Advanced General Relativity and Quantum Gravity

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Chapter 1

Introduction

General relativity is often considered to be quite enigmatic, due to its separation from quantum physics. It is also often mystified due to it being linked to everyday experience like time, as it changes these. The aim of this lecture is to provide a link between quantum theories and general relativity, and at the same time demystify it.

For that purpose, the first part of this lecture will provide a more in-depth introduction to classical general relativity. Especially, it will highlight how general relativity, despite its appearance, is really a theory very similar to other gauge theories like Yang-Mills theories. What makes it look apart is that we are very much caught in a very special solution, and have not the possibility to play around with initial conditions, as in electrodynamics. To this end, special solutions, like black holes and the universe, will be discussed in some detail, to highlight what are physical concepts and what are just auxiliary constructions.

The other challenging question is how to quantize gravity. If one prefers to stick with the idea that quantum field theory is essentially the correct approach to nature, this becomes inevitable. This will be assumed here. What makes this an assumption, and quantum gravity somewhat more shaky than quantum field theory, are two aspects. The one is the total lack of experimental insights about what could be the nature of quantum gravity. This is not a problem of principle, but rather a consequence of the smallness of the parameters of classical general relativity. While it is unclear when this technical barrier can be surpassed, one of the currently best guesses are gravitational waves. They will therefore serve an an intermediary.

The other problem is that, while formally insufficient, standard perturbative quantization of even the simplest quantum gravity theory fails. The reason is the lack of perturbative renormalizability. Furthermore, many decisive concepts of quantum field theory, like four-momentum, requires a very different approach. This made quantization a challenge, as it either requires to perform some kind of non-perturbative quantization or switch to a different approach to quantization. Only within the last few decades, non-perturbative approaches started to yield convincing results. On the other hand, other approaches like loop quantum gravity and string theory, made also progress. But again, without any experimental insight and the classical limit being fairly generally reproducible, no decision is yet possible. Therefore, this lecture aims at giving a brief overview of the various possibilities.

As a consequence, there is a vast amount of literature available on the subject. As usual, the suitability is highly personal. For the sake of completeness, the following books and articles have been used in the preparation of this lecture:

- Ambjorn et al., "Nonperturbative Quantum Gravity", Phys. Rept. 519 p127 (2012)
- A. Asthekar et al., "Loop quantum cosmology: A status report", Class. Quant. Grav. 28 p213001 (2011)
- Freedman et al., "Supergravity", Cambridge
- Hehl et al., "General relativity with spin and torsion", Rev. Mod. Phys. 48, p393 (1976)
- Misner et al., "Gravitation", Freeman
- J. Polchinski, "String Theory", Cambridge
- Straumann, "General relativity", Springer

Chapter 2

General relativity

2.1 Kinematics

2.1.1 Manifold structure

As every theory, also general relativity can be split into a kinematical part, how the physical system is described, and a dynamical part, how things happen. The main difference is that the kinematical part is not embedded in an arena, e. g. the space-time of quantum field theory, but exists without any embedding. This is also one of the major conceptual challenges in understanding general relativity. There is nothing left to stand on and prepare the system. But this is not entirely true, as will be seen. However, what remains true is that general relativity does not allow for the notion of an outside of the system.

The kinematic structure of general relativity is a topological pseudo-Riemannian manifold $\mathbb{M}$. As a manifold, any element of the manifold m is in one-to-one correspondence to an element x of (a patch of) $\mathbb{R}^d$, $m(x) = x^{-1}(m)$ and $x(m) = m^{-1}(x)$ both exist and are well defined. The dimension d is the minimum required to allow for such a one-to-one correspondence. These elements and also their identifiers x will be both called events in this lecture. Moreover, the manifold structure requires that there exists a local isomorphism between the manifold and the $\mathbb{R}^d$ in the sense that overlapping subsets in the manifold are mapped to overlapping subsets in $\mathbb{R}^d$, where the overlap is identical, i. e. contains the same sets of m and x such that $m(x)$ holds for all elements in the overlap. This allows to make a statement about neighborhood relations, though there is not yet any notion of distance. An example of the concept is the surface of a sphere, where points on the surface are the elements of the manifold, and the angular coordinates θ and ϕ form a patch in $\mathbb{R}^2$ on which overlapping subsets are mapped. Note that due to the south pole and the north pole this is not a square patch, but sometimes single points are added along a line.

It is useful to introduce coordinates $X(m) = X(m(x)) = X(x)$ on a manifold. In particular, they can be chosen, e. g., to realize the open set structure in a coordinate language. E. g., for the sphere, coordinates could be chosen to be latitude and longitude. In section 2.2 it will be required that the manifold always allows to at least locally introduce a coordinate system, which is the same as that of Minkowski space-time, i. e. the manifold being locally Lorentzian.

Because only the overlapping set relation needs to be maintained, and the structure of the patch in $\mathbb{R}^d$ can be involved, it is in general not possible to express the coordinates on a manifold globally. To cover it, rather multiple coordinate systems are required, which overlap in both the manifold and the underlying $\mathbb{R}^d$. They are then related by transfer functions, which map the coordinates into each other, i. e. $X_2(X_1) = X_2(X_1^{-1}(x)) = X_2(x)$.

In the sphere case, this can e. g. take the following form. Latitude and longitude are ill-defined at either pole. So, a second coordinate system is needed. Labelling the elements of the manifold set by their coordinates in three dimensions, and the underlying $\mathbb{R}^2$ path by the angle $\phi \in [0, 2\pi)$ and $\theta \in [-\pi/5, 4\pi/5)$, the coordinate systems are given by

$$\begin{aligned} \vec{r} &= \begin{pmatrix} \cos\phi_1 \sin\theta_1 \\ \sin\phi_1 \sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \\ \vec{t} &= \begin{pmatrix} \cos(\theta + 2\pi/5) \\ \sin(\phi)\sin(\theta + 2\pi/5) \\ \cos(\phi)\sin(\theta + 2\pi/5) \end{pmatrix}. \end{aligned}$$

where the first coordinate system lives in the θ strip $[\pi/5, 4\pi/5)$, and the second one in $[-\pi/5, 2\pi/5)$. The overlap exist for the strip $(\pi/5, 2\pi/5)$. In this case, the transfer function is

$$\vec{t}(\vec{r}) = \begin{pmatrix} \cos\left(\cos^{-1}(r_3) + \frac{2\pi}{5}\right) \\ \sin\left(\tan^{-1}\frac{r_1}{r_2}\right)\sin\left(\cos^{-1}(r_3) + \frac{2\pi}{5}\right) \\ \cos\sin\left(\tan^{-1}\frac{r_1}{r_2}\right)\sin\left(\cos^{-1}(r_3) + \frac{2\pi}{5}\right) \end{pmatrix}. \tag{2.1}$$

In the relevant strip both the cosine and the tangens are single-valued. This transfer function between $\vec{r}_2$ and $\vec{r}_1$ is thus invertible and differentiable in the overlap. Thus, it is also possible to express $\vec{r}_2$ as a function of $\vec{r}_1$, and likewise, both ϕ and θ can be uniquely extracted, if so desired.

The minimal set of coordinate system to cover the whole manifold is called an atlas. Note that a coordinate system also offers a reparametrization, e. g. by a rotation. This freedom is independent and applies to each coordinate system separately. Finally, for the example the manifold has been embedded[1] in a higher-dimensional space. This made it easier to express the difference between the coordinates r_i and the underlying $\mathbb{R}^d$. In fact, in such an embedding a three-dimensional coordinate system could be introduced, x, y, and z, which would cover the whole sphere. However, it can be proven that this not possible in general. Hence, it is necessary to stick with transfer functions.

If the transfer functions are, as in the example, differentiable as a function of the parameters, the manifold is called differentiable. In general relativity the manifold will always be required to be differentiable.

The manifold is assumed to be topological, i. e., there exists as distance measure $d(m_1, m_2) = d(m_1(x_1), m(x_2)) \to \mathbb{R}$, which allows to determine a coordinate-independent distance between two elements of the manifold, and thus two events. This is called usually the invariant length element ds, and can be calculated in terms of coordinates, $ds = d(X(x), Y(y))$. It is not assumed that the distance measure is positive-definite, i. e. ds can have either signs. It may also be zero even for any pairings of events. However, it is required that $d(m, m) = 0$. Such a manifold is called pseudo-Riemannian. If $d \geq 0$ would be satisfied, it is called Riemannian.

What will be required axiomatically for general relativity, however, is that the distance measure is locally Minkowski, implying the local existence of a metric. This restricts the possible topological manifolds, as this requires the distance measure to be locally a linear form. This implies that for any pair of events x and y, there exists a δ

[1]A two-dimensional coordinate system would be, e. g., latitude and longitude on the some broad ring, and then shift the meridian for the other broad ring, such that in neither case the north-pole and south-pole of the respective meridian would be part of the covered sphere. However, to avoid confusing both angular systems, it is more useful for now to use the embedding, such that already the dimensionality distinguishes the underlying $\mathbb{R}^2$ and the coordinate system.

such that if $|x-y|<\delta$ in the Euclidean norm[2] then

$$\begin{aligned} d(X(x),Y(y)) &= (X_\mu - Y_\mu)\eta^{\mu\nu}(X_\nu - Y_\nu) \\ X &= x \\ Y &= y \end{aligned} \quad (2.2)$$

where the coordinates are locally introduced trivially and $\eta_{\mu\nu}$ is the usually Minkowski metric of special relativity. Thus, for any fixed coordinate system, this generalizes to the existence of a metric $g_{\mu\nu}(X)$

$$d(X,Y) = (X_\mu - Y_\mu)g^{\mu\nu}(X)(X_\nu - Y_\nu),$$

but $d(X,Y)$ independent of the choice of coordinate system. It is required here that $g_{\mu\nu}(X) = g_{\mu\nu}(Y)$. Thus distances are the same no matter from where it is started to measure. As will be seen in section (2.1.3), this further restricts the possible manifolds.

Such a metric is called compatible. The distances $d(X,Y)$ are called invariant distances. In addition, for the present lecture, it will be defined that $d(X,Y)<0$ is called space-like, $d(X,Y)>0$ is called time-like, and $d(X-Y)=0$ is called light-like. That requires $\eta = \text{diag}(1,-1,-1,-1)$. The local isomorphism always allows to select a patch to introduce a coordinate system without transfer functions in (2.2). This is no longer true if the distance is not infinitesimal. This will be discussed in section 2.1.3.

To give an example, consider again the sphere. Because the sphere is not a pseudo-Riemannian manifold but a Riemannian manifold, the local metric will be the Euclidean metric $\delta_{\mu\nu}$ instead. The distance measure will be chosen such that the distance between two elements of the manifold will be the line distance on the sphere. Thus, going the sphere around once on a grand circle, which will only be looked at in section 2.1.5, will yield a distance of 2π. This requires the usual length element

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \quad (2.3)$$

Consider first the first coordinate system. Because of the embedding, the necessary metric is

$$g^r(\theta,\phi) = \frac{1}{\cos^2\theta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\sin^4\theta}{\sin^2\phi} \end{pmatrix} \quad (2.4)$$

[2]By construction, there is always a sufficiently small patch in which the elements of the manifolds are isomorphic to the patch in the underlying Euclidean space.

which can be expressed in terms of the coordinates as

$$g^r(\vec{r}) = \frac{r_3^2}{r_1^2 + r_2^2 + r_3^2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\frac{(1-r_3^2)^2(r_1^2+r_2^2)}{r_2^2(r_1^2+r_2^2+r_3^2)^2}\end{pmatrix},$$

showing that the singularity is located along the 2-axis. Likewise, using the coordinate system $\vec{t}$ a metric is obtained where the angle θ would be shifted. Since in the line element (2.3) must remain the same in terms of the underlying $\mathbb{R}^2$, this yields

$$\begin{aligned}
g^t_{11} = g^t_{22} &= \frac{\sin^2\phi - \cos^2\theta\sin^2\theta\tan^2\left(\theta - \frac{\pi}{10}\right)}{\sin^2\left(\theta - \frac{\pi}{10}\right)\cos^2\phi - \cos^2(2\phi)}\\
g^t_{33} &= \frac{-\cot^2\phi + \sin^2\theta\left(\csc^2\phi + \tan^2\left(\theta - \frac{\pi}{10}\right)\right)}{\sin^2\left(\theta - \frac{\pi}{10}\right) - \cos^2\phi\sin^2\left(\theta - \frac{\pi}{10}\right)\tan^2\left(\theta - \frac{\pi}{10}\right)\left(\cot^4\phi - 1\right)},
\end{aligned}$$

and all other entries vanishing. This can again be expressed in terms of the coordinates, yielding an entirely coordinate-dependent expression. In particular, in terms of coordinates the invariant distance measure (2.3) differ, but always yield the same result. E. g. in the coordinate system r, the line element take the form

$$ds^2 = r_3^2(dr_1^2 + dr_2^2) - \frac{(r_1^2 + r_2^2)r_3^2(r_3^2 - 1)^2}{r_2^2(r_1^2 + r_2^2 + r_3^2)}dr_3^2 \tag{2.5}$$

and can be calculated now entirely in terms of the coordinates.

In a general manifold, (2.2) will only be possible in an infinitesimally small neighborhood. To extent the concept of distance requires the introduction of paths.

2.1.2 Parallel transport

An important concept is that of a path. A path is a continous sequence of events labeled by a real number τ, $m(\tau)$ or equivalently $x(\tau)$. This creates a path in the manifold $X(x(\tau)) = X(\tau)$. It may be necessary to switch the coordinate system along any finite path. However, for any path there will be a patch around $\tau = \tau_0$ in which a single coordinate system is sufficient. This allows to introduce the derivative of the path in the manifold as $\partial_\tau X(\tau)$, as here the difference of coordinates make sense. This defines a direction, and consequently can be used to attach locally a vector space with vectors X_μ, which is spanned by dimension linearly independent paths, a tangent space. If the number of dimensions of the tangent space is not everywhere the same and equal to the number of dimensions of the manifolds, this is called the a degeneracy or singularity of the manifold. Only at events like black holes such things may occur in general relativity.

Such tangent vectors will change under the change of the coordinate system by a transformation Λ to ΛX. Vectors which change in this way will be called covariant vectors and labeled by an upper index. Quantities, which change by the inverse Λ^{-1} will be called contravariant vectors, and labeled by a lower index.

However, the vectors are now defined in terms of a differential at a single point. Two vectors defined in the tangent space at different events will not be comparable, as they formally belong to different space. To compare vectors V at two different events requires to parallel transport them, taking into account the change of tangent space. While mathematically this remains a space of fixed dimensionality, the base vectors change by the parallel transport along a path, due to the change of directional differentials. Thus, they will only remain the same if these derivatives do not change. This is not the same as an arbitrary base transformation, as it depends on the path taken. Thus, what is necessary is to track how the base vectors change along the path.

Moving an infinitesimal distance, this can at most be an infinitesimal, and thus linear change, in the vector. The most general possibility is quantified by the affine connection Γ,

$$dV^\mu = \Gamma^\mu_{\nu\rho}(X) V^\nu dX^\rho. \tag{2.6}$$

To yield full sense, this equation needs to be divided by $d\tau$, and thus describes how a vector changes, when transported along the path $X(\tau)$. If the boundary of a coordinate system is traversed, this needs to be taken into account. However, it is always possible to first switch to a coordinate system, which covers both infinitesimally separate points. Thus, when traversing some distances which requires a change of coordinate system, the best choice is to move in one coordinate system until entering the overlap region, then stop and switch to the new coordinate system, and then move on in the second coordinate system. This avoids the need to express differentials across boundaries of coordinate systems, and is always possible due to the required overlap.

The affine connection Γ is a feature of the manifold. The path and coordinate system only enters in terms of dX^μ, $d\tau$, and dV^μ is determined by the application. Thus (2.6) cleanly separates the manifold, the coordinate system, and the object. Of course, as tensors of higher rank can be introduced into the tangent space, (2.6) can be generalized to tensors of arbitrary rank. This is done using the corresponding tensor product rules.

Note that the affine connection is, currently, independent from the topology, and thus not related to the metric. It is a measure of how the manifold bends along a path. The antisymmetric part of Γ

$$S^{\mu}_{\nu\rho} = \frac{1}{2}\left(\Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu}\right) \tag{2.7}$$

is Cartan's torsion tensor. Whether S is zero or not has far-reaching consequences, and is a property of the manifold in question. In general relativity proper, it is required to be zero. As

$$S^{\mu}_{\nu\rho}V^{\nu}dX^{\rho} = \frac{1}{2}\left(dV^{\mu} - \Gamma^{\mu}_{\rho\nu}V^{\nu}dX^{\rho}\right)$$

this shows that it describes the vorticity or torsion, of the manifold. This expression vanishes, if for the change dV^{μ} it does not matter whether a left-screw or a right-screw change is made.

Since (2.6) allows to quantify the change of a vector in a given direction, is thus allows to define a covariant derivative on vectors

$$D^{\mu}V_{\nu} = \left(\partial_{\mu}\delta^{\lambda}_{\nu} - \Gamma^{\lambda}_{\mu\nu}\right)V_{\lambda}, \tag{2.8}$$

For a scalar quantity, this reduces to

$$D^{\mu}\phi = \partial^{\mu}\phi,$$

as only the argument needs to be transported, but scalars have no directional properties, and are thus not affected. More general tensors, which are constructed by a tensor product $X_{\mu\nu} = V_{\mu}W_{\nu}$, this yields

$$\begin{aligned} D^{\mu}X_{\nu\rho} &= V_{\nu}D^{\mu}W_{\rho} + \left(D^{\mu}V_{\nu}\right)W_{\rho} = \partial^{\mu}X_{\nu\rho} - V_{\nu}\Gamma^{\lambda}_{\mu\rho}W_{\lambda} - \Gamma^{\lambda}_{\mu\nu}V_{\lambda}W_{\rho} \\ &= \partial^{\mu}X_{\nu\rho} - \Gamma^{\lambda}_{\mu\rho}X_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}X_{\lambda\rho}. \end{aligned}$$

and likewise for higher tensors. This defines the action of D_{μ} on tensors of arbitrary rank.

It is not necessary that when moving a vector along a closed path, it remains unchanged. This information is encoded in whether covariant derivatives can be exchanged for a differentiable function. On a flat manifold, i. e. one on which $\Gamma = 0$, any twice differentiable function will behave as $(D_{\mu}D_{\nu} - D_{\nu}D_{\mu})V = 0$. This is encoded in the Riemann curvature tensor defined as

$$\begin{aligned} \left[D_{\mu}, D_{\nu}\right]V^{\lambda} &= R^{\lambda}_{\rho\mu\nu}V^{\rho} \\ R^{\lambda}_{\rho\mu\nu} &= \partial_{\mu}\Gamma^{\lambda}_{\nu\rho} - \partial_{\nu}\Gamma^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\rho}. \end{aligned} \tag{2.9}$$

It therefore characterizes the manifold. As the Riemann tensor is given entirely in terms of the affine connection, it does not create another independent characterization of the manifold, but it is a combination often useful.

It should be noted that at this point, these are general statements. As the manifold is not specified, it is not yet possible to calculate any of these quantities. Before switching to examples, it is, however, useful to first consider another concept, distances.

2.1.3 Distances

Distances were already introduced as a local concept within an infinitesimal ball. Combining them with parallel transport, it is possible to extend the concept. Given (2.6), an infinitesimal distance can be defined as

$$ds^2 = g_{\mu\nu}(X)dX^\mu dX^\nu, \tag{2.10}$$

where dX^μ are the infinitesimal distances between X and a neighboring element of the manifold in direction μ. The transformation properties under coordinate changes implies that there exist an inverse metric, based on the contravariant vectors, satisfying

$$g^{\mu\nu}(X)g_{\nu\rho}(X) = \delta^\mu_\rho, \tag{2.11}$$

locally. The condition (2.2) implies that there exists always a coordinate transformation that at some specified point X $g_{\mu\nu}(X) = \eta_{\mu\nu}$ and $g^{\mu\nu}(X) = \eta^{\mu\nu}$. However, in general at other points $X' \neq X$ the metric will not be Minkowski. Because coordinate transformations are required to be invertible, the sign of the determinant is coordinate-independent. This implies that $\operatorname{sgn}\det g = \operatorname{sgn}\det\eta$. However, a general coordinate transformation can include a rescaling, and thus $\det g$ is not invariant.

Due to the necessity of the metric to transform under coordinate transformations appropriately to maintain (2.11) invariant, this implies that the metric can be used to raise and lower indices as

$$\begin{aligned} g_{\mu\nu}V^\nu &= V_\mu \\ g^{\mu\nu}V_\nu &= V^\mu. \end{aligned}$$

This holds true for any quantity defined using (2.6), including the affine connection and the Riemann curvature tensor.

For a general manifold, the covariant derivative of the metric will not be zero,

$$Q_{\mu\nu\rho} = D_\mu g_{\nu\rho}.$$

Classical general relativity requires this tensor of non-metricity Q to vanish. The metric is then covariantly constant. If this would not be the case, measuring distances would depend on the point of reference from where to measure. General relativity is based on the experimental result that this is not the case, and thus requires the tensor of non-metricity to vanish as an empirical requirement. This restricts the class of possible manifolds. Thus, the direction of traversing a path will not change the invariant distance.

Now, consider finite distances. There is no a-priori reason why a finite distance will in general not be dependent of the path taken. The total distance traversed can be obtained by moving from point to point along a path $X(\tau)$, this yields that the total distance traveled along the path is given by

$$s=\int ds=\int\limits_{\tau_0}^{\tau_1}\frac{ds(\tau)}{d\tau}d\tau=\int\limits_{\tau_0}^{\tau_1}d\tau\sqrt{g_{\mu\nu}(X(\tau))\frac{dX^\mu(\tau)}{d\tau}\frac{dX^\nu(\tau)}{d\tau}}, \tag{2.12}$$

which generalizes the eigentime. It is invariant under reparametrizations of the parameter τ. To show this invariance, take an arbitrary (but invertible) reparametrization $\tau'=\tau'(\tau)$. This implies

$$\begin{aligned}\dot{\tau}' &= \frac{d\tau'}{d\tau}\\ d\tau &= \frac{d\tau'}{\dot{\tau}'},\end{aligned}$$

yielding the transformation property of the integral measure. For the derivatives follows then

$$\dot{X}^{\mu'}(\tau')=\dot{X}^{\mu}(\tau)\frac{d\tau}{d\tau'}=\dot{X}^{\mu}\frac{1}{\dot{\tau}'}$$

Hence the scalar product changes as

$$\dot{X}^{\mu\prime}\dot{X}_{\mu\prime}=\frac{1}{\dot{\tau}'^2}\dot{X}^\mu\dot{X}_\mu.$$

One power of $\dot{\tau}'$ is removed by the square root, and the remaining one is then compensated by the integral measure. This implies that without the square root the result would not be independent. As usual, this requires that also the integral limits $\tau_{0,1}$ are transformed.

The expression (2.12) is a path-dependent distance. To better understand the implication of this, consider the case of Minkowski metric, and various paths. Start with the path

$$r=(\tau,0,0,0)^T$$

and always $\tau_0 = 0$ and $\tau_1 = 1$. This path is a time-like vector for every value of τ. This yields $s = 1$. If instead the path

$$r = (\tau^2, 0, 0, 0)^T$$

is chosen, but with the same τ_i and thus not being a reparametrization, the result is 3/4. Because the path is traversed at a different rate, the total elapsed eigentime, and thus time-like distance, changes[3].

Consider a path, which is light-like at every value of τ,

$$r = (\tau, 0, 0, \tau)^T.$$

The distance is zero, as the integrand is zero. The distance along a light-like path always vanishes. Again, this can be understood as a light-like vector can never be transformed into a rest frame, and thus no (eigen)time elapses for an object traveling along a light-like trajectory.

Something odd happens for a space-like path,

$$r = (0, 0, 0, \tau)^T.$$

Formally the expression (2.12) becomes imaginary. The reason is that when taking the root an assumption was made on the sign of the invariant distance. A meaningful result is obtained by using

$$s = -\int_{\tau_0}^{\tau_1} d\tau \sqrt{-g_{\mu\nu}\frac{dX^\mu(\tau)}{d\tau}\frac{dX^\nu(\tau)}{d\tau}},$$

taking instead. This yields the expected space-like distance -1 for the path.

What happens if a mixed path is chosen, which is partially time-like and partial space-like (or light-like)? In that case the argument of the root changes sign, leading to a seemingly complex distance. That can be partially avoided by stitching the total distance together from piece-wise definite behavior. While this formally possible, this is actually a conceptually quite non-trivial issue. A pure time-like path is physically sensible, as it determines the causal evolution along a world-line. A purely space-like path answers the question of distance of an observer at fixed eigenzeit. A stitched path will first move along a space-like direction and then hitch onto a worldline, and that possibly multiple times. There is therefore rarely a physics reason to consider this issue.

[3]This is also the resolution to the so-called twin paradox.

Switching back to the general case, this implies that the character of the distance is not only influenced by the path, but also how the metric changes along the line. However, the character of the distance can also be fixed when choosing locally coordinates such that the metric becomes the Minkowski metric. It does not appears sensible to change the nature of the distance, time-like, space-like or light-like, because of this. This leads to the concept of diffeomorphism symmetry.

2.1.4 Diffeomorphism symmetry

It is worthwhile to understand better what is actually meant when talking about coordinate transformations in the present case. Coordinate transformations on the manifold can be arbitrary, $X'(x) = X'(X(x))$, but needs to be invertible, $X(x) = X(X'(x))$. Since the manifold is differentiable, these transformations need to be differentiable themselves, though they may become involved due to the need for transfer functions. Distances, and thus the topological manifold structure, are required to remain invariant. This symmetry is hence called a diffeomorphism symmetry.

Now, any such transformation can necessarily be written as

$$X'(x) = X'(X(x)) = X(x) + \epsilon(X(x)), \tag{2.13}$$

and all changes are encoded in the event-dependent function ϵ. Likewise, the inverse transformation is then given by $-\epsilon(x)$. This also implies

$$\frac{\partial X'}{\partial X} = 1 + \frac{\partial \epsilon}{\partial X},$$

and thus the derivatives of ϵ define the deviation from no transformation.

The important insight is, however, that (2.13) is a local translation. Thus, general coordinate transformation is really a gauging of the global translation symmetry. It appears as if the Lorentz symmetry of the Poincaré group has simply vanished. This appears very strange, especially as the Lorentz group plays an important role for the spin of particles. The latter issue will be postponed until section 2.4.

As for the other, it should be noted that $\epsilon(X)$ can be arbitrarily split as

$$\epsilon_\mu(X) = \Lambda_{\mu\nu}(X)X^\nu + \delta(X). \tag{2.14}$$

Thus, any local translation can always be decomposed into a local rotation and a local translation, but not uniquely. However, this implies that the global Poincaré group is a subgroup of the local coordinate transformations.

However, the idea should not be that special relativity emerges as the limit of (2.14). As general relativity is a classical theory, the structure of the manifold is fixed as function of the initial conditions, in a certain sense, see section 3.1. Thus, special relativity should rather be interpreted as the case where the manifold determined by the initial conditions has a trivial atlas and has invariant distances such that $g = \eta$ can be introduced globally[4]. Because the Minkowski metric is invariant under Poincaré transformations, this symmetry emerges then as a symmetry of the obtained manifold. Thus, the structure of special relativity is a consequence of the dynamics to be introduced in section 2.2. This makes the situation quite different from the classical mechanics limit of special relativity, which is purely kinematical and not affected by the dynamics.

It that sense, diffeomorphism symmetry can be considered to be the gauge symmetry of translations. The gauge-invariant observables, analogues to the electric field and magnetic field of classical electrodynamics, are then the invariant distances, i. e. the topology of the manifold. As will be seen when introducing the dynamics, the metric can be given the role of the gauge field, though it is a rank two tensor now, rather than a rank one tensor in electrodynamics.

Note that this implies that the metric cannot be invariant under a diffeomorphism symmetry. In fact, since distances need (2.10) need to be invariant, it follows that

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu \stackrel{!}{=} g'_{\mu\nu} dX^{\mu'} dX^{\nu'}.$$

The coordinate differentials transform as

$$dX'^\mu = \Lambda^\mu_\nu dX^\nu = \frac{\partial X^\mu}{\partial X^\nu} dX^\nu = \left(\delta^\mu_\nu + \frac{\partial \epsilon^\mu}{\partial X^\nu}\right) dX^\nu$$

requiring

$$g'_{\mu\nu} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu g_{\alpha\beta} \tag{2.15}$$

and likewise its inverse. This is reminiscent of the situation in special relativity. The only difference is that Λ is not a Lorentz transformation, but a diffeomorphism, and that this is a local statement. It is, however, often more convenient to let g transform as Λ and dX^μ with Λ^{-1}, which is a mere shift of definition of Λ.

This resolves also the observation from the end of section 2.1.3. The character of the invariant distances traversed by a path remains invariant under a diffeomorphism symmetry. It is thus always possible to attribute to a path, or at least to parts of path, uniquely whether it is time-like or space-like. It is thus a feature of the path, not the

[4] Of course, in reality this will only be approximately true.

metric, just like in special relativity. The metric, in turn, is dictated by the manifold, and describes how 'fast' paths can be traversed, and thus how much eigentime expires, e. g., when moving from one place to another.

2.1.5 Geodesic distances

So far, distances have been introduced as a path-dependent concept. In fact, the non-vanishing Riemann tensor (2.9) shows that the concept of distance can no longer be path independent on an arbitrary manifold. Thus, the distance between two-events, no matter when it is time-like or space-like, is no longer uniquely defined. In particular, the (absolute) value can often be made arbitrarily large. E. g., on a sphere dense full orbits can be performed on a path between two elements, and therefore the distance can be made arbitrarily large.

It is, however, possible to pose a different question for purely time-like paths and purely space-like paths: What is the path yielding the shortest time-like (space-like) distances between two events and is it a space-like path or a time-like path? The same question does not make sense for purely light-like paths, as their distance is always the same, identically zero. Such shortest path are called geodesics. Again, they do not need to be unique, and their may situations arise, in which it is possible to make the path arbitrarily short or that simply no path between two events exist, or no path of a given characteristic.

Thus, this is a two-step process. The first question is, whether a purely time-like (space-like) path can be found between two events. The second is, what path creates the shortest distance. Both questions will depend, in general, on the manifold and the events in questions. Assume for the moment that a time-like path exists. The question is then which path minimizes the distance (2.12)? The structure is the same as of extremalization of a classical Lagrangian, and thus can be solved using the variational principle. Due to the square root, this is cumbersome. Doing so yields the Euler-Lagrange equations

$$\begin{aligned} 0 &= \frac{1}{d(\tau)} d_\tau \frac{g_{\sigma\nu} \frac{dX^\nu}{d\tau}}{d(\tau)} - \frac{1}{2d(\tau)^2} \frac{\partial g_{\mu\nu}}{\partial X^\sigma} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \\ d(\tau) &= -g_{\alpha\beta} \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau}. \end{aligned}$$

This equation can be recast as

$$d_\tau^2 X^\beta + \frac{g^{\beta\sigma}}{2} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) d_\tau X^\mu d_\tau X^\nu = 0.$$

This gives a conditional equation for a fixed metric. I. e., for a fixed metric, a path between two fixed events need to satisfy this condition to minimize the time-like distance. For g space-time independent, this yields the straight lines of special relativity.

