi}) &= 2. (8.7)
\end{aligned}$$

The CKM matrices are free parameters of the theory, and therefore, as any other parameter, need to be renormalized. This is of course dependent on the given parametrization. However, most renormalization conditions will introduce additional mixings between quarks beyond those present at tree-level, since different species become further connected by radiative contributions. Only very particular renormalization conditions will guarantee that no additional mixing beyond that at tree-level will be generated.

To construct such conditions note that the CKM matrices are given by the transformation matrices (4.5), which transform the fermion fields from weak isospin eigenstates to mass eigenstates as (4.6). To counteract any rotation induced by the renormalization (and mixing under renormalization) of the fermion fields, the CKM matrix element must be renormalized for as

$$\begin{aligned} V_{ij}^{q/l} &= V_{0ij}^{q/l} + \frac{1}{2}\left(\delta\mathcal{Z}_{Lik}^{u/\nu}V_{0kj}^{q/l} + V_{0ik}^{q/l}\delta\mathcal{Z}_{kj}^{Ld/l}\right) \\ \delta\mathcal{Z}_{ij}^{f} &= \frac{1}{2}\left(\delta\mathcal{Z}_{ij}^{Lf} - \delta\mathcal{Z}_{ij}^{Lf\dagger}\right), \end{aligned} \quad (8.8)$$

where the anti-hermitian part must be taken to ensure that the CKM matrix stays unitary. The renormalization constants cannot be the same as the original fermion renormalization constants Z, as the CKM matrix renormalizes independently. If both are taken to be the same, as a result of Slavnov-Taylor identity violation, the gauge invariance of observables is no longer guaranteed. Therefore, the $\mathcal{Z}$ have to be defined as if they were fermion wave-function renormalizations, but with conditions independent from (8.5-8.7). A possibility are the conditions

$$\begin{aligned} \tilde{\Re}\Gamma_{Rij}^{fl}(0)\frac{1-\gamma_5}{2} + \tilde{\Re}\Gamma_{Rij}^{fr}(0)\frac{1+\gamma_5}{2} &= 0 \\ \tilde{\Re}\Gamma_{Rij}^{fL}(0)\frac{1-\gamma_5}{2} + \tilde{\Re}\Gamma_{Rij}^{fR}(0)\frac{1+\gamma_5}{2} &= 0, \end{aligned}$$

which fix a set $\mathcal{Z}^a$ for $a = L, R$, though in (8.8) only $\mathcal{Z}^L$ are necessary. The remaining constants can therefore be discarded. The determination of the $\mathcal{Z}$ from the renormalized fermion correlation functions proceeds in exactly the same way as for the fermion wave-function renormalization conditions themselves, just evaluated at other kinematic configurations. Again, the same also applies to the PMNS matrix.

The electrical charge has to be renormalized as well, in addition to the renormalization conditions imposed on the Weinberg angle and the weak coupling constant implicitly by the masses of the W and Z bosons. This is done in the Thomson limit of electron-photon scattering, yielding the renormalization condition

$$\left(\bar{u}(p)\Gamma_{R\mu}^{Ae\bar{e}}(p,p)u(p)\right)_{p^2=m_e^2} = e\left(\bar{u}(p)\gamma_\mu u(p)\right)_{p^2=m_e^2}.$$

Of course, since the electric charge is universal due to gauge symmetry, this renormalization could be performed by any electromagnetic scattering process. However, this process is the one most readily and precisely measurable. The corresponding renormalized vertex function has the structure

$$\Gamma_{R\mu}^{A\bar{e}e}(p,k) = -e\gamma_\mu + e\Lambda_{R\mu}^{A\bar{e}e}(p,k),$$

where Λ is the vertex correction, which has the generic tensor structure

$$\Lambda_{R\mu}^{A\bar{e}e}(p,q) = \gamma_\mu\Lambda_{RV}^{A\bar{e}} - \gamma_\mu\gamma_5\Lambda_{RA}^{A\bar{e}e} + \frac{(p+k)_\mu}{2m_e}\Lambda_{RS} + \frac{(p-k)_\mu}{2m_e}\gamma_5\Lambda_{RP}^{A\bar{e}e},$$

introducing the (Dirac-)vector, axialvector, scalar, and pseudoscalar parts of the self-energy. Resolving for the charge counter term is a bit more lengthy, but finally permits to obtain the charge counter term as a function of the self-energy of the Z-photon mixed propagator

$$\frac{\delta e}{e}=\left(\frac{1}{2k^2}\Sigma_T^{AA}(k^2)-\frac{\sin\theta_W}{\cos\theta_W}\frac{\Sigma_T^{AZ}(k^2)}{M_Z^2}\right)_{k^2=0},$$

thus showing explicitly that the off-diagonal wave-function renormalization constants, which are fixed by conditions on the off-diagonal self-energies, are indeed not independent, but connected to the charge renormalization, as ascertained.

For the unphysical fields the renormalization conditions will not enter any physical observables. Therefore, they will be renormalized in a convenient way depending for the particular calculation. Since they will not enter the example in the next section, explicit expressions will be skipped here, but the procedure is completely analogous to the one presented for the physical degrees of freedom.

8.1.3 The Nielsen identities

8.2 Amendments for unstable particles

The renormalization conditions so far are valid if the particles are stable. However, most of the particles are not, and often decay on time-scales making it necessary to take this into account. In perturbation theory, this effect is nonetheless beyond leading order, and therefore can often be neglected.

If a particle is unstable, the condition

$$\tilde{\Re}\Gamma_R(M^2)=0 \tag{8.9}$$

will lead to a gauge-dependent renormalization constant, and thus a gauge-dependent mass. In perturbation theory[1] the pole position, defined by the condition

$$\Gamma_R(\mu^2)=-(\mu^2-\tilde{M}^2)=-(\mu^2-M_0^2+\Sigma(\mu))=0 \tag{8.10}$$

is not, to all orders in perturbation theory. It therefore permits to define a reasonable, physically renormalized mass. But since μ is not purely real for a physical particle, in which case everything would coincide with the former discussion, this requires some amendments.

[1]The situation beyond perturbation theory is currently unclear. A viable option is indeed that the analytic structure of gauge-dependent objects, like all elementary fields, actually are not admitting the definition of a pole mass at all. This is a topic of current research.

The mass $\bar{M}$ of an unstable particle is then associated with the real part of μ, while the width $\bar{\Gamma}$ with its imaginary part as

$$\mu = \bar{M}^2 - i\bar{M}\bar{\Gamma}, \tag{8.11}$$

and, in general, $\bar{M} \neq M$. The advantage is that in perturbation theory the width $\bar{\Gamma}$ will be of order $\mathcal{O}(\alpha)$ for a given (set of) coupling constant(s) α, and therefore of higher order. In terms of unrenormalized quantities the renormalization condition reads

$$\Gamma_R(\mu^2) = -(\mu - M_0^2 + \Sigma(\mu))$$

and must equal zero at this point as well. Replacing in this expression μ by (8.11) and expanding the self-energy yields

$$\begin{aligned} & \bar{M}^2 - i\bar{M}\bar{\Gamma} = M_0^2 - \Sigma(\bar{M}^2 - i\bar{M}\bar{\Gamma}) \\ = \quad & M_0^2 - \Sigma(\bar{M}^2) - i\bar{M}\bar{\Gamma}\frac{\partial\Sigma}{\partial k^2}(\bar{M}^2) + \frac{1}{2}\left(\bar{M}\bar{\Gamma}\right)^2\frac{\partial^2\Sigma}{\partial k^4}(\bar{M}^2) + \mathcal{O}(\alpha^4), \end{aligned}$$

and the order emerges since also the self-energy is at least of order $\mathcal{O}(\alpha)$. This expression can be split in its real and imaginary parts, yielding

$$\begin{aligned} \bar{M}^2 &= M_0^2 - \Re\Sigma(\bar{M}^2) - \bar{M}\bar{\Gamma}\Im\frac{\partial\Sigma}{\partial k^2}(\bar{M}^2) + \mathcal{O}(\alpha^3) \\ \bar{M}\bar{\Gamma} &= \Im\Sigma(\bar{M}^2) - \bar{M}\bar{\Gamma}\Re\frac{\partial\Sigma}{\partial k^2}(\bar{M}^2) - \frac{1}{2}\left(\bar{M}\bar{\Gamma}\right)^2\Im\frac{\partial^2\Sigma}{\partial k^4}(\bar{M}^2) + \mathcal{O}(\alpha^4). \end{aligned}$$

These are two non-linearly coupled algebraic equations. Fortunately, in perturbation theory they can simply be solved by iteration, yielding

$$\begin{aligned} \bar{M}^2 &= M_0^2 - \Re\Sigma - \Im\Sigma\Im\frac{\partial\Sigma}{\partial k^2} + \mathcal{O}(\alpha^3) \qquad (8.12) \\ \bar{M}\bar{\Gamma} &= \Im\Sigma\left(1 - \Re\frac{\partial\Sigma}{\partial k^2} + \left(\Re\frac{\partial\Sigma}{\partial k^2}\right)^2 - \frac{1}{2}\Im\Sigma\Im\frac{\partial^2\Sigma}{\partial k^4} + \mathcal{O}(\alpha^3)\right). \end{aligned}$$

This defines the mass counter-term. Implementing the condition (8.9) instead yields from the unrenormalized Green's function

$$M^2 - M_0^2 + \Re\Sigma(\bar{M}^2) = 0, \tag{8.13}$$

where it has been used that the requirement that the real part vanishes is the same as that the real part of (8.10) vanishes. Therefore, (8.10) implies (8.9), leading to (8.13). Comparing (8.13) and (8.12) yields

$$\bar{M}^2 = M^2 - \Im\Sigma\Im\frac{\partial\Sigma}{\partial k^2} + \mathcal{O}(\alpha^3),$$

and the two masses indeed differ by a (gauge-dependent) second-order contribution. A similar relation holds between the width $\bar{M}\bar{\Gamma}$ and the on-shell width defined as

$$M\Gamma = \frac{\Im\Sigma(M^2)}{1+\Re\frac{\partial\Sigma}{\partial k^2}(M^2)},$$

which also yields a second-order difference, but will not be given explicitly here. In case of the W and Z boson the mass differences amount to about 27 MeV and 34 MeV at this order, or about a small fraction of a percent. However, such corrections are relevant for high-precision measurements.

In fact, the experimental result for both masses are better than these shifts, and they therefore have to be taken into account. Furthermore, the radiative corrections to be calculated immanently are also only of the order of a few percent compared to the tree-level values, but experimental precision is much better than this, opening up the possibility for precision tests of the electroweak standard model. In fact, the precision is sufficient to require even higher-order corrections to be calculated, rather then just leading order corrections. Here, however, leading-order effects will suffice to demonstrate the mechanism.

8.3 Radiative corrections

The most direct contribution appearing in the masses of the gauge bosons is the self-energy. At one loop order, the most interesting contribution is the one due to fermionic loops. This contribution is gauge-invariant, and of particular importance. By investigating their contribution to the gauge boson masses and widths, it is possible to determine the number and nature of fermion species coupling to this order to the gauge bosons, permitting, e. g., indirectly the inference of the presence of matter fields. On the other hand, the contribution to the self-energies due to the self-coupling of the gauge bosons and ghost is fixed at this loop order irrespective of the matter content of the theory. Of course, there are also contributions from the Higgs sector, but these are also fixed when the symmetry pattern is assumed to be the one of the standard model Higgs. Only the number and type of matter fields can be varied without changing the theory fundamentally.

To leading order, the matter contribution for a family of doublet of fermions with electromagnetic couplings $q_f^\pm$, vector couplings $v_f^\pm$ and axial couplings $a_f^\pm$ can be obtained from a fermion loop coupled to the external matter fields. In each loop, only one member of the doublet of each flavor can appear.

