Bachelor Thesis

Eigenspectrum of the Faddeev-Popov operator in SU(2)

Searching for Gribov copies outside the first Gribov region

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1. Introduction

When it comes to modern day physics the best theoretical description is the standard model of particle physics. Despite the gravitational interactions, the standard model contains all known interactions, such as the strong and weak nuclear force, electromagnetism, as well as the Higgs interaction.
The theoretical foundation of the standard model are gauge theories. These consist of a gauge sector, described by an abelian or non-abelian Yang-Mills theory. One essential characteristic of non-abelian gauge theories is the existence of transformations which change the field configurations without affecting the observables. These transformations are called gauge transformations and the remaining solutions are called Gribov copies, represented by the gauge orbit $\mathcal{G}(A_\mu)$.
While in abelian gauge theories it is possible to fix a gauge by introducing only local contraints, this is no longer possible for non-Abelian gauge theories.

1.1. Gauge theories

The arguably most famous gauge theory occurs in the classical electromagnetism. The basic building block of this theory is designated by four fundamental relations, called the Maxwell's equations (1).

$$\begin{aligned} \nabla\mathbf{E} &= \rho_{em} \qquad & \nabla\times\mathbf{E} &= -\partial_t\mathbf{B} \\ \nabla\mathbf{B} &= 0 \qquad & \nabla\times\mathbf{B} &= j_{em} + \partial_t\mathbf{E} \end{aligned} \tag{1}$$

Where the fields $\mathbf{E}$ and $\mathbf{B}$ can be expressed by field configurations $\mathbf{A}$ and V.

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad \mathbf{E} = -\partial_t\mathbf{A} - \nabla V \tag{2}$$

These potentials are, however, not unique since adding an arbitrary function of space-time like $(\mathbf{A}\to\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi(x,t) \mid V \to \tilde{\mathrm{V}} = V - \nabla\chi(x,t))$ will not change the observable fields. A transformation of this kind is called a gauge transformation and the theory is said to exhibit a gauge freedom. Nonetheless in this case of abelian gauge theories it is possible to completely fix the gauge by introducing a local constraint like the Landau gauge $(\nabla\mathbf{A} = 0)$.
In relativistic notation the potentials, as well as the densities and derivates are combined.

$$A^\mu = (V,\mathbf{A}) \qquad \partial^\mu = (\partial_t, -\nabla) \qquad j^\mu_{em} = (\rho_{em}, j_{em}) \tag{3}$$

Since it is no longer possible to treat relativistic transformations of both the electric and the magnetic field independently, it is necessary to introduce the field tensor $F^{\mu\nu}$. The Maxwell's equations can now be written as

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= \mu_0 J^\mu \\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \end{aligned} \tag{4}$$

and a gauge transformation is therefore given by

$$A^\mu \to \tilde{A}^\mu = A^\mu - \partial^\mu \chi \tag{5}$$

where again, the gauge can be completely fixed by a local restriction like the Landau gauge ($\partial_\nu A^\nu = 0$).

1.2. Yang-Mills theory

As it was mentioned before gauge theories are inevitable for the standard model. They can be described by abelian and non-abelian Yang-Mills theory. Since without matter fields abelian gauge theories are trivial theories of non-interacting gauge bosons[1], this discussion will be restricted to non-abelian gauge algebras exclusively. The starting point of any such discussion is the classical Lagrangian of Yang-Mills theory.[2]

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \tag{6}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{7}$$

Herein the fields A^a_μ are the gauge fields describing the gluons. In contrast to the abelian case, the field tensor (7) includes the interactions between the gluons. Furthermore, the parameters of the Lagrangian are the coupling constant g, and the structure constant of the associated gauge algebra f^{abc}.

This gauge algebra can, in principle, be any semi-simple Lie algebra. In case of the standard model the underlying symmetry groups are su(3), su(2) and u(1).[2]

For the sake of simplicity this discussion will exclusively consider su(2), as this is the simplest case this problem occurs.[3]

[1]For simplicity referred as gluons from now on

[2]The factor su(3) generates the strong interactions, and the factor su(2) and u(1) yields, after mixing, the weak and electromagnetic interactions.

[3]The su(2) algebra has the total-antisymmetric Levi-Civita Tensor as structure constant, $f^{abc} = \epsilon^{abc}$ with $\epsilon^{abc} = 1$

A gauge transformation in this Lie algebra will not be derived but stated. Due the fact that the Lagrangian is invariant under a infinitesimal gauge transformation, it can be proven that (6) is invariant under a finite gauge transformation given by equation (8).

$$\begin{aligned} A_\mu \to A_\mu^{(h)} &= hA_\mu h^{-1} + h\partial_\mu h^{-1} \\ A_\mu &= \tau_a A_\mu^a \\ h &= \tau_a \phi^a \end{aligned} \tag{8}$$

Where ϕ^a is an arbitrary function of space-time and τ_a are the generators of the so-called gauge algebra. In su(2) the generators are given by the Pauli matrices (9).

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

As a consequence, the set of fields A_μ connected by gauge transformations (8) are represented by the gauge orbit $\mathcal{G}(A_\mu) = \{A_\mu^{(h)}, \forall h\}$.
To be precise, the set of field configurations given by a gauge orbit $\mathcal{G}$ presents all the equivalent representations of a given, fixed space-time configuration A_μ of the gluon field and therefore observed by the same physical reality.

1.2.1. Gauge fixing and Gribov copies

All quantities which do not change under a gauge transformation and thus are equal for every configuration of the gauge orbit, are called gauge-invariant. Nonetheless not every useful quantity is gauge-invariant and therefore it is necessary to fix the gauge.
In abelian gauge theories this can be achieved by introducing a local constraint like the Landau gauge $\partial_\nu A^\nu = 0$ (section 1.1).
This, however, will no longer fix the gauge completely for non-abelian gauge theories due the structure of the group, and their remaining solutions are called Gribov copies.[4] In non-abelian theories gauge-fixing is obstructed by the Gribov-Singer ambiguity, which requires the introduction of an additional non-local constraint. This restriction will be demanded by the topological spectrum of the Faddeev-Popov operator (10).

[4]Gribov copies are the remaining solutions of $\mathcal{G}$, who cannot be connected by a sequence of infinitesimal gauge transformations.