This result can be brought into connection with the parallel transport (2.6). If the vector to be parallel transported is dX^μ itself, this yields

$$d^2X^\mu + \Gamma^\mu_{\nu\rho} dX^\rho dX^\mu = 0. \tag{2.16}$$

But this implies that if a vector should be transported along a geodesic then necessarily

$$\Gamma^\beta_{\mu\nu} = \frac{g^{\beta\sigma}}{2} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) = 0 \tag{2.17}$$

needs to hold. But because the transport cannot depend on the path, this implies that the Christoffel symbols need to be indeed always of this form. There is one catch, though. If (2.17) holds, then Cartan's torsion tensor (2.7) identically vanishes. The reason is the implicit assumption that the statement of transporting along the line of minimal length is the same as parallel transporting. If Cartan's torsion tensor does not vanish, then both are not the same, and moving along a geodesic is not parallel transporting a vector, and both differ by the fact that the right-hand side of (2.17) is not zero. This makes torsion a bit less obvious, and it will be picked up in section 2.4. For the moment, assume that Cartan's torsion tensor vanishes. This is the assumption of general relativity, and is experimentally consistent so far.

Intuitively, what happens is that when moving along a minimizing curve, this is not sufficient to entirely fix the orientation of a vector in directions, suitably defined, transverse to it. If the manifold has zero torsion, there is no preference. The relation (2.17) can be used to restrict the changes to be as minimal as possible. In the presence of torsion, this is not possible, as torsion will uniquely specify the necessary changes, and then (2.17) does no longer hold.

2.2 Dynamics in pure gravity

This sets the stage for general relativity. General relativity is a theory defined on a topological manifold. The topology is such that that it is locally the one of Minkowski space-time, and the induced metric has vanishing torsion and vanishing non-metricity. Thus, (2.17) holds and $D_\mu g^{\mu\nu} = 0$. The manifold is fully specified by giving $d(x, y)$ for all pairs of events.

There is a continuously infinite number of such manifolds. To make it a predictive theory requires to provide a dynamic which allows to connect a (sufficient number) of measurements of the manifold at events with predictions of the manifold elsewhere. This is provided by the Einstein equations. The Einstein equations can, in principle, be formulated entirely in terms of the distances $d(x, y)$. This will play an important role later in section 4.4. This is the so-called Regge calculus. However, in practice, this is often not convenient. It is better to formulate the dynamical principle in terms of the metric.

Given the Riemann tensor (2.9), define the Ricci tensor and curvature scalar as

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha} = g^{\alpha\beta} R_{\beta\mu\nu\alpha} \tag{2.18}$$

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}, \tag{2.19}$$

respectively. Furthermore, define the Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.20}$$

The dynamical equation, known as Einstein equation, is then given by

$$G + \Lambda g = 0. \tag{2.21}$$

Due to the symmetry of g, these are (in four dimensions) ten second-order partial differential equations for g. The quantity Λ is called the cosmological constant. It should be noted that at $\Lambda = 0$ the Minkowski metric $g = \eta$ is a solution to this equation. Moreover, the vanishing of non-metricity and torsion further are constraint equations, which reduce the total number of independent degrees of freedom to only two in four dimensions, and zero in less dimensions.

In principle, the only thing necessary is then to specify sufficiently many initial conditions and then solve the partial differential equations. This turns out to be even conceptually challenging. Thus, solving this equation will be postponed to chapter 3. Of course, if need be, also transfer functions will be involved.

This is it. The entirety of general relativity without matter in encoded in (2.21). It should be noted that, in the absence of matter, Newton's constant does not enter into (2.21). In particular, without cosmological constant, no dimensionful parameter appears, and general relativity is scale-invariant. The cosmological constant breaks this scale invariance and provides an intrinsic scale to the manifold.

It should be noted that the equation (2.21) is covariant constant. The second term is by requirement of the non-metricity to vanish. The first term needs tedious explicit calculation, but finally $D_\mu G^{\mu\nu} = 0$ is found, basically as a consequence of the vanishing non-metricity of the metric. This actually also singles out the Einstein tensor. It is the only second-rank, torsion-free and metric tensor linear in the Riemann tensor. This singles out (2.21) as the unique covariant constant dynamical equation constant in the Riemann tensor. In a sense, it is thus the simplest dynamical equation allowing to determine a manifold itself by initial conditions. Empirically, it is indeed sufficient to describe all phenomena observed so far in general relativity, and which could be uniquely attributed to gravity.

However, it is possible to write down other covariantly constant equations, which are consistent with all observations so far. These are, however, no longer linear in the Riemann tensor, and usually also involve more than just one constant. Still, mathematically they serve the same purpose. Physically, this is an empirical choice. However, at the quantum level, they differ in general, as will be discussed in section 4.3.

2.3 Dynamics in the presence of matter

Pure general relativity (2.21) is a non-trivial theory by itself, similar to Yang-Mills theory. This will be investigated in more detail in chapter 3. However, as a theory of nature, it should also provide coupling to matter.

Covariant constancy has an important implication. For a scalar quantity, it reduces to an ordinary derivative. Thus, it implies that a scalar quantity is constant on the whole manifold. Covariant constancy ensures that this statement also makes sense for tensorial quantities. The physically same vector, i. e. suitable parallel transported, is constant everywhere, if it is covariantly constant. This is well motivated empirical, as no spontaneous generation from nothing is observed. Thus, it is necessary that matter is coupled also in a covariantly constant way to (2.21).

This requires a symmetric rank two tensor, which on Minkowski manifold is conserved in terms of the ordinary derivative. A suitable quantity is the energy-momentum tensor $T_{\mu\nu}$. This leads to the dynamical principle for general relativity with matter,

$$G + \Lambda g = 8\pi\kappa T \tag{2.22}$$

where κ is an additional constant, related to Newton's constant. It should be noted that if distances are given in units of either the cosmological constant or Newton's constant to a suitable power, one of both could be eliminated from (2.22). This will not be done here, as it is useful to characterize non-trivial self-couplings[5] of gravity by Λ and of gravity to matter by κ. In particular, the limit $\kappa \to 0$ is the decoupling of matter from gravity, the situation for matter acting non-gravitational on a fixed, but potentially non-trivial, background manifold. This is the situation of, e. g., ordinary (quantum) field theory when $g = \eta$. Choosing in such a setting $g \neq \eta$ yields (quantum) field theory on curved backgrounds.

As with the left-hand side, their is no general principle to choose T. Rather, any T with the above mentioned properties will do. Eventually experiment allows to constraint and then fix T. In addition, there may exist further equations, which relate the building blocks of T with each other, e. g. Maxwell's equations. While the metric will in generally enter these equations, they are not uniquely fixed by (2.22). They again need to be supplied externally. Some examples will be given in section 2.5, where also a Lagrangian formulation for (2.22) will be given.

2.4 The vielbein formalism and torsion

At the moment, the Lorentz group is absent from the formulation. That appears particularly problematic when considering spin. It is necessary to define a notion of spin in a suitable way before being able to add non-scalar matter. Again, there are more than one possibility to do so. At the moment, and in contrast to (2.22), there is no sufficiently convincing experimental evidence to favor any solution. The one presented here is hence an example. But is one of least complexity and one which opens a straightforward way for quantization.

The key to this is the tangent space. Consider at some point in the manifold m linearly independent[6] paths $X_\mu(m(\tau_\mu))$. Build a coordinate system at this point, which consists out of the vectors defined by the derivatives at this point given by a parameter vector τ, $E_\mu = \partial_{\tau_\mu} X_\mu(m(\tau_\mu))$. The E_μ form then a basis of a vector space at this point. These are often denoted as ∂_μ, as they are really directional derivatives. The set of all such tangent spaces located at every element of the manifold is called the tangent bundle.

[5]If either would be eliminated, the same information would be obtained from the relative sizes of both fields.

[6]Note that in general it may happen that manifolds have degenerate points at which the dimensionality is lower than that of the manifold, if locally not enough independent directions exist. However, with the additional restriction of the topology allowing for a non-degenerate metric, this is forbidden.

It is possible to define a linear transformation by a matrix e_i^μ such that $e_i = e_i^\mu E_\mu$ are a set of orthonormal basis vectors in the tangent space. This requires

$$\begin{aligned}
e_i^\mu e_\mu^j &= \delta_j^i \\
e_i^\mu e_\nu^i &= \delta_\nu^\mu \\
g^{\mu\nu} &= e_i^\mu \eta^{ij} e_j^\nu \\
\eta_{ij} &= e_i^\mu g_{\mu\nu} e_j^\nu \\
e_i e_j &= e_i^\mu e_j^\nu E_\mu E_\nu = e_i^\mu e_j^\nu g_{\mu\nu} = \eta_{ij}.
\end{aligned} \tag{2.23}$$

that is the matrix e transforms the metric locally into the Minkowski metric η. The inverse of e_i^μ yield the basis vectors $\bar{e}^\mu = e_i^\mu dx^i$, which provide infinitesimal movements tangential to the tangent space, yielding the cotangent space. The latter is again attached at every point, and all of them together form the cotangent bundle.

The matrix e is known as the vierbein or tetrad. In dimensions different than four it is also called the vielbein. Since e is by construction invertible this allows to express any tensor either in terms of the coordinates X_μ and non-orthogonal basis vectors E_μ and the metric $g_{\mu\nu}$ or using the orthonormal basis vectors e_i and the tetrad e_μ^i or any mixtures of it.

Because of (2.15) and (2.23) it follows that under a diffeomorphism transformation

$$e_i^{\mu'} = \Lambda_\nu^\mu e_i^\nu.$$

Hence, the tetrad transforms like a vector, and especially forms a linear representation of the diffeomorphism invariance. But (2.23) allows to additionally have locally

$$e_i^{\mu'} = \lambda_i^j e_j^\mu$$

as long as

$$\lambda_i^j \eta^{ik} \lambda_k^l = \eta^{jl}$$

holds locally. But then λ is a local Lorentz transformation. Thus, the introduction of the tangent space added an additional local symmetry, a local Lorentz symmetry.

As long as no matter is present, this allows to recast (2.21) entirely in the tangent space. As a consequence, diffeomorphism symmetry is eliminated entirely in favor of local Lorentz symmetry in the tangent space. There is, at this level, no possibility to eliminate both simultaneously locally. But it is possible to shift them. Eliminating them entirely is in principle also possible, but this would turn (2.21) into an integro-differential equation.

That both can be exchanged can be traced back to the fact that the Christoffel symbols Γ are given by (2.17), and can thus ultimately be expressed in terms of the tetrads as well. This is no longer true if Cartan's torsion tensor (2.7) does not vanish. To understand this, introduce the spin connection

$$\Gamma^{\mu}_{ij} = e^{\rho}_{i} e^{\sigma}_{j} \Gamma^{\mu}_{\rho\sigma}.$$

This quantity transforms nontrivial under both the local Lorentz group and the diffeomorphism group. Expressing Cartan's torsion tensor by it yields

$$S^{\mu}_{\rho\nu} = \frac{1}{2}\Gamma^{\mu}_{ij}\left(e^{i}_{\nu} e^{j}_{\rho} - e^{i}_{\rho} e^{j}_{\nu}\right).$$

Expressing the spin connection in terms of g using (2.17) yields

$$\Gamma^{\beta}_{ij} = \frac{e^{\mu}_{i} e^{\nu}_{j} g^{\beta\sigma}}{2}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right).$$

Because the torsion-free spin connection is symmetric in i and j and is contracted with an antisymmetric object in i and j in the torsion tensor, this show that indeed torsion will invalidate (2.17). But this implies that the spin connection transforms independently from the diffeomorphism symmetry, as it needs now to be non-vanishing, showing how both quantities decouple. In fact, the spin connection becomes an independent object. However, because it is introduced in the tangent space, and Einstein's equation (2.21) are completely formulated in the manifold, it is actually not a dynamical entity. It requires an alteration of these equations to make it dynamical. While this is possible classically, all possibilities come usually with an additional power of Newton's constant, making them far beyond any possibility to measure at the current time. Thus, while logically possible, space-time manifolds with non-trivial torsion are just one more possible extension of general relativity, which do not have experimental support. Thus, they will be neglected for now.

There is, however, even in a torsion-free space-time a very useful application of the spin connection, as it makes the concept of spin transparent. At the moment, only the parallel transport of tensors has been addressed using the covariant derivative (2.8). This restricts to objects which are associated with the translational symmetry. But describing particles requires spin. Spin in flat space-time is identified by representations of the Lorentz group, which has been absent so far. But this can now be addressed. Since now the Lorentz group has been recovered, the assignment of spin can be performed as in particle physics, just that the spin of matter is associated with the tangent space, rather than with the manifold.

Introducing the generators f of the Lorentz group, they can be arranged into an antisymmetric matrices of boost and rotation operators. A quantity ϕ carrying a linear representation indices r, e. g. Dirac indices, will then transform under an infinitesimal Lorentz transformation as

$$(\delta^{rs} + \omega^{ij} f_{ij}^{rs})\phi_s,$$

where ω are the space-time dependent transformation functions. This corresponds therefore to an infinitesimal Lorentz gauge transformation. Note that the contracted indices of the generators and the parameters belong to the tangent space. The entire transformation is defined there. As a consequence, there is also no way how the Lorentz transformation properties, or even the fields, can be defined in the original coordinates. The quantity ϕ_s, e. g. a Dirac spinor, makes only sense in the tangent space. To eliminate again the tangent space and return to the manifold requires to build Lorentz-invariant quantities, e. g. $\phi_s^\dagger \phi^s$, which again are well-defined outside the tangent space. It should be noted that the Lorentz indices r and s are not tangent space indices, and can therefore not be translated using the vierbein.

A further consequence is that the covariant derivative now needs to take care of the comparability of the local Lorentz group. Thus, the covariant derivative becomes

$$D^\mu_{rs}\phi^s = \left(\delta^{rs}\partial^\mu + \Gamma^\mu_{ij} f^{ij}_{rs}\right)\phi^s. \tag{2.24}$$

The so-created object is now a tensor, with one index from the coordinate space and one from the tangent space. It should be noted that, despite that ϕ^s does not carry a coordinate index μ, it does change under a diffeomorphism transformation, as it depends on coordinates. In particular in infinitesimal form

$$\Lambda\phi = (1 + \epsilon^\mu(X)\partial_\mu)\phi(X)$$

where ϵ is the local translation of (2.13), generated by the derivative as the generator of local translation. The affine connection does not appears because ϕ is scalar with respect to the coordinate system on the manifold. Of course, mixed tensors would be possible, but do not add further conceptual complications.

The usual spin of flat-space (quantum) field theory arises, because the metric g becomes constant, $g = \eta$. As a consequence, so do the tetrads, $e^a_\mu = \delta^a_\mu$. Thus, both the Lorentz transformation and the translations of (2.14) become global transformations.

However, a global Lorentz boost in coordinates, Λ_μ, will alter the tetrads, $\Lambda^{\mu\nu}e_\mu$. To maintain $e^a_\mu = \delta^a_\mu$, this requires also a compensating global Lorentz transformation in the tangent space, $\left(\Lambda^{-1}\right)^{ab} e_b$. Therefore, objects carrying a Lorentz index in local orthogonal tangent space need to transform in the same way as those carrying a tensor index in the coordinate space. After all, the tetrads arose from the introduction of an local coordinate change, and this needs to be maintained once having only global rotations alone. Thus, both groups are thereby reduced to a diagonal subgroup, thereby yielding the single Poincaré group of special relativity.

2.5 Lagrangian formulation

For the purpose of a path integral formulation, as well as for manifest coordinate-independent approaches to classical general relativity, it is useful to recast the tensorial Einstein's equation (2.21) in a Lagrangian form. Einstein's equation will then reemerge from a variational principle. It is thus very similar to the action of Maxwell theory, though with the added complication of diffeomorphism symmetry.

The probably most important change is that d^dX is not an invariant volume element. Because diffeomorphism symmetry allows also for a rescaling of coordinates, volume can get rescaled as well. This change needs to be offset. In fact, this can be compensated by the covariant volume element $d^dX\sqrt{-\det g}$. This quantity transforms like an inverse density. Thus, using a Lagrangian density $\mathcal{L}$, the action takes the form

$$S = \int d^dX\sqrt{-\det g}\mathcal{L}. \tag{2.25}$$

A suitable Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2\kappa}\left(R + l\right)$$

where Newton's constant is introduced as a rescaling for latter convenience. Thus, the total action is the manifold-integral of the sum of curvature and the cosmological constant. Note that in Minkowski space-time $R = 0$ and $l = 0$, and the action vanishes identically. Thus, it is necessary to first perform the variational principle before making a choice to obtain Einstein's equation, as usual. This Lagrangian is known as the Einstein-Hilbert Lagrangian. Furthermore, it should be noted that the Lagrangian is quartic in the fields and contains derivatives up to order two. It is thus the equivalent of the field strength tensor squared of Yang-Mills theory, and demonstrates that even pure gravity is self-interacting, and thus a non-trivial theory.

Since R is a curvature density, this implies that Einstein's equation extremalizes total curvature on the manifold. At $l = 0$, a possible extrema is zero, and thus Minkowski space-time is a solution to this problem. Solutions at non-zero l will be discussed in chapter 3.

Adding matter yields

$$\mathcal{L} = \frac{1}{2\kappa}(R + l) + \mathcal{L}_M \tag{2.26}$$

where $\mathcal{L}_M$ is the minimally coupled matter Lagrangian. Similar to the flat space-time gauge theories, this implies for a free, massless scalar field

$$\frac{1}{2}\partial_\mu\phi\partial^\mu \to \frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi = \mathcal{L}_M, \tag{2.27}$$

where the covariant derivative is given by (2.8). Of course, for a scalar field this immediately reduces back to the ordinary derivative. In general, a coordinate tensor field will involved the affine connection. Moreover, if the field has spin, e. g. fermions, this requires to further invoke a term (2.24), as required by the phenomenology.

This implies that even the simplest matter field automatically couples to gravity. This happens due to two effects. One originates from the covariant volume element in (2.25). The other appears from the contraction of the derivatives. Thus, it is impossible for the metric not to interact with matter, even if the matter would be massless as in the present case. Or, matter always interacts with the metric. It is this universal interaction which is known as gravity, as will become evident in section 3.2. Gravity is really just an ordinary gauge interaction, but which couples even to derivatives.

Note that because of dimension also, e. g., a term proportional to $R\phi^2$ can be added to the action. This would imply a direct coupling of the scalar to curvature, and would constitute a mass-like term. Curvature acts like a (space-time-dependent) mass, on top of any tree-level mass. However, any mass term already couples also to the metric by the covariant volume element, and thus gravitates as well. This is the origin of the idea that gravity acts upon mass, while it really couples even to self-interaction terms like ϕ^4, if they are present. Note that this coupling occurs by gauging the argument of the fields, rather than the field itself, which makes it look so different than from ordinary interactions. But since coordinate-dependency can be considered to be a representation of translation symmetry, it is actually not that different, if one thinks of a gauge symmetry with a continous index in an infinite-dimensional representation space rather than the usual finite-dimensional ones.

Having a Lagrangian like (2.26) yields two coupled equations of motion. In their derivation, it is necessary to observe that the action is actually varied, not the Lagrangian alone. As a consequence, the factor $\sqrt{-\det g}$ needs to be included in the Lagrangian equations of motion. One equations following is then Einstein's equation with matter (2.22), where T now becomes

$$T_{\mu\nu} = \frac{\delta\sqrt{-\det g}\mathcal{L}_M}{\delta g^{\mu\nu}} \overset{(2.27)}{=} -\frac{2}{\sqrt{-\det g}}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi,$$

where it was used that

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$$

holds. Thus, the energy momentum tensor stems from the variation of the matter Lagrangian with respect to the metric. In the Minkowski case, this would be the energy-momentum tensor. Of course, if the matter Lagrangian is not that of a scalar field, the affine connection appears, which depends in general relativity on the metric, and its derivatives. Then

$$T_{\mu\nu} = -\partial_\rho\frac{\delta\sqrt{-\det g}\mathcal{L}_M}{\delta\partial_\rho g^{\mu\nu}} + \frac{\delta\sqrt{-\det g}\mathcal{L}_M}{\delta g^{\mu\nu}}.$$

Likewise, the equation of motion for the matter field becomes

$$0 = \partial_\mu\frac{\delta\sqrt{-\det g}\mathcal{L}_M}{\delta\partial_\mu\phi} - \frac{\delta\sqrt{-\det g}\mathcal{L}_M}{\delta\phi} \overset{(2.27)}{=} \partial_\mu\left(g^{\mu\nu}\sqrt{-\det g}\partial_\nu\phi\right). \tag{2.28}$$

At $g = \eta$, this yields the usual plane-wave equation $\partial^2\phi = 0$. However, if g varies, this implies a term sourcing changes in the field as

$$\partial^2\phi = -\left(\partial_\mu g^{\mu\nu}\sqrt{-\det g}\right)\partial_\nu\phi$$

which depends on the changes of the field itself. This equation also holds true if the matter does not backcouple, i. e. Einstein's equation is dropped. Hence, in a static curved space-time even a massless particle will not follow straight lines, but rather move along a line determined by this non-trivial term. Conversely, the presence of a matter field sources a change in the metric, and thus alters the manifold.

Note that both (2.26) and (2.28) are field equations. The metric determines the topology. The pure-gravity equation (2.21) is a self-consistency equation for the topology of the manifold. The manifold itself is given by its set structure. This determines which events have which neighboring relations. In particular, this determines the overall global structure, e. g. of a sphere or a torus[7]. The metric determines the topology, i. e. what is the distance between neighboring events, and thus arbitrary (geodesic) distances on the manifold. The overall manifold structure thus determines which events are neighboring to each other, and given this input, Einstein's equation (2.21) determines the distances. Einstein's equation cannot change which events are neighboring in terms of the sets. However, it can create a topology where the values of the distances do not reflect neighboring relations, i. e. events close by in terms of overlapping sets can have, relative to other pairs of events, very large or very small distances. This is even true for Riemannian manifolds, where the distances are positive semi-definite. The characteristic feature of the neighboring relation is only that there exists always, at least, an infinitesimal neighborhood, in which the distances reflect the neighboring relation, in terms of a Minkowski metric[8].

Once matter is added, at every event a value for the matter field is obtained. The presence of the matter alters the topology, as distances change according to (2.26). However, at the same time the field is affected by (2.28). Thus, the two coupled equations will eventually settle for a self-consistent set of distances and field amplitudes. As both are partial differential equation, this solution will depend on both an initial condition, to be discussed in more detail in section 3.1, as well as the manifold structure in terms of neighboring relations.

Note that the concept of a point particle has not been discussed. The problem is that the distributional character of a point particle's energy momentum tensor makes very challenging to understand the back reaction of the particle on itself. E. g., a massive particle will move, and thereby modify the metric. This changes may spread at the speed of light. Thus, the point particle will eventually interact with this change. That is very similar to the case of the back reaction of an electrically charged point particle in electromagnetism. But the technical challenges involved amplifies in the general relativistic case. Thus, point particles will in the following only be considered as probe particles, i. e. without any (relevant) change on the metric. When using fields,

[7]Sometimes this is also called the topology of the manifold. To avoid confusion with the topology in terms of distances this terminology is not used here. Furtheremore, many neighboring relations can be deformed such that only the number of holes, the so-called genus, of an isomorphic surface is relevant. This is e. g. the case for sphere and torus. A rotational ellipsoid will work the same as the sphere. However, examples like the Klein bottle in three dimensions, which is also a valid manifold in terms of neighboring relations, shows that this is not generally simple.

[8]It may appear at first sight that then the distances should globally be such. However, due to the possibility for transfer functions, this can change quickly radically for finite distances.

this problem does not arise, as the field are not distributions. These are just ordinary coupled partial differential equations.

2.6 Energy, Fourier-space and physical observables

In the context of wave equations switching to Fourier space is often advantegous. As a consequence, usage of momenta is quite common in physics. This is also true already in special relativity, where four momentum conservation becomes a central tenet. This compensates for the loss of individual meaning of energy and three-momentum.

The situation becomes more involved in general relativity. Switching to momentum space is done by a Fourier transformation. However, the exponential factor $\exp(i\eta^{\mu\nu}x_\mu p_\nu)$ needs now to be replaced by $\exp(ig^{\mu\nu}X_\mu P_\nu)$ to ensure that the argument of the exponential factor transforms like a scalar under coordinate transformation. In addition, the volume has to be adapted, leading to

$$f(P) = \int d^d x \sqrt{-\det g} f(X) e^{ig^{\mu\nu}X_\mu P_\nu}. \tag{2.29}$$

While this is formally possible, though in general will be technically involved due to the space-time dependence of the metric, this has far-reaching conceptual problems. Because exactly the requirement of being scalar requires that under a coordinate transformation $P \to \Lambda P$ becomes an event-dependent transformation. Thus, the value of the four momentum as whole, not only of its components, because malleable by diffeomorphism transformations. Thus, just like the vector-potential in electromagnetism or the electric field in Yang-Mills theory, it can no longer be a physical observable. Only the four-momentum squared $P_\mu g^{\mu\nu} P_\nu$ remains invariant, just as distances are.

This is not entirely surprising, after all also coordinates lose their meaning as something, which is at least in principle can be made real. In a sense, in a manifestly covariant description, this does not change too much. After all, already in special relativistic theories quantities should be only dependent on squared quantities. However, The problems now appears when considering a quantity like the scalar product between two different four-momenta. The space-time dependent transformation Λ will not transfer into a momentum-dependent transformation in Fourier space. Likewise the metric needs to be transformed.

What this means can already be seen in position space. Consider the invariant distance, but expand it as,

$$ds^2 = (X_\mu - Y_\mu)g^{\mu\nu}(X_\nu - Y_\nu) = X^2 + Y^2 - 2X_\mu g^{\mu\nu} Y_\nu, \tag{2.30}$$

where it was used that the covariant derivative of the metric vanishes. The full expression is necessarily invariant under a coordinate transformation, by construction. However, because the coordinate transformation is local, the difference $X-Y$ is no longer invariant itself, and the remainder is compensated by the change in metric. As a consequence, an expression like X^2 is not itself invariant, as here the metric changes. Only the combination of all three terms in (2.30) yields the invariance. This especially implies that the term XY is not invariant. Hence, scalar products, and thus angles, are not necessarily an invariant under diffeomorphism transformation. Therefore, a dependency on special Poincaré-invariant quantities is not, in itself, sufficient to be diffeomorphism invariant.

As a consequence, only quantities depending on distances can be invariant. And this makes geodesic distances even more special. They at least have a chance to define path-independent observables, if the geodesic distances are unique.

This does not mean that Fourier space is useless. However, it needs to be considered an auxiliary when calculating eventually diffeomorphism invariant quantities, very much like the vector potential in classical electromagnetism. Of course, once approximations are introduced, it becomes as challenging as in other theories to maintain this invariance.

These issues can, to some extent, be ameliorated in practical cases. This will be discussed in section 3.7.

Chapter 3

Special solutions

Special solutions, like plane electromagnetic waves in classical electrodynamics, are both useful, but also challenging. The main obstacle is again that there is no arena for the special solutions to exist in. Electromagnetic plane waves can be studied, because they are existing in an enclosure of space-time. Special solutions in general relativity do not have such an enclosure. Therefore, in principle, always the whole space-time needs to be considered.

As will be seen, in the absence of matter, solutions can become maximally symmetric, see section 3.3. Some special solutions will address in some sense localized configurations, like black holes in section 3.5. They allow to consider these configurations to be embedded in a specific sense in the maximally symmetric cases without altering their overall character too much. This is probably the closest analogue to the electrodynamics case.

However, other special solutions cannot be considered such. Most prominent of them are those which describe cosmology in section 3.4. Here, everything is, in a sense, special. This makes it also hard to wrap one's mind around their specialness.

In general, most special solutions which do not involve small curvature yield substantial deviations even from the logic of special relativity. This makes them very non-intuitive and difficult to grasp.

3.1 The initial value problem

Einstein's equation, without matter (2.21) or with matter (2.22), are partial differential equations. They are supplemented by conditions of vanishing torsion and covariant constancy. They are thus subject to constraint equations. These reduce the independent degrees of freedom from 16 (in four dimensions) down to 6. Still, in the end only two are independent. This leads to a freedom in the choice of the components, similar to the gauge freedom of classical electrodynamics. Imposing, e. g., the Haywood gauge[1] $\partial^\mu g_{\mu\nu}$ introduces four more constraints, reducing it to the necessary two independent degrees of freedom.

As if solving partial differential equations with constraints is not difficult enough a task[2], there is another severe problem. These differential equations involve expressions like $g^{\mu\nu}\partial_\mu\partial_\nu g_{\rho\sigma}$.

In flat space-time $g = \eta$ this Laplace-Beltrami operator $g^{\mu\nu}\partial_{\mu\nu}\partial_\nu$ has on non-constant modes a positive definite spectrum. In a general metric, this is no longer necessarily true. Likewise the various possible other second-order differential operator appearing are no longer of this type. Since many uniqueness and existence proofs do, however, rely on these properties, they are not applicable to the Einstein's equation. As a consequence, no complete theory of the solution manifold of the most general case of general relativity with (ordinary) matter is available. As a consequence, it is not even clear, which kind of initial values are needed to find a unique solution.

This problem can be largely remedied by restricting the possible initial data. However, the condition will only be possible to realize on certain manifolds, reducing the possible manifolds.

The restriction is that the initial values for the metric g describe a space-like submanifold of one dimension less. I. e., for a submanifold, which is given in terms of a subspace of the underlying $\mathbb{R}^d$, all invariant distances are of the same sign, space-like, and for different elements non-zero. In a visual picture, it is possible that a space-like slice through the universe is possible and specified. If this is the case, it can be shown that the vacuum Einstein's equation evolve such that the whole manifold is separated into such space-like submanifolds, i. e., every element of the manifold belongs to precisely one such space-like sub-manifold. Moreover, any distances between two sub-manifolds are never space-like, and for every element on a given sub-manifold a time-like separated element on a neighboring, in the manifold-sense, sub-manifold can be found. In a visual

[1]Viable gauge conditions, i. e. those for which for every diffeomorphism orbit at least one solution is guaranteed to exist, are not necessarily simple to construct.

[2]An approach like in classical mechanics to incorporate the constraints by switching to generalized coordinates is in general challenging due to the transfer functions. This will usually change the character of the equations to integro-differential ones.

picture, the universe is sliced into space-like slices, separated by time-like distances. This is called a foliation. It is closest to the conventional idea of a universe. While there is no local unique time, there is a global universe time, which essentially counts the slice of the stack. Such initial conditions are therefore called physical, and the decomposition into space-like submanifolds a foliation of the manifold. In a sense, this is also the simplest decomposition into a Riemannian part and a non-Riemannian part of the manifold.

Of course, the space-like submanifolds are still Riemannian manifolds, and can be very involved, and especially still require a non-trivial atlas. An example would be the surface of a four-dimensional sphere, which satisfies these criteria. It is also possible that the submanifold is not simple, and thus describe separate universes. Still, this allows to guarantee a unique solution. If matter is added, this requires it to be ordinary matter, e. g. having a positive energy density in terms of the energy-momentum tensor. Otherwise, such a source term would spoil the evolution for the same reason as arbitrary initial conditions would do.

As a submanifold, such space-like hypersurfaces can be in turned described as before, just with one dimension less. Thus, there exist also a metric, which is called the induced metric h. Of course, the induced metric is fully known in terms of the metric g of the whole manifold. This observation allows a convenient description of the foliation. The induced metric requires to obtain the full metric the difference between neighboring points, in the sense of the full manifold, of two different space-like submanifolds. The connecting quantity is called the lapse function or, when expressed correspondingly, the lapse vector. It encodes locally the time-like separation of each space-like slice, and thus the local passing of time in a suitable rest frame.

In practice, this implies for a curve $X(\tau)$ its tangent vector can be decomposed as

$$\partial_\tau X(\tau) = \alpha n + \beta$$

where n is a future time-like vector of unit length in the tangent space. The function α is then called the lapse function and β, being a vector in the tangent space, is called the shift vector. It is therefore tangential to the space-like hypersurface. This therefore decomposes a direction into one which connects neighboring space-like hypersurfaces, $n\alpha$, as well as a component which describes the shift inside the hypersurface when changing from one to another. Especially, this implies that $n\alpha$ describes the direction in which space-like coordinates can be found, which do not change. Physically, moving along $n\alpha$ is thus the closest resemblance to a worldline in a rest frame.

