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Black Holes as Quantum Phenomena

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Abstract

Though a full quantum theory of gravity is unknown, we can still make general statements about observables that must hold for any theory. In this thesis we discuss why observables must be manifestly diffeomorphism and gauge invariant to be physical. In a general relativistic setting we adopt the Fröhlich-Morchio-Strocchi (FMS) mechanism to the metric field and we elicit its consequences to observables and how flat space quantum field theory emerges. With the observation of gravitational waves and projects like the Event Horizon Telescope, we close in on strongly gravitating systems where quantum effects can potentially influence the observations. With this in mind and clarified what physical observables are we build a physical three point scattering model of a black hole and a scalar field. By applying the FMS mechanism we extract the leading order contribution of the model. This term is not diffeomorphism and gauge invariant anymore but it can be explicitly calculated for different scenarios. In order to do so we discuss a black hole operator which is approximated by its curvature value. Due to the restriction to operate in position space, the interpretation of the results is intricate. Since this is the first approach with the mentioned tools, a plethora of possible extensions is open for future endeavors.

Zusammenfassung

Obwohl eine Quantentheorie der Gravitation derzeit nicht exisitert, ist es möglich allgemeine Aussagen über Observablen zu treffen, die unabhängig von der Theorie erfüllt sein müssen. In dieser Arbeit diskutieren wir warum Observablen eich- und diffeomorphismusinvariant sein müssen, um physikalisch zu sein. In einem allgemein relativistischen Rahmen wenden wir dann den Fröhlich-Morchio-Strocchi (FMS) Mechanismus auf das metrische Feld an und diskutieren seine Konsequenzen für Observablen und wie daraus die Quantenfeldtheorie in flacher Raumzeit auftritt. Durch die Beobachtung von Gravitationswellen und Projekte wie dem Event Horizon Telescope, beobachten wir Systeme mit starken Gravitationsfeldern, bei denen Quanteneffekte die Beobachtung beeinflussen können. In diesem Sinne konstruieren wir ein Dreipunkt-Streumodell von einem schwarzen Loch und einem skalaren Feld. Durch die Anwendung des FMS Mechanismus extrahieren wir den dominierenden Beitrag zur Streufunktion. Dieser Term ist nicht mehr diffeomorphismus- und eichinvariant, aber kann dafür explizit für verschiedene Szenarien berechnet werden. Dazu diskutieren wir einen Operator für ein Schwarzes Loch und wie er durch dessen Raumkrümmung approximiert wird. Da wir nur im Ortsraum arbeiten können, gestaltet sich die Interpretation der Ergebnisse als schwierig. Als eine erste Herangehensweise mit den oben genannten Methoden, bleibt eine Vielzahl an möglichen Erweiterungen für zukünftige Untersuchungen offen.

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It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong.

- Richard P. Feynman

Contents

1 Introduction

The standard model of particle physics in the framework of quantum field theory describes the matter content of our universe amazingly well [1]. But, there are still some open problems left. Most importantly, it does only cover three of the four fundamental forces, i.e. gravity is not covered in this framework [2, 3]. The classical theory that describes gravity is general relativity. Based on this theory, the ΛCDM model describes our universe astoundingly well [4]. This model has also some open questions to it. Most importantly the dark sector, i.e. dark matter and dark energy [5]. They are critical for the structure formation and the evolution of the universe. But, to this date there is no satisfying explanation for them. Apart from these cosmological phenomena, there are objects in the universe that are mysterious on there own: black holes. Though, at first they were only a theoretical solution of Einstein's equations [6, 7, 5], but with astronomical observations like the event horizon telescope [8] or the detection of gravitational waves [9] it became clear that they are rather common in the universe. Since such observations provide the possibility to study gravitation and black holes in more detail, it is interesting to also approach them from a theoretical side. We know that black holes can be created through gravitational collapse at the end of the lifetime of a star [6]. So, they are created from matter that follows the laws of quantum mechanics. Therefore, it is reasonable to ask, if black holes have quantum mechanical properties, i.e. if they are quantum objects. In this sense a black hole can be measured, if there is a corresponding observable that is represented by an operator. Thinking of a black hole as a quantum object provides two options [10]. Either the object is monolithic, i.e. it is essentially an elementary particle. Or, it is a composite object that is build up of a statistical amount of excitations. The latter possibility has its analogy in the description of a neutron star, which can in theory be described by quark operators. As there is yet no experimental evidence that would hint to one of the scenarios, an a priori choice has to be made. Clearly, building a black hole out of a vast amount of constituents poses many technical problems. Most importantly, this conglomerate of particles has to reconstruct macroscopic properties of the black hole, e.g. the emerging of an event horizon. Such a picture would provide a straightforward explanation for the concept of Hawking radiation [11], as it could be considered a quantum tunneling effect or a decay process. Nonetheless, it is technically easier to go with the choice of a monolithic quantum object, which will be done in this thesis. It would be useful to be able to make statements about the connection of the quantum black hole to matter, since a characteristic behavior in a scattering process could hint to the quantum properties of black holes. This is the main objective of this thesis: **Building a black hole-matter scattering model**.