2. Faddeev-Popov operator

In the Gribov-Zwanziger scenario Gribov proposed to restrict to different Gribov regions. Each region is characterized by the domain in field space, where the Faddeev-Popov operator (10) has a certain number of negative eigenvalues. At the boundary, a zero value appears, which becomes negative outside the region.[3]

$$M^{ab} = -\partial_\mu(\partial_\mu \delta^{ab} + g\epsilon^{abc} A^c_\mu) \tag{10}$$

Here g is the gauge coupling and ϵ^{abc} is the structure constant of the su(2) gauge group. Nonetheless, this is not a sufficient restriction for the field configurations as gauge copies exist inside each region.
The aim of this thesis will be to find an admissible field configuration outside the first Gribov region, in fact, searching for an negative eigenvalue λ from the eigenvalue equation of the Faddeev-Popov operator. The equation for an eigenfunction ϕ^a to an eigenvalue λ in Landau gauge is given by:

$$M^{ab}\phi^b = \lambda\phi^a \tag{11}$$

$$-g\epsilon^{abc} A^c_\mu \partial_\mu \phi^b = (\lambda + \Delta)\phi^a \tag{12}$$

3. Switching to momentum space

Since there are no restrictions for the potential A^c_μ so far, it seems convenient to switch from local to momentum space, as the Landau gauge modifies to:

$$\partial_\mu A^c_\mu(x_\mu) \overset{!}{=} 0 \;\rightarrow\; p^\mu A^c_\mu(p^\mu) \overset{!}{=} 0$$

Another noteworthy characteristic of the momentum space is the behavior of the four-Gradient onto functions. Due to the variable transformation, the transformed potential, as well as the eigenfunctions do not depend on space-time anymore and consequently the four-Gradient solely acts on the kernel function of the Fourier transformation.
The potential and the eigenfunction in momentum space are given by (13) and (14).

$$A^c_\mu(x_\mu) = \frac{1}{\sqrt{2\pi}^\mu} \int_{-\infty}^{\infty} A^c_\mu(q^\mu) e^{iq^\mu x_\mu}\, dq^\mu \tag{13}$$

$$\phi^a(x_\mu) = \frac{1}{\sqrt{2\pi}^\mu} \int_{-\infty}^{\infty} \phi^a(p^\mu) e^{ip^\mu x_\mu}\, dp^\mu \tag{14}$$

Transforming the initial eigenvalue equation (12) into momentum space yields:[5]

$$\iint dp^\mu\, dq^\mu \frac{-ig}{\sqrt{2\pi}^\mu} p^\mu \epsilon^{abc} \phi^b(p^\mu) A^c_\mu(q^\mu) e^{ix_\mu(q^\mu+p^\mu)} = \int dp^\mu (\lambda - p^2)\phi^a(p^\mu) e^{ix_\mu p^\mu} \tag{15}$$

By substituting the left side ($t^\mu = p^\mu + q^\mu$; $\tau^\mu = p^\mu \;\rightarrow\; dt^\mu d\tau^\mu = dp^\mu dq^\mu$) and renaming the right side ($t^\mu = p^\mu \rightarrow dt^\mu = dp^\mu$) of equation (15), the kernel functions of the integrals with respect to t^μ are set equal on both sides.

$$\iint dt^\mu\, d\tau^\mu \frac{-ig}{\sqrt{2\pi}^\mu} \tau^\mu \epsilon^{abc} \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) e^{ix_\mu t^\mu} = \int dt^\mu (\lambda - t^2)\phi^a(t^\mu) e^{ix_\mu t^\mu} \tag{16}$$

At this point, the first restriction will be set. Due the fact that both sides of the equation are integrated over the same variable t^μ, it will be demanded that (16) is true for every value t^μ.
Including this restriction, the eigenequation reduces to:

$$\phi^a(t^\mu) = \frac{-ig}{\sqrt{2\pi}^\mu(\lambda - t^2)}\, \epsilon^{abc} \int d\tau^\mu \tau^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) \tag{17}$$

Since the potential A^c_μ has to fulfill the Landau Gauge in momentum space, an even more frugal equation can be obtained by adding a zero.

$$\begin{aligned}\phi^a(t^\mu) = \frac{ig}{\sqrt{2\pi}^\mu(\lambda - t^2)}\, \epsilon^{abc} \Big[-\int d\tau^\mu \tau^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) + \\ + \int d\tau^\mu t^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) - \int d\tau^\mu t^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) \Big]\end{aligned} \tag{18}$$

$$\begin{aligned}\phi^a(t^\mu) = \frac{ig}{\sqrt{2\pi}^\mu(\lambda - t^2)}\, \epsilon^{abc} \Big[-\int d\tau^\mu t^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) + \\ \int d\tau^\mu \underbrace{(t^\mu - \tau^\mu) A^c_\mu(t^\mu - \tau^\mu)}_{\overset{!}{=}0} \phi^b(\tau^\mu) \Big]\end{aligned} \tag{19}$$

This yield the modified eigenvalue equation of the Faddeev-Popov operator in momentum space.

$$\phi^a(t^\mu) = \frac{-ig}{\sqrt{2\pi}^\mu(\lambda - t^2)}\, t^\mu \epsilon^{abc} \int d\tau^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) \tag{20}$$

[5]Note that the borders of the integrals will be neglected till needed.

4. Restrictions

Neither do the different colors decouple nor is the problem fully separable anymore. At this point it is inevitably to prescribe restrictions to both, the involved potential and the eigenfunctions in order to obtain a solution. For the sake of simplicity further investigations are considered in a 2-dimensional space exclusively. This demand will not affect the appearance of Gribov copies and the Gribov-Zwanziger scenario.

4.1. Field configurations

The first step in solving the set of equations (20) is by demanding invariance of the underlying impulse. To fulfill this restriction the external impulse η_μ will be set in a fixed direction, and therefore the potential A^c_μ reduces to a fixed vector times a scalar function depending on the absolute value of its variables.

$$A^c_\mu(t^\mu - \tau^\mu) = \eta_\mu \cdot a^c(|t^\mu - \tau^\mu|) \tag{21}$$

Keep in mind that $\eta_\mu \perp (t^\mu - \tau^\mu)$ as this this fulills the Landau gauge.