This allows to locally decompose the invariant length between two point on neighboring hypersurfaces connected by this path as

$$\begin{aligned} ds^2 &= -(\alpha^2 - h^{rs}\beta_r\beta_s)dx_0^2 + 2h^{rs}\beta_r dx_s dx_0 + h^{rs}dx_r dx_s \\ &= -\alpha^2 dx_0^2 + h^{rs}(dx_r + \beta_s dx_0)(dx_s + \beta_s dx_0), \end{aligned}$$

where the $(d-1)\times(d-1)$ induced metric h in the spatial hypersurface appears as well as r and s label the components with the space-like hypersurface. In particular, this shows that along a pure time-like distance only the lapse function contributes, while $dx_r + \beta_r dx_0$ essentially provide the orthogonal, in the sense of this metric, displacement.

Switching to the orthonormal coordinates of a vielbein, this yields that $e^0 = n$, i. e. one of the vielbeins is just the time-like future direction between the two hypersurfaces. The spatial ones decompose then the induced metric as $e_r^i\delta^{rs}e_s^j = h^{ij}$, as the hypersurface is a Riemannian submanifold.

This particular set of coordinates is well adapted to the foliation of space time. It is therefore well suited to solve the initial value problem. However, choosing suitable coordinates in the remainder space-like hypersurface may still be a non-trivial issue. Also, transfer functions may still be needed.

Another concept, which is useful in the context of discussing the initial value problem, is the concept of Killing vectors and fields. Diffeomorphism symmetry and local Lorentz symmetry are, in a sense, non-existing. They only arise as a consequence of the introduction of coordinates and the tangent space. They are not necessary to describe the manifold. However, the manifold can have additional symmetries. Since the manifold is entirely specified in terms of neighboring relations and distances, any such symmetries can only exist in terms of these, and thus in terms of the distances. Thus, they are also called isometries in this context.

An isometry is obtained if there are path in the manifold, which can be displaced in the manifold, without altering the distances between neighboring points along the curve. If there is a displacement prescription such that this can be achieved for the whole manifold, this is an isometry. It basically expresses that a manifold looks the same when one 'rotates' it along a certain direction. Isometries therefore describe global properties of the manifold. E. g. a torus or a sphere do have this property by rotating around their symmetry axis. This operation does not change distances along curves. Thus, these two manifolds have isometries. There can be multiple such isometries.

While isometries can be defined entirely in terms of the topology of the manifold, it is usually more convenient to utilize coordinates. This is then achieved in terms of a Killing field. An isometry is obtained if, in a fixed coordinate system, the metric will not change under a displacement. As a consequence, for an isometry, there exists always a coordinate system such that the metric becomes independent of a suitably chosen coordinate. In general, this will be a directional derivative

$$\partial_\tau X^\mu \partial_\mu g_{\alpha\rho} = 0. \tag{3.1}$$

Of course, isometries of submanifolds are also possible. In the case of a foliation particularly interesting are isometries determined using the induced metric h. Thus, the direction of an isometry is given in terms of the base vector obtained from $\partial_\tau X_\mu = \xi_\mu$. The directional derivative $\xi_\mu \partial^\mu$ therefore creates a direction under which distances are not changing. This can be immediately seen as the change is independent of the path parametrization, and thus

$$ds^{2'} = (g^{\mu\nu} + \xi_\rho \partial_\rho g^{\mu\nu}) \frac{d(X_\mu + \xi_\mu)}{d\tau} \frac{d(X_\nu + \xi_\nu)}{d\tau} = ds^2$$

by virtue of (3.1).

If ξ indeed determines an isometry and is thus a Killing vector can be tested using the Killing equation,

$$D_\nu \xi_\mu + D_\mu \xi_\nu = 0. \tag{3.2}$$

This is a covariant statement. It is therefore admissible to choose a coordinate system, if the calculation is performed exactly, and prove it in this system. Choosing a coordinate system in which the Killing vector is constant, i. e. one of the coordinate axis is taken to be identically to the direction of the Killing vector, yields

$$D_\mu \xi_\nu = g_{\mu\alpha} \Gamma^\alpha_{\nu\sigma} \xi^\sigma = \frac{1}{2} \left(\xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial X^\nu} + \xi^\sigma \frac{\partial g_{\mu\nu}}{\partial X^\sigma} - \xi^\sigma \frac{\partial g_{\nu\sigma}}{\partial X^\mu} \right) = \frac{\xi^\sigma}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial X^\nu} - \frac{\partial g_{\nu\sigma}}{\partial X^\mu} \right).$$

This is hence an antisymmetric quantity, thus implying Killing's equation.

Killing vectors therefore provide a particularly suited to form a basis. Moreover, they imply the existence of a conserved quantity. Given some curve $X(\lambda)$, then

$$p_\xi = \xi^\mu d_\lambda X_\mu$$

is constant along the curve. This can be seen by taking a derivative with respect to λ,

$$d_\lambda p_\xi = d_\lambda \xi^\mu d_\lambda X_\mu + \xi^\mu d^2_\lambda X_\mu = 0$$

Choosing a coordinate system with ξ_μ a basis vector and where the curve is linearly dependent on λ, this expression vanishes. Thus, the component p_ξ is indeed conserved.

Choosing the same system of coordinates to introduce a Fourier transformation allows to identify this quantity as the momentum component along the direction of the Killing vector, which is conserved.

Physically, this is not entirely surprising. After all, an isometry is really a generalized global translation symmetry, which entails a conserved generalized momentum. More surprising is probably that this is not a build-in symmetry, like in classically mechanics. If the solution to the dynamical equations yield a manifold with Killing vectors, the system has dynamically this symmetry.

It should be noted that the maximal number of Killing vectors is limited. Killing's equation (3.2) is a differential equation for the Killing vectors. Since $D^2\xi = 0$ because of the covariant constancy of the metric, this implies that the Killing vectors can be obtained as solution to a second-order partial differential equation, which has to obey antisymmetry of the first-order derivatives. For an n-component field, this yields at most n components with at most $(n^2 - n)/2 = n(n-1)/2$ independent derivatives. Thus, there can be at most $n(n+1)/2$ independent Killing vectors.

3.2 Identifying gravity

It has been mentioned several times that general relativity is basically the gravity from Newtonian physics. As noted before, Newtonian physics, or even special relativity, need to have flat space-time. It is therefore a requirement on the manifold structure. However, in many cases it is not necessary to consider the case that the whole of space-time has this feature, but often it is sufficient to consider only some small patch to be almost flat. Of course, by construction, there always exists an infinitesimal neighborhood where this is possible, but here a somewhat larger patch is needed.

Assume that in some patch, e. g. earth, the metric in a suitable coordinate system can be written as

$$g = \eta + \gamma$$

where for units in which the metric is dimensionless $|\gamma_{\mu\nu}| \ll 1$ holds. Of course, h itself is not a metric, while η is, as is $\eta + h$. Especially, lowering or raising of indices will not work with γ.

Neglecting quadratic and higher power terms of γ simplifies the Riemann tensor (2.9) as thus the Ricci tensor (2.18) which is needed to determine the Einstein tensor (2.20) and thus the equation of motions for matter. The Ricci tensor takes the form

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu R^\lambda_{\lambda\mu} + \mathcal{O}(\gamma^2). \tag{3.3}$$

Considering further that the aim is to establish the relational to Newton's law of gravity, it is acceptable to neglect time variations of γ.

Newton's law of gravity can be formulated as

$$\partial_i^2 \phi = 4\pi G_N \rho,$$

where ϕ is the gravitational potential, and ρ is the matter density. $G_N = \kappa/(8\pi)$ is then Newton's constant in its usual form. To make contact requires therefore to consider the matter density. The matter density is T_{00} for a classical continuum mechanics system. Thus

$$\partial_i^2 \phi = 4\pi G_N T_{00}. \tag{3.4}$$

Setting $\gamma_{00} = 2\phi$ and combining (3.3) and (3.4) with (2.22) yields

$$\partial_i^2 \frac{\gamma_{00}}{2} = \partial_i^2 \phi = 4\pi G_N \rho - \Lambda. \tag{3.5}$$

Thus, indeed the metric yields the classical gravitational potential. Especially, if the backreaction of the matter on the gravitational potential is neglected, which is justified with γ being small, this is exactly Newton's law of gravity, up to the appearance of the cosmological constant. The latter acts like a constant matter (or energy) density. Only measurement can decide its value. For a large value, it would change Newton's law of gravity. However, its value is measured to be so small, that it can safely be neglected in this context. This also justifies the identification of κ with Newton's constant.

This also shows that the metric is essentially the gravitational potential. There are, however, two important observations to be concluded from this.

First, even at weak gravitational field the equation (3.5) should not be read as the equation of motion for matter, i. e. the equivalent of Newton's second law. This is actually given by (2.28) as Lagrange's second equation. Rather, equation (3.5) and (2.28) are needed to be solved together. In fact, (3.5) can be recast, neglecting Λ, as

$$\phi(\vec{r}') \sim \int d^3r \frac{\rho}{|r - r'|}, \tag{3.6}$$

showing that the gravitational potential, or the metric, is determined by the mass density. In turn, entering γ into (2.28) then shows how the matter density changes under the influence of gravity. The fact that (3.5) is usually given as Newton's gravitational law is that a probe particle, e. g. earth, is considered in the gravitational field of a much larger body, e. g. the sun, who by virtue of (3.6) determines essentially the gravitational potential, and then (3.5) is only considered close to the location of the probe particle. Of course, self-consistently, this will coincide with the solution of (2.28) in this case.

Second, in the whole process never a mention of an inert mass and a gravitational mass was made. This is not due to a tacit identification of both. Rather (2.22) and (2.28) together originate as dynamical equations from a Lagrangian which does not necessitates any such distinction, in fact does not even make it possible. Inertial mass and gravitational mass can only be distinguished if equations (2.22) and (2.28) are not solved simultaneously, and then the masses in either of them can be identified such. Thus, the fact that they form a coupled equation derives from a single Lagrangian establishes that there is not independent existence of either.

The approach here to leading order can be extended in a formal power series beyond leading order. These are so-called post-Newtonian approximations. There are basically perturbation theory in the deviation of the metric from the Minkowski metric. While very important in many practical cases, this subject will not be detailed here for a lack of space.

3.3 Maximally symmetric solutions

Given the usual logic of theoretical physics, the first example of solutions to the initial value problem will be the pure general relativity case (2.21) with the aim to find maximally symmetric solutions. Thus, the aim is to maximize the number of Killing vectors. This could be at most four, or in general the number of dimensions, corresponding to the fact that each Killing vector describes a coordinate of which the metric becomes independent.

As Einstein's equation (2.21) stands, it allows for a rescaling of the cosmological constant. In the absence of any other units, it is always possible to measure the coordinates in suitable powers of it. Thus, Einstein's equations can always be rescaled such that the cosmological constant is either zero or ± 1. Being larger or smaller than one then implies that the distances are small or long compared to the appropriate power of the cosmological constant, which gives the natural scale. But in absence of another scale, this implies that the three cases cannot continuously be deformed into each other. There is always a scale above or below the characteristic one. Also, since Einstein's equations are second-order equation there is no possibility to scale out the sign of the cosmological constant. Thus, it is necessary to treat the three cases independently.

The simplest case arises when the cosmological constant vanishes. There is no characteristic scale in the system. Moreover, employing maximum symmetric requires the metric to be constant, and scalelessness implies that all components have equal size. This yields $g = \eta$ as solution, as the Einstein tensor (2.20) vanishes at $g = \eta$, since the Riemann tensor vanishes as it only depend on derivatives of the metric. Hence, there is a Killing vector for every direction, yielding back momentum conservation of special relativity. In addition, the six Lorentz generators complete the list of the possible ten independent Killing vectors in four dimensions.

At non-vanishing cosmological constant the situation is more involved. Still, the aim is to find a solution with the maximal number of Killing vectors. It is best to do so in a suitable fixed basis. The aim is still to find a foliated space-time. Moreover, to achieve maximal symmetry the spatial part needs to be likewise highly symmetric. In a suitable coordinate system, the metric then takes the Friedmann-Lemaitre-Robertson-Walker (FLRW) form

$$g_{\mu\nu} = -dt^2 + a(t)d^3\Sigma. \tag{3.7}$$

where $d^3\Sigma$ describes a homogeneous and isotropic spatial hypersurface. The only possibilities in general relativity for such a structure is

$$d^3\Sigma = \frac{dr^2}{1-kr^2} + s_k(r)^2 d^2\Omega = \frac{dr^2}{1-kr^2} + s_k(r)^2(d\theta^2 + \sin^2\theta d^2\phi),$$

where θ and ϕ are the usual polar and azimuthal angle. The quantity k determines the curvature of the hypersurface, yielding

$$\begin{aligned} s_{k>0} &= \frac{1}{\sqrt{k}}\sin\left(r\sqrt{k}\right) \\ s_{k=0} &= r \\ s_{k<0} &= \frac{1}{\sqrt{k}}\sinh\left(r\sqrt{-k}\right). \end{aligned}$$

Thus, positive values determine a spherical hypersurface, zero is a flat hypersurface, and a negative value is hyperbolically shaped. It is often customary to rescale $r \to rk^{-\frac{1}{2}}$, and thus distances are measured in units of $k^{-\frac{1}{2}}$. Then k becomes discrete with the three options ± 1 and 0. Note that throughout it will be assumed that the spatial hypersurface has trivial topology, i. e. no complicated boundary conditions, is simply connected and has no holes. The choice of k is part of the initial conditions, as is the fact that the spatial part has no non-trivial boundary structure.

Inserting this form into the definitions yield the curvature scalar (2.19)

$$R = 6\left(\frac{\partial_t^2 a}{a} + \frac{\partial_t a}{a^2} + \frac{k}{a^2}\right)$$

and thus the curvature is both time-dependent and depends on k. In fact, in a static universe, $a(t) = a_0$, k determines entirely the curvature. If the spatial hypersurface is a point, $a = 0$, the curvature diverges.´

The factor a is determined by the solutions of Einstein's equation (2.21), and thus by the cosmological constant,

$$\frac{\partial_t^2 a}{a} = \frac{\Lambda}{3} \tag{3.8}$$

which is solved by

$$a(t) \quad = \quad a_+ e^{-\sqrt{\frac{\Lambda}{3}}t} + a_- e^{\sqrt{\frac{\Lambda}{3}}t}$$

where the prefactors are again determined by the initial conditions. This implies that $a(t)$, which plays the role of a scale factor of the metric, is exponential for $\Lambda > 0$, but periodic for $\Lambda < 0$. I. e. in the first case, called de-Sitter, the universe will have distances changing exponentially with time t. In the second case, called anti-de Sitter, the distances will change periodically. Note that in both cases the spatial structure can be the same, and the prefactor only regulates the time-dependence of distances.

To quantify the behavior of the distances the Hubble parameter is introduced as

$$H(t) = \frac{\partial_t a}{a}, \tag{3.9}$$

it measures the relative change of distances as a function of t. Since both cases describe foliated space-time, the time coordinate is, in principle, eternal. Both solutions are indeed maximal symmetric. This can be seen by the fact that both de Sitter space-time and anti-de Sitter space-time can be described in terms of an embedding in a higher-dimensional Minkowski space time as the surface of a hyperboloid, which differ only by whether time is a closed line around the hyperboloid (anti-de Sitter) or along the open direction of the hyperboloid (de Sitter), with space playing the opposite role. On such a surface there are, of course, the one dimension less symmetry group, and thus giving again a ten-dimensional group, giving ten Killing vectors. Although their interpretation is now less forward than flat Minkowski space-time.

Note that also flat Minkowski space-time fits into this structure, yielding in fact also the two solutions $a(t) = a_0 t + a_1$, rather than just $a(t) = 1$. However, since this is only a time-dependent rescaling, this does not alter the result. Note that the curvature is in general no longer time-independent, but will depend on the interplay and relative size of the different constants.

3.4 Big-bang solutions

The solutions in 3.3 give pure gravity solutions. The natural question is what happens once matter is introduced into the system by the set of coupled equations (2.22) and (2.28). The relevant quantity is then the energy-momentum tensor in (2.22). Assuming for the moment that matter interactions as described by (2.28) are such that they quickly establish local equilibrium, the coupling between both equations will be small, and the energy momentum tensor will have the same isotropy and homogeneity as space-time itself. In a suitable coordinate system the energy-momentum tensor will be of the form $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$, where $\rho(t)$ is the matter density and $p(t)$ is the pressure. Both will be spatially constant on the homogeneous space-like hypersurfaces. The vast difference in coupling strengths between Newton's constant and the typical couplings of the other interactions justify this approximation. In a universe with different values, this would probably not be justified.

The matter density and the pressure are not independent, but related by the equation of state of the matter derived from (2.28). This yields the two coupled equations

$$\partial_t \rho = -3\frac{\partial_t a}{a}(\rho + p^2) \tag{3.10}$$

$$\frac{\partial_t^2 a}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{3.11}$$

where (3.11) is derived from (2.22), and is the version with matter of equation (3.8).

Note that replacing

$$\begin{aligned} \rho &\to \rho + \frac{\Lambda}{8\pi G_N} \\ p &\to p - \frac{\Lambda}{8\pi G} \end{aligned}$$

in (3.10-3.11) would eliminate the cosmological constant. This is equivalent to say that in this setting the cosmological constant behaves like matter satisfying the equation of state

$$\rho_\Lambda = -p_\Lambda \tag{3.12}$$

i. e. like a positive matter density which exerts a negative pressure for a positive cosmological constant, and negative matter density which exerts a positive pressure for a negative cosmological constant. Thus, the cosmological constant tends to blow up the universe in case of the observed positive value.

It follows that the Hubble parameter (3.9) is a convenient quantity for the following it. It takes the form

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a}.$$

This can be used to define a critical density

$$\rho_c = \frac{3H^2}{8\pi G}$$

which implies that the quantity

$$\Omega = \frac{\rho}{\rho_c}$$

is, for k/a^2 negligible, a measure for the fate of the universe. If it is larger than 1, the normalized change of rate yields that the universe will eventually collapse. If it is equal one, it will reach an asymptotic size. And if it is smaller than one, the universe will expand forever. Current measurements for our universe strongly suggest $k \approx 0$ and $\Omega \lesssim 1$, and thus that the universe will expand forever.

What is finally needed to solve the system is the equation of state. Approximating the matter in the universe by a perfect fluid, the equation of state becomes

$$p = w\rho.$$

This implies that the cosmological constant behaves like a perfect fluid with $w = -1$. In particular, it exerts a negative pressure. For matter with thermal energy substantially below the rest energy w is zero, while for ultrarelativistic matter, or massless particles, $w = 1/3$. Inserting these into the coupled system of equations (3.10-3.11) yields at $t \gg \sqrt{G_N}$ the qualitative behavior

$$a_\Lambda(t) \sim e^t \tag{3.13}$$
$$a_{\text{Matter}}(t) \sim t^{\frac{2}{3}} \tag{3.14}$$
$$a_{\text{Radiation}}(t) \sim t^{\frac{1}{2}}. \tag{3.15}$$

at late times.

There are three initial conditions required to solve (3.10-3.11). The results (3.13-3.15) all are adjusted such as to yield an expanding universe. These initial conditions are, of course, entirely based on empirical evidence. Inserting values measured 'now', they yield the existence of a big bang[3], i. e. $a(0) = 0$. In that case, the universe had zero extension at $t = 0$ and also $\rho(t \to 0) \to \infty$ for the case of matter. Hence, this metric has a singularity at $t = 0$, as the metric degenerates at that time. This is, of course, in conflict with the assumptions about the metric laid out for general relativity. Thus, this is seen as an indication that under these conditions the equations (2.22) and (2.28) cease to be valid (alone). Possibly, this can be resolved in quantum gravity, but this is not yet established beyond doubt. At the very least, this implies that within the framework of general relativity, it doe snot make any sense to even consider some $t < 0$, as at $t = 0$ the system leaves its regime of validity. Of course, this can also be artifact of the highly idealized situation treated, as well as the absence of knowledge of the structure of the universe beyond the visible horizon[4]. Changes to the initial conditions or relaxation of assumption can change this.

To unify result, it is possible to combine different types of matter. This yields in terms of the fraction of the total density of each type in units of the critical density at a fixed time t_0

$$\frac{H^2}{H_0^2} = \frac{\Omega_{\text{Radiation}}}{a^4} + \frac{\Omega_{\text{Matter}}}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \tag{3.16}$$

where Ω_k is a suitable normalized version of k and H_0 is the value of H measured at the same fixed time t_0, $H_0 = H(t_0)$, usually today. This equation is, in principle, exactly solvable, given a very lengthy expression in terms of elliptic functions.

However, here just the important properties will be quoted for the observed values of the various Ω_i. These values are determined empirically. The values are $\Omega_k \approx 0$, $\Omega_{\text{Matter}} \approx 0.25$, $\Omega_{\text{Radiation}} \approx 0.01$, and $\Omega_\Lambda \approx 0.74$, where dark matter is included in matter. These values are the ones measured today. This modelling of the universe is also known as the cold-dark matter scenario with cosmological constant, briefly ΛCDM. Cold, because the dark matter is non-relativistic.

With this information, the universe started extremely hot, and was thus radiation

[3]With a too small time interval of about 12 billion years since, compared to the age of the oldest stars of 13.3 billion years. This is corrected by the inflation scenario, yielding a time interval of 13.8 billion years. However, this will not be subject of this lecture. See, e. g., the lectures on beyond-the-standard model physics or astroparticle physics.

[4]Due to the aforementioned inflation scenario, this visible horizon is of order 100 billion light years, rather than 13.8 billion lightyears.

dominated it, yielding an expansion like (3.15). With the matter becoming more and more diluted, the temperature eventually dropped, and the equation became that of matter, slowing down the expansion to (3.14). However, eventually the cosmological constant will take over, yielding an exponentially accelerated expansion of the universe like (3.13). This happens roughly around now. Thus, the ultimate fate of the universe is to become infinitely large, provided that nothing yet unknown kicks in.

3.5 Schwarzschild black hole and Kruskal coordinates

Of course, it is not always reasonable to ask for a complete solution of all of space-time when only a small disturbance is of interest. Similarly as when investigating a water molecule one does not want to investigate the whole ocean. This is also possible in general relativity. To this end, the idea is introduced to consider solutions, which have asymptotically a specific space-time structure. Asymptotic here means at the distances large compared to any other scale of the problem. Since the equations (2.22) and (2.28) are local equations, this should usually be possible. The results will then asymptotically approach one of the maximally symmetric solutions of section 3.3, provide no boundary conditions are imposed. The idea is, of course, that a system of multiple objects can then be patch together from multiple translated copies, just like the vapor can be patched together from individual, non-interacting, water molecules to a very good accuracy. Obviously, this eventually breaks down when the separation is not large enough, as the ocean shows. Mathematically, this can be put into a more precise form, which will depend on the initial conditions, as well as the precise form of the matter Lagrangian.

A paradigmatic example of this approach is the Schwarzschild black hole. It will now also be used to exemplify this process. For this, the cosmological constant will be set to zero, and asymptotically therefore space-time should approach Minkowski space-time, given suitable initial conditions. Furthermore, the situation should be static, i. e. there should exist a Killing vector in the time-like direction. This implies that the formation of the black hole cannot be captured in this way. Finally, only the pure gravity case will be considered, (2.21). This has the nice advantage that it will show that pure gravity alone has already non-trivial solutions.

The foliation implies that the metric will have the form

$$g = -e^{2a(x_i)}dt^2 + h_{ij}dx^i dx^j \tag{3.17}$$

The Killing field makes the metric independent of t. For simplicity, assume furthermore that h will have further space-like isometries, creating an SO(3) invariance. Hence, the spatial submanifold will have spherical symmetry. This is consistent with asymptotically being Minkowski space-time. A suitable ansatz respecting this is

$$h = e^{2b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where factors 2 here and in (3.17) are chosen for later convenience. In particular, the tetrad can now be read off immediately. This also implies $a(x_i) = a(r)$. To obtain asymptotically Minkowski space-time then really just requires $a(r \to \infty) = b(r \to \infty) = 0$.

Constructing the Riemann tensor, the Ricci tensor, the curvature scalar and the Einstein tensor is now only a matter of algebra, albeit tedious. Eventually, the Einstein tensor is given by

$$\begin{aligned} G_0^0 &= -\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} - \frac{2d_r b}{r}\right) \\ G_1^1 &= -\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2d_r a}{r}\right) \\ G_2^2 = G_3^3 &= e^{-2b}\left((d_r a)^2 - d_r a d_r b + d_r^2 a + \frac{d_r a - d_r b}{r}\right). \end{aligned}$$

and all other components vanish identically. In the vacuum, these need to equal zero. Because so need to do all sums, this implies that $d_r(a+b) = 0$ by adding G_{00} and G_{11}. Given the asymptotic conditions on a and b, this implies $a = -b$. This allows to consider only G_{00} without need to tackle the second-order differential equations posed by G_2^2 and G_3^3. For $r > 0$, $G_0^0 = 0$ can be cast as

$$d_r\left(re^{-2n}\right) = 1$$

which is solved by

$$e^{-2b} = 1 - \frac{2m}{r},$$

where m is an integration constant. This yields the solution

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.18}$$

the Schwarzschild solution, describing a Schwarzschild black hole.

This solution has a number of interesting consequences. First, consider the situation at distances $r \geq m$. Using (3.5), this yields a gravitational potential of a mass $8\pi m/\kappa$. Thus, up to a constant, m acts like the mass of a point particle at sufficiently large distance. In this sense, it is possible to think of $8\pi m/\kappa$ as the mass of the black hole. However, this is an effective mass. Neither it is a mass in a particle sense, where this would appear in the Lagrangian, nor is it an intrinsic feature, because it is an integration constant. Given the initial conditions, it describes merely a gravitational field configuration at long distances as if there would be such a mass. But is very different from the concept of mass of particles. In particular, this is a solution to the pure gravity case (2.21). The Schwarzschild solution does not involve matter, and would not, e. g., correspond to the situation of a collapsing star.

It should also be noted hat therefore the gravitational field of a Schwarzschild black hole at long distances cannot be distinguished from that of, say the earth or the sun. That is like in electrodynamics, where a point charge and an extended static charge both have the same long-distance field, and both cannot be distinguished. This allows also the description of how the stars in a galaxy move around their central black hole by, up to dark matter effects, Newtonian dynamics.

The second feature of (3.18) is the spherical singularity on the surface given by $r = 2m$. This is called the event horizon and $2m$ the Schwarzschild radius. It also appears as if the extensions in time directions vanish at the event horizon, as $g_{00} = 0$. Moreover, this singularity is eternal, due to the time-like Killing vector. But this is, actually, illusionary. The metric is only a diffeomorphism-dependent quantity. It is best to consider a -invariant quantity. These are necessarily scalar quantities.

Curvature is described by contractions of the Riemann tensor, as it contains all possible information. There are 14 linearly independent, different possibilities in four dimensions. There can be obtained from contractions of products of the Ricci tensor, including the curvature scalar, as well as

$$K = R^2 \tag{3.19}$$
$$I_1 = \frac{1}{\sqrt{-\det g}}\epsilon^{\lambda\mu}_{\rho\sigma}R^{\rho\sigma\nu\kappa}R_{\lambda\mu\nu\kappa} \tag{3.20}$$
$$I_2 = R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}R^{\lambda\mu}_{\rho\sigma} \tag{3.21}$$
$$I_3 = \frac{1}{\sqrt{-\det g}}R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}\epsilon^{\tau\xi}_{\rho\sigma}R^{\lambda\mu}_{\tau\xi}. \tag{3.22}$$

All other possible contractions can be written as sums of these. Inserting (3.18) into the Riemann tensor and performing the algebra yields a vanishing Ricci tensor, and likewise

I_2 and I_3 vanish. The only non-zero results are the Kretschmann tensor $K = 48m^2/r^6$ and $I_3 = -96m^3/r^6$. This shows that any invariant quantities do not show a singular behavior on the event horizon, but only at the center of the black hole. Thus, a black hole is a curvature singularity, not dissimilar to a point charge. This is where the real singularity structure of a (Schwarzschild) black hole arises from.

But what does the event horizon then signifies? Of course, the singularity shows the loss of validity of the coordinate system, and thus requires a transfer function. But is there a physical consequence?

The answers can be given best by introducing a new coordinate system, which is valid inside the event horizon. A suitable choice are Kruskal coordinates. Obviously, the problem does reside in the r and t coordinates, so θ and ϕ can be left untouched. Introduce new coordinates

$$u = \sqrt{\frac{r}{2m} - 1}e^{\frac{r}{4m}} \cosh\left(\frac{t}{4m}\right) \tag{3.23}$$

$$v = \sqrt{\frac{r}{2m} - 1}e^{\frac{r}{4m}} \sinh\left(\frac{t}{4m}\right). \tag{3.24}$$

This yields a new metric

$$ds^2 = -\frac{32m^3}{r}e^{-\frac{r}{2m}}(dv^2 - du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.25}$$

in which r has to be expressed using (3.23-3.24), which can, however, not be explicitly solved. What holds is

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right)e^{\frac{r}{2m}} \tag{3.26}$$

$$\frac{u}{v} = \tanh\frac{t}{4m}. \tag{3.27}$$

Thus, $r = 2$ is defining a hyperbola in u and v, where the singular parts cancel thereby in the metric. Thus, the coordinates across the horizon are in this way regular.

While the Kruskal coordinates are well-defined inside the event horizon, it is worthwhile to transform, using a second set of transfer functions, back to the Schwarzschild coordinates (3.18), which area again a viable solution of the Einstein equations. However, in this case $r < 2m$, and thus the value of the prefactors of r and t change sign, and thus r and t switch role. This now gives a possible physical interpretation. The event horizon is the location where this switch occurs. However, since the metric needs to be non-degenerate, this is not continuously possible, leading to the coordinate singularity.

Distances, using the transfer functions, are now well-defined across the event horizon. Especially, a path of finite length is possible between the inside of the event horizon and the outside. Thus, the distance can be traversed in finite eigentime. However, attempting to calculate the coordinate time elapsed, it is found that the amount of time diverges when moving towards the horizon. That can be obtained from determining the coordinates of (2.16) for a path connecting the inside and outside, $X_0(\tau =$ event horizon is infinite. Thus, there is no purely time-like (or space-like) curve connecting points on opposite sides of the event horizon. Hence, local light cones are squashed at event horizon. However, inside and outside the event horizon casual connections are possible, but appear space-like on the opposite side. In particular, all time-like trajectories inside and the horizon originate in the singularity, while all outside move asymptotically to the event horizon, as do those inside. Such a situation is called geodesically incomplete, where two (arbitrary) points on the manifold cannot be connected with a unique geodesic. Thus, the inside of an event horizon forms a causal complete universe, as does the outside, but without any possibility to connect causally both.

An interesting implication of this is that the Killing vector, which was time-like on the outside, and thus signifies a static situation, is space-like on the inside. Thus, the inside is not static, but there is effectively one spatial dimension less, as one space-like direction is now static.

There is another interesting feature. As can be seen from (3.26-3.27), the transformation only requires $u > 0$ and $v > 0$. Thus, to cover the manifold only the positive quadrant would be necessary. But the manifold can, but does not need, to be extended, by including also the $u < 0$ and $v < 0$ part. This extends and enlarges the manifold. Likewise, these coordinates will also yield another overlap with another event horizon. However, because of the switch of sign, the behavior is actually reversed around the event horizon. Especially, this shows that the trajectories point outward, rather than inward. This is called a withe hole. There is a transition region between both region between the original manifold, and the extension of it. At $v = 0$, there is a finite transition region from $u > 0$ to $u < 0$, allowing the passing of trajectories. This is called an Einstein-Rosen bridge, or, alternatively, a Schwarzschild throat or a wormhole. It basically states that, if the manifold is extended, something across the event horizon in a suitable trajectory, it will reemerge at a different place through the white hole (or wormhole) at a different place on the extended part of the manifold. However, the situation is very special. Extending it beyond the static case turns out to make this passage unstable or yield other obstacles.