In case of the photon, the relevant self-energy contribution is given by

$$\Sigma_{\mu\nu}(q) = -iq_f^{\pm 2}\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}(\gamma_\mu(\gamma_\rho k^\rho + m_f^\pm)\gamma_\nu(\gamma_\sigma(k^\sigma+q^\sigma)+m_f^\pm))}{(k^2-m_f^{\pm 2})((k+q)^2-m_f^{\pm 2})}.$$

Evaluating the Dirac traces, using that the trace of an odd number of Dirac matrices vanishes and $\mathrm{tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu}$ and

$$\mathrm{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \tag{8.14}$$

yields

$$\begin{aligned}
\Sigma_{\mu\nu}(q) &= -4iq_f^{\pm 2}\int \frac{d^4k}{(2\pi)^4}\frac{2k_\mu k_\nu + k_\mu q_\nu + q_\mu k_\nu - g_{\mu\nu}(k^2 + qk + m_f^{\pm 2})}{(k^2 - m_f^{\pm 2})((k+q)^2 - m_f^{\pm 2})} \\
&= -4iq_f^{\pm 2}\int \frac{d^4k}{(2\pi)^4}\frac{2k_\mu k_\nu + k_\mu q_\nu + q_\mu k_\nu - \frac{1}{2}g_{\mu\nu}((k^2 - m_f^{\pm 2}) + ((k+q)^2 - m_f^{\pm 2}) - q^2)}{(k^2 - m_f^{\pm 2})((k+q)^2 - m_f^{\pm 2})}.
\end{aligned}$$

This expression can be decomposed into standard tensor integrals

$$\begin{aligned}
\Sigma_{\mu\nu}(q) &= \frac{q_f^{\pm 2}}{4\pi^2}(2B_{\mu\nu}(q^2, m_f^\pm, m_f^\pm) + q_\mu B_\nu(q^2, m_f^\pm, m_f^\pm) + B_\mu(q^2, m_f^\pm, m_f^\pm)q_\nu \\
&\quad - \frac{g_{\mu\nu}}{2}(2A_0(m_f^\pm) - q^2 B_0(q^2, m_f^\pm, m_f^\pm)) \\
B_{\mu\nu}(q,m,n) &= \frac{1}{i\pi^2}\int d^4k \frac{k_\mu k_\nu}{(k^2 - m^2)((k+q)^2 - n^2)} \\
B_\mu(q,m,n) &= \frac{1}{i\pi^2}\int d^4k \frac{k_\mu}{(k^2 - m^2)((k+q)^2 - n^2)} \\
B_0(q,m,n) &= \frac{1}{i\pi^2}\int d^4k \frac{1}{(k^2 - m^2)((k+q)^2 - n^2)} \\
A_0(m) &= \frac{1}{i\pi^2}\int d^4k \frac{1}{k^2 - m^2},
\end{aligned}$$

where for one of the A_0 a shift $k \to k - q$ has been performed. Because of Lorentz covariance is is possible to rewrite B_μ and $B_{\mu\nu}$ as

$$\begin{aligned}
B_\mu(q,m,n) &= q_\mu B_1(q^2, m, n) \\
B_{\mu\nu}(q,m,n) &= g_{\mu\nu}B_{00}(q^2, m, n) + q_\mu q_\nu B_{11}(q^2, m, n).
\end{aligned}$$

This can be resolved to yield

$$B_1 = \frac{1}{q^2}q_\mu B^\mu \tag{8.15}$$

$$B_{00} = \frac{1}{3}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)B^{\mu\nu} \tag{8.16}$$

$$B_{11} = \frac{1}{3q^2}\left(3\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)B^{\mu\nu}. \tag{8.17}$$

This can be used to rearrange the self-energy as

$$\begin{aligned}
\Sigma_{\mu\nu}(q) &= \frac{q_f^{\pm 2}}{4\pi^2}\Bigg(\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left(2B_{00} - A_0 + \frac{q^2}{2}B_0\right) \\
&\quad + \frac{q_\mu q_\nu}{q^2}\left(2B_{00} - 2q^2(B_{11} + B_1) + \frac{q^2}{2}B_0 - A_0\right)\Bigg).
\end{aligned} \tag{8.18}$$

It is now possible to reexpress the tensor integrals B_{00} and B_{11} by the scalar integrals B_0 and A_0 by performing the contractions (8.15-8.17). To do this, note that

$$2kq = ((k+q)^2 - n^2) - (k^2 - m^2) - q^2 + n^2 - m^2$$

for arbitrary masses m and n. This yields

$$\begin{aligned} B_1 &= \frac{1}{2iq^2\pi^2}\int d^4k \frac{2kq}{(k^2-m^2)((k+q)^2-n^2)} \\ &= \frac{1}{2q^2}(A_0(m) - A_0(n) - (q^2+m^2-n^2)B_0(q^2,m^2,n^2)). \end{aligned} \quad (8.19)$$

In the same manner

$$\begin{aligned} B_{00} &= \frac{1}{6}(A_0(n) + 2m^2 B_0(q^2,m,n) + (q^2+m^2-n^2)B_1(q^2,m,n)) \\ B_{11} &= \frac{1}{6q^2}(2A_0(n) - 2m^2B_0(q^2,m,n) - 4(q^2+m^2-n^2)B_1(q^2,m,n)), \end{aligned}$$

and B_1 can in turn be replaced by (8.19). Performing all these replacements in the expression for the self-energy (8.18) yields

$$\Sigma^A_{\mu\nu} = -\frac{q_f^{\pm 2}}{4\pi^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{2(2A_0(m_f^\pm) - q^2B_0(q^2,m_f^\pm,m_f^\pm) + 4m_f^{\pm 2}B_0(q^2,m_f^\pm,m_f^\pm))}{6}.$$

The remarkable effect is that the longitudinal contribution vanishes, as it must in covariant gauges[2], as has been employed for the photon field. Furthermore, this requires only two integrals to be evaluated.

The integral A_0 is given by

$$A_0 = \frac{\mu^{4-d}}{i\pi^2}\int \frac{d^dk}{(2\pi)^{d-4}}\frac{1}{k^2-m^2+i\epsilon}, \quad (8.20)$$

which has been continued to arbitrary dimensions[3] d. The arbitrary scale μ appears to keep the mass dimension of the integral fixed. Now dimensional regularization will be performed. I. e., d is taken to be variable (and not necessarily integer) such that the integral is convergent. Only after evaluating the integral d will take its original value. In this case, the integral can be closed in the imaginary plane at infinity. Since the poles are in the second and fourth quadrant, the contour can be closed along the imaginary axis. Equivalently, it is permitted to perform a Wick rotation $k_0 \to ik_0$ to imaginary time, yielding

$$A_0 = \frac{(2\pi\mu)^{4-d}}{i\pi^2} i \int d^dk \frac{1}{k^2+m^2},$$

[2]To this order actually independent of the gauge.

[3]Note that there are subtleties involved if a γ_5 should appear.

where the $i\epsilon$ can now be dropped, as the integral has no longer poles along the integration path. The angular integral can be continued analytically to arbitrary dimensions yielding

$$\begin{aligned} A_0 &= \frac{(2\pi\mu)^{4-d}}{\pi^2}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int\frac{(k^2)^{\frac{d-2}{2}}dk^2}{k^2+m^2} \\ &= -(4\pi\mu^2)^{\frac{4-d}{2}}\Gamma\left(1-\frac{d}{2}\right)m^{d-2}. \end{aligned}$$

The Γ-function has a pole at $d=4$, which would yield the desired results. The second step of dimensional regularization is therefore to expand the result around $d=4$ with a small parameter ϵ, using $\Gamma(4-d+\epsilon)=1/\epsilon-\gamma+\mathcal{O}(\epsilon)$ with $\gamma\approx 0.577$ the Euler constant. The final result is then

$$A_0 = m^2\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\mu^2}+1\right)+\mathcal{O}(\epsilon).$$

The appearing divergence in ϵ is then absorbed into the renormalization constants, as may be other constants, depending on the scheme. It should be noted that the arbitrary scale μ has become part of the result. This is an artifact of the renormalization, and observable quantities may (and in fact do) not depend on it. The process of the appearance of this scale is often termed dimensional transmutation. Using the fact that physical quantities may not depend on this scale gives rise to the renormalization group equations, which will not be discussed here.

The calculation of B_0 can be performed likewise, though more complicated in detail, and yields

$$\begin{aligned} B_0(q^2,m_1,m_2) &= \frac{1}{\epsilon}-\gamma+\ln(4\pi)+2-\ln\frac{m_1m_2}{\mu^2}+\frac{m_1^2-m_2^2}{q^2}\ln\frac{m_2^2}{m_1^2}-\frac{m_1m_2}{q^2}\left(\frac{1}{r}-r\right)\ln r \\ r &= \frac{-q^2+m_1^2+m_2^2-i\varepsilon\pm\sqrt{(q^2-m_1^2-m_2^2+i\varepsilon)^2-4m_1^2m_2^2}}{2m_1m_2}. \end{aligned}$$

The $i\varepsilon$ prescription has no effect for r positive. r becomes negative, if $q^2>(m_1^2+m_2^2)^2$, and thus a negative r is related to particle decay. In this case, the imaginary part becomes $\Im r=\varepsilon\mathrm{sgn}(r-1/r)$. This completes the computation of $\Sigma^A_{\mu\nu}$.

In a similar way, all the self-energy contributions can be assembled. The final result for the four possible self-energies are

$$\begin{aligned}
\Sigma_T^{AA} &= \frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} q_f^{\sigma 2} H(q^2,m_f^\sigma,m_f^\sigma)\\
\Sigma_T^{AZ} &= -\frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} q_f^\sigma v_f^\sigma H(q^2,m_f^\sigma,m_f^\sigma)\\
\Sigma_T^{ZZ} &= \frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} \left(\left(v_f^{\sigma 2}+a_f^{\sigma 2}\right) H(q^2,m_f^\sigma,m_f^\sigma) - 6a_f^{\sigma 2} m_f^{\sigma 2} B_0(q^2,m_f^\sigma,m_f^\sigma)\right)\\
\Sigma_T^{WW} &= \frac{\alpha}{12\pi\sin^2\theta_W}\sum_f \left(H(q^2,m_f^+,m_f^-) - 3m_f^+ m_f^- B_0(q^2,m_f^+,m_f^-)\right).
\end{aligned}$$

The CKM matrix is not appearing explicitly at this order, as it is unitary. The functions H are combinations of the functions A_0 and B_0 as

$$\begin{aligned}
H(q^2,m,m) &= (q^2+2m^2)B_0(q^2,m,m) - 2m^2 B_0(0,m,m) - \frac{q^2}{3}\\
H(q^2,m_+,m_-) &= \frac{3}{2}\left(4B_{00}(q^2,m_+,m_-) - A_0(m_+) - A_0(m_-) + (q^2-(m_+-m_-)^2)B_0(q^2,m_+,m_-)\right).
\end{aligned}$$

Note that in the fermion summation each quark color is counted extra.

The most interesting consequences are to investigate three extreme cases. One is the case of only very light masses, $q \gg m_f^{\pm 2}$. In this case

$$\begin{aligned}
\Sigma_T^{AA} &= \frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} q_f^{\sigma 2} H(q^2,0,0)\\
\Sigma_T^{AZ} &= -\frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} q_f^\sigma v_f^\sigma H(q^2,0,0)\\
\Sigma_T^{ZZ} &= \frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} (v_f^{\sigma^2}+a_f^{\sigma^2}) H(q^2,0,0)\\
\Sigma_T^{WW} &= \frac{\alpha}{12\pi\sin^2\theta_W}\sum_f H(q^2,0,0)\\
H(q^2,0,0) &= q^2\left(B_0(q^2,0,0)-\frac{1}{3}\right) = q^2\left(\Delta - \ln\frac{|q|^2}{\mu^2} + \frac{5}{3} + i\pi\theta(q^2)\right)\\
\Delta &= \frac{1}{\epsilon} - \gamma + \ln(4\pi).
\end{aligned}$$

The contribution of $1/\epsilon$ (or Δ, depending on the scheme) is absorbed in the wave-function renormalization by the condition (8.3). In this kinematic case therefore the momentum dependency is the same for all the gauge bosons, and is only modified by the relative couplings. It is therefore a useful test case to check the relative coupling strengths of the gauge bosons to matter. The imaginary part appears due to the possibility of the gauge bosons to decay into the light fermions.

