Anomalies

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Chapter 1

Anomalies

1.1 Introduction

There is one particular important property of the standard model, which is very much restricting its structure, and which is recurring in extensions of the standard model. That is the absence of anomalies. An anomaly is that some symmetry, which is present on the classical level, is not present when considering the quantum theory. The symmetry is said to be broken by quantum effects. Generically, this occurs if the action of a theory is invariant under a symmetry, but the measure of the path integral is not. Constructing a theory which is at the same time anomaly-free and consistent with the standard model is actually already quite restricting, and therefore anomalies are an important tool to check the consistency of new proposals for physics beyond the standard model. This will be therefore discussed here in some detail.

1.2 Global anomalies

Anomalies fall into two classes, global and local anomalies. Global anomalies refer to the breaking of global symmetries by quantum effects. The most important one of these global anomalies is the breaking of dilatation symmetry. This symmetry corresponds to rescaling all dimensionful quantities, e. g., $x \to \lambda x$. Maxwell theory, massless QED, Yang-Mills theory, and massless QCD are all invariant under such a rescaling, at the classical level, though not the Higgs sector of the standard model. This is no longer the case at the quantum level. By a process called dimensional transmutation, surfacing in the renormalization process, an explicit scale is introduced into the theory, and thereby the quantum theory is no longer scale-invariant. Such global anomalies have very direct consequences. E. g., this dilatation anomaly leads to the fact that the photon is massless in massless QED. Of course, it is also massless in massive QED, but there the breaking of the dilatation symmetry is explicit due to the lepton mass.

Another example is the so-called axial anomaly, which occurs due to the breaking of the global axial symmetry of baryons. A consequence of it is the anomalously large η' mass. While the dilatation anomaly is quite obvious, the chiral anomaly is much more subtle, and therefore deserves some more discussion. In addition, it will be very helpful when generalizing to the local anomalies.

1.2.1 Classical level

To prepare for this, it is worthwhile to consider the situation as it would be without anomalies, i. e. at the classical level. For this purpose, start with a gauge theory with fermions ψ being in some representation R of the gauge Lie group G with generators T and gauge fields in the adjoint representation. The fermionic part of the Lagrangian is then given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu - igT^a A_\mu^a) - m)\psi = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi$$

from which the Dirac equation

$$(i\gamma_\mu D^\mu - m)\psi = 0$$

follows as the equation of motion, and likewise for the anti-fermion.

The current carrying the charge is then

$$j_\mu^a = \bar{\psi}\gamma_\mu T^a \psi.$$

Due to the chiral symmetry, there is also a corresponding axial current

$$j_\mu^{5a} = \bar{\psi}\gamma_5\gamma_\mu T^a \psi.$$

In addition, there are also the singlet currents

$$\begin{aligned} j_\mu &= \bar{\psi}\gamma_\mu\psi \\ j_\mu^5 &= \bar{\psi}\gamma_5\gamma_\mu\psi, \end{aligned}$$

which corresponds to the fermion current and the axial current.

Naively, the divergences of these equations can be calculated using the Dirac equation.

$$\begin{aligned} \partial^\mu j_\mu^a &= -i\bar{\psi}(g\tau^b\gamma_\mu A_b^\mu - m)\tau^a\psi - i\bar{\psi}\tau^a(-g\tau^b\gamma_\mu A_b^\mu + m)\psi \\ &= ig\bar{\psi}\left[\tau^a, \tau^b\right]\gamma_\mu A_b^\mu\psi = -gf^{abc}A_b^\mu\bar{\psi}\gamma_\mu\tau_c\psi = -gf^{ab}{}_c A_b^\mu j_\mu^c. \end{aligned}$$

This implies that the color current is not observed, as long as the current is gauged. For a non-gauge current, like a flavor current, g vanishes, and the current is conserved.

This is not surprising, as a non-Abelian gauge theory has no gauge-invariant charge. However, the current is a gauge-vector, and therefore covariantly conserved

$$D_\mu^{ab} j_b^\mu = 0. \tag{1.1}$$

In the same way, it is possible to calculate the situation of the axial color current. Because of the commutation relation between γ matrices, the result is

$$D_\mu^{ab} j_b^\mu = 2im\bar{\psi}\gamma_5\tau^a\psi = 2mip^a, \tag{1.2}$$

Here, p is the pseudo-scalar density, and not a momentum component. Thus, even in a non-gauge theory this current is only conserved for fermions without a mass term in the Lagrangian.

The calculations for the singlet current is simpler, and yields

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ \partial^\mu j_\mu^5 &= 2im\bar{\psi}\gamma_5\psi = 2imp^0. \end{aligned}$$

Hence, the number of fermion is, a expected, a conserved current. The axial current is only conserved for massless fermions. This is the result that chiral symmetry gets explicitly broken, already classically, by a mass-term.