3.6 Kerr black hole

The Schwarzschild solution was static outside the event horizon. It was also spherically symmetric. It can be shown that all static, i. e. having a time-like Killing vector, spherically symmetric vacuum solutions are exhausted by the Schwarzschild solution. Moreover, it can be shown that all black hole solutions, i. e. solutions with a 'point-like' curvature singularity in general relativity without matter can be parametrized by two quantities, one being the mass parameter m of the Schwarzschild black hole, the other being a likewise analogue of the angular momentum. This is known as the no-hair theorem[5], as no further properties are possible. The second set of solutions are the so-called Kerr-Newman solutions.

As usual, angular momentum requires a specific axis. Therefore, the solutions can be expected to be at most axisymmetric. The aim is therefore to search for static, axisymmetric solutions. The actual derivation, which follows similar lines as for the Schwarzschild solution, turns out to be quite involved, and will be skipped here. In the end, a result is found, which has, as expected, two parameters, m and a. A suitable choice of coordinates are two real-valued parameters r and t, and two angular coordinates θ and ϕ. While it is tempting two think of this as bicylindrical coordinates, they are, of course, much more involved. In particular, even in these Boyer-Lindquist coordinates the metric can not be expressed as a diagonal matrix. This is not surprising, as the rotation will link rotational degrees of freedom with linear ones.

The final result provides in these coordinates the metric

$$\begin{aligned}
ds^2 &= \frac{1}{\Xi}\left(-(\Delta - a^2\sin^2\theta)dt^2 + 2a\sin^2\theta(\Delta - r^2 - a^2)dtd\phi\right.\\
&\quad \left.+\sin^2\theta\left((r^2+a^2)^2 - \Delta^2\sin^2\theta)d\phi^2\right)\right) + \Xi\left(\frac{dr^2}{\Delta} + d\theta^2\right)\\
\Delta &= r^2 - 2mr + a^2\\
\Xi &= r^2 + a^2\cos^2\theta.
\end{aligned}$$

The result is not particularly enlightening in this form. However, it is useful to consider, as for the Schwarzschild black hole, the result at large 'radial' distance r. This yields

$$ds^2 = -\left(1 - \frac{2m}{3}\right)dt^2 - \left(\frac{4am}{r}\sin^2\theta\right)dtd\phi + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \mathcal{O}\left(\frac{1}{r}\right).$$

[5]If electromagnetism is added, electric charge can be also carried. Charged black holes will not be considered here.

Comparing again the g_{00} component to (3.5), this shows that m plays the same role as the mass parameter in the Schwarzschild case. Thus, the gravitational potential of the Kerr-Newman black hole is again the same as that of a point particle of a fixed mass, determined by m.

Using the construction of section 3.7 allows further to understand the significance of a. It shows that it can be interpreted to be the total angular momentum J as $J = am$. Thus, The long-distance field of the Kerr-Newman black hole is that of a spinning point mass.

There are a few general statements to be made about features of this solution. One is the event horizon. As long as $m > a^2$, i. e., the black hole does not rotate too quickly, a event horizon exists, which is again a coordinate singularity. A real curvature singularity exists again only at the center. It is possible to extend coordinates again, with similar consequences as in the Schwarzschild case. However, a second structure arises, the so-called ergosphere. This is an envelope of the event horizon, which coincides at the poles, i. e. the axis of axisymmetry, with the event horizon. For $a \to 0$, it coincides with the event horizon, and bulges outside otherwise. In the region between the event horizon and the ergosphere, and geodesic is dragged long, and any probe particle is forced into an orbit around the black hole. Any particle getting to close will no longer be able to keep a stable orbit, but there path inevitably moves to the event horizon.

If $m < a^2$, the black hole rotates very quickly. In that case, the event horizon ceases to exist, and it is possible to reach the singularity from the outside, then called a naked singularity. Of course, just as with point charges, the process of actually reaching it yields divergencies, indicative of the breakdown of general relativity. Conversely, it is possible to find solutions decreasing a over time, and thus to slow down a spinning black hole, and likewise the opposite. Of course, if a reaches zero, the Kerr-Newman black hole becomes a Schwarzschild black hole. Thus, the Schwarzschild black hole is a special case of the Kerr-Newman black hole.

In turns out that this two-parameter family of black holes has a number of very interesting features. These are known as laws of black hole thermodynamics. However, while they are called thermodynamics, they are called this way due to an analogy. There is no statistical ensemble of objects involved, and the quantities cannot be interpreted as ensemble averages.

The zeroth law introduces the analogue concept of a temperature. Solving the equations of motions of a point particle, it is possible to determine the gravity experienced it by identifying it as the ratio of the acceleration divided by the mass of the probe particle. It is then found that this effective gravity is constant on the whole event horizon. Thinking of the event horizon as the surface of the black hole, this leads to the statement that the surface acceleration at the event horizon

$$a_H = \frac{r_h - m}{r_h^2 + a^2} \tag{3.28}$$

on the black hole is constant, in analogy to the temperature of a heat bath. Here, r_H is the radius of the event horizon, for the Schwarzschild black hole $2m$.

For the next step, the area of the event horizon is obtained from the metric. It yields

$$A_H = 4\pi(r_H^2 + a^2).$$

Expressing the mass as a function of the surface and the angular momentum yields the differential

$$\begin{aligned} dm &= \frac{a_H}{8\pi} dA_h + \Omega_h dJ \\ \Omega_h &= \partial_J \sqrt{\frac{A_h^2}{256\pi^2} + \frac{64\pi^2 J^2}{A_h}}. \end{aligned} \tag{3.29}$$

At first sight, this appears not to be something exceptional. It just states that the mass of the black hole, considered as a function of the event horizon area and its spin, changes when either of these are changed, and in a particularly prescribed way. What makes this to what is called the first law of black hole thermodynamics is by a comparison to the first law of thermodynamics,

$$dU = TdS - pdV,$$

where U is the inner energy, T is the temperature, S the entropy, p the pressure, and V the volume. Especially, this leads to the association of a_H with the temperature and the area with the entropy. Also, as the mathematical structure is the same, it implies that many mathematical features can be taken over. What is not fixed are prefactors. Semiclassical considerations suggest that the analogy is closest when identifying

$$\begin{aligned} T_H &= \frac{\hbar a_H}{2\pi k_B} \\ S &= \frac{k_B A_H}{4\hbar}, \end{aligned}$$

yielding the Hawking temperature and the Bekenstein-Hawking entropy, respectively.

The third law of black hole thermodynamics is then completing the analogy by stating that in a forward time-like direction the area of a black hole, and thus the Bekenstein-Hawking entropy, can never decrease. While this requires a complete dynamical solution to discuss, it is already well motivated with the static Schwarzschild solution. All geodesics on the outside will always point to the event horizon, and not outwards, thus depositing more inside the black hole, and do not take anything away. This also implies that the collision of two black holes can only end in a single, larger black hole, and never in multiple smaller ones.

Of course, the analogy is based on very special cases. However, dynamical (mostly numerical) solutions do support the second law. It appears to also hold once matter with an ordinary equation of state is included. It is certainly consist with all observational data available. Quantum effects seem to allow for changes. Semi-classically, it is allowed that quantum fluctuations lead to the possibility of radiation from a black hole, so-called Hawking radiation. Then the mass decreases. However, the effect is minuscule for black holes of stellar size, and decreases further with increasing mass. It thus remains hypothetical.

There are many further quasiclassical conjectures, which have been based on (3.29). Most notable in this context is that in (3.29) the mass depends at fixed angular momentum only on the surface, not on the volume. This has lead to questions like the information paradox, as the uncertainty principle limits the amount of energy, and thus information, is stored in a fixed phase-space cube. Since the surface has less of these available, it appears tempting to speculate that information is eternally lost by falling into a black hole, and cannot even be recovered by Hawking radiation. However, the latter can evaporate a black hole. So how does this work together. The same is basically true for any conserved quantum number, which lead to the conclusion that global symmetries are all violated by the existence of black holes. However, all of these are based on semiclassical considerations. It appears likely that they will be resolved by a full quantum theory of gravity. These will be considered in chapters 4 and 5. Thus, such ideas will not be pursued further here.

3.7 ADM construction and momenta

As has become clear in sections 3.5 and (3.6), it is sometimes possible to still define concepts of mass and angular momentum by moving sufficiently far away from a disturbance of the metric. This can be systematized in the Arnowitt-Deser-Misner (ADM) construction. This construction allows a definition of energy, momentum, and angular momentum at spatial infinity of an asymptotically flat space-time. This is to a very good approximation a good opportunity to define these quantities for celestial bodies. However, it should be noted that despite its uses this construction is not unique, and even does not yield an unambiguous definition of what these quantities are. It yields, however, one which in the special relativistic case coincides with the usual definitions. The problem in general turns out to be highly non-trivial.

The main problem is that (2.22) does neither allow a unique way of defining the energy momentum tensor of the gravitational field, nor does it allow a coordinate invariant definition. This is expected. After all, the coordinate invariant quantity is the topology, which does not yield any definition of direction. Hence, a directional quantity like the energy-momentum tensor cannot exist without a coordinate system.

To have resemblance with the usual construction in special relativity requires to introduced a suitable orthogonal coordinate system. This can be done using the vielbein construction of section 2.4. First, for matter contain the momentum as

$$T^i = e^j e_j^\mu e_\nu^i T^{\mu\nu},$$

yielding the momentum (density) in the direction of the unit vector e^i. The question is, of course, how to construct the contribution from the metric itself. Due to (2.22), the Einstein tensor behaves like the matter energy-momentum tensor. However, part of it is trivially conserved. This therefore does not allow for a unique definition. A suitable definition is the Landau-Lifshitz construction, yielding at $l = 0$

$$\begin{aligned} t^i_{\text{LL}} &= \frac{1}{16\pi G_N\sqrt{-\det g}}\epsilon^{ijkl}\sum_{\text{permutations}}(\omega_{mj}\omega_k^m e_l - \omega_{jk}\omega_{ml}e^m) \\ \omega_{ij} &= e^k\Gamma^\mu_{kj}e_{\mu i}. \end{aligned}$$

This gives the total four-momentum vector $\tau^i = T^i + t^i_{\text{LL}}$.

From this the angular-momentum density is defined as

$$\begin{aligned} M^{ij} &= \epsilon^{ijkl}X_k\tau_l \\ X_k &= e_k^\mu X_\mu, \end{aligned}$$

again in a very similar way as in flat space-time. Both quantities are not invariant under diffeomorphism, and thus do not provide locally a well-defined quantity.

Define therefore the asymptotic mass and angular momentum as

$$P^i = \int_\sigma d\sigma\sqrt{-\det g}\tau^i \tag{3.30}$$

$$J^{ij} = \int_\sigma d\sigma\sqrt{-\det g}M^{ij}, \tag{3.31}$$

where σ is a hypersurface, in the sense of the initial conditions of section 3.1 a slice of the foliation, at infinity. Because it was required that space-time is asymptotically flat, this leads to a diffeomorphism-invariant quantity. They define the mass and angular momentum of whatever exists in the interior. Albeit lengthy, this ADM definition recovers the results in sections 3.5 and 3.6 for black holes. For $g = \eta$ everywhere this also recovers the special relativistic form of total energy and total momentum.

A particular useful case is the total energy, which reduces to

$$P^0 = -\frac{1}{16\pi G_N}\int d\Sigma\left(\partial_j g_{ij} - \partial_i g_{jj}\right)n^i$$

where n is an outward normal on Σ. This quantity is positive or zero for pure gravity. In particular, choosing a coordinate system in which the energy momentum tensor is diagonal, $\tau = \mathrm{diag}(\rho, \vec{p})$, this implies that $\rho \geq 0$ and $|p_i| \leq \rho$. With matter, this only holds if the matter energy momentum tensor satisfies $T^{\mu\nu}\xi_\mu\xi_\nu \geq 0$ and $T^\mu_\nu\xi^\nu$ non-space-like for any time-like vector ξ. This is called the dominant energy condition. It is satisfied by all known matter. The consequence of this assumption is e. g. that gravity is always attractive. Thus, this dominant energy condition can be used to express the same fact. Classically, any system which violates this condition would be unstable.

Thus, the ADM construction allows the definition of energy, momenta, and angular momenta for isolated systems. Due to the weakness of gravity almost everywhere and always, basically every elementary particle is sufficiently isolated in that sense to allow for a defined four-momentum and angular momentum[6] using (3.30-3.31) with a microscopic σ. It is this mechanism which allows flat-space particle physics to work in the familiar way without including gravity, and by extension non-relativistic physics and thereby chemistry and biology.

[6]As for spin, and angular momentum addition, this requires the combination as a common subgroup described at the end of section 2.4, and then this works as the formulation is in the tangent space.

3.8 Gravitational waves

Apparently, a local change in the metric can propagate over the whole manifold. The big bang cosmology of section 3.4 is probably the most extreme example. However, it can be expected that even a strong local, in the sense of neighborhood relations, disturbance will in general become weaker once it gets distributed over many more events. This expectation is based on Newtonian gravity, and section 3.2 motivates that this could be good enough. Indeed, a more complete treatment of Einstein's equation is indeed consistent with that. Such weak disturbances are found to obey a wave equation, and are therefore called gravitational waves.

Generically, such weak disturbances can be split off the metric as

$$g = g_0 + \gamma,$$

as long as $|\gamma| \ll |g|$, in a suitable norm. Especially, γ can never alter the signature of g_0 and any contractions of γ with itself become negligible. The most common case is that $g_0 = \eta$, which will be pursued here. Note that only g_0 acts as a metric to raise and lower indices, but γ is not. The reason is that

$$1 = g^{-1}g = (g_0+\gamma)^{-1}(g_0+\gamma) \approx g_0^{-1}g_0 - \gamma^{-1}g_0 - g_0^{-1}\gamma + \mathcal{O}(\gamma^2) = 1 - \gamma^{-1}g_0 - g_0^{-1}\gamma + \mathcal{O}(\gamma^2)$$

implying

$$\gamma^{-1}g_0 - g_0^{-1}\gamma = 0$$

and thus

$$\gamma^{-1} = g_0^{-1}\gamma g_0^{-1}.$$

Hence, γ would not yield invariant quantities to this order, and thus is not a metric. Especially, $\gamma_\nu^\nu = g^{\nu\rho}\gamma_{\rho\nu}$.

Neglecting also the cosmological constant and inserting this approximation into Einstein's equation (2.22) yields

$$\partial^2\gamma_{\mu\nu} + \partial_\mu\partial_\nu\gamma_\lambda^\lambda - \partial_\lambda\partial_\mu\gamma_\nu^\lambda - \partial_\lambda\partial_\mu\gamma_\nu^\lambda + \eta_{\mu\nu}\partial^2\gamma_\lambda^\lambda + \eta_{\mu\nu}\partial_\lambda\partial_\sigma\gamma^{\lambda\sigma} = -16\pi G_N T_{\mu\nu}, \quad (3.32)$$

because

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2}\left(\partial_\nu\gamma_\mu^\alpha + \partial_\mu\gamma_\nu^\alpha - \partial^\alpha\gamma_{\mu\nu}\right).$$

As a consequence, the energy momentum tensor fulfills

$$\partial_\nu T^{\mu\nu} = 0$$

and matter does not couple back to gravity.

The equation (3.32) is invariant under

$$\gamma_\mu \to \gamma_{\mu\nu} + \partial_\nu \xi_\mu + \partial_\mu \xi_\nu. \tag{3.33}$$

This implies that γ inherits the property to be a gauge field under diffeomorphisms parametrized by ξ. This is not surprising. After all, (3.32) can be considered to be the first order of a series expansion of g. The diffeomorphism symmetry of g from section 2.1.4 is then inherited order by order. Especially, to linear order at $g_0 = \eta$ only ∂_μ appears, rather than D_μ^h, which appears at higher order or if g_0 is event-dependent.

This freedom can now be used to simplify (3.32), just as is done in electrodynamics. A particularly useful gauge is

$$\partial^\mu \left(\gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma_\lambda^\lambda \right) = 0,$$

which is called the Hilbert gauge[7]. In this gauge (3.32) takes the form

$$\partial^2 \left(\gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma_\lambda^\lambda \right) = \partial^2 h_{\mu\nu} = -16\pi G_N T_{\mu\nu}, \tag{3.34}$$

where the trace-subtracted part h has been identified. This is a wave-equation for h, with a source term on the right-hand side.

The equation (3.34) is a wave-equation. As it is linear, the solution is a superposition

$$h = \epsilon + \omega$$

of the solution $\epsilon_{\mu\nu}$ of the free-wave equation $T = 0$ and the inhomogeneous solution

$$\omega(x) = 4G_N \int d^3\vec{y} \frac{T(x^0 - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|},$$

the usual retarded solution. The latter takes the metric perturbation to be sourced at some instance of time x^0, and gives the solution for all times x^0 afterwards. Of course, ϵ will depend on the boundary conditions.

It is important to note that this solution corresponds to a solution for the vector potential and is gauge-dependent, and does not yet describe a physical quantity. On the other hand, this implies that a matter distribution will source a gravitational wave.

[7] Another choice is $\partial^\mu \gamma_{\mu\nu} = 0$, which is called Haywood gauge.

To do so, it is useful to note that the gauge symmetry (3.33) acting on γ is reducible. Replacing ξ with $\xi + \Xi$ satisfying

$$\partial_\mu \Xi_\nu = -\partial_\nu \Xi_\mu$$

will have no effect in (3.33). Using this freedom to impose $\partial^2 \xi = 0$ and $\partial_\mu \xi^\mu = 0$. Imposing these constraints guarantees that only ξ are admissible, which entails $h_\mu^\mu = 0$. This can therefore be added as a further gauge fixing.

This additional constraints implies $\partial_\mu h^{\mu\nu} = \partial_\mu \gamma^{\mu\nu} = 0$. Hence, in Fourier space[8] this implies that h needs to be transverse, and thus $\partial_\mu \epsilon^{\mu\nu} = 0$. The solutions are then plane waves

$$\epsilon = \Re \left(\epsilon^0_{\mu\nu} e^{\eta^{\mu\nu} k_\mu x_\nu} \right),$$

where the symmetric tensor $\epsilon^0_{\mu\nu}$ is determined by initial conditions and satisfies

$$k^\mu \epsilon^0_{\mu\nu} = 0.$$

This leaves at most six independent quantities to be fixed by initial conditions. However, inserting this into (3.32) at $T = 0$ creates four constraint equations, leaving only two degrees of freedom independent. Note that as a rank two tensor with two independent degrees of freedom, this field carries helicity two, and corresponds to the massless representation of spin two of the Lorentz group. However, this does not originate from carrying a local Lorentz symmetry representation.

To interpret the physical significance, consider the curvature scalar

$$R = \partial_\mu \partial_\nu \gamma^{\mu\nu} - \partial^2 \gamma_\mu^\mu$$

Thus, the curvature of a gravitational wave vanishes.

More interesting is the impact on a geodesic. Consider the case with k along the z-axis. It is then found that a particle moving on a circle on the x-y axis actually moves along an ellipsoid, whose two axes contract and elongate with the period of the gravitational wave. Thus, this a quadrupole movement, and experienced distances change over time. This is also the basis for the detection of gravitational waves. This is not surprising, as the gravitational wave is itself spin two.

Just as with electrodynamics, it is often not-possible to analytically determine the emission of gravitational wave. However, as a spin two quantity is emitted, uint is found

[8] Note that Fourier space in the linearized theory is well defined, as η is used, up to this order, in the Fourier transformation (2.29).

that an matter oscillation of quadrupole nature is generically needed. Other than that, observable gravitational waves require currently very strong sources, such a merging black holes. This can, again, not be described analytically, but only numerically.

Beyond the linear approximation gravitational waves are no longer linear. Thus, the superposition principle fails, and the gravitational wave interacts with itself. This can lead to a wide variety of effects. One is the possibility whether self-stabilized traveling solutions of non-vanishing and non-singular curvature exists, so-called geons. It is found that such solutions are not stable, and dissipate. The other is that a passing gravitational wave could permanently alter the metric of events which it passes. This is indeed found to be possible, and is called the memory effect. Although the picture is rather that the gravitational wave, as a metric fluctuations, is never really done, but has a temporally stable wake. However, this has not yet been detected, but would, e. g., be visible in an alteration of the inference pattern of multiple pulsars after a gravitational wave has passed in between them.

Chapter 4

Quantizing gravity

4.1 Quantization

The success of quantizing classical theories using the correspondence principle suggests that the same should be possible with general relativity. This will be discussed here following a path-integral approach. While it is believed that an approach using canonical quantization should be possible as well, its reliance on a time-evolution operator makes it less obvious how to proceed. There are two fundamental questions to be answered before going into technical details.

The first question is, whether this is necessary. The primary answer to this is Einstein's equation with matter, (2.22). If the right-hand side is determined by a quantum field theory, establishing a connection to the left-hand side appears to mandate the usage of quantum theory for gravity as well. Likewise, the appearance of singularities in black holes or big bang cosmology at short distances suggests their resolution in the same way as of singularities in classical mechanics and electrodynamics as well.

While suggestive, this may not be necessarily so. Consider, e. g., the possibility that in (2.22) the right-hand side is replaced by an expectation value. Of course, this would still be a coupled self-consistency equation. But it would be solved by solving the quantized matter equation in a fixed topology such that (2.22) with an expectation value would be solved. This satisfies the need to include the back reaction without requiring to quantize matter, at the expense of a very involved self-consistency problem, and giving up a unified Lagrangian formalism. However, given the current experimental situation, such a solution cannot be ruled out. Likewise, quantum field theory is the current best description of fundamental interactions. It cannot be excluded that it needs to be replaced by something else when reaching Planckian dimensions. Again, experimentally this cannot be decided yet. Again, this sacrifices simplicity.

This does not mean, as will be seen, that quantizing gravity ordinarily is simple. Far from it. In fact, no quantization is yet available which can rival quantum-field theory in flat space-time in terms of formal qualities. In addition, there is not strong hope yet that experimental evidence will decide in either direction any time soon, and probably not even within the century. Thus, any attempt will likely have to rely on theoretical guidance and the correct sub-Planckian limit only.

The following is thus a highly subjective selection of possibilities to quantize gravity. The present chapter will largely deal with the minimal quantization procedure, i. e. a quantum field theory which quantizes (2.26) using a path integral. Even the next chapter(5) will not move substantially beyond the point, but merely will either replace the quantization procedure lightly, e. g. in loop quantum gravity, or exchange the elementary objects, e. g. strings, without fundamentally altering the concepts, as e. g. connecting a classical theory to expectation values would do. This subjective choice is mainly motivated by the idea that first classifying this near-to-standard approaches with respect to their abilities to correctly describe sub-Planckian physics is a resource-saving strategy before opening up the search space too much. Of course, once any experimental hints should be obtained, this would need to become the guiding principle.

This leaves as a second step in a path-integral approach the question open of suitable integration variables, similarly to the case of flat-space quantum field theory. Again, experiment would be the one to decide the possibilities, in as far as they create different quantum theories. Possible choices would be, e. g., the metric or the tetrad. It is not a sufficient argument that, e. g., the tetrad is not needed if only gravity is involved, as it is then superfluous. Likewise, the vector potential is classically unnecessary when moving to QED. Also that the tetrads are more important due to fermions is insufficient, as they can be, albeit non-linearly, reconstructed[1] even in a quantum theory. Finally, a diffeomorphism-invariant quantity may also be considered, despite the complications it triggers.

Here, not even an argument of resource-saving can be made. At the moment, many of those options appear equally involved. Thus, here the arbitrary choice of the metric as integration variable is made. It appears that many qualitative features will not hinge on this, but quantitatively this may be central eventually.

[1] Though more extended constructions may yield different arguments, see e. g supergravity in section 5.4.

The choice of metric brings with it that the elementary degrees of freedom are not diffeomorphism invariant. Using the tetrad would have the same consequence. Akin to the situation with the vector potential, this implies that it is necessary to either consider only diffeomorphism (and local Lorentz-symmetry) invariant quantities or to gauge fix the calculation. While this choice is conceptually not that different from the situation in flat-space quantum field theory, there are two aspects, which need to be considered in this context.

The first is the one of gauge fixing. While it will be seen that basically a Faddeev-Popov-like approach is feasible, it is actually the gauge condition, which yields a problem. Absence of torsion and covariant constancy eliminate six and four, respectively, degrees of freedom of the sixteen degrees of freedom of the metric. This leaves four more degrees of freedom, which originate from four translation symmetries. It should also be such that the induced breaking fo diffeomorphism symmetry is valid for all coordinate systems[2]. This leaves only the possibility to find a condition of type

$$C_\rho\left(g_{\mu\nu}, \partial_\mu, \{n_\mu\}\right) = 0,$$

where the n are a set of fixed vectors. However, this requires contractions of the metric. Consider the Haywood gauge as a typical example,

$$0 = \partial^\mu g_{\mu\nu} = g^{\mu\rho}\partial_\rho g_{\mu\nu}$$

the source of the problem becomes apparent: The gauge condition is non-linear, and especially involves in general the inverse of the metric. Thus, a gauge condition of this type in quantum gravity can never be linear.

In principle, this is a technical issue, not a conceptual one. However, it needs to be borne in mind in the following.

4.2 Perturbation theory

One of the experimental insights of general relativity is that our universe can very well be described by a fixed metric and small local fluctuations around it. This is a remarkable fact, as the metric is diffeomorphism-dependent. Thus, there exists a preferred gauge

[2]In principle, it appears possible to work with preferred frames. However, this is already in flat-space gauge theories so involved, and not even entirely clear whether possible non-perturbatively, that this will not be considered here.

choice which minimizes fluctuations. Of course, this choice is dependent on both the parameters of general relativity as well as the matter content of the universe. Still, it exists.

A similar situation is already known in particle physics, in terms of the electroweak sector of the standard model. There, also a preferred gauge exist, in which fluctuations become locally small. This is known as the Brout-Englert-Higgs effect. It is, however, in so far different as it does not directly affects the gauge fields, which fluctuate weakly around zero, but a matter field, the Higgs field. Also, in contrast to general relativity, the field is constant, rather the involved metric from section 3.4. Still, conceptually, both situations are equivalent. This suggests to treat it similarly. Thus, a perturbative expansion appears like a sound choice.

However, a perturbative approach faces the problem that the inverse appears and that the split into the dominant part and the fluctuating part does not yield a fluctuation metric, but rather a non-metric field, similar to the situation with gravitational waves in section 3.8.

Assume a split like in section 3.8 as

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \tag{4.1}$$

where $\gamma_m n$ is assumed small enough that it does not affect the signature. It is also for simplicity to expand around the Minkowski metric[3] and setting $\Lambda = 0$ for now. This is partly to avoid the fact that the actual relevant metric from section 3.4 would break many isometries, making things technically even more involved. This yields for the inverse fluctuation field $\gamma^{-1}_{\mu\nu}$

$$\gamma^{\mu\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\rho}g^{\rho\nu}$$

which is a self-consistency equation. Expanding g finally yields

$$\gamma^{\mu\nu} = -\eta^{\rho\nu}\eta^{\mu\sigma}\gamma_{\sigma\rho} + \eta^{\rho\nu}\eta^{\mu\sigma}\gamma_{\sigma\alpha}\eta^{\alpha\beta}\gamma_{\beta\rho} - \mathcal{O}(\gamma^3). \tag{4.2}$$

Thus, the inverse γ is not independent, but entirely determined by γ. At leading order in γ, it becomes linear, and indices are manipulated with the Minkowski metric.

Thus, even when assuming that the expansion (4.2) is justified, the resulting theory will have an infinite number of terms in the Lagrangian. And even if this is constructed

[3] This creates automatically a foliation in which lapse vector and shift vector are only slightly different from their Minkowski counterparts.

only order-by-order in (4.2), this becomes involved. Still, this appears possible, though has not been explored too far[4].

A possible perturbative approach can now be constructed as following. Perform the split (4.1) perturbatively, and perform (4.2) only to leading order. Then quantize the theory with $\gamma_{\mu\nu}$ as integration variable[5]. Since the (equally truncated) gauge symmetry remains, this requires a quantization. However, because

$$\eta^{\mu\rho}\partial_\rho\gamma_{\mu\nu} = 0$$

is now possible and linear, the quantization can be performed essentially following the Faddeev-Popov procedure[6]. This yields a ghost term

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= \bar{c}_\nu\partial_\mu\left(D^{\nu\mu}_\sigma + D^{\mu\nu}_\sigma\right)c^\sigma\\ D^\rho_{\mu\nu} &= \partial_\mu\delta^\rho_\nu - \Gamma^\rho_{\mu\nu}.\end{aligned}$$

Note that the ghosts carry a vector index, as basically one for each of the diffeomorphism symmetries is necessary. This implies that they coupled non-trivially to the affine connection, and thereby non-trivially to the fluctuation metric γ. From this point on onward, the remainder follows as in ordinary perturbation theory. Especially, the existence of a metric η allows to define (four-)momenta by a Fourier transformation. This yields the usual Feynman rules and the possibility to describe (graviton) scattering[7].

As the quantization procedure hinges on the introduction of the Minkowski metric in (4.1). Thus, the quantized theory is background-dependent, and the whole formulation is. As a consequence, diffeomorphism symmetry is explicitly broken, by the truncation of (4.2) even before quantization. Thus, the quantum theory will no longer exhibit full diffeomorphism symmetry, but at best perturbatively. Such a background dependency is often a consequence of an approximate treatment. Even if in a formalism no approximations are made, there can be made arguments, e. g. by the arbitrary size of

[4]In combination with considering as physical states only manifestly diffeomorphism-invariant quantities, this leads to the concept of augmented perturbation theory, which is beyond the scope of this lecture

[5]If the relation would be exact, this would be fine as the path integral is translation invariant. However, due to the use of (4.2), this is actually an inequivalent theory. Essentially, this yields a non-trivial Jacobian, as it is not a pure shift, which is neglected.

[6]Ignoring any potential Gribov-type ambiguities are neglected on the same footing as the remainder terms of (4.2).

[7]Note that the definition of the LSZ formalism hinges on the possibility to define asymptotic states at space-like infinity with respect to each other. This is possible for the Minkowski metric, but a split of (4.1) with a finite spatial manifold could obstruct this. This is beyond the scope of this lecture.

quantum fluctuations, which call into question prescriptions like (4.1). Thus, such a background-dependent approach requires further proof of its validity.

Circumstantial evidence by results in agreement with experiment would be, of course, a good start. Here, however, a serious problem arises. While tree-level calculations work out well, problems arise at loop level. The ratio of free propagation to a tree-level exchange of a graviton is essentially given by the interaction strength of gravity times a free graviton propagator, which is essentially given by the inverse of $G_N E^2$, with E the energy of the graviton. Thus the corresponding ratio is

$$\frac{A_{\text{free}}}{A_{1g}} = \frac{\hbar c^5}{G_N E^2} = \frac{M_P^2}{E^2}$$

where $M_P^2 = \hbar c^5 / G_N$ is the Planck mass, this time in standard units. Since M_P is of the order of 10^{19} GeV, this effect is negligible for typical experimental energies of at most order TeV. However, if the energy becomes much larger than the scale, the ratio of free propagation to exchange of a graviton becomes much smaller than one, indicating the breakdown of perturbation theory.