Additionally the field is restricted to be rotational symmetric, hence switching to a spherical coordinate system seems a convenient choice.

$$t^\mu = \begin{pmatrix} t^1 \\ t^2 \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix} \quad \text{and} \quad \tau^\mu = \begin{pmatrix} \tau^1 \\ \tau^2 \end{pmatrix} = \begin{pmatrix} r\cos\vartheta \\ r\sin\vartheta \end{pmatrix} \tag{22}$$

$$a^c(|t^\mu - \tau^\mu|) = a^c\big(\sqrt{\rho^2 - 2\rho r\cos(\vartheta - \varphi) + r^2}\big) \overset{!}{=} a^c\big(\sqrt{\rho^2 - 2\rho r\cos\vartheta + r^2}\big) \tag{23}$$

Rewriting (20) in spherical coordinates yields equation (24), where the integral with respect to the radial coordinate r ranges over $[0, \infty]$ and the integral with respect to the angular coordinate ϑ ranges over $[0, 2\pi]$.

$$\phi^a(\rho, \varphi) = \underbrace{\frac{-ig}{\sqrt{2\pi}^\mu(\lambda - \rho^2)}\, t^\mu \eta_\mu \epsilon^{abc}}_{:=N} \iint dr d\vartheta\, r\, a^c\big(\sqrt{\rho^2 - 2\rho r\cos\vartheta + r^2}\big)\phi^b(r, \vartheta) \tag{24}$$

4.2. Hermite Polynomial expansion

Due to the structure of (24), it is necessary to state one final restriction. First it will be assumed that the involved functions can be written in terms of a Taylor-series. For the scalar function a^c this is true, since there are no limitations besides rotational symmetry. Nevertheless, this is a rough statement because there is no evidence that the eigenfunctions exhibit analytic behaviour.
Just accepting this assumption the involved functions can be expressed in a Hermite-series with a gaussian pre-factor.[6] [4]
On the basis of the 'Wigner-Eckart theorem' it is possible to separate both, the Lie-algebra su(2) and the Hilbert space in a way that the group theoretical content is only considered in the expansion coefficients of the Hermite-series.
Starting with the eigenfunctions ϕ^b:

$$\phi^b(r,\vartheta) = R^b(r)Y^b(\vartheta) = e^{-r^2} \cdot \sum_{m=0}^{\infty}\sum_{j=-\infty}^{\infty} b^b_{mj} H_m(r) e^{ij\vartheta} \tag{25}$$

where b^b_{mj} are the expansion coefficients and each sum represents the angular and the radial part of the function. Due the fact that the eigenfunctions are real, it is demanded that the expansion coefficients fulfill the relation $b^b_{mj} = b^b_{m-j}$.

$$\begin{aligned}\phi^b(r,\vartheta) &= e^{-r^2}\sum_{m=0}^{\infty}\sum_{j=-\infty}^{\infty} b^b_{mj} H_m(r) e^{ij\vartheta} \\ &= e^{-r^2}\sum_{m=0}^{\infty} H_m(r)\left[b^b_{m0} + \sum_{j=-\infty}^{-1} b^b_{mj} e^{ij\vartheta} + \sum_{j=1}^{\infty} b^b_{mj} e^{ij\vartheta}\right] \\ &= e^{-r^2}\sum_{m=0}^{\infty} H_m(r)\left[b^b_{m0} + \sum_{j=1}^{\infty} b^b_{mj}(e^{-ij\vartheta} + e^{ij\vartheta})\right]\end{aligned} \tag{26}$$

Using the Euler's formula in combination with the binomial identity the complex exponential can get decomposed into a sine and a cosine.

$$\begin{aligned}\phi^b(r,\vartheta) &= e^{-r^2}\sum_{m=0}^{\infty} H_m(r)\left[b^b_{m0} + \sum_{j=1}^{\infty} b^b_{mj} \sum_{p=0}^{j}\binom{j}{p}\cos\vartheta^{j-p}\sin\vartheta^{p}\Big((-i)^p + (i)^p\Big)\right] \\ &= e^{-r^2}\sum_{m=0}^{\infty} H_m(r)\left[b^b_{m0} + 2\sum_{j=1}^{\infty} b^b_{mj} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor}(-1)^p\binom{j}{2p}\cos\vartheta^{j-2p}\sin\vartheta^{2p}\right]\end{aligned} \tag{27}$$

[6] Since Hermite polynomials form a complete orthogonal basis of the Hilbert space.

This leads to the final expression of the eigenfunctions:[7]

$$\phi^b(r,\vartheta) = 2\,e^{-r^2} \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} b^b_{mj} H_m(r) \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^p \binom{j}{2p} \cos\vartheta^{j-2p} \sin\vartheta^{2p} \tag{28}$$

The scalar function a^c could be written in a similar way, but for the sake of simplicity it will be assumed that just one expansion coefficient is non-vanishing ($a^c_0 \neq 0$). In this case it is only necessary to consider H_0 which is equal to one.

$$a^c\big(\sqrt{\rho^2 - 2\rho r\cos\vartheta + r^2}\big) = e^{(-\rho^2 - r^2 + 2\rho r\cos\vartheta)}\; a^c_0 \tag{29}$$

The only degree of freedom left is represented by the coefficient a^c_0.

5. Analytical Treatment

In chapter (4.1) the obtained eigenvalue equation was written in the following way.

$$\phi^a(\rho,\varphi) = N \iint dr d\vartheta\, r\, a^c\big(\sqrt{\rho^2 - 2\rho r\cos\vartheta + r^2}\big)\phi^b(r,\vartheta) \tag{30}$$

Substituting the assumed Hermite-series (28) and (29) into (30) yields

$$\begin{aligned}\phi^a(\rho,\varphi) = 2N \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} a^c_0 \int_0^{\infty} dr\, r\, e^{-(\rho^2+2r^2)} H_m(r) \\ b^b_{mj} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^p \binom{j}{2p} \int_0^{2\pi} d\vartheta\, e^{2\rho r\cos\vartheta} \cos\vartheta^{j-2p} \sin\vartheta^{2p}\end{aligned} \tag{31}$$

where the integrals are now expressed in an explicit way, thus can be solved.
In this context the main goal of this chapter is the analytical treatment of the involved integrals.

[7] This is allowed by introducing modified expansion coefficients $b^b_{mj} \rightarrow b'^b_{mj}$ in a way that $b^b_{m0} = 2 \cdot b'^b_{m0}$ and afterwards renaming $b'^b_{mj} \rightarrow b^b_{mj}$ again.