In a theory like the standard model, where parity is broken, left-handed and right-handed fermions

$$\begin{aligned} \psi_L &= \frac{1-\gamma_5}{2}\psi \\ \psi_R &= \frac{1+\gamma_5}{2}\psi \end{aligned}$$

do not couple in the same way to the gauge-fields

$$\mathcal{L} = \bar{\psi}_L i\gamma_\mu D_L^\mu \psi_L + \bar{\psi}_R i\gamma_\mu D_R^\mu \psi_R,$$

with $D_L \neq D_R$, and no mass term is permitted due to gauge invariance. Thus, the color currents are recombined into covariantly conserved left-handed and right-handed currents as

$$\begin{aligned} j_\mu^{aL} &= \frac{1}{2}(j_\mu^a - j_\mu^{5a}) \\ j_\mu^{aR} &= \frac{1}{2}(j_\mu^a + j_\mu^{5a}) \\ D_L j_\mu^L &= 0 \\ D_R j_\mu^R &= 0, \end{aligned}$$

and a similar recombination for the singlet currents.

1.2.2 One-loop violation

So far, this was the conservation at the classical level, which already requires the fermions to be massless. At the quantum level, this result is expressed by Ward-identities. In particular, take Ward identities for correlation functions of the form

$$T^{ijk}_{\mu\nu\rho} = \langle T j^i_\mu j^j_\nu j^k_\rho \rangle,$$

where i, j, and k can take the values V, A, and P, which require to replace the j by j^a, j^{5a}, and p^a, respectively, and the Lorenz index is dropped in the last case. Calculating the corresponding Ward identities for a local chiral transformation

$$\begin{aligned}\psi' &= e^{i\beta(x)\gamma_5}\psi(x)\\ \bar{\psi}' &= \bar{\psi}e^{i\beta(x)\gamma_5}\end{aligned}$$

yields the expressions

$$\partial^\mu_x T^{VVA}_{\mu\nu\rho}(x,y,z) = \partial^\nu_y T^{VVA}_{\mu\nu\rho}(x,y,z) = 0 \tag{1.3}$$

$$\partial^\rho_z T^{VVA}_{\mu\nu\rho}(x,y,z) = 2mT^{VVP}_{\mu\nu}(x,y,z), \tag{1.4}$$

directly implementing the relations (1.1) and (1.2). This is what should happen, if there would be no anomalies.

To check this, it is possible to calculate the leading-order perturbative correction. Since only fermion fields appear in the vacuum expectation value, this is a vacuum triangle graph, and the coupling is to external currents. In fact, it does not matter at this point whether the external currents are gauged or non-gauged, since to this order this only alters the presence or absence of color matrices at the external vertices. The only relevant part of the external vertices is their Dirac structure.

Evaluating all the Wick contractions yields two Feynman diagrams, which translate to

$$\begin{aligned}&T^{VVA}_{\mu\nu\rho}(p_1,p_2,p_3=-p_1-p_2) = \qquad (1.5)\\ &-i^3\int\frac{d^4k}{(2\pi)^4}\Big(\mathrm{tr}\gamma_\mu(\gamma_\alpha k^\alpha - m)^{-1}\gamma_\nu(\gamma^\beta k_\beta - \gamma_\beta p_2^\beta - m)^{-1}\gamma_\rho\gamma_5(\gamma_\gamma k^\gamma + \gamma_\gamma p_1^\gamma - m)^{-1}\\ &+\mathrm{tr}\gamma_\nu(\gamma_\alpha k^\alpha - m)^{-1}\gamma_\mu(\gamma^\beta k_\beta - \gamma_\beta p_1^\beta - m)^{-1}\gamma_\rho\gamma_5(\gamma_\gamma k^\gamma + \gamma_\gamma p_2^\gamma - m)^{-1}\Big).\end{aligned}$$

This expression is linearly divergent. One of the most important points in anomalies, and in quantum field theories in general, is that the result is independent of the regulator employed. This will be discussed later how to show this. Here, it permits to use a Pauli-Villar regulator with a mass M, which is technically more simple than other possibilities.

Using dimensional regularization makes the result subtle, as it depends on the way the matrix γ_5 is analytically continued. This problem will therefore be avoided here.