This is not cured by higher order effects. E. g., in case of the two-graviton exchange, the corresponding amplitude ratio becomes

$$\frac{A_{2g}}{A_{\text{free}}} \sim (\hbar G_N)^2 \sum_{\text{Intermediate states}} \int_0^E dE' E'^3 \sim \frac{1}{M_P^4} \int dE' E'^3 \to \infty \text{ for } E \to \infty \tag{4.3}$$

This gets even worse with each higher order of perturbation theory. Thus, such a perturbative investigation completely fails for quantum gravity beyond tree-level.

If this is an artifact of the additional assumptions in comparison to flat-space perturbation theory is an open question. While, e. g., in naive power counting the situation should get worse when including more terms in (4.2), this also restores background independence piece-by-piece. Thus, it is conceivable that improving perturbation theory in this or some other way could yield at least a valid perturbative description in the sense of an asymptotic series. However, the technical complexity has restricted this avenue at moment to a most exploratory investigations.

4.3 Beyond the Einstein-Hilbert Lagrangian

It is, of course, entirely possible that the failure of perturbation theory in section 4.2 is not an artifact of the treatment, but rather genuine. A possible argument in favor of this is the appearance of the dimensionful coupling constants κ in the classical Lagrangian, which would make the theory non-renormalizable by power counting. While renormalizability by power counting is again a perturbative statement in itself, it has been reasonable accurate in the past to estimate the validity of a theory at the quantum level. Though deviations in both directions are known.

Taking the argument at face value, this suggests that a different Lagrangian may be necessary to quantize gravity. Of course, such a change cannot alter too much the sub-Planckian physics, which is constrained by observations in the classical regime. Still, this leaves surprisingly many options.

The more extreme case is to add further fields. The most direct example is to make the spin connection of section 2.4 an independent degree of freedom. This introduces torsion into the metric, and breaks the equivalence of the affine connection and the Christoffel symbol. In fact, this is necessary when considering supersymmetry, see section 5.4. The treatment of this option will therefore be relegated to this section. However, the effects of this additional field can be made negligible small in the sub-Planckian regime.

Other fields are not as straight-forward. Of course, it can be attempted to use matter to stabilize the system. Or genuinely new degrees of freedom, akin, but not equivalent, to the metric and the spin connection. Perturbatively, the former cannot cure the loss of power-counting renormalizability. The latter, on the other hand, is extremely speculative, as no objects corresponding to them are (yet) known experimentally. Thus, the options in this direction are limited.

The other option is, of course, to dispose of perturbation theory entirely, and considering the problems as a pure artifact of perturbation theory. This will be taken up in sections 4.4 and 4.5. Indeed, there is substantial circumstantial evidence that this works. However, even if it does work theoretically, again the lack of experimental evidence makes it hard to finally decide. So, for the remainder of this section only perturbative remedies will be considered.

Thus leaves mainly matter fields and alterations of the gravity part. Concerning matter, this implies that the divergent loop integrals need to be canceled. This requires loops integrals of opposite sign. As in flat-space quantum field theories, this requires fermions. Ultimately, this pushes towards supergravity theories to have build-in cancellations, rather than accidental ones. This option will therefore be postponed to section 5.4.

This leaves modification of the Einstein-Hilbert action. Restricting to polynomial actions, this implies that the action can be only a polynomial of the (in four dimensions) 14 independent scalars, which had been briefly introduced in section 3.5. Of these, it has become quite useful to encode the ten build from the Ricci tensor in the Weyl tensor, which is defined as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(R_{\mu\rho}g_{\nu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho} - R_{\nu\sigma}g_{\mu\rho}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R.$$

Adding terms like

$$\mathcal{L}_W = aC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

with an adjustable constant a will provide additional terms at leading order. Judicious choice of the free constants, like of a, allows to modify the properties such that divergencies can be pushed to very high, trans-Planckian scales or partly absorbed. Thereby, it allows the theory be more well behaved.

The simplest such modification is known as $f(R)$ gravity, which, at vanishing cosmological constant, replaces R by some function $f(R)$ in the Einstein-Hilbert Lagrangian[8] (2.25). Assuming that $f(R)$ is a polynomial, truncating it at quadratic order yields Starobinsky gravity. Observationally, the expansion coefficient can be chosen such that no observable consequences arise at sub-Planckian scales. But in general, this gives rise to a massive additional mode in the gravitational waves of section 3.8, which may be observed.

There are many similar extensions, like Gauss-Bonnet gravity or Lovelock gravity, which combined different of these tensors in different ways. In general, none of these completely solves the issue of non-renormalizability, but essentially all can be made consistent with observational constraints. Thus, none of them offers a unique way out.

4.4 Dynamical triangulation

As perturbation theory has even in flat space-time quantum field theory shortcomings even at arbitrarily weak couplings, especially in gauge theories, it appears well possible that non-perturbative methods are required. This leaves the problem to non-perturbatively define a quantum theory of gravity. Even in flat space-time quantum

[8] There is a variant flavor, the so-called Palantini formalism, which derives the Einstein's equation not by varying the metric but the metric and affine connection independently, without introducing torsion. This does so far not yield superior results, and may even be in conflict with sub-Planckian observations. It will therefore not be considered further here.

gauge theories this is a formidable endeavor, and has not been concluded at full mathematical rigor. In this section and section 4.5 two approaches will be briefly discussed.

The mathematically best defined approach in flat space-time quantum gauge theories is the usage of a lattice regularization of space-time and taking a continuum limit. This boils the problem down to show the existence of the limit. This is possible for non-interacting theories and some exactly solvable special cases, mostly two-dimensional theories. In addition, even for theories where the existence of the limit is doubtful, this can often be shown to not affect low-energy properties. Finally, the circumstantial evidence due the success of the approach for the strong interactions and electroweak interactions in comparison to experiment makes it very likely that this approach is viable.

The basic idea is to actually sum over all space-time histories of fields in the path integral, either by analytical methods or, most commonly, by numerical Monte-Carlo methods. This meets two challenges for gravity.

The first that space-time itself needs to be sampled. This requires to sample all possible topologies on a set of events. As usually a numerical approach is needed, this requires to somehow make the effort finite. This can be done by reducing the problem to discrete topologies. For this purpose, Regge calculus is useful.

Regge calculus is basically the extension of the idea of triangulation of surfaces to (hyper-)volumes of space-time on Lorentzian manifolds. It can be shown that that an arbitrary space-time, which is admissible in classical general relativity, can be discretized in terms of simplices, basically the generalization of triangles. However, edges of simplices can be either time-like or space-like, though not light-like. The time-like length and space-like lengths do not need to, and in general do not, coincide, though all space-like edges do. Therefore, isometries are also discretized.

Therefore, at every vertex, where edges of simplices meet, there is a local discrete light-cone. Directing the time-like edges and space-like edges accordingly, this creates foliation, where the time-like edges create the lapse vectors. The coordination number of vertices can then be found to provide the local deficit angle, i. e. how close the angles of a triangle come to π, and thus the type of local curvature. However, in terms of a curvature density the nature of this quantity is not well-defined, and other measures are needed to do so. The path integral over the metric then becomes a sum over such discrete topologies, providing a similar level of mathematical definition as for flat-space quantum gauge theories. The action itself can then be recast into a sum over the number of simplices. In the simplest form, using the Einstein-Hilbert Lagrangian, it reads as

$$S = \sum \left(aN + bN^{1,4} + cN^{2,3}\right),$$

where N is the number of simplices and $N^{1,4}$ and $N^{2,3}$ are simplices with one time-like edge and two time-like edges, respectively. The coefficients are the unrenormalized couplings, which have to be fixed in the same way as in flat space-time quantum gauge theories, and especially cannot be read as tree-level quantities. Adding matter in this formalism is relatively straight-forward, and will not be detailed here.

Such a formulation does not offer a metric, nor does it need one. In principle, it should be possible to reconstruct the (discrete) metric from the information. However, not yet a constructive algorithm to do so is known. On the other hand, geodesic distances and diffeomorphism-invariant quantities can be constructed, though are not often simple to do so. The main reason is that in absence of a metric and coordinates differentials are not easy to reconstruct.

As an example consider the curvature scalar. It is a diffeomorphism-invariant quantity, but usually involves derivatives of the metric to compute. To avoid this, an alternative is the quantum Ricci curvature. It basically considers two (overlapping) $d-1$ dimensional δ-balls B_i and considers all pairwise geodesic distances. Such a ball is constructed by selecting a center event, and include all points, which have a geodesic distance less than δ to it. It then averages over all point-to-point geodesic distances in this overlap, normalized to the volume.

This can be simplified by considering that both balls are identical. Then this is the average over all distances inside the ball. This yields

$$\frac{\bar{d}}{\delta} = \frac{1}{N^2} \sum_{q,q' \in B_\delta} \frac{d(q,q')}{\delta} \approx a + b\delta^2 R + \mathcal{O}(\delta^3)$$

where N is the number of events inside the ball, d is the pairwise distance, and a and b are known constants depending on the dimensionality. Thus, the quadratic coefficient as a function of the sphere size is the local curvature at the center of the ball. While this is a straightforward operative definition, it is still far from the usual expression of the curvature scalar. Corresponding objects can be constructed for the other invariants, though have only been done to a limited extent so far.

With such tools in hand, in principle arbitrary diffeomorphism-invariant quantities can be calculated. Of course, analytical calculations are quite non-trivial in such a construction. However, this lends itself to numerical simulations. For this purpose, a Wick rotation is performed. There are basically two possibilities. One is that the Wick rotation is done for the complete manifold, irrespective of the foliation. This leads to Euclidean dynamical triangulation. Or, it respects the foliation, and thus rotates only the temporal links. This leads to causal dynamical triangulation. As a consequence, the set of discretized manifolds is different in both cases, with causal dynamical triangulations using only a subset of the ones of Euclidean dynamical triangulation. Since this corresponds to a different integration range, these are, in principle, two different quantization prescriptions for the theory. However, similar results are claimed, though for Euclidean dynamical triangulation this requires to add further terms to the Einstein-Hilbert Lagrangian. As these are developing approaches, with causal dynamical triangulation having seen substantially more resources invested, there is no final answer yet. Still, in both cases numerical Monte Carlo calculations of the path integral become possible, akin to lattice calculation in flat-space time.

At any rate, a few common, persistent results are found. Two are of particular relevance.

The first is that a region of bare parameter exists, in which the resulting theory show de Sitter like structures, with the possible existence of a continuum limit, i. e. that the edge lengths of the simplices goes to zero and the discretized manifolds become effectively continous. This is not trivial, as for other values never structures emerge, which are extended. Extension has here to be understood in the sense of maximal geodesic distances possible.

This should not be confused with the having a de Sitter universe. After all, it is a path integral. Thus, it integrates over full manifolds, which in itself are valid[9] complete universes on their own. The result rather states that when averaging over all possible universes with the path integral measure, the expectation values build from correlations of geodesic distances agree with those expected from a single classical de Sitter universe. It is literally the weighted average over all possible universes, and it is the weight factor, which decides this outcome, and in turn the parameters. However, it is also an important insight that it is not possible to create an anti-de Sitter universe in this way, nor Minkowski space-time. Thus, for the time being, only a renormalized positive cosmological constant seems to yield an extended universe.

This same appears to be true once the condition of foliation is relaxed. The dominating field configurations are then those, which are approximately foliated again. Thus,

[9]Not in the sense of classical general relativity, as they are not solutions of Einstein's equation (2.21), just like with flat-space quantum field theory.

it appears that also foliation is a dynamically preferred structure, in agreement with observation of our universe.

The second amazing result comes from the geodesic distances accumulated from random walks. While the underlying manifold is four dimensional, this is not necessarily giving a description of the effective structure of a universe, and especially not of the experienced distances. An alternative is rather to ask what distance s a random walk has accumulated after a fixed set of steps t. In general, this distance is given after t steps of equal length l by

$$s = l\sqrt{\frac{2t}{d}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

Thus, the behavior is always the same, but the prefactor changes from $\sqrt{2/\pi}$, $\sqrt{\pi}/2$, $\sqrt{8/3\pi}$ to $\sqrt{9\pi/32}$ from one to four dimensions, respectively. It is found that it changes continuously as a function of th number of vertices traversed from the value for four dimensions at large distances s to two at short distances s. Thus, the effective dimensionality experienced changes as a function of distance.

While this is remarkable in itself, this also strikingly illustrates how the problems encountered in section 4.2 are resolved. In two space-time dimensions, the quantum theory of general relativity is not only trivial, but actually finite, as Newton's constant becomes dimensionless. Thus, by having an effectively two-dimensional behavior at short distances, ultraviolet fluctuations are suppressed, thus dynamically stabilizing the theory and making it consistent.

Still, why all of this is drawing a remarkably self-consistent scenario, this is not yet firmly established. And, even if this will be the case, it is still not ensured that this is actually the way how quantum gravity is realized in nature.

4.5 Functional renormalization group

The combination of section 4.2 and 4.4 suggest that quantum gravity may only be weakly non-perturbative, and thus approximate methods may still work.

In principle, the gauge-fixed theory of section 4.2 does not need to be treated perturbatively[10], but can also be treated non-perturbatively. As in flat space-time functional

[10]It may be that this is not quite correct due to the existence of obstruction to the type of gauge fixed conditions employed beyond perturbation theory, a so-called Gribov-Singer ambiguity. This is not yet entirely settled, and is mainly due to the potential existence of transfer functions. This issue will be ignored here.

methods are a method to do so.

The method with the biggest success so far is the functional renormalization group. It basically starts out from the reconstruction theorem of the quantum effective action,

$$\Gamma = \int d^dx_1\sqrt{-\det g(x_1)}...d^dx_n\sqrt{-\det g(x_n)}\Gamma^n_{\mu_1...\mu_m}(x_1,...,x_n)g^{\mu_1\mu_2}(x_1)...g^{\mu_{m-1}m}.$$

This is supplemented by a regulator term

$$\Gamma_\Delta = \int d^dx\sqrt{-\det g}g^{\mu\nu}R_{\mu\nu\rho\sigma}g^{\rho\sigma}.$$

Since the situation is gauge-fixed, it is possible to introduce a Fourier space, and also perform a Wick rotation. This allows to introduce a scale Δ, which adjusts the regulator R such that it dampens fluctuation at momenta large compared to Δ, and vanishes for $k \to 0$. Since this regulator has been added arbitrarily, it can be treated like a source. Especially, it can be equipped with a scale $p \equiv \Lambda$, with which it can be switched off. Thus, similarly to sources, correlation functions Γ^n can now be defined by functional derivatives with respect to the regulator at vanishing regulator. This yields a set of self-consistent integro-differential equations. As differential equations, they need furthermore an initial conditions. One possible choice is that at some scale all correlation functions coincide with their tree-level value, thereby having most of them vanishing.

The so obtains equations form an infinite tower, which in most theories cannot be solved exactly, not even numerically. To solve them, it is therefore necessary to perform truncation, i. e. eliminating an infinite amount of information from the unknown Γ^n, usually by setting an infinite number of them to zero, and possibly restricting their dependence on the arguments.

It is now a remarkable result that for the Einstein-Hilbert Lagrangian as initial conditions and constant Γ^n up to $n = 4$ a highly non-trivial solutions arises. While the absence of arguments make this theory formally conformal, its lingering dependence on the renormalization scale can be used as a proxy for momentum dependence, even though strictly speaking only the limit $p \to 0$ is physical. This turns the system into a set of ordinary partial differential equations, which can be solved.

These assumptions yield a running Newton coupling, which vanishes at low scales, and moves to a finite scale at infinity. Thus, the gravitational self-coupling is limited by non-linear effects. Such a feature is known as (ultraviolet) asymptotic freedom. Very much alike to the dimensional reduction in section 4.4 this makes the theory ultraviolet renormalizable, and thus sensible, fixing the problem of perturbation theory in section

4.2. Since the coupling is actually not only bounded but constant at substantially trans-Planckian scales, the theory becomes, in fact, asymptotically conformal, and thus interactions cease at short distances. Though the required regularization prevents it from being ultraviolet finite.

This so-called Reuter fixed point turns out to persist qualitatively beyond this very simple truncation, i. e. when adding further correlation functions and also momentum dependency. When allowing for a cosmological constant, it also shows the interesting behavior to become small on long scales, thereby providing an opportunity to explain why it is small.

While a direct comparison, due to the very different entities, to the results in section 4.4 is not simple, some evidence has accumulated that both are in fact compatible results. This can be mainly gleaned from the kinematic behavior, which can be related to differential operators, which in turn can be related via their gauge-invariant spectrum to the dimensionality of random walks. This indicates also a dimensional reduction behavior. Furthermore, the correlation functions are close to what one would expect for correlation functions on a de Sitter background, furthermore supporting a compatibility.

Thus, two very different approaches appear to converge to the same theory and scenario, which is consistent and explains some observed features. Again, this is encouraging, but not yet a final answer.

4.6 Matter beyond scalars

Adding matter is, in principle, in quantum gravity as straightforward as in general relativity in section 2.3. The Lagrangian gets more terms, and more fields to integrate over. For scalars, this is basically it. Though additional couplings are possible, e. g. to the curvature scalar or other scalars, which have not been possible in flat space-time, as their these vanish.

At the quantum level, they merely add a new level of complications of technical nature. Especially, they make the divergencies in perturbation theory of section 4.2 worse, leading to a breakdown at lower orders. There is also some hints that too much matter will destabilize the asymptotic safety scenario seen in section 4.5. However, given the empirical known matter, this seems to be of little concern, and especially is self-consistent.

The situation for fermions is a bit more involved, as the discussion in section 2.4

showed. Especially, this introduces the auxiliary Lorentz gauge symmetry. Without an independent torsion field, however, it is linked by the spin connection. As a consequence, gauge-fixing diffeomorphism symmetry will also required to gauge fix torsion.

However, this fundamentally alters the nature of spin. It is thus locally possible to rotate spin up to spin down. This implies that spin states can no longer be measure absolutely. It is always necessary to build diffeomorphism and local Lorentz symmetric combinations to obtain meaningful objects involving spin. While this provides a well-defined quantity, this leaves open the question why in a quantum theory spin should be a good observable.

This is still a largely open question. One possibility is to assume a Brout-Englert-Higgs effect taking place in quantum gravity. I. e., the path integral is dominated by some metric configurations g^c, with only small fluctuations γ around it. If this is the case, gauge-fixing can capitalize on this feature. Choosing such a gauge that disturbs this feature as little as possible, it allows for an expansion, after gauge fixing, in such fluctuations. E. g., for g^c being one of the maximally symmetric solutions from section 3.3, this is the case for Haywood gauge.

As a consequence, diffeomorphism-invariant quantities can now be, in a fixed gauge, expanded in the fluctuations, the Fröhlich-Morchio-Strocchi mechanism. This yields, e. g. geodesic distances r

$$\begin{aligned} r &= \left\langle \min_{z(t)} \int_x^y dt g^{\mu\nu} \frac{dz_\mu(t)}{dt}\frac{dz_\nu(t)}{dt} \right\rangle \\ &= \min_{z(t)} \int_x^y dt g^c_{\mu\nu} \frac{dz^\mu(t)}{dt}\frac{dz^\nu(t)}{dt} + \left\langle \min_{z(t)} \int_x^y dt \gamma_{\mu\nu} \frac{dz^\mu(t)}{dt}\frac{dz^\nu(t)}{dt} \right\rangle = r^c + \rho. \end{aligned} \tag{4.4}$$

and curvatures

$$\frac{\left\langle \int d^4x \det(e) R(x) \right\rangle}{\left\langle \int d^4x \det(e) \right\rangle} = 6\Lambda + \mathcal{O}(\gamma)$$

(close to) the classical values r^c. This is in agreement with section 4.4, and therefore appears to a reasonable approximation.

The same is done with torsion and the spin connection. Using a suitable operator, this should give a decent description of an effective spin. However, it turns out that such an operator is not easy to construct with an ordinary fermion only. This will be taken up again in section 5.4. In addition, this will take care of the situation of higher-spin fermions, which face the same problems.

This leaves the case of spin one, and thus necessarily gauge fields. In principle, one would expect them to fall into the same category as before, and thus need to couple to the spin connection, as for fermions. However, take a Abelian field. Its gauge transformation in flat space time is

$$A_\mu \to A_\mu + \partial_\mu g,$$

with some scalar function g. If the gauge field would carry a non-trivial representation of the local Lorentz symmetry, then it need also to transform as

$$A_s \to A_s + (\delta^{rs} + \omega^{ij} f_{ij}^{rs})A_s$$

and thus to carry two indices, $\mu(\sim m$ in tangent space) and s, different from what ordinarily happens, as g (or ∂_μ) do not carry a non-trivial representation. Hence, gauge fields will not couple non-trivially to the spin connection, in contrast to fermions.

That appears odd that they need to be treated differently. However, remembering that gauge fields can be considered to be auxiliary degrees of freedom, which are introduced to localize theories, this makes more sense. They can therefore not independently couple to the spin connection, but only via the intermediary coupling of the fields of which they need to localize their interactions.

Chapter 5

Beyond canonical quantum gravity

Chapter 4 considered the case that quantum gravity is essentially an ordinary quantum field theory. It could, of course, be very different. A few of the ideas will be given here, which relax the conventionality by different degrees.

5.1 Causal sets

One of the important observations was that 'time' on a manifold is a far less well-defined concept. While it is always possibly to locally distinguish future time-like and past time-like, due to the local equivalence of the topology to the Minkowski topology, this is globally not possible. Only the introduction of a foliation in section 3.1 reestablished even the possibility of a global time direction. When this should be maintained at the quantum level is an open question.

However, the local time direction allows for the introduction of a local causal structure. Taking this as a principle of construction leads to the idea of causal sets. A causal set is a set of elements with elements being either not related, or being causally related, the later as cause of effect. Thus, like in the local light cone time-like forward or time-like backward. This generates a directed graph of causal relations, a causal set. This does not yet entail any spatial relations. Note that even in special relativity in fact space-like separations only serve to identify objects, which are not in causal contact.

The first question to pose is whether this yields an equivalent physics to general relativity. This poses the question whether the causal relations can be seen as equivalent to a manifold structure. In principle, such a mapping is possible. The lack of space-like information, however, makes it difficult to state whether such a mapping can be unique.

At the moment, it is assumed that on sufficiently long distances, any ambiguity will wash out, and lead to a unique manifold.

At the same time, the formulation of a dynamical principle requires now a different approach, as there is no direct equivalent to the space-like derivatives appearing in Einstein's equation. This appears to be possible as well, at least at long distances. However, that would be enough, as at sufficiently short distances no experimental results are yet available, which could either choose a manifold structure or a causal set structure.

Causal sets offer a different discretization of the the causal structure compared to Regge calculus. However, in spirit, this allows a path integral construction as well. Instead of an integral over (reggerized) manifolds, the integral is performed over causal sets.

This approach has a different ontological structure than other approach to quantum gravity, by reducing it to the only necessary elements, the cause and effect. Any manifold can certainly be translated also in such a structure, again with possible ambiguities, which are not relevant to the causal structure. It remains to be seen, whether this yields really an equivalent description, or somehow eliminates irrelevant information from (quantum) gravity. In the former case, this would be merely a reformulation of chapter 4.

5.2 Loop-quantum gravity

In contrast to chapter 4, loop quantum gravity goes a step further, and postulates that quantum gravity cannot be canonically quantized. Rather, different variables need to be used for quantization. Especially, the basic requirement is that the degrees of freedom in the path integral to be integrated over are diffeomorphism, i. e. coordinate-transformation, invariant.

This avoids many conceptually tricky problems, which are similar to those arising in (non-)Abelian gauge theories. In fact, a similar reformulation exists also for ordinary non-Abelian gauge theories, and thus it appears in principle possible. In the latter case, the gauge-invariant degrees of freedom are so-called Wilson loops, exponentiated line-integrals over gauge-fields. In the same way the new variables are loop integrals over the metric, and thus the name. However, the downside is that the ensuing theory is much more involved, and contains a substantial, probably infinite, number of degrees of freedom and potential non-localities. This makes work with this theory, even at the perturbative level, very much more involved. In particular, it may even be only possible in a genuine non-perturbative way.

It starts by defining the Ashtekar variables, which are the set of the normalized tetrads on the space-like hypersurfaces, i. e. indices only running from 1 to 3, and the time-like ones unchanged,

$$\epsilon_\mu^i = \sqrt{-\det h} e_\mu^i,$$

where h is the metric on the spatial hypersurface, chiral spin connection

$$\begin{aligned} A_\mu^i &= \Gamma_\mu^i + \beta K_\mu^i \\ \Gamma_\mu^i = &= \epsilon^{ijk}\Gamma_{\mu jk} \\ K_\mu^i &= K_{ij} e_\mu^j \\ K_{ij} &= \frac{\delta \mathcal{L}_{\mathrm{EH}}}{\delta h^{ij}} \end{aligned}$$

which turn out to be conjugated to each other, with the normalized tetrads playing the role of the canonical conjugate momenta. As the spatial part carries an SU(2) representation, such spatial variables are SU(2) valued.

From this holonomies are constructed as

$$h_\mu[\mathcal{C}] = \exp\left(-i \int\limits_{\mathcal{C}} ds A_\mu^i \tau^i\right), \tag{5.1}$$

where the τ^i are the Pauli matrices. This expression is known as a Wilson line in gauge theories. They are diffeomorphism and local Lorentz invariant, and thus offers the observable.

The Hamilton density can now be expressed in terms of the Asthekar variables as

$$\begin{aligned} H &= \frac{\epsilon_k^{ij} F_{ab}^k \epsilon_i^a \epsilon_j^b}{\sqrt{-\det h}} - 2\frac{\beta^2+1}{\beta^2}\frac{\epsilon_i^a\epsilon_j^b - \epsilon_j^a\epsilon_i^b}{\sqrt{-\det h}}(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) \\ F_{ab}^k &= \partial_a A_b^k - \partial_b A_a^k + \epsilon^{cdk} A_a^c A_b^d. \end{aligned}$$

With such a Hamilton function in hand, the system can be promoted to a quantum theory by taking this to be the Hamilton operator, with wave functions $\Psi(A)$.

The wave functions can be rewritten in a basis using the holonomies (5.1), as

$$\Psi[\mathcal{C}] = \int dA_i^\mu \Psi^i(A) h_\mu[\mathcal{C}].$$

This acts similar to a Fourier transformation, and can be likewise be applied.

As a consequence, the wave function, and other objects, can be expanded in terms of the loops of the holonomies, allowing to interpret solutions as networks of loops. As they are associated with SU(2), these are also known as spin networks or spin foams, and roughly corresponds to trajectories of spin 1/2 particles in the space-time structure.

While this process is technically challenging, the expansion in terms of holonomies gives a manifestly diffeomorphism-invariant quantization and observables. It thus never involves the metric in any way. However, the technical complexity has limited progress in this respect.

5.3 Non-commutative geometry

One further possibility to quantize gravity is to postulate the existence of a minimal length, similar to the postulate of a minimal phase space volume $\Delta x \Delta p \sim \hbar$ in ordinary quantum mechanics. This is also similar to the idea of a maximum speed in general relativity. As there, the existence of such a minimal length, which is typically of the order of the Planck length 10^{-20} fm, has profound consequences for the structure of space-time. Especially, coordinate operators do no longer commute, just like coordinate and momenta do not commute in quantum mechanics, i. e.

$$[X_\mu, X_\nu] = i\theta_{\mu\nu}, \tag{5.2}$$

where θ is an antisymmetric constant matrix. Thus, distances along different directions can no longer be measured simultaneously. This immediately forbids the emergence of singularities, as the uncertainty in position will wash out the possibility to localize a single point, and thus a singularity. It thus motivates the absence of black holes in quantum gravity.

The same effect can be reached by postulating canonical commutation relations for coordinates, in addition to the ones between coordinates and momenta. Thus, this ansatz is called non-commutative geometry. Since there is a minimal length, there is also a maximal energy, and hence all quantities become inherently finite, and renormalization is no longer necessary. On the downside of this approach, besides an enormous increase in technical complexity, is that in general relativity neither coordinates nor energies themselves are any longer physical entities, like in special relativity or in quantum (field) theories. Thus, the precise physical interpretation of a non-commutative geometry is not entirely clear. Furthermore, so far it was not possible to establish a non-commutative theory which, in a satisfactory manner, provides a low-energy limit sufficiently similar to the standard model. Particularly cumbersome is that it is very hard to separate the ultraviolet regime where the non-commutativity becomes manifest and the infrared, where the coordinates should again effectively commute. This problem is known as IR-UV mixing. In particular, this can inject the violation of diffeomorphism symmetry and local Lorentz symmetry inherent in (5.2) into low-energy physics.

An important advantage of non-commutative theories is that it is one of the possibilities that they allow to circumvent the Coleman-Mandula theorem, which forbids any non-trivial combination of gravity and internal symmetry groups. It thus would be a possible path towards a genuine unification of the forces. However, the technical obstructions and the consequences of IR-UV mixing has so far made it not possible to obtain a working implementation consistent with low-energy physics.

5.4 Supergravity

A more fundamental shift is based on the idea to change the very nature of space-time itself, by altering the mathematical algebraic structure underlying it. This can be achieved by altering the Poincaré algebra. By gauging it, this will by extension alter the nature of general relativity. One of the possibilities to do so is supersymmetry.

The conventional Poincaré algebra reads[1]

$$\begin{aligned}
[P_\mu, P_\nu] &= 0\\
[M_{\mu\nu}, M_{\kappa\lambda}] &= \eta_{\mu\kappa} M_{\nu\Lambda} - \eta_{\nu\lambda} M_{\mu\lambda} + \eta_{\nu\lambda} M_{\mu\kappa}\\
[M_{\mu\nu}, P_\kappa] &= \eta_{\mu\kappa} P_\nu - \eta_{\nu\kappa} P_\mu.
\end{aligned}$$

Herein, it should be kept in mind that in the general case, where both symmetries become gauge symmetries, the translation symmetry will be connected to the diffeomorphism symmetry, and the Lorentz symmetry with the one in the tangent space, see section 2.4.

This algebra is then extended by introducing a set of new, anticommuting generators

[1] In four dimensions, the Lorentz algebra part can be reformulated in terms of two SU(2) algebras. This will not be done here.

Q, satisfying the algebra extension

$$\begin{aligned}
\{Q_a, Q_b^\dagger\} &= (\sigma^\mu)_{ab}P_\mu \\
\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[M_{\rho\sigma}, Q_a] &= i\sigma_a^{\mu\nu b}Q_b \\
[T_R, Q_a] &= -i(\sigma_2)_a^b Q_b \\
\sigma_{\mu\nu} &= \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \\
\sigma_\mu &= (1, \sigma_i)_\mu \\
\bar{\sigma}_\mu &= (1, -\sigma_i)_\mu,
\end{aligned}$$

where σ_i are the Pauli matrices. Thus, the Q transform as a Weyl spinor under rotation. There is an additional global symmetry, the R-symmetry, generated by T_R, which is an Abelian phase symmetry. It is possible to introduce multiple independent Q charges, which will not be pursued here.

This can be interpreted as adding additional fermionic dimensions to space-time, i. e. dimensions in which coordinates are given in terms of Grassmann numbers. The supersymmetry generators Q act then as translations in these directions. However, due to the non-trivial mixing, they will also always act as translations in the ordinary dimensions, and it is not possible to unmix both.

Also, the mixing with the Lorentz generators cannot be undone. Thus, representations need to be always happen in the full algebra. A consequence is that irreducible representations of the Lorentz generators are not irreducible representations of supersymmetry. Without going into detail, the new irreducible representations are contain combinations of some irreducible representation of the Lorentz symmetry, e. g. the spin 0 and spin 1/2 or the spin 1/2 and the spin 1. These so-called supermultiplets describe fields, which are dependent on both the ordinary representations and the new additional coordinates. In particle physics terms, a supersymmetric theory contains necessarily both bosons and fermions, which can transform into each other. A more detailed discussion of these aspects can be found in the lecture 'beyond the standard model'.