5.1. Spherical integral treatment

Without further ado, the angular integral from equation (31) will be represented by $\mathcal{Q}^b_{mj}$.

$$
\begin{aligned}
\mathcal{Q}^b_{mj} &= b^b_{mj} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^p \binom{j}{2p} \int_0^{2\pi} d\vartheta \, e^{2\rho r \cos\vartheta} \cos\vartheta^{j-2p} \sin\vartheta^{2p} \\
&= 2\, b^b_{mj} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^p \binom{j}{2p} \int_0^{\pi} d\vartheta \, e^{2\rho r \cos\vartheta} \cos\vartheta^{j-2p} \sin\vartheta^{2p}
\end{aligned}
\tag{32}
$$

Where the last term is equal because the involved arguments are symmetric at π. Since the summation integer j appears exclusively in this spherical integral, it is possible to split the integral for even and odd integers j. This step will affect the upper bound of the summation with respect to p.

$$
\begin{aligned}
\mathcal{Q}^b_{mj} = 2\Bigg[& b^b_{m(2j+1)} \sum_{p=0}^{\lfloor \frac{2j+1}{2} \rfloor} (-1)^p \binom{2j+1}{2p} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \cos\vartheta^{2j+1-2p} \sin\vartheta^{2p} \\
& + b^b_{m(2j)} \sum_{p=0}^{\lfloor \frac{2j}{2} \rfloor} (-1)^p \binom{2j}{2p} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \cos\vartheta^{2j-2p} \sin\vartheta^{2p} \Bigg] \\
= 2 \sum_{p=0}^{j} (-1)^p \binom{2j}{2p} \Bigg[& b^b_{m(2j+1)} \frac{2j+1}{2(j-p)+1} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \cos\vartheta^{2(j-p)+1} \sin\vartheta^{2p} \\
& + b^b_{m(2j)} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \cos\vartheta^{2(j-p)} \sin\vartheta^{2p} \Bigg]
\end{aligned}
\tag{33}
$$

At this point, the equation was reshaped in a way that a factor of 2 occurs in front of the integers j and p. Note that the exponent value of the cosine cannot be negative, since the maximum value of p is equal to j. By using the pythagorean identity ($\cos\vartheta^2 = 1 - \sin\vartheta^2$) with, again the binomial identity, the integrals of equation (33) can be further simplified:

$$
\begin{aligned}
\mathcal{Q}^b_{mj} = 2 \sum_{p=0}^{j} \sum_{q=0}^{j-p} (-1)^{p+q} \binom{2j}{2p} \binom{j-p}{q} \Bigg[& b^b_{m(2j)} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \sin\vartheta^{2(p+q)} \\
& + b^b_{m(2j+1)} \frac{2j+1}{2(j-p)+1} \int_0^{\pi} d\vartheta e^{2\rho r \cos\vartheta} \sin\vartheta^{2(p+q)} \cos\vartheta \Bigg]
\end{aligned}
\tag{34}
$$

Integrals like this are directly solvable and yield modified Bessel functions of the first kind. A general overview about modified Bessel functions and how they contribute to the solution of such integrals is shown in appendix (A).
The final result for $\mathcal{Q}^b_{mj}$ can be written as:

$$\begin{aligned}\mathcal{Q}^b_{mj} = 2\sum_{p=0}^{j}\sum_{q=0}^{j-p}(-1)^{p+q}\binom{2j}{2p}\binom{j-p}{q}\pi\Bigg[b^b_{m(2j)}\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}}\prod_{v=0}^{p+q-1}(2v+1)\\ +\frac{b^b_{m(2j+1)}}{2\rho r}\frac{2j+1}{2(j-p)+1}\Bigg((2(p+q)-1)\frac{I_{p+q-1}(2\rho r)}{(2\rho r)^{p+q-1}}\prod_{v=0}^{p+q-2}(2v+1)\\ -2(p+q)\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}}\prod_{v=0}^{p+q-1}(2v+1)\Bigg)\Bigg]\end{aligned} \quad (35)$$

Where I_α denotes the modified Bessel functions of the first kind of order α.[8]

5.2. Radial integral treatment

So far the spherical integral of the eigenvalue equation (31) was completely solved and the solution yielded three similar terms which are constructed by I_α respectively. By replacing the former integral through the final expression of $\mathcal{Q}^b_{mj}$, the eigenvalue equation transforms to:

$$\begin{aligned}\phi^a(\rho,\varphi) = 2N\sum_{j=0}^{\infty}\sum_{p=0}^{j}\sum_{q=0}^{j-p}\sum_{m=0}^{\infty}a^c_0(-1)^{p+q}\binom{2j}{2p}\binom{j-p}{q}\frac{\pi}{\rho}e^{-\rho^2}\\ \Bigg[\rho\prod_{v=0}^{p+q-1}(2v+1)\,b^b_{m(2j)}\int_0^\infty dr\,e^{-2r^2}H_m(r)(2r)\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}}\\ +\frac{2j+1}{2(j-p)+1}(2(p+q)-1)\prod_{v=0}^{p+q-2}(2v+1)\,b^b_{m(2j+1)}\int_0^\infty dr\,e^{-2r^2}H_m(r)\frac{I_{p+q-1}(2\rho r)}{(2\rho r)^{p+q-1}}\\ -\frac{2j+1}{2(j-p)+1}2(p+q)\prod_{v=0}^{p+q-1}(2v+1)\,b^b_{m(2j+1)}\int_0^\infty dr\,e^{-2r^2}H_m(r)\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}}\Bigg]\end{aligned} \quad (36)$$

where the solution of the spherical integral contributes a function of the form $\frac{I_\alpha(2\rho r)}{(2\rho r)^\alpha}$ to the remaining integral with respect to r.

[8]Note that modified Bessel functions behave like exponentially growing functions.