To test the vector Ward identity, the expression can be multiplied with p_1^μ. To simplify the so obtained expression it is useful to employ

$$\gamma_\mu p_1^\mu = -(\gamma_\mu k^\mu - \gamma_\mu p_1^\mu - m) + (\gamma_\mu k^\mu - m),$$

yielding

$$\begin{aligned} & p_1^\mu T^{VVA}_{\mu\nu\rho}(p_1, p_2, p_3 = -p_1 - p_2) = \\ & -i^3 \int \frac{d^4k}{(2\pi)^4} \Big(\mathrm{tr} - (\gamma_\alpha k^\alpha - m)^{-1} \gamma_\nu (\gamma^\beta k_\beta - \gamma_\beta p_2^\beta - m)^{-1} \gamma_\rho \gamma_5 \\ & \mathrm{tr}(\gamma_\gamma k^\gamma + \gamma_\gamma p_1^\gamma - m)^{-1} \gamma_\nu (\gamma^\beta k_\beta - \gamma_\beta p_2^\beta - m)^{-1} \gamma_\rho \gamma_5 \\ & + \mathrm{tr}(\gamma_\gamma k^\gamma + \gamma_\gamma p_2^\gamma - m)^{-1} \gamma_\nu (\gamma_\alpha k^\alpha - m)^{-1} \gamma_\rho \gamma_5 \\ & + \mathrm{tr} - (\gamma_\gamma k^\gamma + \gamma_\gamma p_2^\gamma - m)^{-1} \gamma_\nu (\gamma^\beta k_\beta - \gamma_\beta p_1^\beta - m)^{-1} \gamma_\rho \gamma_5 + (m \to M) \Big) . \end{aligned} \tag{1.6}$$

This rather length expression is now a finite integral. It is therefore permissible to reshuffle the momenta like $k \to k + p_2$ in the first term and $k \to k + p_2 - p_1$ in the second term. Then, the first and third and second and fourth term cancel each other, and likewise this happens for the regulator. Thus, the vector Ward identity is fulfilled. The result for the second identity in (1.3) works in the same way.

The situation changes drastically for the axial Ward identity (1.4). The expression (1.5) is still divergent, so before doing anything, it will again be regulated using a Pauli-Villar regulator, to make it well-defined. To evaluate (1.4) requires multiplication with $p_3 = -p_1 - p_2$, which can rewritten as

$$\begin{aligned} \gamma_\mu p_3^\mu \gamma_5 &= (\gamma_\mu k^\mu - \gamma_\mu p_2^\mu - m)\gamma_5 + \gamma_5(\gamma_\mu k^\mu + \gamma_\mu p_1^\mu - m) + 2m\gamma_5 \\ &= (\gamma_\mu k^\mu - \gamma_\mu p_1^\mu - m)\gamma_5 + \gamma_5(\gamma_\mu k^\mu + \gamma_\mu p_2^\mu - m) + 2m\gamma_5. \end{aligned}$$

This yields

$$\begin{aligned} & p_3^\rho T^{VVA}_{\mu\nu\rho}(p_1, p_2, p_3 = -p_1 - p_2) = 2i \int \frac{d^4k}{(2\pi)^4} \\ & \Big(m\mathrm{tr} \left(\gamma_\mu (k_\alpha \gamma^\alpha - m)^{-1} \gamma_\nu (\gamma_\beta k^\beta - \gamma_\beta p_2^\beta - m)^{-1} \gamma_5 (\gamma_\gamma k^\gamma + p_1^\gamma \gamma_\gamma - m)^{-1} \right) \\ & m\mathrm{tr} \left(\gamma_\mu (k_\alpha \gamma^\alpha - m)^{-1} \gamma_\nu (\gamma_\beta k^\beta - \gamma_\beta p_1^\beta - m)^{-1} \gamma_5 (\gamma_\gamma k^\gamma + p_2^\gamma \gamma_\gamma - m)^{-1} \right) \\ & M\mathrm{tr} \left(\gamma_\mu (k_\alpha \gamma^\alpha - M)^{-1} \gamma_\nu (\gamma_\beta k^\beta - \gamma_\beta p_2^\beta - M)^{-1} \gamma_5 (\gamma_\gamma k^\gamma + p_1^\gamma \gamma_\gamma - M)^{-1} \right) \\ & M\mathrm{tr} \left(\gamma_\mu (k_\alpha \gamma^\alpha - M)^{-1} \gamma_\nu (\gamma_\beta k^\beta - \gamma_\beta p_1^\beta - M)^{-1} \gamma_5 (\gamma_\gamma k^\gamma + p_2^\gamma \gamma_\gamma - M)^{-1} \right) \Big) \end{aligned}$$

There are two remarkable facts to be observed. The first is that this expression is finite. The projection with p_3 drops out the divergent terms. This can be seen using the Dirac matrix identity

$$\text{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 = -4i\epsilon_{\mu\nu\rho\sigma}. \tag{1.7}$$

Because of the anti-symmetry of the ϵ-symbol, any term containing two or more factors of k vanishes. Hence, the numerator is reduced by two powers of k, making the integral finite. This did not work in (1.6) as there one index less was uncontracted. However, the regulator still had to be present in te first place to make this projection well-defined. The second is that this expression, except for the regulator, is identical to T^{VVP} up to a factor of m, which is obtained by replacing $\gamma_\rho\gamma_5$ in (1.5).