In the opposite case of very massive fermions, or equivalently low energy, $q^2 \ll m_f^2$, the behavior is given by

$$\begin{aligned}
\Sigma_T^{AA} &= \frac{\alpha}{3\pi}\sum_{f,\sigma=pm} q_f^{\sigma 2} q^2\left(\Delta - \ln\frac{m_f^{\sigma 2}}{\mu^2} + \frac{q^2}{5m_f^{\sigma 2}}\right) + \mathcal{O}(q^6)\\
\Sigma_T^{AZ} &= -\frac{\alpha}{3\pi}\sum_{f,\sigma=\pm} q_f^{\sigma} v_f^{\sigma}\left(\Delta - \ln\frac{m_f^{\sigma 2}}{\mu^2} + \frac{q^2}{5m_f^{\sigma 2}}\right) + \mathcal{O}(q^6)\\
\Sigma_T^{ZZ} &= -\frac{2\alpha}{\pi}\sum_{f,\sigma=\pm} a_f^{\sigma^2} m_f^{\sigma 2}\left(\Delta - \ln\frac{m_f^{\sigma 2}}{\mu^2}\right) + \mathcal{O}(q^2)\\
\Sigma_T^{WW} &= -\frac{\alpha}{8\pi\sin^2\theta_W}\sum_f\left(m_f^{+2}\left(\Delta - \ln\frac{m_+^2}{\mu^2}\right) + m_f^{-2}\left(\Delta - \ln\frac{m_f^{-2}}{\mu^2}\right) + \frac{1}{2}(m_f^{+2} + m_f^{-2})\right.\\
&\left. - \frac{m_f^{+2}m_f^{-2}}{m_f^{+2} - m_f^{-2}}\ln\frac{m_f^{+2}}{m_f^{-2}}\right) + \mathcal{O}(q^2).
\end{aligned}$$

The leading corrections are in case of the Z and W boson self-energies proportional to the masses. This is therefore an indirect possibility to measure these in low-energy electroweak processes.

A third possibility is offered by the W-boson self-energy. Since in it the masses of the two doublet members are not occurring symmetrically, the case $m^{+2} \gg q^2 \gg m^{-2}$ yields an asymmetric result of

$$\Sigma_T^{WW} = -\frac{\alpha}{8\pi\sin^2\theta_W}\sum_f m_f^{+2}\left(\Delta - \ln\frac{m_f^{+2}}{\mu^2} + \frac{1}{2}\right) + \mathcal{O}(q^2).$$

Therefore this offers an indirect possibility to determine the mass of the heavier doublet member indirectly. Such a case is, e. g., given by the bottom and top doublet, with an energy difference in the range of tens of GeV. Similarly, if a new matter quark or lepton is found, this would offer one possibility to determine the (possible) mass of its doublet partner. In this context it is particularly useful that the mass appears as a multiplicative factor instead of only as the argument of a logarithm. It is therefore much simpler to obtain it than, e. g., in the case of the Higgs of which the mass in all standard model processes accessible appears only logarithmically.

8.4 The hierarchy problem

As noted, the mass of the Higgs particle receives radiative corrections, most notably by the tadpoles. Though the tadpole contribution are removed in the standard renormalizations scheme, this requires a very precise cancellation of the counter-terms and the tadpole itself, and only a minor change would induce large corrections to the Higgs mass. This is known as the hierarchy problem.

This appears at first a very superficial problem, as it affects a renormalization-group-dependent quantity, which is unphysical. However, the problem propagates from this point, and actually many quantities in the Higgs sector turn out to be rather sensitive to the parameters of the Higgs model, usually quadratically, in strong contrast to the remainder of the standard model where the sensitivity on the parameters is usually logarithmically only.

Since it is assumed that the standard model is just the low-energy effective theory of an underlying high-energy theory, this has substantial implications. The hierarchy problem actually becomes one because under the assumption of new physics only at the GUT or Planck scale of 10^{15} or 10^{19} GeV, respectively, small changes can accumulate during the renormalization-group evolution large effects because of the hierarchy problem. Thus, the problem can be recast into the question why the scale of new physics is so large. If it would be in the TeV range, the problem would be essentially non-existent, as long as the existence of a potential fine-tuning problem is not regarded as a problem itself.

This motivated a large number of proposals, most notably supersymmetry and technicolor, of new physics to cure this problem, though, of course, nature could just be like this.

However, there are still the possibility that non-perturbative effects can cure this. One possibility is that the asymptotic safety moderates the growth of the Higgs mass. However, so far this does not seem to be the case. The other is that all observables, especially the correct bound states rather than the elementary states, do not show fine-tuning. Though the mass spectrum indeed seems not to exhibit this, it still appears to be very sensitive to the parameters, and thus the hierarchy problem remains even beyond perturbation theory.

Chapter 9

Anomalies

9.1 Introduction

There is one particular important property of the standard model, which is very much restricting its structure, and which is recurring in extensions of the standard model. That is the absence of anomalies. An anomaly is that some symmetry, which is present on the classical level, is not present when considering the quantum theory. The symmetry is said to be broken by quantum effects. Generically, this occurs if the action of a theory is invariant under a symmetry, but the measure of the path integral is not. Constructing a theory which is at the same time anomaly-free and consistent with the standard model is actually already quite restricting, and therefore anomalies are an important tool to check the consistency of new proposals for physics beyond the standard model, but also for the standard model itself. As in the standard model it originates from the electroweak sector, it will be discussed here in some detail.

9.2 Global anomalies

Anomalies fall into two classes, global and local anomalies. Global anomalies refer to the breaking of global symmetries by quantum effects. The most important one of these global anomalies is the breaking of dilatation symmetry. This symmetry corresponds to rescaling all dimensionful quantities, e. g., $x \to \lambda x$. Maxwell theory, massless QED, Yang-Mills theory, and massless QCD are all invariant under such a rescaling, at the classical level, though not the Higgs sector of the standard model. This is no longer the case at the quantum level. By dimensional transmutation an explicit scale is introduced into the theory, and thereby the quantum theory is no longer scale-invariant. Such global anomalies have very direct consequences. E. g., this dilatation anomaly leads to the fact that the photon is massless in massless QED. Of course, it is also massless in massive QED, but there the breaking of the dilatation symmetry is explicit due to the lepton mass.

Another example is the so-called axial anomaly, which occurs due to the breaking of the global axial symmetry of quarks. A consequence of it is the anomalously large η' mass. While the dilatation anomaly is quite obvious, the chiral anomaly is much more subtle, and therefore deserves some more discussion. In addition, it will be very helpful when generalizing to the local anomalies.

9.2.1 Classical level

To prepare for this, it is worthwhile to consider the situation as it would be without anomalies, i. e. at the classical level. For this purpose, start with a gauge theory with fermions ψ being in some representation R of the gauge Lie group G with generators T and gauge fields in the adjoint representation. The fermionic part of the Lagrangian is then given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu - igT^a A_\mu^a) - m)\psi = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi$$

from which the Dirac equation

$$(i\gamma_\mu D^\mu - m)\psi = 0$$

follows as the equation of motion, and likewise for the anti-fermion.

The current carrying the charge is then

$$j_\mu^a = \bar{\psi}\gamma_\mu T^a \psi.$$

Due to the chiral symmetry, there is also a corresponding axial current

$$j_\mu^{5a} = \bar{\psi}\gamma_5\gamma_\mu T^a \psi.$$

In addition, there are also the singlet currents

$$\begin{aligned} j_\mu &= \bar{\psi}\gamma_\mu\psi \\ j_\mu^5 &= \bar{\psi}\gamma_5\gamma_\mu\psi, \end{aligned}$$

which corresponds to the fermion current and the axial current.

Naively, the divergences of these equations can be calculated using the Dirac equation.

$$\begin{aligned} \partial^\mu j_\mu^a &= -i\bar{\psi}(g\tau^b\gamma_\mu A_b^\mu - m)\tau^a\psi - i\bar{\psi}\tau^a(-g\tau^b\gamma_\mu A_b^\mu + m)\psi \\ &= ig\bar{\psi}\left[\tau^a, \tau^b\right]\gamma_\mu A_b^\mu\psi = -gf^{abc}A_b^\mu\bar{\psi}\gamma_\mu\tau_c\psi = -gf^{ab}{}_c A_b^\mu j_\mu^c. \end{aligned}$$

This implies that the color current is not observed, as long as the current is gauged. For a non-gauge current, like a flavor current, g vanishes, and the current is conserved.

This is not surprising, as a non-Abelian gauge theory has no gauge-invariant charge. However, the current is a gauge-vector, and therefore covariantly conserved

$$D_\mu^{ab} j_b^\mu = 0. \tag{9.1}$$

In the same way, it is possible to calculate the situation of the axial color current. Because of the commutation relation between γ matrices, the result is

$$D_\mu^{ab} j_b^\mu = 2im\bar{\psi}\gamma_5\tau^a\psi = 2mip^a, \tag{9.2}$$

Here, p is the pseudo-scalar density, and not a momentum component. Thus, even in a non-gauge theory this current is only conserved for fermions without a mass term in the Lagrangian.

The calculations for the singlet current is simpler, and yields

$$\begin{aligned}\partial_\mu j^\mu &= 0\\ \partial^\mu j_\mu^5 &= 2im\bar{\psi}\gamma_5\psi = 2imp^0.\end{aligned}$$

Hence, the number of fermion is, a expected, a conserved current. The axial current is only conserved for massless fermions. This is the result that chiral symmetry gets explicitly broken, already classically, by a mass-term.

In a theory like the standard model, where parity is broken, left-handed and right-handed fermions

$$\begin{aligned}\psi_L &= \frac{1-\gamma_5}{2}\psi\\ \psi_R &= \frac{1+\gamma_5}{2}\psi\end{aligned}$$

do not couple in the same way to the gauge-fields

$$\mathcal{L} = \bar{\psi}_L i\gamma_\mu D_L^\mu \psi_L + \bar{\psi}_R i\gamma_\mu D_R^\mu \psi_R,$$

with $D_L \neq D_R$, and no mass term is permitted due to gauge invariance. Thus, the color currents are recombined into covariantly conserved left-handed and right-handed currents as

$$\begin{aligned}j_\mu^{aL} &= \frac{1}{2}(j_\mu^a - j_\mu^{5a})\\ j_\mu^{aR} &= \frac{1}{2}(j_\mu^a + j_\mu^{5a})\\ D_L j_\mu^L &= 0\\ D_R j_\mu^R &= 0,\end{aligned}$$

and a similar recombination for the singlet currents.

9.2.2 One-loop violation

So far, this was the conservation at the classical level, which already requires the fermions to be massless. At the quantum level, this result is expressed by Ward-identities. In particular, take Ward identities for correlation functions of the form

$$T^{ijk}_{\mu\nu\rho} = \langle T j^i_\mu j^j_\nu j^k_\rho \rangle,$$

where i, j, and k can take the values V, A, and P, which require to replace the j by j^a, j^{5a}, and p^a, respectively, and the Lorenz index is dropped in the last case. Calculating the corresponding Ward identities for a local chiral transformation

$$\begin{aligned} \psi' &= e^{i\beta(x)\gamma_5}\psi(x) \\ \bar{\psi}' &= \bar{\psi}e^{i\beta(x)\gamma_5} \end{aligned}$$

yields the expressions

$$\partial^\mu_x T^{VVA}_{\mu\nu\rho}(x,y,z) = \partial^\nu_y T^{VVA}_{\mu\nu\rho}(x,y,z) = 0 \tag{9.3}$$

$$\partial^\rho_z T^{VVA}_{\mu\nu\rho}(x,y,z) = 2mT^{VVP}_{\mu\nu}(x,y,z), \tag{9.4}$$

directly implementing the relations (9.1) and (9.2). This is what should happen, if there would be no anomalies.

To check this, it is possible to calculate the leading-order perturbative correction. Since only fermion fields appear in the vacuum expectation value, this is a vacuum triangle graph, and the coupling is to external currents. In fact, it does not matter at this point whether the external currents are gauged or non-gauged, since to this order this only alters the presence or absence of color matrices at the external vertices. The only relevant part of the external vertices is their Dirac structure.

Evaluating all the Wick contractions yields two Feynman diagrams, which translate to

$$\begin{aligned} &T^{VVA}_{\mu\nu\rho}(p_1,p_2,p_3=-p_1-p_2) = \qquad (9.5)\\ &-i^3\int\frac{d^4k}{(2\pi)^4}\Big(\mathrm{tr}\gamma_\mu(\gamma_\alpha k^\alpha-m)^{-1}\gamma_\nu(\gamma^\beta k_\beta-\gamma_\beta p_2^\beta-m)^{-1}\gamma_\rho\gamma_5(\gamma_\gamma k^\gamma+\gamma_\gamma p_1^\gamma-m)^{-1}\\ &+\mathrm{tr}\gamma_\nu(\gamma_\alpha k^\alpha-m)^{-1}\gamma_\mu(\gamma^\beta k_\beta-\gamma_\beta p_1^\beta-m)^{-1}\gamma_\rho\gamma_5(\gamma_\gamma k^\gamma+\gamma_\gamma p_2^\gamma-m)^{-1}\Big). \end{aligned}$$

This expression is linearly divergent. One of the most important points in anomalies, and in quantum field theories in general, is that the result is independent of the regulator employed. This will be discussed later how to show this. Here, it permits to use a Pauli-Villar regulator with a mass M, which is technically more simple than other possibilities.