The mixing of the two algebras implies than when moving to general relativity it is necessary that also the supersymmetry generators generate a local gauge symmetry. The manifold then becomes a supermanifold, with the additional fermionic dimensions as well. The highly non-trivial question is then how to deal with the connection to the local Lorentz symmetry. It is here where the tetrad formulation of section 2.4 comes into play. Considering now the tetrad as the decisive degree of freedom, the Lorentz symmetry involved can be the local Lorentz symmetry, creating a non-trivial link between the previously separate manifold symmetry and tangent space symmetry. The Pauli matrices then link both, and carry indices of both.

However, this implies that for invariant distances the metric needs to be extended itself, by adding a metric acting in the fermionic directions. As this needs still to create a bosonic topology, the metric needs to become a supermetric, with partly fermionic components. This eventually yields the simplest supergravity theory

$$\begin{aligned}
\mathcal{L} &= \frac{\det e}{2\kappa}\left(e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu\Psi_\rho\right)\\
R_{\mu\nu ab} &= \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\omega_{\nu\ b}^{\ c} - \omega_{\nu ac}\omega_{\mu\ b}^{\ c}\\
D_\nu &= \partial_\nu + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\\
\omega_{\nu ab} &= 2e_{\mu[a}\partial_{[\nu}e_{b]}^{\mu]} - e_{\mu[a}e_{b]}^{\sigma}e_{\nu c}\partial^\mu e_\sigma^c
\end{aligned}$$

where the brackets around the indices indicates that the expression has to be antisymmetrized with respect to the same-type indices. It is seen that the covariant derivative couples gravity and the fermionic part of the metric, Ψ, which is a spin 3/2 Rarita-Schwinger field. This theory is therefore coupled. It is noteworthy that the spin connection now appears explicitly, though not dynamically, a sign of the non-trivial connections between diffeomorphism symmetry and local Lorentz symmetry.

The, now local, supersymmetry transformations of the fields are, without proof,

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu\\
\delta\Psi_\mu &= D_\mu\epsilon,
\end{aligned}$$

with the spinor 1/2 field $\epsilon(x)$, which is the gauge transformation functions, and thus arbitrary. This collects both transformations.

A quantization of this simplest supergravity theory proceeds than as with ordinary gravity. Especially, perturbation theory turns out to be much better behaved. However, technically, it is substantially more complicated than ordinary gravity.

5.5 String theory

The following will discuss the quantization of the simplest possible string system, the simple, non-interacting, bosonic string. This will still be a formidable task, and will yield a number of rather generic properties of string theories, like the natural appearance of gravitons, the need for additional dimensions, and the problems encountered with, e.

g., tachyons. In particular the natural appearance of the graviton makes string theories rather interesting, given the intrinsic problems of quantum gravity. Further advantages of more sophisticated string theories are that they have generically few parameters, are not featuring space-time singularities such as black holes on a quantum level, and often have no need for renormalization, thus being consistent ultraviolet completions. The price to be paid is that only rather complicated string theories even have a chance to resemble the standard model, their quantization beyond perturbation theory is not yet fully solved, and it is unclear how to identify a string theory which has a vacuum state which is compatible with standard model physics. Furthermore, in general genuine string signatures usually only appear at energy levels comparable to the Planck scale, making an experimental investigation, or even verification of stringy properties of physics, almost impossible with the current technology.

The basic idea behind string theory is to try something new. The problem leading to the divergencies in section 4.2 is that with ever increasing energy ever shorter distances are probed, and by this ever more gravitons are found. This occupation with gravitons is then what ultimately leads to the problem. The ansatz of string theory is then to prevent such an effect. This is achieved by smearing out the interaction over a space-time volume. For a conventional quantum field theory such an inherent non-locality usually comes with the loss of causality. String theories, however, are a possibility to preserve causality and smear out the interaction in such a way that the problem is not occurring.

However, the approach of string theory actually goes (and, as a matter of fact, has to go) a step further. Instead of smearing only the interaction, it smears out the particles themselves. Of course, this occurs already anyway in quantum physics by the uncertainty principle. But in quantum field theory it is still possible to speak in the classical limit of a world-line of a particle. In string theory, this world line becomes a world sheet. In fact, string theories can also harbor world volumes in the form of branes. However, a dynamical theory of such branes, called M(atrix)-theory, is still not known, despite many efforts. One of the problems in formulating such a theory is that internal degrees of freedom of a world volume are also troublesome, and can once more give rise to consistency problems. String theory seems to be singled out to be theory with just enough smearing to avoid the problems of quantum field theory and at the same time having enough internal rigidity as to avoid new problems. The details of this are beyond the scope of this lecture, which thus only introduces string theory.

One feature of string theory is that there is usually no consistent solution in four space-time dimensions, but typically more are required. How many more is actually a dynamical property of the theory: It is necessary to solve it to give an answer. In perturbation theory, it appears that ten dimensions are required, but beyond perturbation theory indications have been found that rather eleven dimensions are necessary. Anyway, the number is usually too large. Thus, some of the dimensions have to be hidden, which can be performed by compactification, as with the setup for large extra dimensions. Indeed, as has been emphasized, large extra dimensions are rather often interpreted as a low-energy effective theory of string theory.

Since the space-time geometry of string theory is dynamic, as in case of quantum gravity, the compactification is a dynamical process. It turns out that already classically there are a huge number of (quasi-)stable solutions having a decent compactification of the surplus dimensions, but all of them harbor a different low-energy physics, i. e., a different standard model. To have the string theory choose the right vacuum, thus yielding the observed standard model, turns out to be complicated, though quantum effects actually improve the situation. Nonetheless, this problem remains a persistent challenge for string theories. This is known as the landscape problem.

Here, these problems will be left aside in favor for a very simple string theory. This theory will exhibit many generic features of string theory, despite requiring 26 (large) dimensions and, at least perturbatively, will not have a stable vacuum state. The latter will be signaled by the existence of a tachyon, a particle traveling faster than the speed of light, which is another generic, though beatable, problem of string theories.

To give a more intuitive picture for the peculiarities and properties of string theory in the following a point particle and its quantization will be compared step-by-step to the quantization of the string theory.

5.5.1 Classic string theories

In the following the number of dimensions will be D, and the Minkowski metric will take the form

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 & \\ & 1 & & 0 \\ 0 & & \ddots & \\ & 0 & & 1 \end{pmatrix}.$$

It is actually a good question why the signature of the Minkowski metric should be like this, and also string theory so far failed to provide a convincing answer. But before turning to string theory, it makes sense to set the stage with a relativistic point particle.

5.5.1.1 Point particle

To become confident with the concepts take a classical particle moving along a world line in D dimensions. Classically, a trajectory is described by the $D-1$ spatial coordinates $x_i(t)$ as a function of time $t = x_0$. More useful in the context of string theory is a redundant description in terms of D functions $X_\mu(\tau)$ of a variable τ, which strictly monotonously increases along the world line. A natural candidate for this variable is the eigentime, which thus parametrizes the world line of the particle.

The simplest Poincare-invariant action describing a free particle of mass m in terms of the eigentime is then given by

$$S_{pp} = -m \int d\tau \sqrt{-\partial_\tau X^\mu \partial^\tau X_\mu}. \tag{5.3}$$

This thus tells that the minimum action is obtained for the minimum (geodetic) length of the world line. Variation along the world line

$$\delta \dot{X}_\mu \equiv \delta \partial_\tau X^\mu = \partial_\tau \delta X^\mu$$

yields the equation of motion as

$$\begin{aligned}
\delta S_{pp} &= -m \int d\tau \left(\sqrt{-\dot{X}_\mu \dot{X}^\mu} - \sqrt{-\left(\dot{X}^\mu + \delta \dot{X}^\mu\right)\left(\dot{X}_\mu + \delta \dot{X}_\mu\right)} \right) \\
&= -m \int d\tau \left(\sqrt{-\dot{X}_\mu \dot{X}^\mu} - \sqrt{-\left(\dot{X}_\mu \dot{X}^\mu + 2\dot{X}^\mu \delta \dot{X}_\mu\right)} \right) \\
&= -m \int d\tau \left(\sqrt{-\dot{X}_\mu \dot{X}^\mu} - \sqrt{-\dot{X}_\mu \dot{X}^\mu \left(1 + 2\frac{\dot{X}^\mu \delta \dot{X}_\mu}{\dot{X}_\mu \dot{X}^\mu}\right)} \right) \\
&\overset{\text{Taylor}}{=} m \int d\tau \frac{\dot{X}^\mu \delta \dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}
\end{aligned}$$

where in the last line use has been made of the infinitesimality of $\delta \dot{X}_\mu$ and the square root has been Taylor-expanded.

Defining now the D-dimensional normalized speed as

$$u^\mu = \frac{\dot{X}^\mu}{\sqrt{-\dot{X}_\mu \dot{X}^\mu}} \tag{5.4}$$

yields the equation of motion after imposing the vanishing of the action under the variation and a partial integration as

$$m\dot{u}^{\mu} = 0 \tag{5.5}$$

This is nothing else then the equation of motion for a free relativistic particle, which of course reduces to the one of Newton in the limit of small speeds. This also justifies the interpretation of m as the rest mass of the particle.

With τ the eigentime the action is indeed Poincare-invariant. This can be seen as follows. A Poincare transformation is given by

$$X'^{\mu} = \Lambda^{\mu}_{\ \nu} X^{\nu} + a^{\mu}.$$

Inserting this expression for the argument of the square root yields

$$\begin{aligned} & \partial_\tau \left(\Lambda^{\mu}_{\ \nu} X^{\nu} + a^{\mu}\right) \partial_\tau \left(\Lambda_{\mu}^{\ \omega} X_{\omega} + a^{\mu}\right) \\ = \; & \left(\Lambda^{\mu}_{\ \nu} \Lambda_{\mu}^{\ \omega}\right) \partial_\tau X^{\nu} \partial_\tau X_{\omega}. \end{aligned}$$

Since the expression in parenthesis is just δ_{ν}^{ω} because of the (pseudo-)orthogonality of Lorentz transformations, this makes the expression invariant. Since the eigentime is invariant by definition, this shows the invariance of the total action.

Additionally, it is also reparametrization invariant, i. e., it is possible to transform the eigentime to a different variable without changing the contents of the theory, as it ought to be: Physics should be independent of the coordinate systems imposed by the observer. This is what ultimately leads to the diffeomorphism (diff) invariance of general relativity.

To show this invariance also for the action (5.3) take an arbitrary (but invertible) reparametrization $\tau' = f(\tau)$. This implies

$$\begin{aligned} \dot{\tau}' &= \frac{d\tau'}{d\tau} \\ d\tau &= \frac{d\tau'}{\dot{\tau}'}, \end{aligned}$$

yielding the transformation property of the integral measure. For the functions follows then

$$\dot{X}^{\mu'}(\tau') = \dot{X}^{\mu}(\tau)\frac{d\tau}{d\tau'} = \dot{X}^{\mu}\frac{1}{\dot{\tau}'}$$

Hence the scalar product changes as

$$\dot{X}^{\mu\prime}\dot{X}_{\mu\prime} = \frac{1}{\dot{\tau}'^2}\dot{X}^{\mu}\dot{X}_{\mu}.$$

One power of $\dot{\tau}'$ is removed by the square root, and the remaining one is then compensated by the integral measure.

Showing this explicitly for the action (5.3) was rather tedious, and it is useful to rewrite the action. For this purpose it is useful to introduce a metric along the world line. Since the world line is one-dimensional, this metric is only a single function $\gamma_{\tau\tau}(\tau)$ of the eigentime. This yields a trivial example of a tetrad η

$$\eta(\tau) := (-\gamma_{\tau\tau}(\tau))^{\frac{1}{2}},$$

which in general is a set of N (by definition positive) orthogonal vectors on a manifold. However, the manifold is just one-dimensional for a world line, and thus the tetrad is again only a scalar. In analogy to the string case, $\gamma_{\tau\tau}$ can also be denoted as the world-line metric.

Taking the tetrad as an independent function a new action is defined as

$$S'_{pp} = \frac{1}{2}\int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\eta} - \eta m^2\right).$$

Under a reparametrization $\tau \to \tau'(\tau)$ it is defined that the functions X and η transform as

$$\begin{aligned} X(\tau) &= X(\tau'(\tau)) \\ \eta'(\tau') &= \eta(\tau)\frac{d\tau}{d\tau'} = \frac{1}{\dot{\tau}'}\eta(\tau) \end{aligned} \tag{5.6}$$

This makes the expression invariant under diffeomorphisms: The transformation of η (5.6) takes care of the extra factor of $\dot{\tau}'$, and also makes the second expression invariant.

To show that the new action is indeed equivalent to the old, and that η is thus just an auxiliary function, can be shown by using the equation of motion for η. Using the Euler-Lagrange equation this time yields

$$\begin{aligned} 0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\eta}} - \frac{\partial L}{\partial \eta} &= \frac{\dot{X}^\mu \dot{X}_\mu}{\eta^2} + m^2 \\ &\implies \eta^2 = -\frac{\dot{X}^\mu \dot{X}_\mu}{m^2}. \end{aligned}$$

Thus knowledge of X determines η completely, since no derivatives of η appear. Inserting this expression into 5.6 leads to

$$\begin{aligned}
S'_{pp} &= \frac{1}{2}\int d\tau\left(\frac{\dot{X}^\mu\dot{X}_\mu}{\sqrt{-\frac{\dot{X}^\mu\dot{X}_\mu}{m^2}}}-\sqrt{-\frac{\dot{X}^\mu\dot{X}_\mu}{m^2}}m^2\right)\\
&= \frac{m}{2}\int d\tau\left(-\frac{-\dot{X}^\mu\dot{X}_\mu}{\sqrt{-\dot{X}^\mu\dot{X}_\mu}}-\sqrt{-\dot{X}^\mu\dot{X}_\mu}\right)\\
&= -m\int d\tau\sqrt{-\dot{X}^\mu\dot{X}_\mu}=S_{pp}.
\end{aligned}$$

Thus S'_{pp} is indeed equivalent to S_{pp}. However, one advantage remains to be exploited. By separation of the mass S'_{pp} can also be applied to the case of $m=0$ directly, which is only possible in a limiting process for the original action S_{pp}.

5.5.1.2 Strings

For strings, the world line becomes a world sheet. As a consequence, at any fixed eigentime τ the string has an extension. This extension can be infinite or finite. In the latter case, the string can be closed, i. e., its ends are connected, or open. In string theories usually only finite strings appear, with lengths L of size the Planck length. Furthermore, open strings have usually to have their ends located on branes. This is not necessary for the simple case here, which will be investigated both for open and closed strings.

Analogous to the eigentime then an eigenlength σ can be introduced. Both parameters together describe any point on the world-sheet. The functions X_μ describing the position of the points of the world-sheet are therefore functions of both parameters, $X_\mu = X_\mu(\sigma,\tau)$. Furthermore, as for the point particle, these functions should be reparametrization invariant

$$X^\mu(\sigma,\tau)=X^\mu(\sigma'(\sigma,\tau),\tau'(\sigma,\tau)) \tag{5.7}$$

such that the position of the world sheet is not depending on the parametrization.

Derivatives with respect to the two parameters will be counted by Latin indices a, ...,

$$\begin{aligned}
\partial_{a,b,\ldots} &= \partial_\tau,\partial_\sigma\\
\partial_0 &= \partial_\tau\\
\partial_1 &= \partial_\sigma.
\end{aligned}$$

It is then possible to define the induced metric on the world sheet as

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu,$$

as a generalization of $h_{\tau\tau} = \dot{X}^\mu \dot{X}_\mu$, which as a metric has already been used to define the action (5.3) in analogy to the Einstein-Hilbert action. $\sqrt{-\det h_{ab}} d\tau d\sigma$ is then an infinitesimally element of the world sheet area.

The simplest possible Poincare-invariant action which can be written down for this system is the Nambu-Goto action

$$S_{NG} = \int_M d\tau d\sigma \mathcal{L}_{\text{NG}}$$

in which M is the world-sheet of the string and $\mathcal{L}_{\text{NG}}$ is the Nambu-Goto Lagrangian

$$\mathcal{L}_{\text{NG}} = -\frac{1}{2\pi\alpha'}\sqrt{-\det h_{ab}} = -\frac{1}{2\pi\alpha'}\sqrt{\partial_\tau X_\mu \partial_\sigma X^\mu \partial_\sigma X_\rho \partial_\tau X^\rho - \partial_\tau X_\mu \partial_\tau X^\mu \partial_\sigma X^\rho \partial_\sigma X_\rho},$$

again the direct generalization of the point-particle action. In particular, the minimum area of the world sheet minimizes the action.

The constant α' is the so-called Regge slope, having dimension Mass squared. In principle, it could be set to one in the following for the non-interacting string, but due to its importance in the general case, it will be left explicit. The Regge slope can be associated with the string tension T as $T = 1/(2\pi\alpha')$.

The Nambu-Goto action has two symmetries. One is diffeomorphism invariance. This can be seen directly, as in the case of the point particle, except that now the Jacobian appears. The second invariance is Poincare invariance, which leaves the world-sheet parameters τ and σ invariant. However, the functions X_μ transform as

$$\begin{aligned} X'^\mu &= \Lambda^\mu{}_\nu X^\nu + a^\mu \\ \partial_a \Lambda^\mu{}_\nu X^\nu \partial_b \Lambda_\mu{}^\gamma X_\gamma &= \overbrace{\Lambda^\mu{}_\nu \Lambda_\mu{}^\gamma}^{\delta_\nu^\gamma} \partial_a X^\nu \partial_b X_\gamma = \partial_a X^\mu \partial_b X_\mu. \end{aligned}$$

Thus, the induced metric is Poincare invariant, and hence also the action as well as the Lagrangian and any other quantity constructed from it is.

It is once more rather cumbersome to use an action involving a square root. To construct a simpler action, it is useful to introduce a world-sheet metric $\gamma_{ab}(\tau, \sigma)$. This metric is taken to have a Lorentz signature for some chosen coordinate system

$$\gamma_{ab} = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}.$$

Thus, this metric is traceless, and has a determinant smaller zero. With it the new action, the Brink-Di Vecchia-Howe-Deser-Zumino or Polyakov action,

$$S_P = -\frac{1}{4\pi\alpha'}\int\limits_M d\tau d\sigma\left(-\gamma\right)^{\frac{1}{2}}\gamma^{ab}h_{ab} \tag{5.8}$$

is constructed, where γ denotes $\det\gamma_{ab}$.

As in case of the point particle, the world-sheet metric γ_{ab} has to have a non-trivial transformation property under diffeomorphisms,

$$\frac{\partial\omega'^c}{\partial\omega^a}\frac{\partial\omega'^d}{\partial\omega^b}\gamma'_{cd}\left(\tau',\sigma'\right) = \gamma_{ab}\left(\tau,\sigma\right),$$

where the variables ω denote either σ and τ, depending on the index. This guarantees that for all invertible reparametrizations, which are continuous deformations of the identity transformation, the metric is still traceless and has negative determinant.

To obtain the relation of the Polyakov action to the Nambu-Goto action it is again necessary to obtain its equation of motion. This is most conveniently obtained using the variational principle. For this, the general relation for determinants of metrics

$$\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma\gamma_{ab}\delta\gamma^{ab}$$

is quite useful.

Abbreviating the Polyakov Lagrangian by L_P and performing a variation with respect to γ yields

$$\begin{aligned}\delta S_P &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(L_P - \left(-\gamma-\delta\gamma\right)^{\frac{1}{2}}\left(\gamma^{ab}+\delta\gamma^{ab}\right)h_{ab}\right)\\ &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(L_P - \left(-\gamma+\gamma\gamma^{cd}\delta\gamma_{cd}\right)^{\frac{1}{2}}\left(\gamma^{ab}+\delta\gamma^{ab}\right)h_{ab}\right)\\ &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(L_P - \left(-\gamma\right)^{\frac{1}{2}}\left(1-\gamma^{cd}\delta\gamma_{cd}\right)^{\frac{1}{2}}\left(\gamma^{ab}+\delta\gamma^{ab}\right)h_{ab}\right).\end{aligned}$$

Expanding the term with indices cd up to first order in the variation leads to

$$\begin{aligned}\delta S_P &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(L_P - \left(-\gamma\right)^{\frac{1}{2}}\left(1-\frac{1}{2}\gamma^{cd}\delta\gamma_{cd}\right)\left(\gamma^{ab}+\delta\gamma^{ab}\right)h_{ab}\right)\\ &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(L_P - \left(-\gamma\right)^{\frac{1}{2}}\left(\gamma^{ab}+\delta\gamma^{ab}-\frac{1}{2}\gamma^{cd}\gamma^{ab}\delta\gamma_{cd}\right)h_{ab}\right).\end{aligned}$$

The second term is again the Polyakov Lagrangian, canceling the zero-order term. Then only

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(-\gamma\right)^{\frac{1}{2}}\left(h_{ab}-\frac{1}{2}\gamma_{ab}\gamma_{cd}h^{cd}\right)\delta\gamma^{ab}$$

is left.

The condition that this expression should vanish yields the equation of motion for the world-sheet metric as

$$h_{ab} = \frac{1}{2}\gamma_{ab}\gamma_{cd}h^{cd} \tag{5.9}$$

Division of each side by its determinant finally yields

$$\begin{aligned}
\frac{h_{ab}}{(-h)^{\frac{1}{2}}} &= \frac{1}{2}\frac{\gamma_{ab}\left(\gamma_{cd}h^{cd}\right)}{\left(\det -\frac{1}{2}\gamma_{ab}\gamma_{cd}h^{cd}\right)^{\frac{1}{2}}} \\
&= \frac{1}{2}\frac{\gamma_{ab}\left(\gamma_{cd}h^{cd}\right)}{\left(\left(\frac{1}{2}\gamma_{cd}h^{cd}\right)^2 \det -\gamma_{ab}\right)^{\frac{1}{2}}} \\
&= \frac{\gamma_{ab}}{(-\gamma)^{\frac{1}{2}}}
\end{aligned}$$

In the second line it has been used that $\gamma_{cd}h^{cd}$ is a scalar, permitting it to pull it out of the determinant. The result implies that h and γ are essentially proportional.

Inserting this result in the Polyakov action yields

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \gamma^{ab}\gamma_{ab}\left(-h\right)^{\frac{1}{2}} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(-h\right)^{\frac{1}{2}} = S_{NG}$$

showing that it is indeed equivalent to the Nambu-Goto action, where the fact that the diffeomorphism invariant quantity $\gamma^{ab}\gamma_{ab}$ is two, due to the Lorentz signature of γ, has been used.

The Polyakov action thus retains the Poincare and diffeomorphism invariance of the Nambu-Goto action. The Poincare invariance follows since γ is Poincare invariant, since it is proportional to the Poincare-invariant induced metric, thus

$$\gamma^{ab\prime} = \Lambda\gamma^{ab} = \gamma^{ab}.$$

The diffeomorphism invariance follows directly from the transformation properties of the world-sheet metric, in total analogy with the point-particle case, but considerably more lengthy since track of both variables has to be kept.

The redundancy introduced with the additional degree of freedom γ grants a further symmetry. This is the so-called Weyl symmetry, given by

$$\begin{aligned}
X'^{\mu}\left(\tau,\sigma\right) &= X^{\mu}\left(\tau,\sigma\right) \\
h'_{ab} &= h_{ab} \\
\gamma'_{ab} &= e^{2\omega(\tau,\sigma)}\gamma_{ab}.
\end{aligned}$$

for arbitrary functions $\omega(\tau, \sigma)$. The origin of this symmetry comes from the unfixed proportionality of induced metric and the world-sheet metric. The expression of γ in terms of the induced metric h is invariant under this transformation,

$$\frac{\gamma'_{ab}}{(-\gamma')^{\frac{1}{2}}} = \frac{\gamma_{ab}e^{2\omega}}{(-\gamma')^{\frac{1}{2}}} = \frac{\gamma_{ab}e^{2\omega}}{(-\gamma e^{4\omega})^{\frac{1}{2}}} = \frac{\gamma_{ab}}{(-\gamma)^{\frac{1}{2}}}.$$

Also the action is invariant. To see this note that γ_{ab} is indeed a metric. Since $\gamma_{ab}\gamma^{ab}$ has to be a constant, as noted before, this implies that

$$\gamma'^{ab} = e^{-2\omega}\gamma^{ab}.$$

As a consequence, the expression appearing in the action transforms as

$$(-\gamma')^{\frac{1}{2}}\gamma'^{ab} = \left(-\gamma e^{4\omega}\right)^{\frac{1}{2}}\gamma^{ab}e^{-2\omega} = (-\gamma)^{\frac{1}{2}}\gamma^{ab}.$$

Thus, the Weyl invariance is indeed a symmetry.

The Polyakov action can also be viewed with a different interpretation. Promoting the world-sheet indices to space-time indices and taking the indices μ to label internal degrees of freedom, then the Polyakov action just describes D massless Klein-Gordon fields X_μ (with internal symmetry group SO($D-1$,1)) in two space-time dimensions with a non-trivial metric γ, which is dynamically coupled to the fields. This is an example of a duality of two theories, which plays an important role for more complicated theories. E. g., dualities between certain string theories on certain background metrics with so-called supergravity theories, the AdS/CFT correspondence, had an enormous impact recently on both string theory and quantum field theory.

5.5.2 Quantized theory

5.5.2.1 Light cone gauge

As the Poincare and Weyl symmetry introduce a gauge symmetry, it is easier to perform the quantization in a fixed gauge. Particularly useful for this purpose in the present context is the light-cone gauge. Though this gauge is not keeping manifest Poincare covariance, it is very useful (similar to the case of quantizing electrodynamics in Coulomb rather than linear covariant gauges). Proving that the theory is still covariant after quantization is non-trivial, but possible. Hence, this will not be shown here.

To formulate light-cone gauge light-cone coordinates are useful. They are introduced by the definitions

$$\begin{aligned} x^{\pm} &= \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right) \\ x^i &= x^i,\ i=2,...,D-1, \end{aligned}$$

and thus mix the time-coordinate and one, now distinguished, spatial coordinate. Since the zero-component is the only one involving a non-positive sign in the metric this yields the following relation between covariant and contravariant light-cone coordinates

$$\begin{aligned} x_{\pm} &= \frac{1}{\sqrt{2}}\left(x_0 \mp x_1\right) \\ x_- &= -x^+ \\ x_+ &= -x^- \\ x_i &= x^i. \end{aligned}$$

This implies the metric

$$a^\mu b_\mu = a^+ b_+ + a^- b_- + a^i b_i = -a^+ b^- - a^- b^+ + a^i b^i,$$

which is equivalent to the conventional one

$$\begin{aligned} -a^+ b^- - a^- b^+ + a^i b^i &= -\frac{1}{2}\left(a^0 + a^1\right)\left(b^0 - b^1\right) - \frac{1}{2}\left(a^0 - a^1\right)\left(b^0 + b^1\right) + a^i b^i \\ &= -a^0 b^0 + a^1 b^1 + a^i b^i = a^\mu b_\mu. \end{aligned}$$

Aim of the gauge fixing is to restore the original number of independent degrees of freedom. In case of the point particle this amounts to remove the eigentime τ. This is most conveniently done by the condition

$$\tau \equiv x^+,$$

thus being the light-cone gauge condition for the point particle. This is more convenient than the more conventional choice $\tau = x^0$. With this x^+ corresponds to the time and p^- to the energy . Correspondingly, x^- and p^+ are now longitudinal degrees of freedom while x^i and p^i are transverse ones. This immediately follows from the scalar product

$$\frac{\partial}{\partial a^+}\left(-a^+ b^- + ...\right) = -b^-,$$

and correspondingly for the derivative with respect to x^+ which produces p^-.

5.5.2.2 Point particle

To demonstrate the principles, it is once more convenient to first investigate the point particle. However, one should be warned that the resulting theory is actually flawed due to the appearance of unphysical (non-normalizable) states. It should therefore be taken rather as a mathematical than a physical discussion.

Returning to the parametrization of the point particle of section 5.5.1.1, the gauge condition to fix the diffeomorphism invariance becomes

$$X^+(\tau) = \tau.$$

The action is given by equation (5.6), thus

$$\begin{aligned}
S'_{pp} &= \frac{1}{2}\int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\eta} - \eta m^2\right) \\
&= \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\left(-\dot{X}^+\dot{X}^- - \dot{X}^-\dot{X}^+ + \dot{X}^i\dot{X}^i\right) - \eta^2 m\right) \\
&= \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\left(-2\dot{X}^-\dot{\tau} + \dot{X}^i\dot{X}^i\right) - \eta^2 m\right) \\
&= \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\left(-2\dot{X}^- + \dot{X}^i\dot{X}^i\right) - \eta m^2\right).
\end{aligned}$$

As usual, the Lagrangian yields the canonical conjugated momenta by the expression

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu}$$

yielding

$$\begin{aligned}
P_- &= -\frac{1}{\eta} \\
P_i &= \frac{\dot{X}^i}{\eta}
\end{aligned}$$

With this the Hamiltonian can be readily constructed as

$$\begin{aligned}
H &= \sum P\dot{Q} - L \\
&= P_-\dot{X}^- + P_i\dot{X}^i - L \\
&= -\frac{\dot{X}^-}{\eta} + \eta P_i P_i + \frac{\dot{X}^-}{\eta} - \frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{2}\eta m^2 \\
&= \eta P_i P_i - \frac{1}{2}\eta P_i P_i + \frac{1}{2}\eta m^2 = \frac{P^i P^i + m^2}{2P^+}.
\end{aligned}$$

Where it has been used that

$$P^+ = -P_- = \frac{1}{\eta},$$

and it is thus possible to remove η and P_- from the expression.

In this result the variable X^+ is no longer a dynamical variable, and thus the gauge is fixed. Furthermore it follows that $P_\eta = 0$, since the Lagrangian does not depend on $\dot{\eta}$. Hence η is not a dynamical variable. This was expected, since it was already in the classical case only used to make the Lagrangian more easily tractable

For the quantization then the usual canonical commutation relations are imposed as

$$\begin{aligned} \left[P_i, X^j\right] &= -i\delta_i^j \\ \left[P_-, X^-\right] &= -i \end{aligned}$$

The relations for P_+ is provided by the other relations, since P^- is the energy and thus

$$H = P^- = -P_+. \tag{5.10}$$

That is essentially the relativistic mass-shell equation, implying once more that P^+ is not an independent degree of freedom. The resulting Hamilton operator is the one of a $D-2$-dimensional harmonic oscillator, but supplemented with the additional unconstrained degree of freedom P_-. The spectrum of this is known, being a relativistic scalar (with all its sicknesses) and states $|k_-, k_i\rangle$.

5.5.2.3 Open string

Again, the first step is to fix the gauge. For that purpose first the permitted range for the world-sheet parameters have to be chosen, which will be

$$\begin{aligned} -\infty &\leq \tau \leq +\infty \\ 0 &\leq \sigma \leq L. \end{aligned} \tag{5.11}$$

Thus, L is the length of the string. Again, it is chosen that

$$\tau = X^+. \tag{5.12}$$

This deals again with the diffeomorphism degree of freedom. To also take care of the Weyl freedom a second condition is necessary, which will be chosen to be

$$\partial_\sigma \gamma_{\sigma\sigma} = 0 \tag{5.13}$$

$$\det \gamma_{ab} = -1 \tag{5.14}$$

The conditions (5.12-5.14) fixes these degrees of freedom completely, provided that the world-sheet is parametrized by the eigenvariables in such a way that one and only one set of eigentime and eigenlength correspond to a given point on the world sheet. In the case of the point particle, it can be shown that this condition is actually superfluous, since even in case of a double backing world line this would not contribute to a path integral. For string theory, this is something not yet really simply understood.