The term involving the regulator can then be easily calculated, as when removing the regulator in the end, the external momenta and masses can always be neglected, and the integral becomes a simple tadpole integral. The final result is thus

$$\begin{aligned} ip_3^\rho T^{VVA}_{\mu\nu\rho}(p_1,p_2) &= 2miT^{VVP}_{\mu\nu}(p_1,p_2) + \lim_{M\to\infty} 8iM^2\epsilon_{\mu\nu\rho\sigma}p_\rho^1 p_\sigma^2 \times \frac{i}{16\pi^2}\frac{-1}{2M^2} \\ &= 2miT^{VVP}_{\mu\nu}(p_1,p_2) + \frac{1}{2\pi^2}\epsilon^{\mu\nu\rho\sigma}p_\rho^1 p_\sigma^2 \end{aligned} \tag{1.8}$$

Thus, the Ward identity (1.4) is violated. The anomaly is both finite and independent of the masses of the involved particles. It is also independent of the structure of the external interaction, except for its Lorentz structure. The only thing changes is the appearance of corresponding pre-factor a^{abc} of the coupling matrices T^a in charge space, which turn out to be

$$a^{abc} = \frac{1}{2}\text{tr}\left(\left\{T^a, T^b\right\}T^c\right), \tag{1.9}$$

a result which will become significant later. This is not the only anomaly, and a similar result holds for the case of three axial currents.

Without proof, it should be noted here that there is still a certain regulator dependency. It is possible by symmetries to add a finite term of form $C\epsilon_{\mu\nu\rho\sigma}(p_1 - p_2)^\sigma$ to the counter-term in (1.6). Though C can be tuned to absorb the anomaly, this term will also contribute to the vector identities, and induce there an anomaly for $C \neq 0$. Thus, it is only possible to shift the anomaly around, without removing it.

The most well-known consequence of this anomaly is the decay of a neutral pion into two photons. This is precisely of the type investigate here, where the photons play the role of the vector currents. The axial current is related to the pion field by a QCD relation

$$\partial^\mu j_\mu^a = \frac{f_\pi}{\sqrt{2}}M_\pi^2\pi^a, \tag{1.10}$$

where a is an isospin index, counting the three pions, $a = 0, \pm$, where only $a = 0$ is relevant because of charge conservation. Since there are no massless hadrons, there can be no pole in the corresponding amplitude T^{VVA}, and thus the product with p_ρ has to vanish. As a consequence, the amplitude T^{VVP}, describing the transition, would vanish as well, because of the Ward identity, and therefore the pion would usually not decay into two photons, if at rest. However, due to the anomaly, this is not necessary, as the anomaly can balance the Ward identity. Hence, the pion at rest can decay into two photons, due to the anomaly, a process indeed observed in experiment.

1.3 Local anomalies

In contrast to the global anomalies, the local anomalies are a more severe problem. A local anomaly occurs, when a quantum effect breaks a local gauge symmetry. The consequence of this would be that observable quantities depend on the gauge, and therefore the theory makes no sense. Thus, such anomalies may not occur. There are two possibilities how such anomalies can be avoided. One is that no such anomalies occurs, i. e., the path integral measure must be invariant under the symmetry. The second is by anomaly cancellation, i. e., some parts of the measure are not invariant under the symmetry, but the sum of all such anomalous terms cancel. It is the latter mechanism which makes the standard model anomaly-free. However, the price to pay for this is that the matter content of the standard model has to follow certain rules. It is thus rather important to understand how this comes about. Furthermore, any chiral gauge theory beyond the standard model faces similar, or even more severe, problems.

Already the classical result (1.2) is already indicating that the current is only covariantly conserved,. The latter equation implies already that only for massless fermions there will be no gauge anomaly. However, this is not a problem, as only zero-mass fermions are admitted to the standard model anyway, and all apparent fermion masses are generated by the Higgs effect. But for the standard model this is still modified. Due to the parity violation, it is necessary to consider a current for left-handed and right-handed fermions separately, where the corresponding left-handed and right-handed covariant derivatives for the left-handed and right-handed currents appear.

In principle, it is possible to do the same one-loop calculation in a gauge theory, and the final result is quite similar. However, it may still be questioned whether this is an artifact of perturbation theory. It is not, and to show this it is useful to derive the local anomaly for gauge theories using a different approach. In a path integral approach, this becomes particularly clear, as it can be shown that the anomaly stems from the fact that the path-integral measure for fermions, $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is not invariant under chiral gauge transformations, and therefore the anomaly arises. It is, of course, invariant under vectorial gauge transformations, and thus theories like QCD need not to be considered, as will be confirmed below. This also shows that the anomaly is a pure quantum phenomenon, as the measure is part of the quantization process.