With the lack of a more fundamental theory we want to focus on observables. In order

to do so, it is necessary to clarify what properties an observable needs to hold to be actually physical [12, 10]. After elucidating this, we use the property of being physical as a guiding principle to construct our scattering model. It is important to note that in generally curved spacetimes a global momentum-space representation does not exist. Which means that we need to consider the observables in position space. This complicates this endeavor. The interpretation of a propagator in momentum space is straightforward, since divergences of the very same provides the mass(es) of particles (scattering states) [2]. In position space such an interpretation is lacking. There, the mass of a particle shows itself in the asymptotic behavior of spacelike distances (see 3.4.1) [13]. For this reason, it is not possible to determine the mass exactly.
Building a black hole scattering model requires a black hole operator. Since an exact expression for such an object is probably lacking until a quantum gravity theory is found, we need to make a priori assumptions. The quantity exhibiting a singular behavior, which is characteristic for a black hole, is the spacetime curvature [7, 5, 6]. We therefore assume that it is the proper quantity to use as a black hole operator. Considering the metric a quantum field, the curvature will also show quantum fluctuations. Since we can't describe or determine them, we make the assumption that the full curvature is dominated by a classical part. This classical part is the well-known spacetime curvature of a black hole described by general relativity. Hence, we can approximate the curvature operator by the classical curvature. Further details about this are discussed in section 5.

A property of a theory that describes everything is that it most likely[1] contains general relativity and quantum field theory in the form of low energy effective theories [3]. This is motivated by the fact that both theories describe our universe in an astoundingly precise way. Only in extreme situations they break down, e.g. distances at the Planck length scale and strong gravitational fields, like in the vicinity of a black hole singularity. We take this as a starting point and assume that both theories are sufficient for a leading order investigation. This means, that the matter we include is coupled to gravity. In this context, it would be possible to use tools of quantum field theory on curved spacetime [14, 11]. Then, a choice has to be made which type of matter. The best case scenario would be to use standard model particles. Except the Higgs particle, every particle contained in the standard model possesses spin. This would become very complicated in a treatment that is located on curved spacetime. So, what is left is the standard model Higgs particle, which is a complex scalar field gauged under the $SU(2)$ group [12]. Gauging the matter field of our model would make the discussion unnecessarily intricate. Hence, for this first

[1] Of course, it is possible that a theory of everything is based on completely different assumptions as our current theories.

encounter we will choose the simple case of a real and massive Klein-Gordon field.
Since general relativity describes our universe very well on large scales, it will in a sense be the playground of this enterprise. Assuming that a path integral quantization of gravity is possible, it can be done in different formalisms, e.g. metric formalism or Palatini formalism [15]. The most straightforward way is to choose the metric as the dynamical quantum field. This means that the metric is subject to quantum fluctuations, which are empirically small. This provides a very interesting perception. On a classical level the dynamical variable of spacetime is the metric. Solving its equation of motion, i.e. Einsteins field equations, provides a solution for a given physical scenario. If the equations are applied to the whole universe, we get a cosmological model [7, 16, 5]. The solution is then a metric that describes our universe. Therefore, the value of a full quantum metric seems to be dominated by a metric that is obtained from a classical solution of general relativity superimposed with quantum fluctuations. This combination of a dominant value in combination with quantum fluctuations is not new. It occurs in the Higgs sector of the standard model and is actually the foundation of the Brout-Englert-Higgs effect [12, 2]. Hence, it is only pragmatic to adopt methods and tools that are well established in this sector. The most important one is the Fröhlich-Morchio-Strocchi mechanism (FMS) [12, 10]. This will then provide an ansatz to simplify our considered model.