A way to get an intuition for the significance of these gauge condition is by the use of the invariant length. The choice of $\tau = X^+$ is of course always possible. Then start by the definition

$$f = \gamma_{\sigma\sigma}\left(\frac{1}{-\det\gamma_{ab}}\right)^{\frac{1}{2}}.$$

Now perform a reparametrization which leaves τ invariant. This implies

$$f' = f\frac{d\sigma}{d\sigma'}.$$

because of the transformation properties of the γ_{ab}. Hence, the length element $dl = f d\sigma$ is invariant under this reparametrization. Therefore, it can be considered as an invariant length-element, since it is not changing under a change of the eigenlength of the string. In fact, this can be used to define the σ coordinate, by setting it equal to $\int dl$ along the world sheet,

$$\sigma = \int\limits_0^\sigma dl.$$

As a consequence, f can no longer depend on σ, since dl is σ-independent. Secondly, it is then possible to make a Weyl-transformation to rescale $\det\gamma$ such that it becomes -1, yielding (5.14), and fixing the Weyl invariance. Since f is Weyl-invariant by construction, this implies that $\partial_\sigma\gamma_{\sigma\sigma}$ trivially vanishes, yielding (5.13). Thus, in this coordinate system the gauge condition are fulfilled, and therefore are a permitted choice.

Since γ is by construction symmetric, these gauge condition permit to rewrite it in a simpler way. It then takes the form

$$\begin{aligned}\gamma &= \begin{pmatrix}\gamma^{\tau\tau} & \gamma^{\tau\sigma}\\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma}\end{pmatrix}\\ &= \begin{pmatrix}-\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma)\\ \gamma_{\tau\sigma}(\tau,\sigma) & \gamma_{\sigma\sigma}^{-1}(\tau)\left(1-\gamma_{\tau\sigma}^2(\tau,\sigma)\right)\end{pmatrix},\end{aligned}$$

thereby eliminating two of the four variables in γ_{ab}, and also reducing their dependence on the world sheet parameters.

It is furthermore useful to define the average and variation of the X^- coordinate for the following as

$$\begin{aligned} Z^-(\tau) &= \frac{1}{L}\int_0^L d\sigma X^-(\tau,\sigma) \\ Y^-(\tau,\sigma) &= X^-(\tau,\sigma) - Z^-(\tau). \end{aligned}$$

This is the starting point to rewrite the action in a more useful form.

Start by rewriting the Lagrangian as

$$\begin{aligned} L_P &= -\frac{1}{4\pi\alpha'}\int_0^L d\sigma \overbrace{(-\gamma)^{\frac{1}{2}}}^{=1} \gamma^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= -\frac{1}{4\pi\alpha'}\int_0^L d\sigma \left(\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^\mu\partial_\sigma X_\mu + \gamma_{\tau\sigma}\partial_\tau X^\mu \partial_\sigma X_\mu + \gamma_{\tau\sigma}\partial_\sigma X^\mu\partial_\tau X_\mu - \gamma_{\sigma\sigma}\partial_\tau X^\mu \partial_\tau X_\mu\right). \end{aligned}$$

Now, it is useful to investigate the expressions piece-by-piece. Start with

$$\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^\mu\partial_\sigma X_\mu = \frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i$$

where it has been used that

$$\partial_\sigma X^+ = \partial_\sigma \tau = 0$$

follows trivially from the gauge conditions. Next, use furthermore that

$$\gamma_{\tau\sigma}(\overbrace{\partial_\tau X^+\partial_\sigma X_+}^{=-\partial_\tau\tau\partial_\sigma X^-} + \partial_\tau X^- \overbrace{\partial_\sigma X_-}^{=-\partial_\sigma X^+} + \partial_\tau X^i\partial_\sigma X^i) = \gamma_{\tau\sigma}\left(-\partial_\sigma X^- + \partial_\sigma X^i \partial_\tau X^i\right)$$

and

$$-\gamma_{\sigma\sigma}\left(\partial_\tau X^+\partial_\tau X_+ + \partial_\tau X^-\partial_\tau X_- + \partial_\tau X^i\partial_\tau X^i\right) = -\gamma_{\sigma\sigma}\left(-2\partial_\tau X^- + \partial_\tau X^i\partial_\tau X^i\right).$$

Reinserting everything into the Lagrangian yields

$$\begin{aligned} L_P = &-\frac{1}{4\pi\alpha'}\int_0^L d\sigma\Big(\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i - 2\gamma_{\tau\sigma}\left(-\partial_\sigma X^- + \partial_\tau X^i\partial_\sigma X^i\right) \\ &-\gamma_{\sigma\sigma}\left(-2\partial_\tau X^- + \partial_\tau X^i\partial_\tau X^i\right)\Big). \end{aligned}$$

Employing now the relations for the average and variation this yields

$$\begin{aligned} L_P &= -\frac{1}{4\pi\alpha'}\Big(\gamma_{\sigma\sigma}2L\partial_\tau Z^- + \int\limits_0^L d\sigma\Big(\gamma_{\sigma\sigma}\left(-\partial_\tau X^i\partial_\tau X^i\right) \\ &\quad +2\gamma_{\tau\sigma}\left(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i\right) + \frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i\Big)\Big) \end{aligned}$$

In the resulting expression there is no τ-derivative of Y^- appearing, which is thus a non-dynamical field, behaving like a Lagrange factor for $\gamma_{\tau\sigma}$, which therefore is fixed to

$$\partial_\sigma\gamma_{\tau\sigma} = 0, \tag{5.15}$$

and thus does not depend on σ.

Returning to the boundary condition of this open[2] string yields

$$\left(\partial_\sigma X^\mu\right)(\tau, 0) = \left(\partial_\sigma X^\mu\right)(\tau, L) = 0, \tag{5.16}$$

because otherwise the fields would not be continuously differentiable at the boundaries, which is imposed like for wave-functions. These are von Neumann conditions in the terms of the large extra dimensions. This is also obtained by varying the Polyakov action. First, vary with respect to the fields to obtain

$$-\frac{1}{2\pi\alpha'}\int\limits_{-\infty}^{\infty} d\tau\left(-\gamma\right)^{\frac{1}{2}}\partial_\sigma X^\mu\delta X^\mu|_{\sigma=0}^{\sigma=L}.$$

Since this has to vanish for arbitrary variations of the fields, this implies the boundary condition (5.16).

On the other hand, when varying the original action with respect to the fields, this yields

$$\begin{aligned} \delta S_P &= S_P + \frac{1}{4\pi\alpha'}\int d\tau d\sigma\gamma_{ab}\partial_a\left(X^\mu+\delta X^\mu\right)\partial_b\left(X_\mu+\delta X_\mu\right) \\ &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma\gamma_{ab}\left(\partial_a X^\mu\partial_b\delta X_\mu + \partial_\alpha\delta X^\mu\partial_b X_\mu\right). \end{aligned}$$

Since variation and differentiation are independent, they can be exchanged,

$$\partial_a\delta X^\mu = \delta\partial_a X^\mu.$$

[2]No cyclicity of any function on σ has been imposed, which would be one possibility to implement a closed string.

Doing a partial integration, keeping an appearing boundary term yields

$$\begin{aligned} \delta S_P &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma \gamma_{ab}\left(\partial_a\partial_b X^\mu \delta X_\mu + \partial_b\partial_a X_\mu \delta X^\mu\right) \\ &\quad -\frac{1}{4\pi\alpha'}\int d\tau \gamma_{ab}\left(\partial_a X^\mu \delta X_\mu + \partial_b X_\mu \delta X^\mu\right)|_0^L \end{aligned} \tag{5.17}$$

Note that in the boundary term as a shorthand notation one of the indices is uncontracted. This is of course always the σ-index for which the total integration has been performed. However, the last expression must vanish under variation, implying once more the von Neumann condition (5.16)

$$\gamma_{ab}\left(\partial_a X^\mu + \partial_b X^\mu\right) = 0.$$

Incidentally, this also implies for $\mu = +$ and $a = \tau$ and $b = \sigma$ that $\gamma_{\tau\sigma}$ vanishes on the boundary.

Since for $\mu = -$ the fields are non-dynamical, this implies that $\partial_\sigma X^- = 0$ and that therefore X^- only depends on τ.

To obtain some further useful results, the variation can be repeated after the gauge has been fixed. This yields

$$\begin{aligned} \delta S_P &= S_P + \frac{1}{4\pi\alpha'}\int d\tau d\sigma \Big(\gamma_{\sigma\sigma}\left(2\partial_\tau\left(X^- + \delta X^-\right) - \partial_\tau\left(X^i + \delta X^i\right)\partial_\tau\left(X^i + \delta X^i\right)\right) \\ &\quad + 2\gamma_{\tau\sigma}\left(\partial_\sigma\left(X^- + \delta X^-\right) - \partial_\tau\left(X^i + \delta X^i\right)\partial_\sigma\left(X^i + \delta X^i\right)\right) \\ &\quad + \frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma\left(X^i + \delta X^i\right)\partial_\sigma\left(X^i + \delta X^i\right)\Big). \end{aligned}$$

Expanding the result and dropping $\mathcal{O}\left(\delta^2\right)$ terms and annihilating a term of type S_P just leaves

$$\begin{aligned} \delta S_P &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma \Big(\gamma_{\sigma\sigma}\left(2\partial_\tau \delta X^- - 2\partial_\tau X^i \partial_\tau \delta X^i\right) \\ &\quad + 2\gamma_{\tau\sigma}\left(\partial_\sigma \delta X^- - \partial_\tau X^i \partial_\sigma \delta X^i - \partial_\tau \delta X^i \partial_\sigma X^i\right) + 2\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^i \partial_\sigma \delta X^i\Big) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \delta S_P &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma \Big(\left(2\gamma_{\tau\sigma}\partial_\sigma \delta X^-\right) + \left(-2\gamma_{\tau\sigma}\partial_\tau X^i \partial_\sigma \delta X^i + 2\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma \delta X^i \partial_\sigma X\right) \\ &\quad + \left(\gamma_{\sigma\sigma}2\partial_\tau \delta X^- - 2\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau \delta X^i\right) - \left(\gamma_{\tau\sigma}\partial_\tau \delta X^i \partial_\sigma X^i\right)\Big). \end{aligned}$$

After partial integration of the first term this yields once more that $\partial_\sigma \gamma_{\tau\sigma}$ still vanishes at the end of the string.

The second term in parentheses yields after partial integration

$$-\partial_\sigma\left(2\gamma_{\tau\sigma}\partial_\tau X^i\right)+\partial_\sigma\left(\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\right)=-2\gamma_{\tau\sigma}\partial_\sigma\partial_\tau X^i+\frac{1-\gamma_{\tau\sigma}^2}{\gamma_{\sigma\sigma}}\partial_\sigma^2 X^i.$$

Again, this boundary term has to vanish. The second does this, if the derivative of the X^i with respect to σ does so at the boundary, again yielding (5.16). Since this is not the case for the τ-derivative, this again requires $\gamma_{\tau\sigma}=0$ at the boundary of the string. Hence, this implies that both the function and its first derivative vanishes on the boundary. Because of the equation of motion for $\gamma_{\tau\sigma}$ (5.15), this implies

$$\gamma_{\tau\sigma}\equiv 0,$$

and it can be dropped everywhere.

This eliminates one degree of freedom, leaving only

$$Z^-(\tau)\,,\ \gamma_{\sigma\sigma}(\tau)\,,\ X^i(\tau,\sigma)\,,$$

which is a rather short list. Furthermore, this simplifies the Polyakov Lagrangian to

$$L_P=-\frac{L}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau Z^-+\frac{1}{4\pi\alpha'}\int_0^L d\sigma\left(\gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i-\frac{1}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i\right),$$

which will now serve as the starting point for quantization. It should be noted that the gauge-fixing was the reason for eliminating the degrees of freedom, reducing the set to a one more manageable for the following.

The first step for quantization is then the calculation of the canonical momenta

$$\begin{aligned}P_- &= -P^+=\frac{\partial L_P}{\partial\left(\partial_\tau Z^-\right)}=-\frac{L}{2\pi\alpha'}\gamma_{\sigma\sigma} \\ \Pi^i &= \frac{\delta L_P}{\delta\left(\partial_\tau X^i\right)}=\frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i=\frac{P^+}{L}\partial_\tau X^i.\end{aligned}\tag{5.18}$$

From this the Hamiltonian is immediately constructed to be

$$\begin{aligned}H &= P_-\partial_\tau Z^-+\int_0^L d\sigma\Pi^i\partial_\tau X^i-L \\ &= \frac{L}{4\pi\alpha' P^+}\int_0^L d\sigma\left(2\pi\alpha'\Pi^i\Pi^i+\frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right)\end{aligned}\tag{5.19}$$

This is the Hamiltonian for $D-2$ free fields X^i and the conserved quantity P^+, as can be seen form the equations of motion

$$\begin{aligned}
\partial_\tau Z^- &= \frac{\partial H}{\partial P^-} = \frac{H}{P^+} \\
\partial_\tau P^+ &= -\frac{\partial H}{\partial Z^-} = 0 \\
\partial_\tau X^i &= \frac{\delta H}{\delta \Pi^i} = 2\pi\alpha' c \Pi^i && (5.20) \\
\partial_\tau \Pi^i &= -\frac{\delta H}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i, && (5.21)
\end{aligned}$$

where a partial integration has been performed in (5.21) and c is defined as

$$c := \frac{L}{2\pi\alpha' P^+}.$$

Inserting (5.20) in (5.21) yields the wave equation for X^i

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i,$$

where c takes the role of the wave speed. Thus, the transverse degrees of freedom form waves along the string.

Since P^+ and L are constants of motion, so is c. Thus, given the boundary conditions for the open string, the equations of motions can be solved, yielding

$$\begin{aligned}
\hat{X}^i(\tau,\sigma) &= \hat{Z}^i + \frac{\hat{P}^i}{P^+}\tau + i(2\alpha')^{\frac{1}{2}} \sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{\alpha_n^i}{n} e^{-\frac{\pi i n c \tau}{L}} \cos\frac{\pi n \sigma}{L} && (5.22) \\
\alpha_{-n}^i &= \alpha_n^{i+}. && (5.23)
\end{aligned}$$

The relation (5.23) applies since the X^i are real functions. For the purpose at hand also the center-of-mass variables

$$\begin{aligned}
\hat{Z}^i(\tau) &= \frac{1}{L}\int_0^L d\sigma \hat{X}^i(\tau,\sigma) \\
\hat{P}^i(\tau) &= \int_0^L d\sigma \Pi^i(\tau,\sigma) = \frac{P^+}{L}\int_0^L d\sigma \partial_\tau X^i(\tau,\sigma)
\end{aligned}$$

have been introduced. Thus, the center-of-mass of the string follows a free, linear trajectory in space, which overlays the transverse motions of the oscillations transverse to the string. Herein $\hat{Z}^i$ and $\hat{P}^i$ in (5.22) have to be taken at $\tau = 0$, and will become Schrödinger operators in the quantization procedure to come now.

The quantization procedure is started as usually with imposing equal-time canonical commutation relations

$$\begin{aligned} \left[Z^-, P^+\right] &= -i \\ \left[X^i(\sigma), \Pi^j(\sigma')\right] &= i\delta^{ij}\delta(\sigma - \sigma') \end{aligned}$$

Performing a Fourier expansion this is equivalent to the relations

$$\begin{aligned} \left[\hat{X}^i, \hat{P}^j\right] &= i\delta^{ij} \\ \left[\alpha^i_m, \alpha^j_n\right] &= m\delta^{ij}\delta_{m,-n} \end{aligned} \tag{5.24}$$

Here, a non-standard, though useful, normalization of (5.24) has been performed.

The natural consequence is now that every transverse component behaves as a harmonic oscillator with a non-standard normalization. The corresponding creation and annihilation operators are then given for $m > 0$

$$\begin{aligned} \alpha^i_m &= \hbar\sqrt{m}a \\ \alpha^i_{-m} &= \hbar\sqrt{m}a^\dagger \\ -1 &= \left[a^\dagger, a\right] \end{aligned} \tag{5.25}$$

where m gives the oscillator level for direction i. So far, so standard.

Defining now the momentum vector $k = (k^+, k^i)$ the state $|0, k\rangle$ of lowest excitation has the properties

$$\begin{aligned} P^+ |0, k\rangle &= k^+ |0, k\rangle \\ P^i |0, k\rangle &= k^i |0, k\rangle \\ \alpha^i_m |0, k\rangle &= 0 \text{ for } m > 0 \end{aligned} \tag{5.26}$$

Therefore k is the center-of-mass momentum. Higher excited states are then denoted by $|N, k\rangle$ and can be constructed as

$$|N, k\rangle = \left(\prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{\left(\alpha^i_{-n}\right)^{N_{in}}}{\sqrt{(n^{N_{in}}N_{in})!}}\right)|0, k\rangle\,,$$

just as ordinary oscillator states. Therefore, N_{in} are the occupation numbers for each direction and level. In particular, these can be interpreted as internal degrees of freedom, while the motion of the center-of-mass corresponds to a particle like behavior of the whole string. As will be discussed below, from this point of view every state corresponds to a certain particle with a certain spin.

The total set of states (5.22) forms the Hilbert space of a single string, H_1. In particular, $|0,0\rangle$ is not the vacuum, but merely a momentum-zero string with no internal excitations, except zero-point oscillations: A quantum-mechanical string always quivers. The vacuum is devoid of a string, its Hilbert-space H_0 is denoted by the single state $|\mathrm{vac}\rangle$. However, none of the operators so far can mediate between H_0 and H_1, but only act inside H_1. Since there are no interactions, an N-string Hilbert space can be build just as a product space of H_1s as

$$h_n = |\mathrm{vac}\rangle \oplus H_1 \oplus ... \oplus H_n.$$

where the states are implicitly symmetrized, yielding a Fock space, since the string states are bosonic, given that there creation and annihilation operators fulfill bosonic canonical commutation relations, (5.25).

Since the states are just free states, it is straightforward to construct the number-state version of the Hamiltonian. For this purpose, it is necessary to calculate the explicit form of the canonical momentum operators Π^i first as

$$\begin{aligned}\Pi^i &= \frac{P^+}{L}\left(\partial_\tau X^i\right)\\ &= \frac{P^+}{L}\left(\frac{\hat{P}^i}{P^+}+\frac{\pi c}{L}\left(2\alpha'\right)^{\frac{1}{2}}\sum_{n=-\infty,n\neq 0}^{n=+\infty}\alpha_n^i e^{-\frac{\pi i n c\tau}{L}}\cos\frac{\pi n\sigma}{L}\right).\end{aligned}$$

In addition, also $\partial_\sigma X^i$ is required, and is given by

$$\partial_\sigma X^i = -\frac{i\pi}{L}\left(2\alpha'\right)^{\frac{1}{2}}\sum_{n=-\infty,n\neq 0}^{n=+\infty}\alpha_n^i e^{-\frac{\pi i n c\tau}{L}}\sin\frac{\pi n\sigma}{L}.$$

Putting everything together yields the Hamiltonian

$$
\begin{aligned}
&\frac{L}{4\pi\alpha' P^+}\int_0^L d\sigma\left(2\pi\alpha\Pi^i\Pi^i+\frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right)\\
=\;&\frac{L}{4\pi\alpha' P^+}\Big(2\pi\alpha' P^iP^i+\int_0^L d\sigma\\
&\Big(\frac{\pi}{4\alpha' LP^+}\sum_{n=-\infty n\neq 0}^{n=+\infty}\alpha_n^i e^{-\frac{\pi inc\tau}{L}}\cos\frac{\pi n\sigma}{L}\sum_{m=-\infty,m\neq 0}^{m=+\infty}\alpha_m^i e^{-\frac{\pi mnc\tau}{L}}\cos\frac{\pi m\sigma}{L}\\
&-\frac{\pi}{4\alpha' LP^+}\sum_{n=-\infty n\neq 0}^{n=+\infty}\alpha_n^i e^{-\frac{\pi inc\tau}{L}}\sin\frac{\pi n\sigma}{L}\sum_{m=-\infty,m\neq 0}^{m=+\infty}\alpha_m^i e^{-\frac{\pi imc\tau}{L}}\sin\frac{\pi m\sigma}{L}\Big)\Big),
\end{aligned}
$$

where in the integration σ was replaced by $\pi\sigma/L$. Since sine and cosine are orthogonal, the integrations can be performed explicitly. Those over cos yield $\pi\delta_{n-m}$, while those over sin yield $-\pi\delta_{n-m}$. This leads to

$$
H=\frac{P^iP^i}{2P^+}+\frac{1}{2P^+\alpha'}\sum_{n=-\infty n\neq 0}^{n=+\infty}\alpha_n^i\alpha_{-n}^i,
$$

and finally by rearranging to

$$
H=\frac{P^iP^i}{2P^+}+\frac{1}{2P^+\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i+A.
$$

This Hamiltonian is already in normal order, and A is a (divergent) constant which appears in the process of normal ordering.

The actual value of A can be determined by explicitly verifying the Lorentz covariance of the result, since the Hamiltonian is just the energy, and thus a zero-component of a four-vector. However, in light-cone gauge this is far from trivial, and this will therefore be done here only in a rather sketchy way.

First, consider the zero-point energy. Every oscillator will have a zero-point energy of $\omega/2 = 1/(2P^+\alpha')$, while the transverse momenta P^i will be 0. In total, at zero excitation, it should be expected that

$$
\langle 0,0|H|0,0\rangle = A,
$$

due to the normal ordering. Due to the non-standard normalization, each oscillator actually contributes $n\omega/2$ of vacuum energy to this value. These oscillations appear for

$D-2$ dimensions. Rewriting A as ωA this yields[3]

$$A=\frac{D-2}{2}\sum_{n=1}^{\infty}n,$$

which is, of course, infinite. However, in contrast to normal quantum mechanics or quantum field theory, the vacuum energy is not necessarily irrelevant, but may couple to gravity. It is therefore necessary to maintain Lorentz invariance when treating it, and it cannot be absorbed just in a redefinition of the zero-point energy, as in quantum mechanics.

To regularize the result Lorentz-invariantly, it is necessary to include a cut-off function

$$\begin{aligned} & e^{-\frac{\varepsilon|k_\sigma|}{\sqrt{\gamma_{\sigma\sigma}}}} \\ k_\sigma &= \frac{n\pi}{L}, \end{aligned}$$

in the sum, and taking the limit $\varepsilon\to 0$ only after summation. The factor of $\sqrt{\gamma_{\sigma\sigma}}^{-1}$ is required to maintain the effects of reparametrization invariance correctly. The reason for this is simple. Outside light-cone gauge, the string length is not fixed, but can be changed by a reparametrization. Therefore, k_σ, which depends on the length of the string, changes under such transformations. Including the function of $\gamma_{\sigma\sigma}$ exactly cancels this effect.

Inserting this expression into the sum permits to evaluate it exactly, yielding

$$\begin{aligned} A &= \frac{D-2}{2}\sum_{n=1}^{\infty}ne^{-\frac{\varepsilon|k_\sigma|}{\sqrt{\gamma_{\sigma\sigma}}}} \\ &= \frac{D-2}{2}\left(\frac{2LP^+\alpha'}{\varepsilon^2\pi}-\frac{1}{12}+O(\varepsilon)\right), \end{aligned}$$

where the second line of (5.18) has been used. The first term is proportional to L and can therefore be absorbed in the action by an additional term proportional to

$$-\int d\sigma\,(-\gamma)^{\frac{1}{2}}=-L.$$

This is a constant, and therefore is not changing the action. In fact, the value of the action has to be regularized itself by a similar expression, and also regularized by $e^{-\varepsilon}$.

[3] With standard normalization, n would be replaced by 1, changing nothing qualitatively.

Thus, by appropriately selecting the pre-factors, both terms cancel. Since also the last term vanishes in the limit of $\varepsilon \to 0$, the only thing remaining is

$$A = \frac{2-D}{24}, \tag{5.27}$$

which is known as the Casimir energy, and can be traced back to the fact that the string is only of finite length. Thus, the string has indeed a non-zero vacuum energy. In contrast to the first contribution, this constant, non-divergent term cannot be naturally absorbed by a counter term in the action without spoiling Lorentz invariance.

Having now obtained the Hamiltonian and the state space, it is about time to determine the properties of the physical state space. In particular, the question is whether the string excitations can be interpreted as particle states, the original motivation to study it. For that purpose the primary object is of course whether the states satisfy the energy-momentum relation of a point particle, and if yes, what are their masses.

The corresponding operator for the rest mass is just given by the mass-shell equation, where it is to be used that $P^- = H$ to yield

$$m^2 = 2P^+H - P^iP^i, \tag{5.28}$$

as a result of the light-cone equation

$$m^2 = P^+P^- + P^-P^+ - P^iP^i.$$

Inserting the result (5.27) into (5.28) for the lowest-energy state yields

$$\begin{aligned} m^2 &= 2P^+\left(\frac{P^iP^i}{2P^+} + \frac{1}{2P^+\alpha'}(N+A)\right) - P^iP^i \\ &= \frac{1}{\alpha'}\left(N + \frac{2-D}{24}\right). \end{aligned}$$

That is quite an important result, as it implies that the mass is only dependent on the state sum N defined as

$$N = \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} nN_{in}$$

and the space-time dimensionality D. Thus, mass becomes an intrinsic property rather than an external parameter as in the standard model. The importance of the Regge slope is now also clear, as it links as constant of proportionality the number of a state and its rest mass.

The lowest state is of course $N = 0$, hence $|0, k\rangle$, and this yields

$$m^2 = \frac{2 - D}{24\alpha'}$$

Since for any phenomenologically relevant string theory $D > 2$ the rest mass of the lowest state is imaginary, $m^2 < 0$. Thus it is a tachyon. That is of course unfortunate, since interpreting this as a particle is very problematic. E. g., constructing a theory of such a non-interacting scalar tachyon yields a potential energy proportional to $m^2\phi^2/2$. Hence, the vacuum state is unstable. Of course, this would be the lowest approximation, and it could still be that the bosonic string theory is nonetheless stable, but this is unknown so far. Fortunately, in particular in supersymmetric string theories tachyons usually do not appear, so they provide a possibility to circumvent this problem without having to deal with it explicitly.

The first non-tachyonic state is obtained for the state $\alpha^i_{-1} |0, k\rangle$ with $N = 1$. Its mass reads

$$m^2 = \frac{26 - D}{24\alpha'}. \tag{5.29}$$

Since there are $D-2$ ways to obtain $N = 1$, this state is $D-2$-times degenerate. To be still Lorentz invariant, these transverse modes must form a representation of SO$(D-2)$ for a massless particle and SO$(D-1)$ for a massive particle. The former follows because there is no rest-frame for a massless particle, and the minimum momentum is at least $P^\mu = \left(E, E, \vec{0}\right)$, thus having less symmetry then the one for a massive particle in the rest frame being $P^\mu = (m, \vec{0})$.

As a consequence, in $D = 4$ massive bosonic particles have integer spin $j > 0$ as representations of SO(3) with $2j + 1$-fold degeneracies. Massless particles, however, are denoted by their helicity forming a representation of the group SO(2), having only one state with positive helicity. Because of CPT symmetry the number of states is actually doubled, since a state with positive helicity can be transformed by CPT into one with negative helicity. Put it in another view, the lowest non-trivial representation of SO(3) is 3-dimensional, a spin-1 state with three magnetic quantum numbers. For SO(2), the lowest non-trivial representation has actually only two possible magnetic quantum numbers, either 1 or -1. However, CPT guarantees that if one exists, then so does the other.

Going back to D dimensions there are thus $D-1$ states for massive bosonic particles, but only $D - 2$ for massless ones. Since the degeneracy for the $N = 1$ states is $D - 2$, this implies that their mass must be zero. From this immediately follows that the theory is only Lorentz-invariant in $D = 26$ dimensions, since otherwise (5.29) would not yield zero. This implies also $A = -1$, due to (5.27).

Hence, this indirect inference yields that the consistency of the string theory with Lorentz and CPT invariance requires a certain number of dimensions, different to quantum field theories, which at least in principle can be formulated in any number of space-time dimensions. Note that this is actually a quantum effect, since only quantization yields the mass-dimension relation (5.29).

A more formal argument will be given below, when it can be done simultaneously for both the open and the closed string, which will be analyzed now.

5.5.2.4 Closed string spectrum

A closed string is obtained when instead of open boundaries periodic boundaries are imposed. In this case the light-cone gauge conditions become.

$$\begin{aligned} X^\mu(\tau, L) &= X^\mu(\tau, 0) \\ \partial_\sigma X^\mu(\tau, L) &= \partial_\sigma X^\mu(\tau, 0) \\ \gamma_{ab}(\tau, L) &= \gamma_{ab}(\tau, 0) \end{aligned}$$

Similarly, it is then possible to quantize the closed string as the open string. However, this provides another ambiguity, since the zero position of σ can now be anywhere along the string. Consequently, a shift of the zero point is another symmetry of the system as

$$\sigma' = \sigma + s(\tau).$$

To fix it requires another gauge condition, which is conveniently chosen as

$$\gamma_{\tau\sigma}(\tau, 0) = 0$$

This implies that lines of constant τ are orthogonal to lines of constant σ at $\sigma = 0$. This reduces the problem to translations about one string length as

$$\sigma' = \sigma + s(\tau) \mod L. \tag{5.30}$$

Nonetheless, this is sufficient to start.

Up to the formulation of the Hamiltonian then everything is as for the open string case. Of course, the solutions to the equations of motion are now different, respecting the new boundary conditions. They read

$$X^i(\tau, \sigma) = X^i + \frac{P^i}{P^+}\tau + i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n=-\infty n\neq 0}^{\infty} \left(\frac{\alpha_n^i}{n} e^{-\frac{2\pi i n(\sigma + c\tau)}{L}} + \frac{\beta_n^i}{n} e^{\frac{2\pi i n(\sigma - c\tau)}{L}}\right),$$

in analogy to point quantum mechanics of a particle in a periodic box. As a consequence, there are now two independent sets of Fourier coefficients, α and β. These corresponds to oppositely directed waves along the string with α being those running in the left direction and β to the right direction.

Nonetheless, quantization proceeds as usual with the canonical quantization conditions

$$\begin{aligned}
\left[Z^-, P^-\right] &= -i \\
\left[X^i, P^i\right] &= i\delta^{ij} \\
\left[\alpha^i_m, \alpha^j_n\right] &= m\delta^{ij}\delta_{m,-n} \\
\left[\beta^i_m, \beta^j_n\right] &= m\delta^{ij}\delta_{m,-n}.
\end{aligned}$$

Thus, the system is again that of a set of free oscillators with a superimposed center-of-mass motion. The eigenstates are thus

$$|N, R, k\rangle = \left(\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\prod_{r=1}^{\infty}\frac{\left(\alpha^i_{-n}\right)^{N_{in}}\left(\beta^i_{-n}\right)^{R_{in}}}{\left(n^{N_{in}}N_{in}!r^{R_{in}}R_{in}!\right)^{\frac{1}{2}}}\right)|0, 0, k\rangle\,.$$

Herein N counts the number of left-moving states and R the number of right-moving states. It is then possible to obtain again the Hamiltonian in number-operator form, and to obtain the mass-shell equation as

$$m^2 = 2P^+H - P^iP^i = \frac{2}{\alpha'}\left(N + R + A + B\right),$$

and in the same way as previously also

$$A = B = \frac{2 - D}{24}$$

is obtained.