1.3.1 Anomalies as a quantum effect

To see that this is a relevant effect, it is important to remember how Ward identities are obtained in general. Any well-defined symmetry transformation should leave the partition function unchanged, i. e.

$$0 = \delta Z = \delta \int \mathcal{D}\phi e^{iS+i\int d^4xj\phi}, \tag{1.11}$$

where ϕ is for simplicity a non-Grassmann field, which changes under the transformation as $\phi \to \phi + \epsilon f(\phi, x)$, with f some arbitrary function and ϵ infinitesimal. Performing the variation yields

$$0 = \int \mathcal{D}\phi e^{iS+i\int d^4xj\phi} \int d^4x \left(i \left(\frac{\delta S}{\delta\phi} + j \right) f + \frac{\delta f}{\delta\phi} \right), \tag{1.12}$$

where the first two terms come from the exponent. At the classical level, the source term vanishes, and the derivative of the action just gives the equations of motion, yielding the classical Ward identities. The third term is new in the quantum theory, and gives the contribution of the Jacobian,

$$\det \frac{\phi + \epsilon f}{\delta\phi} = \det \left(1 + \epsilon \frac{\delta f}{\delta\phi} \right) \approx 1 + \epsilon \frac{\delta f}{\delta\phi} + \mathcal{O}(\epsilon^2).$$

This is a genuine quantum contribution. It will be the source of the anomaly. Here it also becomes evident that the term anomaly is actually a misnomer. There is nothing anomalous about them. They are just a quantum effect.

To obtain Ward identities from (1.12), it is sufficient to derive with respect to the source some number of times, and then set the sources to zero at the end, yielding

$$0 = \left\langle T\Pi_l\phi_l \frac{\delta f}{\delta\phi} \right\rangle + i \left\langle T\Pi_l\phi_l \frac{\delta S}{\delta\phi} f \right\rangle + \sum_k \langle T\Pi_{l<k}\phi_l f \Pi_{m>k}\phi_m \rangle . \tag{1.13}$$

In this way an anomaly surfaces in Ward identities in the full quantum theory. This also shows that an anomaly is not a perturbative effect, since this is an exact result. However, it is still possible that the Jacobian is actually one, and a deviation from one in the one-loop calculation is just an artifact of perturbation theory.

1.3.2 Full expression for the anomaly

To check this, rotate first to Euclidean time, by replacing $t \to it$ and correspondingly in all covariant quantities the time component by i-times the time component and in all contravariant quantities the time components by $-i$-times the time components. Then expand the fermion fields in orthonormal eigenfunctions ψ_n of the Dirac operator,

$$\begin{aligned} \psi(x) &= \sum_n a_n \psi_n(x) \\ \bar{\psi}(x) &= \sum_n \psi_n^+(x) \bar{b}_n, \end{aligned}$$

which satisfy

$$i\gamma_\mu D^\mu \psi_n = \lambda_n \psi_n \quad (1.14)$$

$$-i\gamma_\mu D^\mu \psi_n^+ = \lambda_n \psi_n^+. \quad (1.15)$$

This permits to rewrite the path integral as an infinite product of integrations over the coefficients,

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \Pi_m da_m d\bar{b}_m, \quad (1.16)$$

keeping in mind that these differentials are Grassmannian.

Now, a local chiral transformation $\beta(x)$

$$\psi \to e^{i\beta(x)\gamma_5} \psi,$$

then corresponds to a linear transformation of the coefficients

$$a_m \to C_{mn} a_n = a_n',$$

which yields the Jacobian

$$\Pi_m da_m' d\bar{b}_m' = \frac{1}{(\det C)^2} \Pi_m da_m d\bar{b}_m,$$

or, formally,

$$\mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \frac{1}{(\det C)^2} \mathcal{D}\psi \mathcal{D}\bar{\psi}.$$

This determinant can be rewritten as

$$\frac{1}{(\det C)^2} = e^{-2\operatorname{tr}\ln C} = e^{-2\operatorname{tr}\delta C}, \quad (1.17)$$

where in the last equality it was assumed that β is infinitesimal, and thus $C = 1 + \delta C$ is close to one. In this case, δC can be evaluated starting from

$$a_m' \psi_m = (1 + i\beta\gamma_5) a_n \psi_n$$

which can be reduced using the orthonormality of the eigenstates of the Dirac equation to

$$a'_m = \int d^4x \psi_m^+ (1 + i\beta\gamma_5)\psi_n a_n = (1 + \delta c_{mn}) a_n. \tag{1.18}$$

Inserting this result into (1.17) yields for the Jacobian of the infinitesimal transformation

$$J = \exp\left(-2i \int d^4x \beta \psi_m^+ \gamma_5 \psi_m\right), \tag{1.19}$$

where the trace has been evaluated.