Using dimensional regularization makes the result subtle, as it depends on the way the matrix γ_5 is analytically continued. This problem will therefore be avoided here.

To test the vector Ward identity, the expression can be multiplied with p_1^μ. To simplify the so obtained expression it is useful to employ

$$\gamma_\mu p_1^\mu = -(\gamma_\mu k^\mu - \gamma_\mu p_1^\mu - m) + (\gamma_\mu k^\mu - m),$$

yielding

$$\begin{aligned}
& p_1^\mu T^{VVA}_{\mu\nu\rho}(p_1,p_2,p_3=-p_1-p_2) = \\
& -i^3 \int \frac{d^4k}{(2\pi)^4} \Big(\mathrm{tr} - (\gamma_\alpha k^\alpha - m)^{-1}\gamma_\nu(\gamma^\beta k_\beta - \gamma_\beta p_2^\beta - m)^{-1}\gamma_\rho\gamma_5 \\
& \mathrm{tr}(\gamma_\gamma k^\gamma + \gamma_\gamma p_1^\gamma - m)^{-1}\gamma_\nu(\gamma^\beta k_\beta - \gamma_\beta p_2^\beta - m)^{-1}\gamma_\rho\gamma_5 \\
& + \mathrm{tr}(\gamma_\gamma k^\gamma + \gamma_\gamma p_2^\gamma - m)^{-1}\gamma_\nu(\gamma_\alpha k^\alpha - m)^{-1}\gamma_\rho\gamma_5 \\
& + \mathrm{tr} - (\gamma_\gamma k^\gamma + \gamma_\gamma p_2^\gamma - m)^{-1}\gamma_\nu(\gamma^\beta k_\beta - \gamma_\beta p_1^\beta - m)^{-1}\gamma_\rho\gamma_5 + (m \to M) \Big) .
\end{aligned} \tag{9.6}$$

This rather lengthy expression is now a finite integral. It is therefore permissible to reshuffle the momenta like $k \to k + p_2$ in the first term and $k \to k + p_2 - p_1$ in the second term. Then, the first and third and second and fourth term cancel each other, and likewise this happens for the regulator. Thus, the vector Ward identity is fulfilled. The result for the second identity in (9.3) works in the same way.

The situation changes drastically for the axial Ward identity (9.4). The expression (9.5) is still divergent, so before doing anything, it will again be regulated using a Pauli-Villar regulator, to make it well-defined. To evaluate (9.4) requires multiplication with $p_3 = -p_1 - p_2$, which can rewritten as

$$\begin{aligned}
\gamma_\mu p_3^\mu \gamma_5 &= (\gamma_\mu k^\mu - \gamma_\mu p_2^\mu - m)\gamma_5 + \gamma_5(\gamma_\mu k^\mu + \gamma_\mu p_1^\mu - m) + 2m\gamma_5 \\
&= (\gamma_\mu k^\mu - \gamma_\mu p_1^\mu - m)\gamma_5 + \gamma_5(\gamma_\mu k^\mu + \gamma_\mu p_2^\mu - m) + 2m\gamma_5.
\end{aligned}$$

This yields

$$\begin{aligned}
& p_3^\rho T^{VVA}_{\mu\nu\rho}(p_1,p_2,p_3=-p_1-p_2) = 2i \int \frac{d^4k}{(2\pi)^4} \\
& \Big(m\mathrm{tr}\left(\gamma_\mu(k_\alpha\gamma^\alpha - m)^{-1}\gamma_\nu(\gamma_\beta k^\beta - \gamma_\beta p_2^\beta - m)^{-1}\gamma_5(\gamma_\gamma k^\gamma + p_1^\gamma\gamma_\gamma - m)^{-1}\right) \\
& m\mathrm{tr}\left(\gamma_\mu(k_\alpha\gamma^\alpha - m)^{-1}\gamma_\nu(\gamma_\beta k^\beta - \gamma_\beta p_1^\beta - m)^{-1}\gamma_5(\gamma_\gamma k^\gamma + p_2^\gamma\gamma_\gamma - m)^{-1}\right) \\
& M\mathrm{tr}\left(\gamma_\mu(k_\alpha\gamma^\alpha - M)^{-1}\gamma_\nu(\gamma_\beta k^\beta - \gamma_\beta p_2^\beta - M)^{-1}\gamma_5(\gamma_\gamma k^\gamma + p_1^\gamma\gamma_\gamma - M)^{-1}\right) \\
& M\mathrm{tr}\left(\gamma_\mu(k_\alpha\gamma^\alpha - M)^{-1}\gamma_\nu(\gamma_\beta k^\beta - \gamma_\beta p_1^\beta - M)^{-1}\gamma_5(\gamma_\gamma k^\gamma + p_2^\gamma\gamma_\gamma - M)^{-1}\right)\Big)
\end{aligned}$$

There are two remarkable facts to be observed. The first is that this expression is finite. The projection with p_3 drops out the divergent terms. This can be seen using the Dirac matrix identity

$$\mathrm{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 = -4i\epsilon_{\mu\nu\rho\sigma}. \tag{9.7}$$

Because of the anti-symmetry of the ϵ-symbol, any term containing two or more factors of k vanishes. Hence, the numerator is reduced by two powers of k, making the integral finite. This did not work in (9.6) as there one index less was uncontracted. However, the regulator still had to be present in the first place to make this projection well-defined. The second is that this expression, except for the regulator, is identical to T^{VVP} up to a factor of m, which is obtained by replacing $\gamma_\rho\gamma_5$ in (9.5).

The term involving the regulator can then be easily calculated, as when removing the regulator in the end, the external momenta and masses can always be neglected, and the integral becomes a simple tadpole integral. The final result is thus

$$\begin{aligned} ip_3^\rho T^{VVA}_{\mu\nu\rho}(p_1,p_2) &= 2miT^{VVP}_{\mu\nu}(p_1,p_2) + \lim_{M\to\infty} 8iM^2\epsilon_{\mu\nu\rho\sigma}p^1_\rho p^2_\sigma \times \frac{i}{16\pi^2}\frac{-1}{2M^2} \\ &= 2miT^{VVP}_{\mu\nu}(p_1,p_2) + \frac{1}{2\pi^2}\epsilon^{\mu\nu\rho\sigma}p^1_\rho p^2_\sigma \end{aligned} \tag{9.8}$$

Thus, the Ward identity (9.4) is violated. The anomaly is both finite and independent of the masses of the involved particles. It is also independent of the structure of the external interaction, except for its Lorentz structure. The only thing changing is the appearance of corresponding pre-factor a^{abc} of the coupling matrices T^a in charge space, which turn out to be

$$a^{abc} = \frac{1}{2}\mathrm{tr}\left(\left\{T^a, T^b\right\}T^c\right), \tag{9.9}$$

a result which will become significant later. This is not the only anomaly, and a similar result holds for the case of three axial currents.

Without proof, it should be noted here that there is still a certain regulator dependency. It is possible by symmetries to add a finite term of form $C\epsilon_{\mu\nu\rho\sigma}(p_1 - p_2)^\sigma$ to the counter-term in (9.6). Though C can be tuned to absorb the anomaly, this term will also contribute to the vector identities, and induce there an anomaly for $C \neq 0$. Thus, it is only possible to shift the anomaly around, without removing it.

The most well-known consequence of this anomaly is the decay of a neutral pion into two photons. This is precisely of the type investigate here, where the photons play the role of the vector currents. The axial current is related to the pion field by a QCD relation

$$\partial^\mu j^a_\mu = \frac{f_\pi}{\sqrt{2}}M_\pi^2\pi^a, \tag{9.10}$$

where a is an isospin index, counting the three pions, $a = 0, \pm$, where only $a = 0$ is relevant because of charge conservation. Since there are no massless hadrons, there can be no pole in the corresponding amplitude T^{VVA}, and thus the product with p_ρ has to vanish. As a consequence, the amplitude T^{VVP}, describing the transition, would vanish as well, because of the Ward identity, and therefore the pion would usually not decay into two photons, if at rest. However, due to the anomaly, this is not necessary, as the anomaly can balance the Ward identity. Hence, the pion at rest can decay into two photons, due to the anomaly, a process indeed observed in experiment.

9.3 Local anomalies

In contrast to the global anomalies, the local anomalies are a more severe problem. A local anomaly occurs, when a quantum effect breaks a local gauge symmetry. The consequence of this would be that observable quantities depend on the gauge, and therefore the theory makes no sense. Thus, such anomalies may not occur. There are two possibilities how such anomalies can be avoided. One is that no such anomalies occurs, i. e., the path integral measure must be invariant under the symmetry. The second is by anomaly cancellation, i. e., some parts of the measure are not invariant under the symmetry, but the sum of all such anomalous terms cancel. It is the latter mechanism which makes the standard model anomaly-free. However, the price to pay for this is that the matter content of the standard model has to follow certain rules. It is thus rather important to understand how this comes about. Furthermore, any chiral gauge theory beyond the standard model faces similar, or even more severe, problems.

Already the classical result (9.2) indicates that the current is only covariantly conserved,. The latter equation implies that only for massless fermions there will be no gauge anomaly. However, this is not a problem, as only zero-mass fermions are admitted to the standard model anyway, and all apparent fermion masses are generated by the Higgs effect. But for the standard model this is still modified. Due to the parity violation, it is necessary to consider a current for left-handed and right-handed fermions separately, where the corresponding left-handed and right-handed covariant derivatives for the left-handed and right-handed currents appear.

In principle, it is possible to do the same one-loop calculation in a gauge theory, and the final result is quite similar. However, it may still be questioned whether this is an artifact of perturbation theory. It is not, and to show this it is useful to derive the local anomaly for gauge theories using a different approach. In a path integral approach, this becomes particularly clear, as it can be shown that the anomaly stems from the fact that the path-integral measure for fermions, $\mathcal{D}\psi\mathcal{D}\bar{\psi}$, is not invariant under chiral gauge transformations, and therefore the anomaly arises. It is, of course, invariant under vectorial gauge transformations, and thus theories like QCD need not to be considered, as will be confirmed below. This also shows that the anomaly is a pure quantum phenomenon, as the measure is part of the quantization process.

9.3.1 Anomalies as a quantum effect

To see that this is a relevant effect, it is important to remember how Ward identities are obtained in general. Any well-defined symmetry transformation should leave the partition function unchanged, i. e.

$$0 = \delta Z = \delta \int \mathcal{D}\phi e^{iS+i\int d^4xj\phi}, \tag{9.11}$$

where ϕ is for simplicity a non-Grassmann field, which changes under the transformation as $\phi \to \phi + \epsilon f(\phi, x)$, with f some arbitrary function and ϵ infinitesimal. Performing the variation yields

$$0 = \int \mathcal{D}\phi e^{iS+i\int d^4xj\phi} \int d^4x \left(i\left(\frac{\delta S}{\delta\phi} + j\right) f + \frac{\delta f}{\delta\phi} \right), \tag{9.12}$$

where the first two terms come from the exponent. At the classical level, the source term vanishes, and the derivative of the action just gives the equations of motion, yielding the classical Ward identities. The third term is new in the quantum theory, and gives the contribution of the Jacobian,

$$\det \frac{\delta(\phi + \epsilon f)}{\delta\phi} = \det \left(1 + \epsilon\frac{\delta f}{\delta\phi}\right) \approx 1 + \epsilon\frac{\delta f}{\delta\phi} + \mathcal{O}(\epsilon^2).$$

This is a genuine quantum contribution. It will be the source of the anomaly. Here it also becomes evident that the term anomaly is actually a misnomer. There is nothing anomalous about them. They are just a quantum effect.

To obtain Ward identities from (9.12), it is sufficient to derive with respect to the source some number of times, and then set the sources to zero at the end, yielding

$$0 = \left\langle T\Pi_l\phi_l\frac{\delta f}{\delta\phi}\right\rangle + i\left\langle T\Pi_l\phi_l\frac{\delta S}{\delta\phi}f\right\rangle + \sum_k \left\langle T\Pi_{l<k}\phi_l f\Pi_{m>k}\phi_m\right\rangle. \tag{9.13}$$

In this way an anomaly surfaces in Ward identities in the full quantum theory. This also shows that an anomaly is not a perturbative effect, since this is an exact result. However, it is still possible that the Jacobian is actually one, and a deviation from one in the one-loop calculation is just an artifact of perturbation theory.