Remember that this expression is still divided for even and odd integer j, which can be checked by looking at the expansion coefficients b^b_{mj}.
From here on, the focus is set on the calculation of the three remaining integrals. First, it is necessary to rewrite the involved Hermite polynomials and the hyperbolic Bessel functions in their explicit forms.
The explicit expression of an Hermite polynomial can be obtained by using the floor function.

$$H_n(x) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^l \, n!}{l!(n-2l)!}(2x)^{n-2l} \tag{37}$$

On the other hand, a Modified Bessel function of order α can be expressed by an infinite sum given by

$$I_\alpha(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(s+\alpha+1)}\left(\frac{x}{2}\right)^{2s+\alpha} \tag{38}$$

where the Gamma function is an extension of the factorial function $\Gamma(s+\alpha+1) = (s+\alpha)!$. The three remaining integrals will be designated by $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$. Expressing the involved H_n and I_α in their explicit ways yields:

$$\begin{aligned} \mathcal{R}_1 &= b^b_{m(2j)} \int_0^\infty dr\, e^{-2r^2}(2r)H_m(r)\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}} \\ &= \sum_{s=0}^{\infty}\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^l \, m!}{l!(m-2l)!}\frac{b^b_{m(2j)}}{s!(s+p+q)!}\int_0^\infty dr\, e^{-2r^2}\frac{(2r)^{m-2l+1}}{(2)^{p+q}}(\rho r)^{2s} \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{R}_2 &= b^b_{m(2j+1)} \int_0^\infty dr\, e^{-2r^2}H_m(r)\frac{I_{p+q-1}(2\rho r)}{(2\rho r)^{p+q-1}} \\ &= \sum_{s=0}^{\infty}\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^l \, m!}{l!(m-2l)!}\frac{b^b_{m(2j+1)}}{s!(s+p+q-1)!}\int_0^\infty dr\, e^{-2r^2}\frac{(2r)^{m-2l}}{(2)^{p+q-1}}(\rho r)^{2s} \end{aligned} \tag{40}$$

$$\begin{aligned} \mathcal{R}_3 &= b^b_{m(2j+1)} \int_0^\infty dr\, e^{-2r^2}H_m(r)\frac{I_{p+q}(2\rho r)}{(2\rho r)^{p+q}} \\ &= \sum_{s=0}^{\infty}\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^l \, m!}{l!(m-2l)!}\frac{b^b_{m(2j+1)}}{s!(s+p+q)!}\int_0^\infty dr\, e^{-2r^2}\frac{(2r)^{m-2l}}{(2)^{p+q}}(\rho r)^{2s} \end{aligned} \tag{41}$$

Besides the pre factors the integrals of $\mathcal{R}_2$ and $\mathcal{R}_3$ are identical and the one for $\mathcal{R}_1$ just differs by a factor r. There is indeed a way to solve those integrals analytically. The way how they are solved is shown in appendix (B) and their results are stated.

$$\mathcal{R}_1 = \sum_{s=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^l(2m)!}{l!(2(m-l))!} \frac{1}{s!(s+p+q)!} \frac{\rho^{2s}}{(2)^{p+q}} (2)^{2(m-l)+1} \left[\frac{2\, b^b_{(2m+1)(2j)}(2m+1)}{(2(m-l)+1)} \frac{(2(m-l+s+1)-1)!!}{2(4)^{m-l+s+1)}} \sqrt{\frac{\pi}{2}} + b^b_{(2m)(2j)} \frac{(m-l+s)!}{(2)^{m-l+s+2}} \right] \tag{42}$$

$$\mathcal{R}_2 = \sum_{s=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^l(2m)!}{l!(2(m-l))!} \frac{1}{s!(s+p+q-1)!} \frac{\rho^{2s}}{(2)^{p+q-1}} (2)^{2(m-l)} \left[\frac{2\, b^b_{(2m+1)(2j+1)}(2m+1)}{(2(m-l)+1)} \frac{(m-l+s)!}{(2)^{m-l+s+2}} + b^b_{(2m)(2j+1)} \frac{(2(m-l+s+1)-1)!!}{2(4)^{m-l+s}} \sqrt{\frac{\pi}{2}} \right] \tag{43}$$

$$\mathcal{R}_3 = \sum_{s=0}^{\infty}\sum_{l=0}^{m} \frac{(-1)^l(2m)!}{l!(2(m-l))!} \frac{1}{s!(s+p+q)!} \frac{\rho^{2s}}{(2)^{p+q}} (2)^{2(m-l)} \left[\frac{2\, b^b_{(2m+1)(2j+1)}(2m+1)}{(2(m-l)+1)} \frac{(m-l+s)!}{(2)^{m-l+s+2}} + b^b_{(2m)(2j+1)} \frac{(2(m-l+s+1)-1)!!}{2(4)^{m-l+s}} \sqrt{\frac{\pi}{2}} \right] \tag{44}$$

Note that in order to obtain those solutions it was again needed to differ between even and odd summation integer m. Reinserting the solutions back into the equation yields the final form of the initial eigenvalue equation (31), where both integrals are completely solved and the remaining equation depends on ρ and φ.

$$\begin{aligned} \phi^a(\rho,\varphi) = \; & 2N \sum_{j=0}^{\infty}\sum_{p=0}^{j}\sum_{q=0}^{j-p}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{s=0}^{\infty} a^c_0 (-1)^{p+q} \binom{2j}{2p}\binom{j-p}{q} \frac{\pi}{\rho} e^{-\rho^2} \\ & \frac{(-1)^l(2m)!}{l!(2(m-l))!} \frac{1}{s!\,\Gamma(s+p+q)} \frac{(2)^{2(m-l)}}{(2)^{p+q}} \rho^{2s} \Bigg\{ \frac{2\rho}{(s+p+q)} \prod_{v=0}^{p+q-1}(2v+1) \\ & \left[\frac{b^b_{(2m+1)(2j)}(2m+1)}{(2(m-l)+1)} \frac{(2(m-l+s+1)-1)!!}{(4)^{m-l+s+1}} \sqrt{\frac{\pi}{2}} + b^b_{(2m)(2j)} \frac{(m-l+s)!}{(2)^{m-l+s+2}} \right] \\ & + \frac{2j+1}{2(j-p)+1} \left[(2(p+q)-1) \prod_{v=0}^{p+q-2}(2v+1) - \frac{p+q}{(s+p+q)} \prod_{v=0}^{p+q-1}(2v+1) \right] \\ & \left[\frac{b^b_{(2m+1)(2j+1)}(2m+1)}{(2(m-l)+1)} \frac{(m-l+s)!}{(2)^{m-l+s}} + b^b_{(2m)(2j+1)} \frac{(2(m-l+s)-1)!!}{(4)^{m-l+s}} \sqrt{\frac{\pi}{2}} \right] \Bigg\} \end{aligned} \tag{45}$$