Unfortunately, the expression, as it stands, is ill-defined. It is necessary to regularize it. A useful possibility to make the expression well-defined is by replacing the trace over the eigenstates as

$$\psi_m^+ \gamma_5 \psi_m \to \lim_{\tau\to 0} \psi_m^+ \gamma_5 e^{-\lambda_m^2 \tau} \psi_m, \tag{1.20}$$

where the limit has to be performed at the end of the calculation only. Expanding the Gaussian and using the relations (1.14-1.15), this expression can be rewritten as

$$\lim_{\tau\to 0} \psi_m^+ \gamma_5 e^{-\lambda_m^2 \tau} \psi_m = \lim_{\tau\to 0} \operatorname{tr}\left(\gamma_5 e^{-\tau(\gamma_\mu D^\mu)^+ \gamma_\nu D^\nu}\right). \tag{1.21}$$

The exponential can be rewritten as

$$(\gamma_\mu D^\mu)^+ \gamma_\nu D^\nu = -D_\mu D^\mu + \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] F_{\mu\nu}^a \tau_a. \tag{1.22}$$

The limit is still ill-defined. It is necessary to rewrite the expression in a suitable way. This is achieved by the heat-kernel regularization.

For a differential operator, here given by $\Delta = (\gamma_\mu D^\mu)^+ \gamma_\nu D^\nu$, it is possible to define a heat-kernel as

$$(\partial_\tau + \Delta_x) G(x, y, \tau) = 0 \tag{1.23}$$

$$G(x, y, 0) = \delta(x - y). \tag{1.24}$$

Which is solved by the formal expression

$$G(x, y, \tau) = e^{-\Delta_x \tau} = \sum_m e^{-\tau\lambda_m} \psi_m^+(y) \psi_m(x).$$

This is already the expression (1.21). Without proof, it can now be shown that this heat kernel can be expanded for small τ as

$$G(x, y, \tau) \to_{\tau\to 0} \frac{1}{(4\pi\tau)^2} \exp^{-\frac{(x-y)^2}{4\tau}} \sum_{j=0}^{\infty} a_j(x, y) \tau^j.$$

Inserting this expansion into (1.19) yields

$$\ln J = -2i \lim_{\tau\to 0} \frac{1}{(4\pi\tau)^2} \int d^4x\beta \sum_j \tau^j \mathrm{tr}\gamma_5 a_j.$$

For $\tau \to 0$, the first term does not contribute, as a_0 has to be equal to one because of the condition (1.24). Terms with $j > 2$ will be irrelevant, because of the powers of τ. This leaves only $j = 1$ and $j = 2$. For these terms follows from the requirement that the expansion satisfies (1.23) a descent equation

$$-\Delta a_{j-1} = j a_j.$$

Since $a_0 = 1$, a_1 can be obtained algebraically from (1.22). Since all resulting terms have at most two γ matrices, the trace will vanish. Similarly, for a_2 only those terms can contribute to the trace where at least four γ matrices appear, which implies only the term quadratic in $F_{\mu\nu}$ will contribute. Which is precisely what is necessary to cancel the pre-factor.

Thus, the remainder is just

$$J = \exp\left(-\frac{i}{32\pi^2}\int d^4x\beta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F_{a\rho\sigma}\right). \quad (1.25)$$

Hence, the Jacobian is non-trivial, and will contribute in the Ward identities (1.13). However, this is still a rather complicated expression, which does not yet look like the one-loop result.

That this coincides with the one-loop anomaly can be obtained by an explicit calculation. Since this was for the global case, take β to be constant. The integral can then be rewritten as

$$\int d^4x \mathrm{tr}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(iA^a_\nu\partial_\rho A^a_\sigma + \frac{2}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right). \quad (1.26)$$

Since the pertruabtive case was the Abelian case, the second term can be dropped. The first term is then for the global case just two external fields, e. g. playing the roles of the photon field in the pion decay, and two momenta in Fourier space, which, after relabeling, yield the desired one-loop expression. Hence, indeed the full and the one-loop anomaly coincide. In gauge theories there are also anomalies in box and pentagon graphs with an odd number of axial insertions, which are again one-loop exact.

To obtain the final result including all color factors requires then just an explicit calculation, inserting the Jacobian (1.25) into the Ward identity (1.13). This will yield (1.8) with (1.9) inserted.