9.3.2 Full expression for the anomaly

To check this, rotate first to Euclidean time, by replacing $t \to it$ and correspondingly in all covariant quantities the time components by i-times the time components and in all contravariant quantities the time components by $-i$-times the time components. Then expand the fermion fields in orthonormal eigenfunctions ψ_n of the Dirac operator[1],

$$\begin{aligned}\psi(x) &= \sum_n a_n \psi_n(x) \\ \bar{\psi}(x) &= \sum_n \psi_n^\dagger(x) \bar{b}_n,\end{aligned}$$

which satisfy

$$i\gamma_\mu D^\mu \psi_n = \lambda_n \psi_n \tag{9.14}$$

$$-i\gamma_\mu D^\mu \psi_n^\dagger = \lambda_n \psi_n^\dagger. \tag{9.15}$$

This permits to rewrite the path integral as an infinite product of integrations over the coefficients,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \Pi_m da_m d\bar{b}_m, \tag{9.16}$$

keeping in mind that these differentials are Grassmannian.

Now, a local chiral transformation $\beta(x)$

$$\psi \to e^{i\beta(x)\gamma_5}\psi,$$

then corresponds to a linear transformation of the coefficients

$$a_m \to C_{mn} a_n = a'_n,$$

which yields the Jacobian

$$\Pi_m da'_m d\bar{b}'_m = \frac{1}{(\det C)^2} \Pi_m da_m d\bar{b}_m,$$

or, formally,

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \frac{1}{(\det C)^2} \mathcal{D}\psi\mathcal{D}\bar{\psi}.$$

This determinant can be rewritten as

$$\frac{1}{(\det C)^2} = e^{-2\mathrm{tr}\ln C} = e^{-2\mathrm{tr}\delta C}, \tag{9.17}$$

[1]The spectrum is actually continuous, but for simplicity it will be treated here as if it would be discrete.

where in the last equality it was assumed that β is infinitesimal, and thus $C = 1 + \delta C$ is close to one. In this case, δC can be evaluated starting from

$$a'_m \psi_m = (1 + i\beta\gamma_5) a_n \psi_n$$

which can be reduced using the orthonormality of the eigenstates of the Dirac equation to

$$a'_m = \int d^4x \psi_m^\dagger (1 + i\beta\gamma_5)\psi_n a_n = (1 + \delta c_{mn}) a_n.$$

Inserting this result into (9.17) yields for the Jacobian of the infinitesimal transformation

$$J = \exp\left(-2i \int d^4x \beta \psi_m^\dagger \gamma_5 \psi_m\right), \tag{9.18}$$

where the trace has been evaluated.

Unfortunately, the expression, as it stands, is ill-defined. It is necessary to regularize it. A useful possibility to make the expression well-defined is by replacing the trace over the eigenstates as

$$\psi_m^\dagger \gamma_5 \psi_m \to \lim_{\tau \to 0} \psi_m^\dagger \gamma_5 e^{-\lambda_m^2 \tau} \psi_m, \tag{9.19}$$

where the limit has to be performed at the end of the calculation only. Expanding the Gaussian and using the relations (9.14-9.15), this expression can be rewritten as

$$\lim_{\tau \to 0} \psi_m^\dagger \gamma_5 e^{-\lambda_m^2 \tau} \psi_m = \lim_{\tau \to 0} \mathrm{tr}\left(\gamma_5 e^{-\tau(\gamma_\mu D^\mu)^\dagger \gamma_\nu D^\nu}\right). \tag{9.20}$$

The exponential can be rewritten as

$$(\gamma_\mu D^\mu)^\dagger \gamma_\nu D^\nu = -D_\mu D^\mu + \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] F_{\mu\nu}^a \tau_a. \tag{9.21}$$

The limit is still ill-defined. It is necessary to rewrite the expression in a suitable way. This is achieved by the heat-kernel regularization.

For a differential operator, here given by $\Delta = (\gamma_\mu D^\mu)^\dagger \gamma_\nu D^\nu$, it is possible to define a heat-kernel as

$$(\partial_\tau + \Delta_x) G(x, y, \tau) = 0 \tag{9.22}$$

$$G(x, y, 0) = \delta(x - y). \tag{9.23}$$

Which is solved by the formal expression

$$G(x, y, \tau) = e^{-\Delta_x \tau} = \sum_m e^{-\tau\lambda_m} \psi_m^\dagger(y) \psi_m(x).$$

This is already the expression (9.20). Without proof, it can now be shown that this heat kernel can be expanded for small τ as

$$G(x,y,\tau) \to_{\tau\to 0} \frac{1}{(4\pi\tau)^2}\exp^{-\frac{(x-y)^2}{4\tau}}\sum_{j=0}^{\infty}a_j(x,y)\tau^j.$$

Inserting this expansion into (9.18) yields

$$\ln J = -2i\lim_{\tau\to 0}\frac{1}{(4\pi\tau)^2}\int d^4x\beta\sum_j \tau^j \mathrm{tr}\gamma_5 a_j.$$

For $\tau \to 0$, the first term does not contribute, as a_0 has to be equal to one because of the condition (9.23). Terms with $j > 2$ will be irrelevant, because of the powers of τ. This leaves only $j = 1$ and $j = 2$. For these terms follows from the requirement that the expansion satisfies (9.22) a descent equation

$$-\Delta a_{j-1} = ja_j.$$

Since $a_0 = 1$, a_1 can be obtained algebraically from (9.21). Since all resulting terms have at most two γ matrices, the trace will vanish. Similarly, for a_2 only those terms can contribute to the trace where at least four γ matrices appear, which implies only the term quadratic in $F_{\mu\nu}$ will contribute. Which is precisely what is necessary to cancel the pre-factor.

Thus, the remainder is just

$$J = \exp\left(-\frac{i}{32\pi^2}\int d^4x\beta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F_{a\rho\sigma}\right). \tag{9.24}$$

Hence, the Jacobian is non-trivial, and will contribute in the Ward identities (9.13). However, this is still a rather complicated expression, which does not yet look like the one-loop result.

That this coincides with the one-loop anomaly can be obtained by an explicit calculation. Since this was for the global case, take β to be constant. The integral can then be rewritten as

$$\int d^4x\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(iA^a_\nu\partial_\rho A^a_\sigma + \frac{2}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right). \tag{9.25}$$

Since the perturbative case was the Abelian case, the second term can be dropped. The first term is then for the global case just two external fields, e. g. playing the roles of the photon field in the pion decay, and two momenta in Fourier space, which, after relabeling, yield the desired one-loop expression. Hence, indeed the full and the one-loop anomaly coincide. In gauge theories there are also anomalies in box and pentagon graphs with an odd number of axial insertions, which are again one-loop exact.

To obtain the final result including all color factors requires then just an explicit calculation, inserting the Jacobian (9.24) into the Ward identity (9.13). This will yield (9.8) with (9.9) inserted.

The actual form of the anomaly in (9.24) and (9.25) has a very particular meaning, it is the so-called topological charge density, a concept which will be discussed in more detail in the context of baryon number violation in section 9.5 and especially 9.5.2.

9.3.3 Anomaly cancellation

However, for the standard model it is more interesting to consider the case that left-handed and right-handed fermions are coupled differently to and/or with different gauge fields. Due to the different sign of γ_5 in the corresponding projector, this will reemerge as a different sign of the anomaly, yielding

$$k^\rho T^{V^aV^bA^c}_{\mu\nu\rho}(p,q,k)=2mT^{V^aV^bP^c}_{\mu\nu}(p,q,k)+\frac{\operatorname{tr}\left\{\tau^a_L,\tau^b_L\right\}\tau^c_L-\operatorname{tr}\left\{\tau^a_R,\tau^b_R\right\}\tau^c_R}{2}\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma,$$

were L and R indicate the representation of the left-handed and right-handed fermions. As a consequence, the classical gauge symmetry is broken by the anomaly, and results will depend on the choice of gauge. This can be directly understood form this expression: the left-hand side should vanish, if there is no massless pseudo-scalar particle in the theory, which is true for the standard model. On the right-hand side, the first term will indeed do so, if the fermion mass is zero. This is already required due to parity violation in the standard model. But for the second term this is not obvious.

There are now two possibilities how to obtain an anomaly-free theory. Either, the theory is anomaly-free, if each of the remaining terms is individually zero, or they cancel. Indeed, the expression $\operatorname{tr}\{\tau^a,\tau^b\}\tau^c$, the so-called symmetric structure constant, is zero for all (semi-)simple Lie groups, except for SU($N\geq 3$) and U(1). Unfortunately, these are precisely those appearing in the standard model, except for the SU(2) of weak isospin. For the group SU(3) of QCD, this is actually not a problem, since QCD is vectorial, and thus[2] $\tau_L=\tau_R$, and the terms cancel for each flavor individually. Thus remains only the part induced by the hypercharge.

In this case, each generation represents an identical contribution to the total result, as the generations are just identical copies concerning the generators. It is thus sufficient to consider one generation. The right-handed contributions are all singlets under the weak isospin, and thus they only couple vectorially to electromagnetism, and therefore yield zero. The contributions from the left-handed doublets contain then the generators of the

[2] Actually, unitarily equivalent is sufficient.

weak isospin, τ^a, and the electric charge $Q = \tau^3 + \mathbf{1}y/2$. The possible combinations contributing are

$$\mathrm{tr}\tau^a\{\tau^b,\tau^c\} \tag{9.26}$$

$$\mathrm{tr}Q\{\tau^a,\tau^b\} \tag{9.27}$$

$$\mathrm{tr}\tau^a Q^2 \tag{9.28}$$

$$\mathrm{tr}Q^3. \tag{9.29}$$

The contribution (9.26) vanishes, as this is a pure SU(2) expression. The term (9.29) is not making a difference between left and right, and is therefore also vanishing. It turns out that (9.27) and (9.28) lead to the same result, so it is sufficient to investigate (9.28). Since the isospin group is SU(2), the anti-commutator of two Pauli matrices just gives a Kronecker-δ times a constant, yielding in total

$$\mathrm{tr}Q\{\tau^a,\tau^b\} = \frac{1}{2}\delta^{ab}\sum_f Q_f,$$

where Q_f is the electric charge of the member f of the generation in units of the electric charge. It has to vanish to prevent any gauge anomaly in the standard model, which is fulfilled:

$$\sum_f Q_f = (0-1) + N_c\left(\frac{2}{3}-\frac{1}{3}\right) = -1 + \frac{N_c}{3} = 0.$$

Therefore, there is no gauge anomaly in the standard model. However, this is only possible, because the electric charges have certain ratios, and the number of colors N_c is three. This implies that the different sectors of the standard model, the weak isospin, the strong interactions, and electromagnetism, very carefully balance each other, to provide a well-defined theory. Such a perfect combination is one of the reasons to believe that the standard model is part of a larger theory, which imposes this structure.

9.4 Witten anomalies

There is actually a further possible anomaly for fermions, the so-called Witten anomaly, which is also connected to the parity violation in the standard model. It is also a gauge anomaly, and has therefore to be canceled as well. This occurs in the standard model if the number of weak fermion states is even. This would not be the case, if, e. g., there would be a single triplet of fermions charged under the weak isospin. In technicolor theories, or other theories beyond the standard model, this is a constraint, as in such theories multiplets with an odd number of fermions may appear, e. g. when the chirally coupled fermions are additionally charged under different gauge groups or representations, leading to an odd number of fermions. This has then to be canceled by additional fermions. This is a problem exclusively applying to the Sp(N) gauge groups, and to SU(2) of the weak interactions because SU(2)≈Sp(1), as well as O($N < 6$) groups, except for SO(2).