However, in this case the values of N and R are restricted, since all physical states have to be invariant under the residual gauge freedom (5.30). To see this, the operator for translations on the string is useful. To obtain it, the simplest starting point is the energy-momentum tensor on the world-sheet. It is given by

$$\begin{aligned}
T^{ab} &= -4\pi\,(-\gamma)^{-\frac{1}{2}}\,\frac{\delta L}{\delta\gamma_{ab}} \qquad (5.31)\\
&= -\frac{4\pi}{(-\gamma)^{\frac{1}{2}}}\frac{\delta}{\delta\gamma_{ab}}\left(-\frac{1}{4\pi\alpha'}\,(-\gamma)^{\frac{1}{2}}\,\gamma^{cd}\partial_c X^\mu\partial_d X_\mu\right)\\
&= \frac{1}{\alpha'\,(-\gamma)^{\frac{1}{2}}}\left(\frac{\delta\,(-\gamma)^{\frac{1}{2}}}{\delta\gamma_{ab}}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu+(-\gamma)^{\frac{1}{2}}\,\frac{\delta\gamma^{cd}}{\delta\gamma_{ab}}\partial_c X^\mu\partial_d X_\mu\right)\\
&= \frac{1}{\alpha'\,(-\gamma)^{\frac{1}{2}}}\left(-\frac{1}{2\,(-\gamma)^{\frac{1}{2}}}\frac{\delta\gamma}{\delta\gamma_{ab}}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu+(-\gamma)^{\frac{1}{2}}\,\partial_a X^\mu\partial_b X_\mu\right)\\
&= \frac{1}{\alpha'\,(-\gamma)^{\frac{1}{2}}}\left(-\frac{1}{2\,(-\gamma)^{\frac{1}{2}}}\gamma\gamma^{ab}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu+(-\gamma)^{\frac{1}{2}}\,\partial_a X^\mu\partial_b X_\mu\right)\\
&= \frac{1}{\alpha'}\left(\partial_a X^\mu\partial_b X_\mu-\frac{1}{2}\gamma^{ab}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu\right)\\
&= \frac{1}{\alpha'}\left(\partial^a X^\mu\partial^b X_\mu-\frac{1}{2}\gamma^{ab}\partial_c X^\mu\partial^c X_\mu\right).
\end{aligned}$$

To argue that this indeed is an energy-momentum tensor[4], it is necessary to show that it has the necessary properties of an energy-momentum tensor, in particular it has to be conserved and traceless, and its $\tau\tau$-component must equal the Hamilton operator.

Start with its conservation. The elements of the energy-momentum tensors appear to be not invariant under diffeomorphisms, since the appearing expressions for γ^{ab} are not, since it seems there are no compensating factor of $\det\gamma$. However, the expression in terms of the Lagrangian is, so there must be a hidden invariance. This is in fact only possible if the energy-momentum tensor is a constant, which would imply its conservation.

Using (5.9), this can be obtained explicitly

$$\begin{aligned}
\partial_a T^{ab} &= \frac{1}{\alpha'}\partial_a\left(\partial^a X^\mu\partial^b X_\mu-\frac{1}{2}\gamma^{ab}\partial_c X^\mu\partial^c X_\mu\right)\\
&= \frac{1}{\alpha'}\left(\partial_a h^{ab}-\partial_a\left(\frac{1}{2}\gamma^{ab}\gamma^{cd}h_{cd}\right)\right)\\
&= \frac{1}{\alpha'}\left(\partial_a h^{ab}-\partial_a h^{ab}\right)=0,
\end{aligned}$$

and thus the energy-momentum tensor is conserved.

[4] In quantum field theory this is already a non-obvious fact, lest in string theory.

The next condition is the one of tracelessness. Calculating the trace $T^a_{\ a}$ explicitly yields

$$\begin{aligned}\gamma_{ab}\frac{\delta L}{\delta\gamma_{ab}} &= \frac{1}{(-\gamma)^{\frac{1}{2}}}\left(\gamma_{ab}\partial^a X^\mu\partial^b X_\mu - \partial^c X^\mu\partial_c X_\mu\right)\\ &= \frac{1}{(-\gamma)^{\frac{1}{2}}}\left(\partial^a X^\mu\partial_a X_\mu - \partial^c X^\mu\partial_c X_\mu\right) = 0 \\ &= \gamma_{ab}\frac{T^{ab}}{(-\gamma)^{\frac{1}{2}}},\end{aligned} \tag{5.32}$$

where it has been used that $\gamma^{ab}\gamma_{ab} = 2$. Finally, this yields

$$T^a_{\ a}\frac{1}{(-\gamma)^{\frac{1}{2}}} = 0,$$

confirming that the energy-momentum tensor is indeed traceless. Incidentally, this shows that the classical energy-momentum tensor vanishes when the equations of motions are fulfilled, by virtue of (5.32) and the fact that the Lagrange function is not depending on the τ-derivatives of γ_{ab}.

Using (5.9), this could also be shown more directly as

$$\begin{aligned}T^a_{\ a} &= \frac{1}{\alpha'}\left(\partial^a X^\mu\partial_a X_\mu - \frac{1}{2}\gamma^a_{\ a}\partial^c X^\mu\partial_c X_\mu\right)\\ &= \frac{1}{\alpha'}\left(\partial^a X^\mu\partial_a X_\mu - \frac{1}{2}\gamma^a_{\ a}\gamma_{cd}\partial^c X^\mu\partial^d X_\mu\right)\\ &= \frac{1}{\alpha'}\left(\partial^a X^\mu\partial_a X_\mu - \partial^a X^\mu\partial_a X_\mu\right) = 0,\end{aligned}$$

and thus the same result.

Finally, the $\tau\tau$ component should be the Hamiltonian. To show this, it is simpler to go backwards. By reexpressing the Hamiltonian (5.19) as a function of $\partial_\sigma X^\mu$ and $\partial_\tau X^\mu$ it becomes

$$\begin{aligned}H &= \frac{L}{4\pi\alpha' P^+}\int_0^L d\sigma\left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right)\\ &= \frac{L}{4\pi\alpha' P^+}\int_0^L d\sigma\left(2\pi\alpha'\frac{P^+}{L}\frac{P^+}{L}\partial_\tau X^i\partial_\tau X^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right).\end{aligned}$$

Using now (5.18) changes this to

$$\begin{aligned}
H &= \frac{1}{4\pi\alpha'}\int\limits_0^L d\sigma\left(2\pi\alpha'\frac{P^+}{L}\partial_\tau X^i\partial_\tau X^i+\frac{1}{2\pi\alpha'}\frac{L}{P^+}\partial_\sigma X^i\partial_\sigma X^i\right)\\
&= \frac{1}{4\pi\alpha'}\int\limits_0^L d\sigma\left(2\pi\alpha'\frac{\gamma_{\sigma\sigma}}{2\pi\alpha'}\partial_\tau X^i\partial_\tau X^i+\frac{1}{2\pi\alpha'}\frac{2\pi\alpha'}{\gamma_{\sigma\sigma}}\partial_\sigma X^i\partial_\sigma X^i\right)\\
&= \frac{1}{4\pi\alpha'}\int\limits_0^L d\sigma\left(\gamma_{\sigma\sigma}\partial_\tau X^\mu\partial_\tau X_\mu+\frac{1}{\gamma_{\sigma\sigma}}\partial_\sigma X^\mu\partial_\sigma X_\mu\right).
\end{aligned}$$

The expansion of i to μ in the last line was permitted because this is only an addition of zero in the second term and also a zero in the first term by virtue of the boundary conditions after exchange of integration and differentiation.

To bring the $\tau\tau$ component of the energy-momentum tensor into the same form it can be expressed as

$$\begin{aligned}
T^{\tau\tau} &= \frac{1}{\alpha'}\left(\partial^\tau X^\mu\partial^\tau X_\mu-\gamma^{\tau\tau}\partial_\tau X^\mu\partial^\tau X_\mu-\gamma^{\tau\tau}\partial_\sigma X^\mu\partial^\sigma X_\mu\right)\\
&= \frac{1}{\alpha'}\Big(\partial^\tau X^\mu\partial^\tau X_\mu-\frac{1}{2}\left(\gamma^{\tau\tau}\gamma_{\tau\tau}\right)\partial_\tau X^\mu\partial_\tau X_\mu\\
&\quad-\frac{1}{2}\gamma^{\tau\tau}\gamma_{\tau\sigma}\partial_\tau X^\mu\partial_\sigma X_\mu-\frac{1}{2}\gamma^{\tau\tau}\gamma_{\tau\sigma}\partial_\sigma X^\mu\partial_\tau X_\mu-\frac{1}{2}\gamma^{\tau\tau}\gamma_{\sigma\sigma}\partial_\sigma X^\mu\partial_\sigma X_\mu\Big).
\end{aligned}$$

Because of the gauge condition $\gamma^{\tau\tau}$ and $-\gamma^{\sigma\sigma}$ are related, and yielding that the square of $\gamma^{\tau\tau}$ is -1, because otherwise the gauge condition for the determinant would be violated, given that $\gamma_{\tau\sigma}$ vanishes. This yields

$$\begin{aligned}
T^{\tau\tau} &= \frac{1}{\alpha'}\left(\frac{1}{2}\partial_\tau X^\mu\partial_\tau X_\mu+\frac{1}{2}\partial_\sigma X^\mu\partial_\sigma X_\mu\right)\\
&= \frac{1}{2\alpha'}\left(\gamma^{\sigma\sigma 2}\partial_\tau X^\mu\partial_\tau X_\mu+\partial_\sigma X^\mu\partial_\sigma X_\mu\right)\\
&= \frac{1}{2\alpha'}\gamma^{\sigma\sigma}\left(\gamma^{\sigma\sigma}\partial_\tau X^\mu\partial_\tau X_\mu+\frac{1}{\gamma^{\sigma\sigma}}\partial_\sigma X^\mu\partial_\sigma X_\mu\right),
\end{aligned}$$

which concludes

$$H=-\frac{1}{2\pi}\int\limits_0^L d\sigma\gamma^{\sigma\sigma}T^{\tau\tau}$$

where the factor $\gamma^{\sigma\sigma}$ is actually part of the measure to make the expression diffeomorphism invariant, and thus shows the correct relation between the Hamiltonian and the energy-momentum tensor.

Hence, it is permitted to use this expression for the energy-momentum tensor to obtain the operator of linear translation. It is given by the $\sigma\tau$ component, as in case of classical mechanics. Since $\gamma_{\tau\sigma}=0$ this component is given by

$$T^{\sigma\tau}=\frac{1}{\alpha'}\left(\partial^\sigma X^\mu\partial^\tau X_\mu\right)=2\pi c\Pi^i\partial_\sigma X^i,$$

since the $\pm$ have a vanishing σ component each. Integrating yields the operator as

$$\begin{aligned}P &= -\int_0^L d\sigma\Pi^i\partial_\sigma X^i\\ &= \frac{2\pi}{L}\left(\sum_{n=1}^\infty\left(\alpha^i_{-n}\alpha^i_n-\beta^i_{-n}\beta^i_n\right)+A-B\right)\\ &= \frac{2\pi}{L}\left(N-R\right).\end{aligned}$$

The residual gauge freedom is essentially giving that the coordinates hop around the string by an integer times L, permitting to turn left-moving into right-moving modes. This can be restricted by enforcing

$$N=R. \tag{5.33}$$

Thus, the expectation value of translations along the string is zero, and any physical state has a localized coordinate system on the string. With other words, the number of left and right moving modes must be the same.

The lowest state is given again by

$$m^2=\frac{2}{\alpha'}2\frac{2-D}{24}=\frac{2-D}{6\alpha'},$$

and is therefore again a tachyon. The lowest excited state is given by $|1,1,k\rangle$

$$m^2=\frac{26-D}{6\alpha'}.$$

However, in contrast to the previous case, it is not constructed by a single creation operator with just one space-time index, but by two as

$$|1,1,k\rangle=\alpha^i_{-1}\beta^j_{-1}\left|0,0,k\right\rangle,$$

and therefore is a tensor state e_{ij}. As in the case of large extra dimensions, this state can be separated as

$$e^{ij}=\frac{1}{2}\left(e^{ij}+e^{ji}-\frac{2}{D-2}\delta^{ij}e^{kk}\right)+\frac{1}{2}\left(e^{ij}-e^{ji}\right)+\frac{1}{D-2}\delta^{ij}e^{kk}.$$

The first term is traceless symmetric, the second antisymmetric and the third scalar. Furthermore, the occupation numbers N_{in} and R_{in} can vary freely as long as $N = R$ is fulfilled. Therefore, the number of states is substantially increased with respect to the open string spectrum at the same N. Whether it is necessarily massless, and thus again $D = 26$, is not a trivial question, but will turn out to be correct. This time, the helicity of the state will be useful to show this will yield a graviton, an axion, and a dilaton.

To verify the assignment of spin, a little more formal investigation is useful. Note that it is always possible to obtain a spin algebra from creation and annihilation operators, when summing over oscillators, called the Schwinger representation. In case of the open string, the corresponding operators are given by

$$S^{ij} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i \right) \tag{5.34}$$

and the ones for the closed string are completely analogous, just requiring that it is now necessary to sum over both, left-moving and right-moving modes. The two indices already indicate that these will be the corresponding n-dimensional generalization of the spin.

That (5.34) are indeed spin operators can be shown by explicitly calculating the corresponding algebra. Start by evaluating the commutator as

$$\begin{aligned}
\left[S^{ij}, S^{kl}\right] &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm} \left[\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i, \alpha_{-m}^k \alpha_m^l - \alpha_{-m}^l \alpha_m^k\right] \\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm} \left(\left[\alpha_{-n}^i \alpha_n^j, \alpha_{-m}^k \alpha_m^l - \alpha_{-m}^l \alpha_m^k\right] \right. \\
&\quad \left. - \left[\alpha_{-n}^j \alpha_n^i, \alpha_{-m}^k \alpha_m^l - \alpha_{-m}^l \alpha_m^k\right] \right) \\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm} \left(\left[\alpha_{-n}^i \alpha_n^j, a_{-m}^k \alpha_m^l\right] - \left[\alpha_{-n}^i \alpha_n^j, \alpha_{-m}^l \alpha_m^k\right] \right. \\
&\quad \left. - \left[\alpha_{-n}^j \alpha_n^i, \alpha_{-m}^k \alpha_m^l\right] + \left[\alpha_{-n}^j \alpha_n^i, \alpha_{-m}^l \alpha_m^k\right] \right).
\end{aligned}$$

It is simpler to evaluate each of the four terms individually. For this the relation

$$[ab, c] = a\,[b, c] + [a, c]\,b$$

for double commutators is quite useful, as well as the quantization conditions (5.24) are necessary. In the following the summation is kept implicit. This yields for the first term

$$\begin{aligned}\left[\alpha^i_{-n}\alpha^j_n,\alpha^k_{-m}\alpha^l_m\right] &= \alpha^i_{-n}\left[\alpha^i_n,\alpha^k_{-m}\alpha^l_m\right]+\left[\alpha^i_{-n},\alpha^k_{-m}\alpha^l_m\right]\alpha^j_n\\ &= \alpha^i_{-n}\alpha^k_{-m}\left[\alpha^j_n,\alpha^l_m\right]+\alpha^i_{-n}\left[\alpha^j_n,\alpha^k_{-m}\right]\alpha^l_m\\ &\quad+\alpha^k_{-m}\left[\alpha^i_{-n},\alpha^l_m\right]\alpha^i_n+\left[\alpha^i_{-n},\alpha^k_{-m}\right]\alpha^l_m\alpha^j_n\\ &= \alpha^i_{-n}\alpha^k_{-m}n\delta^{jl}\delta_{n,-m}+\alpha^i_{-n}\alpha^l_m n\delta^{jk}\delta_{n,m}\\ &\quad-\alpha^k_{-m}\alpha^j_n n\delta^{il}\delta_{-n,-m}-\alpha^l_m\alpha^j_n n\delta^{ik}\delta_{-n,m}\\ &= n(\alpha^i_{-n}\alpha^k_n\delta^{jl}+\alpha^i_{-n}\alpha^l_n\delta^{jk}-\alpha^k_{-n}\alpha^j_n\delta^{il}-\alpha^l_{-n}\alpha^j_n\delta^{ik}),\end{aligned}\tag{5.35}$$

for the second term

$$\begin{aligned}\left[\alpha^i_{-n}\alpha^j_n,\alpha^l_{-m}\alpha^k_m\right] &= \alpha^i_{-n}\left[\alpha^j_n,\alpha^l_{-m}\alpha^k_m\right]+\left[\alpha^i_{-n},\alpha^l_{-m}\alpha^k_m\right]\alpha^j_n\\ &= \alpha^i_{-n}\alpha^l_{-m}\left[\alpha^j_n,\alpha^k_m\right]+\alpha^i_{-n}\left[\alpha^j_n,\alpha^l_{-m}\right]\alpha^k_m\\ &\quad+\alpha^l_{-m}\left[\alpha^i_{-n},\alpha^k_m\right]\alpha^j_n+\left[\alpha^i_{-n},\alpha^l_{-m}\right]\alpha^k_m\alpha^j_n\\ &= \alpha^i_{-n}\alpha^l_{-m}n\delta^{jk}\delta_{n,-m}+\alpha^i_{-n}\alpha^k_m n\delta^{jl}\delta_{n,m}\\ &\quad-\alpha^l_{-m}\alpha^j_n n\delta^{ik}\delta_{-n,-m}-\alpha^k_m\alpha^j_n n\delta^{il}\delta_{-n,m}\\ &= n(\alpha^i_{-n}\alpha^l_n\delta^{jk}+\alpha^i_{-n}\alpha^k_n\delta^{jl}-\alpha^l_{-n}\alpha^j_n\delta^{ik}-\alpha^k_{-n}\alpha^j_n\delta^{il}),\end{aligned}\tag{5.36}$$

for the third term

$$\begin{aligned}\left[\alpha^j_{-n}\alpha^i_n,\alpha^k_{-m}\alpha^l_m\right] &= \alpha^j_{-n}\left[\alpha^i_n,\alpha^k_{-m}\alpha^l_m\right]+\left[\alpha^j_{-n},\alpha^k_{-m}\alpha^l_m\right]\alpha^i_n\\ &= \alpha^j_{-n}\alpha^k_{-m}\left[\alpha^i_n,\alpha^l_m\right]+\alpha^j_{-n}\left[\alpha^i_n,\alpha^k_{-m}\right]\alpha^l_m\\ &\quad+\alpha^k_{-m}\left[\alpha^j_{-n},\alpha^l_m\right]\alpha^i_n+\left[\alpha^j_{-n},\alpha^k_{-m}\right]\alpha^l_m\alpha^i_n\\ &= \alpha^j_{-n}\alpha^k_{-m}n\delta^{il}\delta_{n,-m}+\alpha^j_{-n}\alpha^l_m n\delta^{ik}\delta_{n,m}\\ &\quad-\alpha^k_{-m}\alpha^i_n n\delta^{jl}\delta_{-n,-m}-\alpha^l_m\alpha^i_n n\delta^{jk}\delta_{-n,m}\\ &= n(\alpha^j_{-n}\alpha^k_n\delta^{il}+\alpha^j_{-n}\alpha^l_n\delta^{ik}-\alpha^k_{-n}\alpha^i_n\delta^{jl}-\alpha^l_{-n}\alpha^i_n\delta^{jk}),\end{aligned}\tag{5.37}$$

and finally the fourth

$$\begin{aligned}\left[\alpha^j_{-n}\alpha^i_n,\alpha^l_{-m}\alpha^k_m\right] &= \alpha^j_{-n}\left[\alpha^i_n,\alpha^l_{-m}\alpha^k_m\right]+\left[\alpha^j_{-n},\alpha^l_{-m}\alpha^k_m\right]\alpha^i_n\\ &= \alpha^j_{-n}\alpha^l_{-m}\left[\alpha^i_n,\alpha^k_m\right]+\alpha^j_{-n}\left[\alpha^i_n,\alpha^l_{-m}\right]\alpha^k_m\\ &\quad+\alpha^l_{-m}\left[\alpha^j_{-n},\alpha^k_m\right]\alpha^i_n+\left[\alpha^j_{-n},\alpha^l_{-m}\right]\alpha^k_m\alpha^i_n\\ &= \alpha^i_{-n}\alpha^l_{-m}n\delta^{ik}\delta_{n,-m}+\alpha^j_{-n}\alpha^k_m\delta^{ij}\delta_{n,m}\\ &\quad-\alpha^l_{-m}\alpha^i_n n\delta^{jk}\delta_{-n,-m}-\alpha^k_m\alpha^i_n n\delta^{jl}\delta_{-n,m}\\ &= n(\alpha^j_{-n}\alpha^l_n\delta^{ik}+\alpha^j_{-n}\alpha^k_n\delta^{ij}-\alpha^l_{-n}\alpha^i_n\delta^{ik}-\alpha^k_{-n}\alpha^i_n\delta^{jl}).\end{aligned}\tag{5.38}$$

Combining (5.35-5.38) permits to drop the summation over m. In addition, for every δ each term appears twice, reducing the total expression to

$$\begin{aligned}\left[S^{ij},S^{kl}\right] &= -2\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^{i}\alpha_{n}^{k}\delta^{jl}+\alpha_{-n}^{i}\alpha_{n}^{l}\delta^{jk}+\alpha_{-n}^{j}\alpha_{n}^{k}\delta^{il}+\alpha_{-n}^{j}\alpha_{n}^{l}\delta^{ik}\\ &\quad -\alpha_{-n}^{k}\alpha_{n}^{j}\delta^{il}-\alpha_{-n}^{l}\alpha_{n}^{j}\delta^{ik}-\alpha_{-n}^{k}\alpha_{n}^{i}\delta^{jl}-\alpha_{-n}^{l}\alpha_{n}^{i}\delta^{jk}).\end{aligned}$$

Reordering, expanding -1 to i^2, and combing terms with the same δ permits to reconstruct spin operators. Finally, the result becomes

$$\left[S^{ij},S^{kl}\right]=2i\left(\delta^{jl}S^{ik}+\delta^{jk}S^{il}+\delta^{il}S^{jk}+\delta^{ik}S^{il}\right).$$

Thus, indeed the operators satisfy a spin algebra. If all indices are different then the commutator vanishes. Since furthermore all diagonal elements of the S^{ij} vanish only elements with the same indices remain. For example this leaves

$$\left[S^{12},S^{23}\right]=2iS^{13}.$$

The commutator hence contains always the two unequal indices in the same order. Since the spin operator is antisymmetric by definition also the correct exchange property for the arguments of the commutator is obtained, completing the construction.

To see how the helicity emerges investigate first the 23 component of the spin operator, being the one relevant in a four-dimensional sub-space. The helicity of the lowest excitation of the open string is then given by

$$\langle 1,k|\,S^{23}\,|1,k\rangle=-i\sum_{n=1}^{\infty}\frac{1}{n}\,\langle 1,k|\left(\alpha_{-n}^{2}\alpha_{n}^{3}-\alpha_{-n}^{3}\alpha_{n}^{2}\right)|1,k\rangle=\langle 1,k|\,i\,|1,k\rangle=i.$$

Thus the value is 1. For the lowest excitation of the closed string, the value is found analogously to be two. Thus the lowest excitation of the open string is a vector particle while the one of the closed string is rather a graviton, in accordance with the previous considerations.

Comparing all results a number of interesting observations are obtained. Since vector particles always harbor a gauge symmetry, the open string already furnishes a gauge theory. Since it is non-interacting, this gauge theory has to be non-interacting as well, leaving only a U(1) gauge theory. A more detailed calculation would confirm this. Therefore, it is admissible to call the state $|1,k\rangle$ a photon.

Similarly, a spin 2 particle couples to a conserved tensor current. Since the only one available is the energy-momentum tensor, the symmetric contribution of the lowest excitation of the closed string can be interpreted as a graviton. The antisymmetric particle can be given the meaning of an axion, as it is equivalent to a 2-form gauge boson. Finally, the scalar particle is then the dilaton, as in the case of large extra dimensions.

Calculating the helicity gives already the correct result for the photon and the graviton. Indeed, for the axion and the dilaton a value of zero is obtained, as they would have also in a generic quantum field theory of these particles.

It should be noted that it can be shown that a string theory turns out to be only consistent if it at least contains the closed string, with the open string being an optional addition. Thus, the graviton is there in any string theory.

5.5.3 Virasaro algebra

5.5.3.1 The algebra

The property of being consistent only in a certain number of dimensions can be linked to an algebraic structure, the Virasaro algebra. For this, it is useful to not use a particular gauge, but rather a more general setting. For the following, this essentially boils down to use instead of the canonical commutation relations (5.24)

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \tag{5.39}$$

and thus to permit quantized oscillations in all directions. Of course, this is to be expected: These oscillations are the same as in light cone gauge, as the remainder coordinate functions are completely determined by the reduced set of spatial directions due to the present symmetries, and therefore were not needed to be given explicitly.

The starting point for the construction of the algebra is then the Fourier expansion of the diagonal elements of the world-sheet energy momentum tensor. For this purpose, it is useful to set the string length to 2π, to avoid a proliferation of factors of L. Classically, for the open string, its is defined as

$$T_{aa} = \alpha' \sum_{n=-\infty}^{\infty} L_n e^{-in\xi^a},$$

(no summation over a implied) where the L_n are the expansion modes and the ξ^a are the momenta along the directions σ and τ on the world-sheet. Because of the two different movement directions on the closed string, the modes for the τ and σ directions are different,

$$
\begin{aligned}
T_{\tau\tau} &= 4\alpha' \sum_{n=-\infty}^{\infty} \tilde{L}_n e^{-in\xi^\tau} \\
T_{\sigma\sigma} &= 4\alpha' \sum_{n=-\infty}^{\infty} L_n e^{-in\xi^\sigma}.
\end{aligned}
$$

These modes can be expressed in terms of the Fourier coefficients α as

$$
L_m = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma e^{im\sigma} T_{\tau\tau}\Bigg|_{\tau=0} = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma e^{-im\sigma} T_{\sigma\sigma}\Bigg|_{\tau=0} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha^\mu_n
$$

for the open string and

$$
\begin{aligned}
L_m &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha^\mu_n \\
\tilde{L}_m &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \beta^\mu_{m-n}\beta^\mu_n
\end{aligned}
$$

for the closed string. Note that the energy momentum tensor vanishes by being the equation of motion (5.31) of a cyclic variable. From the vanishing of the energy momentum tensor then follows $L_m = \tilde{L}_m = 0$ for all m, the so-called Virasaro constraints. Since L_m is not differing between open and closed strings, it will not be differentiated in the following between both, except for the presence or absence of the second mode $\tilde{L}_m$.

When now quantizing the system, there appears an ordering problem for L_0, as α_{-n} is not commuting with α_n, see (5.39). Thus, an ambiguity arises, and therefore the quantum version of L_0 and $\tilde{L}_0$ are defined as

$$
L_0 = \frac{\alpha_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = a + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n,
$$

and similarly for $\tilde{L}_0$. The constant a can be determined when observing that the mass operator m^2, defined to be the Hamiltonian minus P^2, is given by

$$
m^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} \alpha_n\alpha_{-n} = -\frac{1}{\alpha'}\left(a - \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n\right),
$$

since the same operator ordering problem arises. Since the mass is invariant under the gauge choice the value of a can be read off (5.27), as then $-A = a = 1$.

The Virasaro algebra is now given by the algebra of the operators L_m. For $m+n\neq 0$, it can be straightforwardly, albeit tediously, shown that

$$[L_m,L_n]=(m-n)L_{m+n},$$

using the canonical commutator relations for the αs (5.24). However, it is more complicated if $m+n=0$. It is direct to show that for any m

$$[L_m,\alpha_n^\mu]=-n\alpha_{m+n}^\mu. \tag{5.40}$$

holds. The commutator is now given by

$$\begin{aligned}
[L_m,L_n] &= \frac{1}{2}\Bigg(\sum_{p=-\infty}^{-1}\left((m-p)\alpha_p^\mu\alpha_{m+n-p}^\mu+p\alpha_{n+p}^\mu\alpha_{m-p}^\mu\right)\\
&\quad+\sum_{p=0}^{\infty}\left(p\alpha_{m-p}^\mu\alpha_{n+p}^\mu+(m-p)\alpha_{m+n-p}^\mu\alpha_p^\mu\right)\Bigg)\\
&= \frac{1}{2}\Bigg(\sum_{p=-\infty}^{-1}(m-p)\alpha_p^\mu\alpha_{m+n-p}^\mu+\sum_{p=-\infty}^{n-1}(p-n)\alpha_p^\mu\alpha_{n+m-p}^\mu\\
&\quad+\sum_{p=n}^{\infty}(p-n)\alpha_{n+m-p}^\mu\alpha_p^\mu+\sum_{p=0}^{\infty}(m-p)\alpha_{m+n-p}^\mu\alpha_p^\mu\Bigg)
\end{aligned}$$

Now it remains to bring the terms all in the same order as necessary for the definitions of the L_m. This is again a somewhat tedious exercise, and ultimately yields

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{d}{12}(m^3-m)\delta_{m+n}$$

where the last term is called the central extension of the algebra.

5.5.3.2 Physical states

One of the main advantages of the Virasaro algebra is to permit a simple identification of physical states, and to check that only physical states of a string theory contribute to observables. As in quantum mechanics and in quantum field theory, a state p is considered to be physical if it has a positive norm and positive semi-definite inner product with other physical states q,

$$\begin{aligned}
\langle p|p\rangle &> 0\\
|\langle p|q\rangle|^2 &\geq 0.
\end{aligned}$$

There may exist other states in a theory. One such class are states with zero inner product with any physical state p, so-called spurious states,

$$\langle p|z\rangle = 0, \tag{5.41}$$

These spurious state then do not contribute to any observable. What is not permitted are states with negative norm or overlaps, so-called ghost states g, as these would spoil any probability interpretation of the theory.

Physical states can now be shown to behave as

$$L_{m>0}|p\rangle = 0 \tag{5.42}$$

$$(L_0 - a)|p\rangle = 0, \tag{5.43}$$

while spurious states obey besides the second condition (5.43) also (5.41) for all physical states. The correctness of this assignment follows from the fact that the conditions (5.42) and (5.43) can be shown to correspond to the vanishing of the quantized world-sheet energy momentum tensor, and thus imply the satisfaction of the equations of motion.

Since the adjoint of L_m is L_{-m}, spurious states can be written as

$$|z\rangle = \sum_{n>0} L_{-n}|\chi_n\rangle,$$

where the χ_n satisfy

$$(L_0 - a + n)|\chi_n\rangle = 0.$$

This implements both conditions for spurious states (5.43) and (5.41) by construction. Since for $m < -2$ the L_m can be rewritten, using the Virasaro algebra, in terms of L_{-1} and L_{-2}, this can be simplified to

$$|z\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle.$$

A state can be both physical and spurious. By construction, it follows that such states have zero scalar product with any physical states including themselves, i. e., they have zero inner norm. Such states are called null states.

Such null states n can be constructed using spurious states of the form

$$|n\rangle = L_{-1}|\chi_1\rangle.$$

Such a state fulfills all conditions of being physical, except

$$L_1|\chi_1\rangle = L_1L_{-1}|\chi_1 >= 2L_0|\chi_1\rangle = 2(a-1)|\chi_1\rangle, \tag{5.44}$$

using the Virasaro algebra. Only since $a = 1$, the state is physical. Given the definition of L_{-1}, it actually follows that $|\chi\rangle = |0, k\rangle$, i. e. the state where the string has no internal excitations. Incidentally, this implies that the tachyon is not a physical state. Furthermore, this implies that any physical state is actually an equivalence class of states

$$|p\rangle \sim |p\rangle + |n\rangle,$$

as no measurement can differentiate between the original state and the one where an arbitrary zero norm state has been added. In fact, an infinite number of such null states can be constructed. These are required to cancel in any physical process contribution from negative norm states, the ghost states, very similar to the situation in gauge theories. This is, however, beyond the present scope. In fact, in light-cone gauge such states do not arise, implying that the theory is well-defined. the reason for their appearance here is that the covariant formulation is not fully fixing the reparametrization invariance, as it explicitly contains unphysical degrees of freedom, the additional $X^{\pm}$, just like in an ordinary gauge theory.