6. Results

Beforehand, it is worthwhile to summarize (45) in the following way:

$$\begin{aligned} \phi^a(\rho,\varphi) &= 2Na_0^c\,\mathcal{P}^b_{mj}(\rho) \\ \lambda\,\phi^a(\rho,\varphi) &= \rho^2\phi^a(\rho,\varphi) - \frac{2ig}{\sqrt{2\pi}^{\mu}}t^{\mu}\eta_{\mu}\epsilon^{abc}a_0^c\,\mathcal{P}^b_{mj}(\rho) \end{aligned} \tag{46}$$

Remember that the underlying impulse η was not specified so far. Thus every direction is admissible and for this reason the final discussion will be restricted to $\eta = \binom{1}{0}$.

$$\lambda\,\phi^a(\rho,\varphi) = \rho^2\phi^a(\rho,\varphi) - \frac{ig}{\pi}\cos\varphi\,\rho\,\epsilon^{abc}a_0^c\,\mathcal{P}^b_{mj}(\rho) \tag{47}$$

This alters the angular dependency of the last term on the right side to $\cos\varphi$. Therefore it is assumed that the angular dependency of the wavefunctions exclusively behave like $\cos\varphi$, which can be achieved by stating that the expansion coefficients b^a_{mj} are set zero for every $j \neq 1$.
In doing so equation (47) is true for every angle φ. Since the radial dependency of ϕ^a is given by an Hermite-series which forms a complete orthogonal basis, it is possible to project this equation onto every polynomial H_u which yields an infinite set of linear equations. Before calculating this overlap a few properties of Hermite polynomials are needed.[4]
First, note that Hermite polynomials are orthogonal with respect to a weight function $w(x) = e^{-x^2}$.

$$\int_{-\infty}^{\infty} dx\,H_m(x)H_n(x)e^{-x^2} \;=\; 2^n n!\sqrt{\pi}\,\delta_{nm}$$

Furthermore these polynomials constitute an Appell sequence, thus they satisfy the identity:

$$H'_n(x) \;=\; 2nH_{n-1}(x)$$

With these relations the projection of the eigenvalue equation on any polynomial H_u can be calculated. For the sake of simplicity the projection of the three terms in (47) will be treated individually.

Starting with $\lambda\phi^a$, the inner product with H_u is given by:

$$\begin{aligned}\lambda \int_{-\infty}^{\infty} d\rho\, H_u\, \phi^a =&\ \lambda \cos\varphi \sum_{m=0}^{\infty} b^a_{m1} \int_{-\infty}^{\infty} d\rho\, H_u H_m e^{-\rho^2} \\ =&\ \lambda \cos\varphi\, b^a_{u1}\, 2^u\, u!\sqrt{\pi}\end{aligned} \tag{48}$$

The projection of the second term $\rho^2\phi^a$ can be solved through partial Integration, as the derivations of H_m can be rewritten using the Appell sequence. The calculation of the occurring integral is shown below.

$$\begin{aligned}&\int_{-\infty}^{\infty} d\rho\, \rho e^{-\rho^2} (\rho H_u H_m) = \\ &= \int_{-\infty}^{\infty} d\rho\, \frac{e^{-\rho^2}}{2} \big[H_u H_m + \rho\big(2m H_u H_{m-1} + 2u H_{u-1} H_m\big)\big] \\ &= \sqrt{\pi}\, \big[2^{u-1} u!(1{+}2m)\delta_{um} + m(m{-}1)2^u u!\, \delta_{u(m-2)} + 2^{u-2} u!\, \delta_{(u-2)m}\big]\end{aligned} \tag{49}$$

Using the solution of (49), the inner product of $\rho^2\phi$ with H_u can be written as:

$$\begin{aligned}&\int_{-\infty}^{\infty} d\rho\, H_u\, \rho^2\phi^a = \cos\varphi \sum_{m=0}^{\infty} b^a_{m1} \int_{-\infty}^{\infty} d\rho\, \rho e^{-\rho^2} (\rho H_u H_m) \\ &= \cos\varphi\sqrt{\pi}\, \big[2^{u-1} u!\, b^a_{u1}(1{+}2u) + 2^u (u{+}2)!\, b^a_{(u+2)1} + 2^{u-2} u!\, b^a_{(u-2)1}\big]\end{aligned} \tag{50}$$

Last but not least the only thing outstanding is the inner product of the last term $-\frac{ig}{\pi}\cos\varphi\epsilon^{abc}a^c_0\,\rho\mathcal{P}^b_{m1}(\rho)$ with H_u. Since the expansion coefficients are restricted to be zero for any $j \neq 1$, the explicit expression of $\mathcal{P}^b_{m1}(\rho)$ simplifies to:

$$\begin{aligned}\mathcal{P}^b_{m1}(\rho) = -\pi \sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{s=0}^{\infty}\ \rho^{2s-1} e^{-\rho^2} \frac{(-1)^l (2m)!}{l!(2(m{-}l))!} \frac{s}{s!\, s!} 2^{2(m-l)} \\ \left[\frac{b^b_{(2m+1)(1)}(2m{+}1)}{(2(m{-}l){+}1)} \frac{(m{-}l{+}s)!}{(2)^{m-l+s}} + b^b_{(2m)(1)} \frac{(2(m{-}l{+}s){-}1)!!}{(4)^{m-l+s}} \sqrt{\frac{\pi}{2}}\right]\end{aligned} \tag{51}$$

Note that the ρ dependency is similar to the calculation shown in appendix (B). For that reason the overlap can be calculated by using the explicit expression of an Hermite polynomial.