The reason can be most easily illustrated by considering the path-integral with the fermions integrated out. For n Weyl fermions, the expression is

$$Z = \int \mathcal{D}A_\mu (\det i\gamma_\mu D^\mu)^{\frac{n}{2}} e^{iS}, \tag{9.30}$$

with S the usual gauge-field action. The problem arises, as it can be proven that for each gauge-field configuration of a gauge theory with an affected gauge group there exists a gauge-transformed one such that

$$(\det i\gamma_\mu D^\mu)^{\frac{1}{2}} = -(\det i\gamma_\mu D^{\mu'})^{\frac{1}{2}},$$

where $'$ denotes gauge-transformed. The proof is somewhat involved, but essentially boils down to the fact that the determinant has to be defined in terms of a product of eigenvalues. For Sp(N) gauge theories as well as the groups O($N < 6$) it can then be shown that there exist gauge-transformations, which are topologically non-trivial, in the sense that one of the non-zero eigenvalues changes sign. Mathematically, the reason is that the fourth homotopy group of these groups is non-trivial and actually is Z_2 or Z_2^2. Hence, the integrand of the path integral (9.30) exists twice on each gauge orbit, but with opposite signs. Thus, the partition function vanishes, and all expectation values become ill-defined 0/0 constructs. Thus, such a theory is ill-defined, as there is no continuous deformation of the gauge group possible to introduce a suitable definition, similar to L'Hospital's rule.

In the standard model, the problem does not arise, because the number of Weyl flavors of the fermions is even since for every left-handed doublet there are two right-handed singlets. One could also hope that, since the gauge group of the standard model is actually S(U(3)×U(2))≈SU(3)/Z_3×SU(2)/Z_2×U(1), this problem would not arise. The reason for this division is that only for this particular gauge group the matter field representation becomes single-valued, as is necessary for them to be meaningful. However, because SU(2)/Z_2 ≈SO(3) instead of Sp(2), this does not help, as the fourth homotopy group of SO(3) is also non-trivial, and the problem persists,

Thus, adding further sectors to the standard model, or embedding it in a grand-unified theory, must respect this fact, to avoid triggering the Witten anomaly.

9.5 Baryon number violation

9.5.1 Instantons

Our current understanding of the origin of the universe indicates that it emerged from a big bang, a space-time singularity where everything started as pure energy. Why is then not an equal amount of matter and anti-matter present today, but there is a preference for matter? CP violation explains that there is indeed a preference for matter over anti-matter, but the apparent conservation of lepton and baryon number seems to indicate that this is only true for mesons and other states which do not carry either of these quantum numbers. This impression is wrong, as, in fact, there is a process violating baryon (and lepton) number conservation in the standard model. Unfortunately, both this process and CP violation turn out to be quantitatively too weak to explain with our current understanding of the early evolution of the universe the quantitative level of the asymmetry between matter and anti-matter. Thus, the current belief is that so far undiscovered physics is responsible for the missing (large$\sim 10^9$) amount.

It is nonetheless instructive to understand how baryon number violation comes about in the standard model. Lepton number violation proceeds in the same way, but is even more suppressed, due to the much smaller masses.

The basic ingredient is a classical field configuration of Yang-Mills theory. Define the matrix-valued field strength tensor $F_{\mu\nu} = \tau^a F^a_{\mu\nu}$, with τ^a the Pauli matrices. To proceed further, it is useful to make the formal replacement $it \to t$, which can be undone at the end. This is an analytic continuation from Minkowski space-time to Euclidean space-time, as now all components of the metric have the same sign.

The Bianchi identity and the antisymmetry of the field strength tensor implies

$$\begin{aligned} 0 &= \frac{1}{2}\epsilon_\sigma{}^{\rho\mu\nu} D_\rho F_{\mu\nu} = D_\rho \tilde{F}_{\sigma\rho} \\ \tilde{F}_{\sigma\rho} &= \frac{1}{2}\epsilon_{\sigma\rho}{}^{\mu\nu} F_{\mu\nu}, \end{aligned}$$

where $\tilde{F}$ is called the dual field-strength tensor. This implies that the (inhomogeneous) Maxwell equation

$$D_\mu F^{\mu\nu} = 0, \tag{9.31}$$

are trivially solved by (anti-)self-dual solutions

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}, \tag{9.32}$$

as these convert the equation (9.31) into the trivial Bianchi identity. The self-duality equations (9.32) have the advantage of being only first-order differential equations instead of second-order differential equations, and are therefore easier to solve.

Furthermore, classical solutions have to have a finite amount of energy, and therefore their behavior at large distances is constrained. Especially, since the Lagrangian can be written as the sum of the squares of the electric and magnetic field strength, both these fields must vanish. This can only occur if the potential becomes at large distances gauge-equivalent to the vacuum, i. e. it has the form

$$A_\mu^a \tau^a = A_\mu = ig(x)\partial_\mu g^{-1}(x),$$

where $g = g^a \tau_a$ is an arbitrary function. Since all choices of g are gauge-equivalent, any choice will do. One possibility which turns out to be technically convenient is

$$\begin{aligned} g(x) &= \frac{x^\mu \tau_\mu}{|x|} \\ \tau_\mu &= (1, i\tau^a). \end{aligned} \tag{9.33}$$

The simplest extension of this is a multiplication with a function $f(x^2)$ which becomes 1 at large distances,

$$\begin{aligned} A_\mu^a \tau^a &= if(x^2)g(x)\partial_\mu g^{-1}(x) = 2f(x^2)\tau_{\mu\nu}\frac{x^\nu}{x^2} \\ \tau_{\mu\nu} &= \frac{1}{4i}(\tau_\mu \bar{\tau}_\nu - \tau_\nu \bar{\tau}_\mu) \\ \bar{\tau}_\mu &= (1, -i\tau^a). \end{aligned} \tag{9.34}$$

The matrices $\tau_{\mu\nu}$ are called ’t Hooft symbols. Thus, this ansatz mixes non-trivially the weak isospin and space-time.

Plugging this in into the self-duality equation (9.32) yields a first-order differential equation for $f(x^2)$,

$$x^2 \frac{df}{dx} - f(1 - f)) = 0.$$

The solution to this equation, which can be obtained by separation of variables, is

$$f(x^2) = \frac{x^2}{x^2 + \lambda^2},$$

where λ is an integration constant. The function indeed goes to one at large distances, as required. The structure described by this field configuration is now localized in space-time at the origin, and extended over a range of size λ. Such a localized event in space and time is called an instanton. Solving the equation with the other sign in the self-duality equation (9.32) yields a similar result, though with some small differences, and is called an anti-instanton.

Going back to Minkowski space-time, the field configuration will have a singularity at $x^2 = -\lambda^2$. This is called a sphaleron, and hence a violent event in space-time. Note that the gauge coupling does not explicitly appear in the calculation. This result can therefore not be obtained perturbatively, and the presence of instantons is a non-perturbative effect.

Seeing now that these indeed create baryon number violation is unfortunately technically very complicated, so here only the most important steps will be sketched.

9.5.2 Relation between topology and anomalies

One highly non-trivial, but very fundamental, insight needed is that instantons turn out to be very much connected with the anomalies of section 9.3. This can be seen in the following way.

There is an interesting twist for the quantity making up the Jacobian

$$\frac{1}{64\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F_{a\rho\sigma}=-\frac{i}{512\pi^4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(iA^a_\nu\partial_\rho A^a_\sigma+\frac{2}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)$$

Evidently, this is a total derivative, and hence can be cast into a surface integral at infinity. It is therefore independent of the internal structure of the space-time it is integrated over, but depends only on the contribution from the boundary. Furthermore, the expression has the same color structure as the usual Lagrangian, and the Lorentz indices do not play a role in gauge transformations of the field-strength tensor. Hence, this quantity is gauge-invariant. Thus, it is an observable quantity. It is the so-called topological charge, or Chern class of the gauge field configuration. Furthermore, the quantity is evidently invariant under any continuous distortions of the gauge fields inside the volume. It is less obvious that this is true for any continuous deformations of the gauge fields on the boundary, and that all of these possible deformations fall into distinct classes, the so-called Chern classes, such that the integral is an integer k, characterizing this class. This fact is stated here without proof. However, by explicit calculation it can be verified that the (anti-)instanton field configuration (9.34) yields $(-)1$.

Since this quantity was obtained from the chiral transformation properties of the fermions, it suggest itself that it is connected to properties of the Dirac operator, and this is indeed the case. This topological charge is equal to the difference of the number of the left-handed n^- and right-handed n^+ zero modes of the (necessarily in the present context massless) Dirac operator D_μ, $\gamma_\mu D^\mu\psi=0$, called the index of the Dirac operator. This is the celebrated Atiyah-Singer index theorem.

To see this, note first that because γ_5 anti-commutes with the other γ_μ it follows that that for any eigenmode of the Dirac operator ψ_m to eigenvalue λ_m that

$$i\gamma_\mu D^\mu\gamma_5\psi_m=-i\gamma_5\gamma_\mu D^\mu\psi_m=-\lambda_m\gamma_5\psi_m.$$

Hence, every non-zero eigenmode is doubly degenerate, and therefore the index is the same if all eigenmodes are included.

Start with an expression for this difference,

$$n^+ - n^- = \int d^4x \sum_{m,\lambda_m=0} \psi_m^\dagger \gamma_5 \psi_m.$$

The inserted γ_5 will guarantee the correct counting. It is possible to use a very similar trick as before when regularizing the sums when doing the path integral calculation in section 9.3.2. The additional eigenvalues can be added as

$$\int d^4x \sum_m \psi_m^\dagger \gamma_5 \psi_m e^{-\lambda_m^2 \tau},$$

as the γ_5 symmetry ensures that all added terms vanish. But this is precisely expression (9.19), and thus this will lead to the same result as in section 9.3.2. Thus, the final answer is

$$n^+ - n^- = k = \frac{1}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{a\rho\sigma}$$

Hence, the anomaly has a certain connection to the topology of the gauge-fields.

This is in as far remarkable as the topology of gauge fields is an intrinsic property of Yang-Mills theory, and thus existing without any fermions, and hence in anomaly-free theories. At the same time, anomalies also exist without gauge fields, e. g. in the form of global anomalies. They are tied to the path-integral measure. It is the unique property of the covariant derivative in the form of the Dirac operator for fermions which ties both effects together in the presented way. Other realizations than minimal coupling will not have this property, or at least in a different way. This connection is therefore deeply ingrained in the gauge formulation.

9.5.3 Instantons and baryon number violation

In fact, now any instanton induces via the global anomaly interactions between fermions. Especially, it can connect fermions of different types, as long as all are charged and all are affected by the anomaly. Especially, this yields that instantons effectively create an interaction which involves, besides the gauge fields also three quarks and one lepton, and thus permits baryon number violation. However, this integration still conserves fermion number, and as a consequence the change in the baryon number must be offset by the same change in lepton number. Still, this implies that reducing the baryon number by one can be offset by a change in lepton number by one, which implies that a proton can be converted, with the involvement of gauge fields, into a positron. Hence, baryon (and lepton number) are not conserved in the standard model.

To see how this work, note that because the chiral structure of the global anomalies take in the standard model the form

$$\partial_\mu J_B^\mu = \partial_\mu J_L^\mu = \frac{g^2}{16\pi^2}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{a\rho\sigma},$$

where here the weakly charged quark and lepton currents are noted. It is here very important that the carrier of the baryon and lepton number are not gauge-invariant states. Integrating yields

$$\Delta Q_B = \Delta Q_L = \frac{g^2}{32\pi^2}\int d^d x \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{a\rho\sigma}$$

and thus the anomaly indeed induces a change in both the baryon number and the lepton number.

The rate with which this happens is, however, small. It can be calculated by determining the transition rate, in leading order, of quarks to leptons in the background of an instanton field. Though a straightforward calculation, this is rather lengthy, and thus only the result will be quoted here. It finally turns out to be suppressed exponentially by $\exp(-c/\alpha_W)$, where c is a number, and α_W is the weak isospin fine-structure constant. Since the latter is small, the suppression is huge, and the life-time of the proton in the standard model exceeds the current upper experimental limit for the decay in any channel of 2.1×10^{28} years by many orders of magnitude. The appearance of the exponential can be intuitively understood, as it stems from the action of the instanton, which appears explicitly in the path integral expression.

Hence, the baryon number violation in the standard model is not able to explain the fact that so much more baryons than anti-baryons exist. However, this is a statement about the current state of the universe, and especially its temperature, and it may change in earlier times. Especially, it can be shown that the effect becomes exponentially enhanced with temperature. Still, the temperatures necessary to make this a sufficiently effective process have been available for too short a time in the early universe, given the other parameters of the standard model.

Chapter 10

Restoration at finite temperature

Just as the magnetization in a magnet can be removed by heating it up, so can the Higgs condensate melt at high temperatures, making the symmetry manifest once more. This process is different in nature from the effectively manifest symmetry at large energies. However, there is not necessarily a phase transition associated with the melting of the condensate. In fact, it is in general not a symmetry restoration, as in case of a global symmetry to be discussed here as well. This is due to the fact that the symmetry is just hidden, not broken. As such, there is no local gauge-invariant order parameter associated with it. Only gauge-dependent order parameters can be local, but in their case the temperature where the symmetry becomes manifest once more is in general gauge-dependent. Only a transition which would be indicated by a non-local gauge-invariant order parameter could in principle mark a true phase transition.