Structure of the thesis

The first thing to be discussed in section 2 is the theory content used to build the model. This also includes a discussion about the quantization of gravity and the assumptions that need to be made. In order to discuss a scattering model, it is necessary to discuss observables first. The distinction between physical and non-physical observables and the building of physical correlation functions will be discussed in section 3. The next step is to implement the afore-mentioned tools that are copied from the Higgs sector. This is done in section 4. It will provide the means to do actual calculations. After introducing all needed ingredients, the scattering model is constructed in section 5. It contains also a discussion about the representation of a black hole via an operator. The result of this section is a final model that is applied to different scattering scenarios in section 6. At the end, a conclusion is made and an outlook for future work is given in section 7.

2 Theory setup

Although, this thesis mainly focuses on observables, it is necessary to discuss the underlying theory content. The important contribution from the theory setup will be a few central assumptions [10] discussed in this section. Those contribute to the necessary properties that observables must fulfill to be physical.

2.1 Classical theory

As mentioned in the introduction, the combination of general relativity and quantum field theory poses an adequate description of many physical situations. Considering an underlying theory, these two will most likely be contained in a limit [3]. Therefore, we assume that those theories are (low energy) effective theories of something more fundamental. Taking this under consideration, it is sufficient for the following to write the action as

$$S(g_{\mu\nu}, \phi) = \frac{1}{2\kappa} \int d^4x \sqrt{|g(x)|} R(x) + \frac{1}{2} \int d^4x \sqrt{|g(x)|} \left(g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi^2(x) \right) \tag{1}$$

The first term constitutes the Einstein-Hilbert action of general relativity [3], where $\kappa = 8\pi G$ and G is the gravitational constant. The metric $g_{\mu\nu}$ describes the dynamics of spacetime. The signature of the metric is of Lorentzian type and chosen to be $(- + ++)$ and $g = det(g_{\mu\nu})$. The second term represents the action of a massive Klein-Gordon field that is minimally coupled to gravity [17]. Since the field has no orientation in spacetime, there is no need for a covariant derivative. Of course, this could be extended to more intricate matter fields, e.g. a Dirac field that possesses spin. This would make the model way more complex, since the vierbein formalism of gravity [18, 19] has to be introduced. Moreover, it would also be possible to consider internal symmetries and a gauging of the matter field[2] [10]. These two extensions would provide the possibility to incorporate standard model particles. But, it would make this endeavor more extensive than necessary and will not be done here.

2.1.1 Diffeomorphisms

The action (1) is generally covariant. This means that it is invariant under general diffeomorphism transformations, i.e. active local coordinate transformations [5]. This symmetry poses constraints on physical quantities. Most importantly, any object that transforms under a general diffeomorphism can not be physical [10], since this would mean that the object is depending on a coordinate system artificially implemented by an observer. This applies for any tensor with spacetime indices, since the characteristic

[2] E.g. a complex scalar field gauged under SU(2) would resemble the standard model Higgs field[12].

property of the very same is the correct transformation behavior under coordinate transformations. Therefore, physically measureable quantities need to be tensors of order zero, i.e. scalars with respect to spacetime.
Diffeomorphism invariance has another important implication. Consider quantities like the Ricci scalar $R(x)$ or the scalar field $\phi(x)$. They are evaluated at some argument x. Usually, one would consider the argument to be a set of coordinates. In a generally curved spacetime this makes only sense, if the quantities occur as integrands and the integral ranges over the whole spacetime. The value of the integration would then be a valid physical quantity, since it is completely diffeomorphism invariant. An example would be the action (1) itself. If the quantities appear not as integrands, using coordinates as arguments is problematic, since they are not invariant under general diffeomorphisms. This means, that the arguments of such objects should not be coordinates, if the object is to be invariant. Instead, we will consider the arguments as labels for spacetime events [10]. If more than one of them appear in a context, they are considered ennumerable and neighboring relations can be defined. Though, this formalism is not useful for practical calculations, it needs to be introduced for conceptual clarity. A solution how to handle the problem of practical usability is discussed in section 3.3.

2.2 Quantum theory

A naive perturbative quantization of general relativity is problematic, since it leads to nonrenormalizable divergences [3]. Therefore, we need to make several assumptions to proceed [10]. First of all, it is assumed that the quantization of the theory via path integral approach works. The partition function is then written as

$$Z = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi e^{iS(g_{\mu\nu},\phi)} \tag{2}$$

If different or additional matter fields are considered, the path integral would have the same structure. In that sense ϕ could be seen as a placeholder for all matter fields. This approach assumes the metric to be the suitable quantum degree of freedom. Other possibilities like the Palatini formalism or the vierbein formalism exist [15, 18]. Note, that the integration goes over all possible Lorentzian metrics. This means that the integration contains all metrics that are covered by arbitrary diffeomorphisms, including different coordinate systems to the same physical scenario. In this regard it is assumed that all diffeomorphism orbits have the same size.
Since we are not going beyond leading order, these assumptions are sufficient. Going further would require a mechanism like asymptotic safety to take affect in order to make the theory well defined in the path integral approach [20]. Besides that, it is most likely

necessary to extend the gravity action by higher orders in the Riemann tensor, i.e. higher derivatives of the metric [21]. This would probably alter the results of this work only in a quantitative way and therefore it is not important for the conceptual line of argument.