The inner product of $-\frac{ig}{\pi}\cos\varphi\epsilon^{abc}a_0^c\,\rho\mathcal{P}^b_{m1}(\rho)$ with H_u is therefore given by

$$
\begin{aligned}
&-\cos\varphi\frac{ig}{\pi}\epsilon^{abc}a_0^c\int_{-\infty}^{\infty}d\rho\,H_u\,\rho\mathcal{P}^b_{m1}(\rho) \;= \\
&=\cos\varphi\;ig\epsilon^{abc}a_0^c\sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{s=0}^{\infty}\frac{(-1)^l(2m)!}{l!(2(m-l))!}\frac{s}{s!\,s!}2^{2(m-l)} \\
&\left[\frac{b^b_{(2m+1)(1)}(2m+1)}{(2(m-l)+1)}\frac{(m-l+s)!}{2^{(m-l+s)}}+b^b_{(2m)(1)}\frac{(2(m-l+s)-1)!!}{4^{(m-l+s)}}\sqrt{\frac{\pi}{2}}\right] \\
&u!\sum_{t=0}^{\lfloor\frac{u}{2}\rfloor}\frac{(-1)^t\,2^{(u-2t)}}{t!(u-2t)!}\frac{(2(s+\frac{u}{2}-t)-1)!!}{2^{(s+\frac{u}{2}-t)}}\sqrt{\pi}
\end{aligned}
\tag{52}
$$

where the only criterion to obtain this non-vanishing solution is that the order of H_u is said to be even. For uneven u this overlap vanishes.

Finally, back in equation (47) these overlaps form a system of linear equations. Before putting the solutions (48),(50) and (52) back into (47) it seems convenient to use the following notation:

$$
\begin{aligned}
\Lambda(u) &= \frac{1}{2}(1+2u) \\
\alpha(u) &= (u+2)(u+1) \\
\beta_1(m,u) &= ig\sum_{l=0}^{m}\sum_{s=0}^{\infty}\frac{(-1)^l(2m)!}{l!(2(m-l))!}\frac{s}{s!\,s!}2^{2(m-l)}\frac{(2m+1)}{(2(m-l)+1)}\frac{(m-l+s)!}{2^{(m-l+s)}} \\
&\quad\sum_{t=0}^{\lfloor\frac{u}{2}\rfloor}\frac{(-1)^t\,2^{(-2t)}}{t!(u-2t)!}\frac{(2(s+\frac{u}{2}-t)-1)!!}{2^{(s+\frac{u}{2}-t)}} \\
\beta_2(m,u) &= ig\sum_{l=0}^{m}\sum_{s=0}^{\infty}\frac{(-1)^l(2m)!}{l!(2(m-l))!}\frac{s}{s!\,s!}2^{2(m-l)}\frac{(2(m-l+s)-1)!!}{4^{(m-l+s)}}\sqrt{\frac{\pi}{2}} \\
&\quad\sum_{t=0}^{\lfloor\frac{u}{2}\rfloor}\frac{(-1)^t\,2^{(-2t)}}{t!(u-2t)!}\frac{(2(s+\frac{u}{2}-t)-1)!!}{2^{(s+\frac{u}{2}-t)}}
\end{aligned}
\tag{53}
$$

where the functions depend on the order of the overlap H_u. Note that the sum with respect to s in β_1 and β_2 converges, which may not be obvious at first sight. With this notation any overlap of H_u with the eigenvalue equation (47) can be streamlined yielding two similar equations with respect to even or odd u.

At this point one last configuration will be set. Due to the fact that the expansion coefficient a_0^c of the potential A_μ^c in momentum space can be chosen freely, it will be demanded that just one color is charged, meaning that the expansion coefficient a_0^c is zero for any other color than $c = 3$. The final set of linear equations can therefore be written as follows.

For odd u:

$$\begin{aligned} \lambda b_{u1}^1 &= \Lambda(u) b_{u1}^1 + \alpha(u) b_{(u+2)1}^1 + \frac{1}{4} b_{(u-2)1}^1 \\ \lambda b_{u1}^2 &= \Lambda(u) b_{u1}^2 + \alpha(u) b_{(u+2)1}^2 + \frac{1}{4} b_{(u-2)1}^2 \\ \lambda b_{u1}^3 &= \Lambda(u) b_{u1}^3 + \alpha(u) b_{(u+2)1}^3 + \frac{1}{4} b_{(u-2)1}^3 \end{aligned} \tag{54}$$

For even u:

$$\begin{aligned} \lambda b_{u1}^1 &= \Lambda(u) b_{u1}^1 + \alpha(u) b_{(u+2)1}^1 + \frac{1}{4} b_{(u-2)1}^1 + a_0^3 \sum_{m=0}^{\infty} \left[\beta_1(m,u) b_{(2m+1)1}^2 + \beta_2(m,u) b_{(2m)1}^2 \right] \\ \lambda b_{u1}^2 &= \Lambda(u) b_{u1}^2 + \alpha(u) b_{(u+2)1}^2 + \frac{1}{4} b_{(u-2)1}^2 - a_0^3 \sum_{m=0}^{\infty} \left[\beta_1(m,u) b_{(2m+1)1}^1 + \beta_2(m,u) b_{(2m)1}^1 \right] \\ \lambda b_{u1}^3 &= \Lambda(u) b_{u1}^3 + \alpha(u) b_{(u+2)1}^3 + \frac{1}{4} b_{(u-2)1}^3 \end{aligned} \tag{55}$$

This set of equations can indeed be solved and their eigenvalues λ are either real and positive or they occur in complex conjugated pairs where the real part can be negative. Their associated expansion coefficients b_{m1}^a are then complex numbers, in a way that $b_{m1}^1 = i b_{m1}^2$.
However, since the eigenfunctions are said to be real, those negative λ's contribute no solution to the initial eigenvalue problem because their associated expansion coefficients are complex.
In fact, it would be possible to obtain an admissible solution with complex expansion coefficients, if they would fulfill the relation $b_{m1}^1 = (b_{m2}^2)^*$. But since this is not the case, the obtained solution will not be observed by a physical reality.