Studying the phase transition would thus require non-perturbative methods. This has been done, e. g. using lattice methods. In the end, it is found that the picture in perturbation theory, discussed here, gives the qualitatively right idea. However, for the actual mass of the Higgs the transition turns out to be quite weak, which is a serious problem for cosmology, a topic which will not be detailed here further.

Again, it is quite useful to first study the case of a global symmetry, then of an Abelian local symmetry before going to the electroweak theory.

10.1 Global symmetry

A useful starting point is given by a rewriting of the Lagrangian (3.7)

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}(6\lambda f^2-\mu^2)\eta^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}(2\lambda f^2-\mu^2)\chi^2 \\ &\quad -\sqrt{2}\lambda f\eta(\eta^2+\chi^2) - \frac{1}{4}\lambda(\eta^2+\chi^2)^2 - \mu^2 f^2 + \lambda f^4. \qquad (10.1)\end{aligned}$$

In this case the explicit zero-energy contribution is kept for reasons that will become apparently shortly, but will be essentially the same as when treating non-relativistic Bose-Einstein condensation. Only terms linear in the fields have been dropped, as they will not contribute in the following. The situation is similar as before, but now the condensate f has not been specified by the minimization of the classical potential, but is kept as a free quantity, which will take its value dynamically.

To investigate the thermodynamic behavior it is useful to analyze the thermodynamic potential Ω in analogy to the non-relativistic case as

$$\Omega(T,f) = -P(T,f) = -T\ln\frac{Z}{V},$$

where P is the pressure, T the temperature, and V the volume. Z is the generating functional. For the following purposes, it is sufficient to use the so-called mean-field approximation. In this case, the interaction terms are neglected. Without going into details, the thermodynamic potential can be evaluated directly, since the functional integral becomes Gaussian. It reads

$$\begin{aligned}
\Omega(T,f) &= -\mu^2 f^2 + \lambda f^4 \\
&\quad + \int \frac{d^3p}{(2\pi)^3}\left(\frac{\omega_1^2+\omega_2^2}{2} + T\left(\ln\left(1-e^{-\frac{\omega_1}{T}}\right) + \ln\left(1-e^{-\frac{\omega_2}{T}}\right)\right)\right) \\
\omega_1 &= \sqrt{6\lambda f^2 - \mu^2 + p^2} = \sqrt{m_\eta^2 + p^2} \\
\omega_2 &= \sqrt{2\lambda f^2 - \mu^2 + p^2} = \sqrt{m_\chi^2 + p^2}.
\end{aligned} \qquad (10.2)$$

The frequencies ω consists of the momenta and the masses of the particles after hiding the symmetry, which is dependent on the value of the condensate f. There are three contributions. The first outside the integral is the classical contribution. The second are the first two terms inside the integral. They are the contributions from quantum fluctuations. The third term represents thermal fluctuations.

To recover the results from section 3.2.1, the second term must be neglected and the zero-temperature limit taken. This yields

$$\Omega(0,f) = -\mu^2 f^2 + \lambda f^4.$$

As in section 3.2.1, this potential has a minimum at non-zero f, $f = \mu^2/(2\lambda)$. Inserting this into the Lagrangian (10.1) makes it equivalent to (3.7), with a massive Higgs boson and a massless Goldstone boson.

Something new happens at finite temperature. At small temperature, little other happens than that it is possible to excite Higgs bosons or Goldstone bosons, which then form a thermal bath of non-interacting bosons, and the total pressure is just the sum of their respective pressures. However, the value of f will become temperature-dependent: At each temperature it will take the value which minimizes the thermodynamic potential.

When going to higher temperatures, it is useful to make a high-temperature expansion for the thermodynamic potential. High temperature requires here T to be larger than the scale of the zero-temperature case, which is given by the condensate, which is of order $\mu/\sqrt{\lambda}$. In this case, it is possible to obtain an expansion for Ω. The leading terms up to order $\mathcal{O}(1)$ are given by

$$\Omega(T,f) = \lambda f^4 + \left(\frac{1}{3}\lambda T^2 - \mu^2\right) f^2 - \frac{\pi^2}{45}T^4 - \frac{\mu^2 T^2}{12}. \tag{10.3}$$

This results exhibits one interesting feature. The term of order f^2 has a temperature-dependent coefficient, which changes sign at[1] $T_c^2 = 3\mu^2/\lambda$. As a consequence, the shape of the thermodynamic potential as a function of f changes. Below T_c, it has a minimum away from zero, as at zero temperature. With increasing temperature, this minimum moves to smaller and smaller temperatures, and arrives at zero at T_c. Hence, at T_c, the value of f changes from a non-zero to a zero value, and the symmetry becomes manifest once more. Above T_c, the minimum stays at zero, and for all higher temperatures the symmetry is manifest.

Replacing f with its temperature dependent value in (10.3) yields the expressions

$$\begin{aligned}
\Omega_{T<T_c} &= \tfrac{\mu^2 T^2}{12} - \left(\tfrac{\pi^2}{45} + \tfrac{\lambda}{36}\right) T^4 \overset{T=T_c}{=} -\frac{\pi^2\mu^2}{5\lambda^2} \\
\Omega_{T>T_c} &= \tfrac{\mu^4}{4\lambda} - \tfrac{\pi^2}{45}T^4 - \tfrac{\mu^2 T^2}{12} \overset{T=T_c}{=} -\frac{\pi^2\mu^2}{5\lambda^2},
\end{aligned}$$

which coincide at T_c. Also their first derivatives with respect to the temperature equal at T_c

$$\begin{aligned}
\frac{d\Omega_{T<T_c}}{dT} &= -(8\pi^2 T^2 + 10\lambda T^2 - 15\mu^2)\tfrac{T}{90} \overset{T=T_c}{=} -\frac{8\pi^2 + 5\lambda}{\sqrt{300}}\sqrt{\frac{\mu^2}{\lambda}} \\
\frac{d\Omega_{T>T_c}}{dT} &= -(8\pi^2 T^2 + 15\mu^2)\tfrac{T}{90} \overset{T=T_c}{=} -\frac{8\pi^2 + 5\lambda}{\sqrt{300}}\sqrt{\frac{\mu^2}{\lambda}},
\end{aligned} \tag{10.4}$$

[1]Note that strictly speaking using the high-temperature expansion at this temperature is doubtful. For the purpose here it will be kept since it makes the mechanisms more evident then the rather technical calculations necessary beyond the high-temperature expansion. The qualitative outcome, however, is not altered, at least within the first few orders of perturbation theory.

but their second derivatives do not

$$\begin{aligned}\frac{d^2\Omega_{T<T_c}}{dT^2} &= \tfrac{\mu^2}{6}-(4\pi^2+5\lambda)\tfrac{T^2}{15} \overset{T=T_c}{=} & -\frac{(25\lambda+24\pi^2)\,\mu^2}{30\lambda}\\ \frac{d^2\Omega_{T>T_c}}{dT^2} &= -\tfrac{8\pi^2T^2+5\mu^2}{30} \overset{T=T_c}{=} & -\frac{(5\lambda+24\pi^2)\,\mu^2}{30\lambda}.\end{aligned} \tag{10.5}$$

Thus, a phase transition of second order occurs at T_c. Note that at very large temperatures only the term $\pi^2T^4/45$ is relevant, which is precisely the one of a free non-interacting gas of two boson species, a Stefan-Boltzmann-like behavior.

As stressed previously repeatedly, it is possible that quantum effects could modify the pattern considerably or even melt the condensate. It is therefore instructive to investigate the leading quantum corrections to the previous discussion.

This is also necessary for another reason. If the symmetry becomes manifest once more at large temperatures, the mass of Higgs-like excitations become tachyonic, indicating a flaw of the theory. That can be seen directly by reading off the condensate-dependent masses of the excitations being as usual

$$\begin{aligned} m_\eta^2 &= 6\lambda f^2-\mu^2 = & -\mu^2\theta(T-T_c)+(2\mu^2-\lambda T^2)\theta(T_c-T)\\ m_\chi^2 &= 2\lambda f^2-\mu^2 = & -\mu^2\theta(T-T_c)-\frac{\lambda T^2}{3}\theta(T_c-T).\end{aligned}$$

Furthermore, also the Goldstone theorem is violated, as the mass of the Goldstone boson χ is no longer zero[2]. Both problems are fixed by quantum corrections, demonstrating the importance of quantum fluctuations even in the high-temperature phase.

In the expression for the free energy (10.2) the zero-point energy, and thus the quantum fluctuations have been neglected. Using a cutoff-regularization with cutoff Λ their contribution can be determined as

$$\int\frac{d^3p}{(2\pi)^3}\frac{\omega}{2}=\frac{1}{64\pi^2}\left(2m^2\Lambda^2-m^4\ln\frac{\Lambda^2}{m^2}-\frac{m^4}{2}\right)+\mathcal{O}\left(1,\frac{1}{\Lambda}\right).$$

where the constant terms $\mathcal{O}(1)$ do not depend on the mass. This contribution is quadratically divergent and has to be regulated. This can be done by introducing into the Lagrangian (10.1) the necessary counter-terms

$$\delta\mu^2(\eta^2+\chi^2)-\delta\lambda(\eta^2+\chi^2)^2.$$

[2]In a full quantum treatment, the role of the Goldstone boson could be played at finite temperature by some composite excitation instead. However, at the mean-field level no such excitations are available, and thus the Goldstone theorem is violated.

Repeating the calculation for the free energy yields at zero temperature the expression

$$\begin{aligned}\Omega(0,f) &= -(\mu^2+\delta\mu^2)f^2+(\lambda+\delta\lambda)f^4 \\ &+\frac{1}{64\pi^2}\left(2(m_\eta^2+m_\chi^2)\Lambda^2-m_\eta^4\ln\frac{\Lambda^2}{m_\eta^2}-m_\chi^4\frac{\Lambda^2}{m_\chi^2}-\frac{m_\eta^4}{2}-\frac{m_\chi^2}{2}\right).\end{aligned}$$

To determine the renormalization constants two conditions will be implemented. One is that the free energy is finite when the cutoff is send to infinity. The second is that the Goldstone boson mass is zero, equivalent to requiring that $f=\mu^2/(2\lambda)$, and required by the Goldstone theorem. Both conditions can be satisfied by the choice

$$\begin{aligned}\delta\mu^2 &= \frac{\lambda\Lambda^2}{4\pi^2}+\frac{\lambda\mu^2}{4\pi^2}\ln\frac{\Lambda^2}{2\mu^2}+\mu^2\frac{\delta\xi}{\lambda} \\ \delta\lambda &= \frac{5\lambda^2}{8\pi^2}\ln\frac{\Lambda^2}{2\mu^2}+\delta\xi.\end{aligned}$$

Herein the contribution $\delta\xi$ is not determined by these conditions, and can be set at will by other renormalization conditions. This indicates that both conditions are not independent. This fixes the thermodynamic potential at zero temperature. It can be shown that no new counter terms are necessary at non-zero temperature. Therefore, the high-temperature expansion can be performed as previously.

Performing once more a high-temperature expansion is possible. However, in this case also higher-order terms have to be kept, since the vacuum energy has now contributions of order $\mathcal{O}(m^4\ln(m^2/\mu^2))$. At higher order in the high-temperature expansion terms of order $\mathcal{O}(m^4\ln(m^2/T^2))$ appear, which combine to relevant terms. The result is

$$\begin{aligned}\Omega(T,f) &= -\frac{\pi^2}{45}T^4-\frac{\mu^2T^2}{12}-\frac{(m_\eta^3+m_\chi^3)T}{12}+\frac{\mu^4}{32\pi^2}\ln\frac{8\pi^2T^2e^{-2\gamma+\frac{3}{2}}}{\mu^2} \\ &-\mu^2f^2\left(1+\frac{\delta\xi}{\lambda}+\frac{\lambda}{4\pi^2}\ln\frac{8\pi^2T^2e^{-2\gamma+1}}{\mu^2}-\frac{\lambda T^2}{3\mu^2}\right) \\ &+\lambda f^4\left(1+\frac{\delta\xi}{\lambda}+\frac{5\lambda}{8\pi^2}\ln\frac{8\pi^2T^2e^{-2\gamma+1}}{\mu^2}\right).\end{aligned}$$

The critical temperature can be determined again as the point where f vanishes, yielding

$$T_c^2=\frac{3\mu^2}{\lambda}\left(1+\frac{\delta\xi}{\lambda}+\frac{\lambda}{4\pi^2}\ln\frac{24\pi^2e^{-2\gamma+1}}{\lambda}\right).$$

To attach a final value it would be necessary to determine the value for $\delta\xi$ by some other renormalization condition. To order λ, which is the current order, the final result for T_c will then not depend on this renormalization prescription. One obvious possibility would be to give T_c its (hypothetically) experimentally measured value, as T_c may not depend on the renormalization process: As a physical observable, it is renormalization-group invariant.