1.3.3 Anomaly cancellation

However, for the standard model it is more interesting to consider the case that left-handed and right-handed fermions are coupled with different gauge fields. Due to the different sign of γ_5 in the corresponding projector, this will reemerge as a different sign of the anomaly, yielding

$$k^\rho T^{V^aV^bA^c}_{\mu\nu\rho}(p,q,k)=2mT^{V^aV^bP^c}_{\mu\nu}(p,q,k)+\frac{\operatorname{tr}\left\{\tau_L^a,\tau_L^b\right\}\tau_L^c-\operatorname{tr}\left\{\tau_R^a,\tau_R^b\right\}\tau_R^c}{2}\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma,$$

were L and R indicate the representation of the left-handed and right-handed fermions. As a consequence, the classical gauge symmetry is broken by the anomaly, and results will depend on the choice of gauge. This can be directly understood form this expression: the left-hand side should vanish, if there is no massless pseudo-scalar particle in the theory, which is true for the standard model. On the right-hand side, the first term will indeed do so, if the fermion mass is zero. This is already required due to parity violation in the standard model. But for the second term this is not obvious.

There are now two possibilities how to obtain an anomaly-free theory. Either, the theory is anomaly-free, if each of the remaining terms is individually zero, or they cancel. Indeed, the expression $\operatorname{tr}\{\tau^a,\tau^b\}\tau^c$, the so-called symmetric structure constant, is zero for all (semi-)simple Lie groups, except for SU($N\geq 3$) and U(1). Unfortunately, these are precisely those appearing in the standard model, except for the SU(2) of weak isospin. For the group SU(3) of QCD, this is actually not a problem, since QCD is vectorial, and thus[1] $\tau_L=\tau_R$, and the terms cancel for each flavor individually. Thus remains only the part induced by the hypercharge.

In this case, each generation represents an identical contribution to the total result, as the generations are just identical copies concerning the generators. It is thus sufficient to consider one generation. The right-handed contributions are all singlets under the weak isospin, and thus they only couple vectorially to electromagnetism, and therefore yield zero. The contributions from the left-handed doublet contain then the generators of the weak isospin, τ^a, and the electric charge $Q=\tau^3+\mathbf{1}y/2$. The possible combinations contributing are

$$\operatorname{tr}t^a\{\tau^b,\tau^c\} \quad (1.27)$$

$$\operatorname{tr}Q\{\tau^a,\tau^b\} \quad (1.28)$$

$$\operatorname{tr}\tau^aQ^2 \quad (1.29)$$

$$\operatorname{tr}Q^3. \quad (1.30)$$

[1]Actually, unitarily equivalent is sufficient.

The contribution (1.27) vanishes, as this is a pure SU(2) expression. The term (1.30) is not making a difference between left and right, and is therefore also vanishing. It turns out that (1.28) and (1.29) lead to the same result, so it is sufficient to investigate (1.29). Since the isospin group is SU(2), the anti-commutator of two Pauli matrices just gives a Kronecker-δ times a constant, yielding in total

$$\mathrm{tr}Q\{\tau^a,\tau^b\} = \frac{1}{2}\delta^{ab}\sum_f Q_f,$$

where Q_f is the electric charge of the member f of the generation in units of the electric charge. It has to vanish to prevent any gauge anomaly in the standard model, which is fulfilled:

$$\sum_f Q_f = (0-1) + N_c\left(\frac{2}{3}-\frac{1}{3}\right) = -1+\frac{N_c}{3} = 0.$$

Therefore, there is no gauge anomaly in the standard model. However, this is only possible, because the electric charges have certain ratios, and the number of colors N_c is three. This implies that the different sectors of the standard model, the weak isospin, the strong interactions, and electromagnetism, very carefully balance each other, to provide a well-defined theory. Such a perfect combination is one of the reasons to believe that the standard model is part of a larger theory, which imposes this structure.

1.4 Relation to topology

There is an interesting twist for the quantity making up the Jacobian

$$\frac{1}{64\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F_{a\rho\sigma} = -\frac{i}{512\pi^4}\int d^4x\mathrm{tr}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(iA^a_\nu\partial_\rho A^a_\sigma + \frac{2}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)$$

Evidently, this is a total derivative, and hence can be cast into a surface integral at infinity. It is therefore independent of the internal structure of the space-time it is integrated over, but depends only on the contribution from the boundary. Furthermore, the expression has the same color structure as the usual Lagrangian, and the Lorentz indices do not play a role in gauge transformations of the field-strength tensor. Hence, this quantity is gauge-invariant. Thus, it is an observable quantity. It is the so-called topological charge, or Chern class of the gauge field configuration. Furthermore, the quantity is evidently invariant under any continuous distortions of the gauge fields inside the volume. It is less obvious that this is true for any continuous deformations of the gauge fields on the boundary, and that all of these possible deformations fall into distinct classes, the so-called

Chern classes, such that the integral is an integer k, characterizing this class. This fact is stated here without proof.

Since this quantity was obtained from the chiral transformation properties of the fermions, it suggest itself that it is connected to properties of the Dirac operator, and this is indeed the case. This topological charge is equal to the difference of the number of the left-handed n^- and right-handed n^+ zero modes of the (necessarily in the present context massless) Dirac operator D_μ, $\gamma_\mu D^\mu\psi = 0$, called the index of the Dirac operator. This is the celebrated index theorem.