7. Summary and Discussion

Summarizing, a general expression for the eigenvalue equation of the Faddeev-Popov operator in momentum space was derived.

$$\phi^a(t^\mu) = \frac{-ig}{\sqrt{2\pi}^\mu(\lambda - t^2)} \, t^\mu \epsilon^{abc} \int d\tau^\mu \phi^b(\tau^\mu) A^c_\mu(t^\mu - \tau^\mu) \tag{56}$$

After restricting the field to be rotational symmetric and setting the external impulse in a fixed direction, the potential A^c_μ was reduced to a scalar function depending on the absolute value of its qualities. Afterwards it was assumed that both, the eigenfunctions ϕ^a and the obtained scalar function a^c exhibit analytic behaviour. By using separation of variables for $\phi^a(r, \vartheta) = R^a(r)Y^a(\vartheta)$, it was possible to expand ϕ^a and a^c in a formal Hermite-series.

$$\begin{aligned} \phi^b(r,\vartheta) &= 2\, e^{-r^2} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} b^b_{mj} H_m(r) \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^p \binom{j}{2p} \cos\vartheta^{j-2p} \sin\vartheta^{2p} \\ a^c\big(\sqrt{\rho^2 - 2\rho r\cos\vartheta + r^2}\big) &= e^{(-\rho^2 - r^2 + 2\rho r\cos\vartheta)} \; a^c_0 \end{aligned} \tag{57}$$

Where the potential function a^c was set to the easiest case just considering $H_0 = 1$. From there the integral of (56) was completely solved and their solution converges. In conclusion the overlap of any H_u with the obtained eigenvalue equation was calculated, producing a infinite set of systems of linear equations with coupled colors.

This set of equations is solvable, yielding either real or complex eigenvalues λ. The real eigenvalues are exclusively positive and the complex eigenvalues occur in complex conjugated pairs. It is indeed possible for these complex eigenvalues to have a negative real part, but since their associated expansion coefficient b^a_{m1} are said to be real, this leads to a contradiction.

In conclusion one can state that at least one of the introduced restrictions was to harsh, altering the problem to a fictive calculation without any reference to the observed reality.

A. Modified Bessel functions of the first kind

Modified Bessel functions[9] are solutions of Bessel's differential equation (58) which have purely imaginary arguments.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{58}$$

Bessel's equation arises when finding separable solutions to Laplace's equation for an arbitrary complex number α, the order of the Bessel function.
There is, in fact, a way that modified Bessel functions occur as solutions for specific integrals. [1]

$$\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta} = \pi I_0(2\rho r) \tag{59}$$

Here, I_0 is the modified Bessel Function of the first kind of order 0. Starting from there it is rather possible to obtain such functions of different orders through integrals like:

$$\begin{aligned}
\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\sin\vartheta^2 &= \pi\frac{I_1(2\rho r)}{2\rho r}\\
\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\sin\vartheta^4 &= 3\pi\frac{I_2(2\rho r)}{(2\rho r)^2}\\
\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\sin\vartheta^6 &= 15\pi\frac{I_3(2\rho r)}{(2\rho r)^3}\\
\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\sin\vartheta^8 &= 105\pi\frac{I_4(2\rho r)}{(2\rho r)^4}
\end{aligned} \tag{60}$$

It is obvious that there is a pattern, as the order of I_α rises with the exponent value of the sine. Therefore it is possible to generalize these integrals to obtain a modified Bessel function of any order α.

$$\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\sin\vartheta^{2\alpha} = \pi\frac{I_\alpha(2\rho r)}{(2\rho r)^\alpha}\prod_{v=0}^{\alpha-1}(2v+1) \tag{61}$$

This generalization solves the first integral of equation (34).

[9] also referred as hyperbolic Bessel functions

On the contrary, with the help of partial integration, the second integral can be solved in a similar way.

$$\begin{aligned} \int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\cos\vartheta\sin\vartheta^{2\alpha} &= \underbrace{\frac{e^{2\rho r\cos\vartheta}}{-2\rho r}\cos\vartheta\sin\vartheta^{2\alpha-1}\Big|_0^\pi}_{=0} + \\ &+ \int_0^\pi d\vartheta \frac{e^{2\rho r\cos\vartheta}}{2\rho r}\Big((2\alpha-1)\sin\vartheta^{2(\alpha-1)}\cos\vartheta^2 - \sin\vartheta^{2\alpha}\Big) \\ &= \frac{1}{2\rho r}\int_0^\pi d\vartheta e^{2\rho r\cos\vartheta}\Big((2\alpha-1)\sin\vartheta^{2(\alpha-1)} - 2\alpha\sin\vartheta^{2\alpha}\Big) \\ = \frac{\pi}{2\rho r}\Bigg((2\alpha-1)\frac{I_{\alpha-1}(2\rho r)}{(2\rho r)^{\alpha-1}}\prod_{v=0}^{\alpha-2}(2v+1) &- 2\alpha\frac{I_\alpha(2\rho r)}{(2\rho r)^\alpha}\prod_{v=0}^{\alpha-1}(2v+1)\Bigg) \end{aligned} \tag{62}$$

B. Calculating the radial integral

Even though the integrals in chapter (5.2) are not the same, they just differ by a factor r. Furthermore they are completely solvable and the result can be generalized whereby the analytic solution distinguishes for even and odd exponent values. [1]

$$\begin{aligned} \int_0^\infty dr\, e^{-pr^2} r^{2n} &= \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \\ \int_0^\infty dr\, e^{-pr^2} r^{2n+1} &= \frac{n!}{2p^{n+1}} \end{aligned} \tag{63}$$

These expressions are true for $[p > 0]$.
Hence the approach to solve them is identical and the calculation will be delineated by a generalized integral $\mathcal{B}$.

$$\mathcal{B} = \sum_{s=0}^{\infty} \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \int_0^\infty dr\, e^{-2r^2} r^{m-2l} r^{2s} \tag{64}$$

Obviously the only thing left is to differentiate between even or odd exponent values of r. Since the integers l and s are even anyways, only the integer m needs to be dismantled.

$$\begin{aligned} \mathcal{B} &= \sum_{s=0}^{\infty} \left[\sum_{l=0}^{\lfloor \frac{2m+1}{2} \rfloor} \int_0^\infty dr\, e^{-2r^2} r^{2(m-l)+1} r^{2s} + \sum_{l=0}^{\lfloor \frac{2m}{2} \rfloor} \int_0^\infty dr\, e^{-2r^2} r^{2(m-l)} r^{2s} \right] \\ &= \sum_{s=0}^{\infty} \sum_{l=0}^{m} \left[\int_0^\infty dr\, e^{-2r^2} r^{2(m-l+s)+1} + \int_0^\infty dr\, e^{-2r^2} r^{2(m-l+s)} \right] \\ &= \sum_{s=0}^{\infty} \sum_{l=0}^{m} \left[\frac{(m-l+s)!}{(2)^{m-l+s+2}} + \frac{(2(m-l+s)-1)!!}{2(4)^{m-l+s}} \sqrt{\frac{\pi}{2}} \right] \end{aligned} \tag{65}$$

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