To obtain the corrections for the masses, it is necessary to calculate the corresponding self-energies. Without going into the details, the result to the present order in λ is given at high temperatures and after renormalization by

$$\Pi_\eta = \Pi_\chi = \frac{\lambda T^2}{3},$$

and thus momentum independent. It is therefore a correction to the mass. The complete mass to this order is therefore

$$\begin{aligned} m_\eta^2 &= 2\mu^2\left(1-\frac{\lambda T^2}{3\mu^2}\right)\theta(T_c-T)+\frac{1}{3}\lambda\left(T^2-\frac{3\mu^2}{\lambda}\right)\theta(T-T_c) \\ m_\chi^2 &= \frac{\lambda}{3}\left(T^2-\frac{3\mu^2}{\lambda}\right)\theta(T-T_c). \end{aligned}$$

These results yield a number of interesting observation. First, since T_c is larger[3] than $3\mu^2/\lambda$, the mass of the Higgs is always positive, stabilizing the system. Secondly, in this case the mass of the Goldstone boson is always zero below the phase transition temperature, in agreement with the Goldstone theorem. Above the phase transition, the masses of both particles degenerate, and the symmetry is manifest once more also in the spectrum. These properties are generic for symmetries hiding by a condensate which thaws with increasing temperature. Also that the mean-field approximation is in general insufficient is a lesson which should be kept duly in mind. Of course, at the present time much more sophisticated methods are available to treat this problem, though they are in general very complicated.

10.2 Abelian case

As was visible in the previous case of the global symmetry, quantum fluctuations are important to obtain a consistent result. Hence the use of the unitary gauge with its lack of renormalizability is not recommending itself for calculations at finite temperature. Instead, the 't Hooft gauge is very useful. The gauge-fixed Lagrangian is then given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta-\frac{m_\eta^2}{2}\eta^2+\frac{1}{2}\partial_\mu\chi\partial^\mu\chi-\frac{m_\chi^2+e^2f^2\xi^2}{2}\chi^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\left(e^2f^2-\frac{1}{2\xi}\right)A^\mu A_\mu \\ &-\sqrt{2}\lambda f\eta(\eta^2+\chi^2)-\frac{\lambda}{4}(\eta^2+\chi^2)^2+eA^\mu(\eta\partial_\mu\chi-\chi\partial_\mu\eta) \\ &+e^2A_\mu A^\mu\left(\sqrt{2}f\eta+\frac{1}{2}(\eta^2+\chi^2)\right), \end{aligned}$$

[3]It is not obvious that $\delta\xi$ cannot be negative and large, thus making the improved estimate for T_c smaller than before. However, it turns out not to be the case at this order for any renormalization prescription.

with the distinct gauge-dependent masses at tree-level. After condensation, the tree-level masses are still given by

$$
\begin{aligned}
m_\eta^2 &= 2\mu^2 \\
m_\chi^2 &= 0,
\end{aligned}
$$

with the parameter μ of the Lagrangian with manifest symmetry.

Again going through the mean-field calculations (introducing only a mean-field for the Higgs fields η condensate , but neither for the χ nor for the photon), it is possible to obtain once more a high-temperature expansion for the free energy. One should note that the ghost fields have to be included in this calculation, even when taking the Landau-gauge limit, as they contribute two negative degrees of freedom to the pressure, canceling the one of the time-like and scalar photon. The full expression at mean-field level then reads

$$
\Omega(f,T) = \lambda f^4 + \left(\left(\frac{\lambda}{3} + \frac{e^2}{4}\right)T^2 - \mu^2\right) f^2 - \frac{\mu^2 T^2}{12} - \frac{2\pi^2}{45}T^4.
$$

First, the Stefan-Boltzmann contribution proportional to T^4 is now twice as large, as the photon also contributes two scalar degrees of freedom. Furthermore, the leading high-temperature behavior is not altered otherwise by the presence of the interaction. At the level of the mean-field approximation there is still only a non-interacting gas of two scalars and two photon polarizations left. However, the term crucial for the phase transition, the one of order T^2, is affected by the interactions. Thus, the interactions have an influence on the phase transition. At mean-field level they just shift the phase transition temperature.

Beyond mean-field, their impact is more relevant. At one-loop level, the phase structure becomes dependent on the relative size of λ and e.

As long as λ is larger[4] than e, the situation is found to be in agreement as when the photon field would be absent. In particular, the condensate melts at a (now also e-dependent) critical temperature, and above the phase transition both the scalars and the photon become massless again.

If λ is smaller than e, the critical temperature becomes lower than the effective mass of the photon. As a consequence, the second-order phase transition may in fact become first order. Even more drastic, if λ is below $3e^4/(32\pi^2)$, the phase transition temperature decreases to zero, and spontaneous breaking is not occurring at all. In that case, the stronger photon-scalar interactions stabilize the vacuum, and no condensate can form.

An important insight is found in the case of $\mu = 0$. At first sight no condensation is possible. However, the interactions mediated by the photon may still be sufficiently strong

[4]If e and λ are approximately equal, even higher order corrections can have a qualitative impact.

and attractive enough that the scalars condense, yielding the same physics as if μ would not be zero, and λ would be sufficiently large. In this case, the condensation is a genuine quantum effect, as only due to (one-)loop quantum corrections spontaneous condensation of the Higgs occurs.

10.3 The electroweak case

Also in the electroweak case it is useful to use the Lagrangian in the 't Hooft gauge, given by (4.7) and the gauge-fixing term (5.5). Though straight-forward, the calculation in the present case is much more cumbersome than in the previous case. For the present purpose, the masses of the fermions (and thus the CKM matrices) can be neglected. The imprecision of this is comparable to the effects of going to the next order. The final result at mean-field level is

$$\begin{aligned}\Omega(f,T) &= \frac{\mu^2}{4\lambda}-\frac{\lambda f^4}{4}+\left(\frac{379}{360}+\frac{2}{15}\right)\pi^2T^4\\ &+\frac{T^2}{24}((3+\xi)m_Z^2+(6+2\xi)m_W^2+m_\eta^2+2m_\phi^2+m_\chi^2-6m_{c^A}^2-2m_{c^B}^2).\end{aligned}$$

At first sight, this expression appears to be gauge-dependent due to the explicit appearance of the gauge-parameter ξ. However, the masses of the elementary particles are also gauge-dependent. Replacing them by the more elementary expressions in terms of the condensate, it is seen that this gauge-dependence is spurious. The result is then

$$\Omega(f,T)=-\frac{427}{360}\pi^2T^4-\frac{f^2}{2}\left(\mu^2-\frac{T^2}{4}\left(2\lambda+\frac{3g^2}{4}+\frac{g'^2}{4}\right)\right)+\frac{\lambda f^4}{4}+\frac{\mu^4}{4\lambda}-\frac{\mu^2T^2}{6}.$$

Extremalizing this expression with respect to f yields the critical temperature for the electroweak standard model in this approximation as

$$T_c^2=\frac{4\mu^2}{2\lambda+\frac{3g^2}{4}+\frac{g'^2}{4}}.$$

The temperature dependence of the condensate f and the pressure then read, respectively,

$$\begin{aligned}f^2 &= \frac{\mu^2}{\lambda}\left(1-\frac{T^2}{T_c^2}\right)\theta(T_c-T)\\ P &= \begin{cases}\frac{427\pi^2T^4}{360}+\frac{\mu^2}{4\lambda}\left(1-\frac{T^2}{T_c^2}\right)^2+\frac{\mu^2f^2T^2}{6}-\frac{\mu^4}{4\lambda}, & T\leq T_c\\ \frac{427\pi^2T^4}{360}+\frac{\mu^2T^2}{6}-\frac{\mu^4}{4\lambda}, & T\geq T_c\end{cases}.\end{aligned}$$

The corresponding phase transition is at mean-field level thus of second order, as the second derivative of the pressure exhibits a discontinuity. For a Higgs mass of 100 GeV the critical temperature is about 200 GeV. For the actual value of the Higgs mass of 125 GeV, it is only slightly higher. Thus the transition temperature is of the same order as the Higgs condensate, and about three orders of magnitude larger than the corresponding temperature in QCD.

Note that all problems with consistency of the mean-field approach pertain also to the full electroweak standard model. Therefore, for a consistent treatment at least leading order corrections have to be included. As previously, they do not change the phase transition temperature, but may change its order.

10.4 Implications for the early universe

The relevance of such a temperature is only given in the early universe, or perhaps during a collapse to a black hole. Here, the more certain case of the early universe will be treated.

To assess the relevance for the early universe it is necessary to add equations which describe its development. For the present purpose simplified versions of the Einstein equations are sufficient. Adding energy conservation gives

$$\begin{aligned} \left(\frac{dR}{dt}\right)^2 &= \frac{8\pi G}{3}\epsilon R^2 \\ \frac{d(\epsilon R^3)}{dR} &= -3PR^2, \end{aligned} \tag{10.5}$$

where G is Newton's constant, ϵ is the energy density, R is the scale factor, essentially given by the Ricci curvature scalar, t the proper time, and P is the pressure. To close the equations, an equation of state is necessary, which is given as a function of the pressure by thermodynamic relations as

$$\epsilon = -P + T\frac{\partial P}{\partial T}.$$

Since the sicknesses of the mean-field approximation are not too problematic for this estimate, it is sufficient to use it for obtaining the corresponding energy density as

$$\epsilon = \begin{cases} \left(\frac{1281\pi^2}{360} + \frac{\mu^4}{4\lambda T_c^4}\right)T^4 + \left(1 - \frac{3\mu^2}{\lambda T_c^2}\right)\frac{\mu^2 T^2}{6}, & T \leq T_c \\ \frac{1281\pi^2}{360}T^4 + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{4\lambda}, & T \geq T_c \end{cases}.$$

Rewriting equation (10.5) in terms of the temperature yields the ordinary differential equation

$$R\frac{d\epsilon}{dT} + 3\frac{dR}{dT} = -3P.$$

Imposing as a boundary condition that R should be one at the phase transition yields

$$\begin{aligned} R^3 &= \begin{cases} \frac{T_c}{T}\frac{T_c^2-b^2}{T^2-b^2}, & T \leq T_c \\ \frac{T_c}{T}\frac{T_c^2+a^2}{T^2+a^2}, & T \geq T_c \end{cases} \\ a^2 &= \frac{30\mu^2}{427\pi^2} \\ b^2 &= \frac{a^2T_c^2(1-r)}{a^2+rT_c^2} \\ r &= \frac{4\lambda}{6\lambda+\frac{9g^2}{4}+\frac{3g'^2}{4}}. \end{aligned}$$

For a Higgs mass about 100 GeV the characteristic parameters are $r = 0.22$, $a = 6$ GeV and $b = 10$ GeV. Thus, the dominant behavior is that R behaves like T_c^3/T^3, up to some small modifications close to the phase transition, and thus drops essentially in the electroweak domain.

An interesting consequence is obtained if R^3 is multiplied by the entropy

$$s = \frac{\partial P}{\partial T} \sim T^3.$$

An elementary calculation yields thus that sR^3 is constant. Since s is in units inverse length cubed, this is just the statement that entropy is conserved since R only describes the expansion of a unit length over time. Hence, the electroweak interactions at mean-field level conserve entropy. In particular, this is a consequence of the second order phase transition, which is not permitting latent heat or supercooling. Finally, inserting the numbers shows that R increases somewhat slower around the phase transition. Thus, the expansion of the universe slows down during the electroweak phase transition.

Of course, all of this is just an estimate. That it can never be fully correct is seen by the fact that R diverges at the finite temperature $T = b$, much above the QCD phase transition (and nowadays) temperature. This is an artifact of the high-temperature expansion involved. To obtain the correct behavior down to the QCD phase transition would require more detailed (non-perturbative) calculations.

Index