To see this, note first that because γ_5 anti-commutes with the other γ_μ it follows that that for any eigen-mode of the Dirac operator ψ_m to eigenvalue λ_m that

$$i\gamma_\mu D^\mu \gamma_5 \psi_m = -i\gamma_5\gamma_\mu D^\mu \psi_m = -\lambda_m \gamma_5 \psi_m.$$

Hence, every non-zero eigen-mode is doubly degenerate, and therefore the index is the same if all eigenmodes are included.

Start with an expression for this difference,

$$n^+ - n^- = \int d^4x \sum_{m,\lambda_m=0} \psi_m^+ \gamma_5 \psi_m.$$

The inserted γ_5 will guarantee the correct counting. It is possible to use a very similar trick as before when regularizing the sums when doing the path integral calculation in section 1.3.2. The additional eigenvalues can be added as

$$\int d^4x \sum_m \psi_m^+ \gamma_5 \psi_m e^{-\lambda_m^2 \tau},$$

as the γ_5 symmetry ensures that all added terms vanish. But this is precisely expression (1.20), and thus this will lead to the same result as in section 1.3.2. Thus, the final answer is

$$n^+ - n^- = k = \frac{1}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F_{a\rho\sigma}$$

Hence, the anomaly has a certain connection to the topology of the gauge-fields.

This is in as far remarkable as the topology of gauge fields is an intrinsic property of Yang-Mills theory, and thus existing without any fermions, and hence in anomaly-free theories. At the same time, anomalies also exist without gauge fields, e. g. in the form of global anomalies. They are tied to the path-integral measure for gauge theories. It is the unique property of the covariant derivative in the form of the Dirac operator for fermions which ties both effects together in the presented way. Other realizations than minimal coupling will not have this property, or at least in a different way. This connection is therefore deeply ingrained in the gauge formulation.

1.5 Witten anomalies

There is actually a further possible anomaly for fermions, the so-called Witten anomaly, which is also connected to the parity violation in the standard model. It is also a gauge anomaly, and has therefore to be canceled as well. This occurs in the standard model if the number of weak fermion states is even. This would not be the case, if, e. g., there would be a single triplet of fermions charged under the weak isospin. In technicolor theories, or other theories beyond the standard model, this is a constraint, as in such theories multiplets with an odd number of fermions may appear, e. g. when the chirally coupled fermions are additionally charged under different gauge groups or representations, leading to an odd number of fermions. This has then to be canceled by additional fermions. This is a problem exclusively applying to the Sp(N) gauge groups, and to SU(2) of the weak interactions because SU(2)≈Sp(1), as well as O($N < 6$) groups, except for SO(2).

The reason can be most easily illustrated by considering the path-integral with the fermions integrated out. For n Weyl fermions, the expression is

$$Z = \int \mathcal{D}A_\mu (\det i\gamma_\mu D^\mu)^{\frac{n}{2}} e^i S, \tag{1.31}$$

with S the usual gauge-field action. The problem arises, as it can be proven that for each gauge-field configuration of a gauge theory with an affected gauge group there exists a gauge-transformed one such that

$$(\det i\gamma_\mu D^\mu)^{\frac{1}{2}} = -(\det i\gamma_\mu D^{\mu\prime})^{\frac{1}{2}},$$

where $'$ denotes gauge-transformed. The proof is somewhat involved, but essentially boils down to the fact that the determinant has to be defined in terms of a product of eigenvalues. For Sp(N) gauge theories as well as the groups O($N < 6$) it can then be shown that there exist gauge-transformations, which are topologically non-trivial, such that one of the non-zero eigenvalues changes sign. Mathematically, the reason is that the fourth homotopy group of these groups is non-trivial and actually is Z_2 or Z_2^2. Hence, the integrand of the path integral (1.31) exists twice on each gauge orbit, but with opposite signs. Thus, the partition function vanishes, and all expectation values become ill-defined 0/0 constructs. Thus, such a theory is ill-defined, as there is no continous deformation of teh aguge group possible to introduce a suitable definition, similar to L'Hospital's rule.

In the standard model, the problem does not arise, because the number of Weyl flavors of the fermions is even since only Dirac fermions appear. One could also hope that, since the gauge group of the standard model is actually S(U(3)×U(2))≈SU(3)/Z_3×SU(2)/Z_2×U(1), this problem would not arise. The reason for this division is that only for this particular gauge group the matter field representation becomes single-valued, as is necessary for them to be meaningful. However, because $SU(2)/Z_2 \approx SO(3)$ instead of Sp(2), this does not help, as the fourth homotopy group of SO(3) is also non-trivial, and the problem persists,

Thus, adding further sectors to the standard model, or embedding it in a grand-unified theory, must respect this fact, to avoid triggering the Witten anomaly.