Gauge fixing

The Einstein-Hilbert action of general relativity is based on general covariance, i.e. the invariance under general diffeomorphisms [7]. This local coordinate transformations are analogous to local gauge transformations in gauge theories like Yang-Mills theory. The difference is that the former relates to external symmetries, i.e. they affect spacetime at every event differently. Whereas the latter relates to internal symmetries, i.e. they affect internal vector spaces at every event differently [2]. Similar as in a Yang-Mills theory, the integral over the gauge group of the metric would diverge [12]. This problem is overcome either by the use of a lattice formulation or by gauge fixing. The latter is in principle considered here. We employ gauge fixing not only to overcome the problem of a diverging integral, but also for another reason that is discussed in section 4.1.1.

Expectation value

As usual in a quantum theory, physical quantities are represented by operators and their expectation value. With respect to the path integral (2) the expectation value of an observable $O(g_{\mu\nu}, \phi)$ is given by [2]

$$\langle O(g_{\mu\nu}, \phi) \rangle = \frac{\int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \; O(g_{\mu\nu}, \phi) \; e^{iS(g_{\mu\nu},\phi)}}{\int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \; e^{iS(g_{\mu\nu},\phi)}} \tag{3}$$

The definitions (2) and (3) are given for completeness, since the thesis operates on the level of observables without actually considering any path integral.

3 Observables and correlation functions

After clarifying the theory setup, it is now necessary to clarify what physical and non-physical observables are and especially what they are in the context of gauge theories[3] [12, 10]. This will provide a basic guideline to build the scattering model we are after. The following discussion is about local symmetry transformations. Introducing conserved quantities like a charge or spin would need an additional discussion about global symmetries and how to construct observables that carry such conserved quantum numbers [12, 10].

3.1 Physical observables

A consequence of the integration over all Lorentzian metrics in (2) is that the expectation value of gauge variant operators must vanish without gauge fixing [10]. This is similar as the integration over a gauge group in Yang-Mills theory and can be understood in the following schematic way [12]. The path integral is essentially a sum over all possible metrics, where each metric has the same weight, since the weight factor, i.e. the action, is invariant. This means that no metric is distinct. An integration over a non-invariant quantity can than be seen as the integration of a vector over a sphere. With no direction distinguished, the integration will yield a vanishing value.
Therefore, expectation values of gauge variant operators vanish. Due to this trivial value, the expectation values are then in a sense invariant under gauge transformations. But since the value vanishes, this can't pose a physical meaningful quantity. Hence, gauge variant operators don't represent physical observables. This statement can also be motivated in a less technical way. When working with gauge theories it must be clear, that the actual gauge part of the theory is artificially introduced to localize the theory. Concerning observables that should be measurable, this means that they must be represented by operators that do not transform under diffeomorphsims or gauge transformations. This can't be stressed enough: **for an observable to be physical it must be invariant under diffeomorphism and gauge transformations** [12, 10]. No matter the considered theory, this fact can be used as a guiding principle to define and build observables. Even if the underlying theory is unknown, this concept provides a means of discussion of observables and phenomenology of the theory.
The standard model only contains gauge transformations that act on internal vector spaces [2], i.e. an according invariance is denoted internal symmetry. On the other hand, the diffeomorphisms of generel relativity apply to spacetime itself. A corresponding invariance is denoted external symmetry.

[3] For the rest of the thesis, the term gauge also includes diffeomorphisms.

3.1.1 Internal symmetries

Although this work does not explicitly consider internal symmetries, a short discussion about them will illustrate the concept. In order to emphasize the notion of physical observables, we consider the example of a single real and ungauged scalar field[4]. A valid physical observable would be

$$O_1(x) = \phi(x) \tag{4}$$

Considering this an observable is possible as long as the scalar field is not gauged [12]. When a gauge symmetry is introduced, the field is in some representation of the gauge group and as such not invariant. In order to build a physical operator from such a gauged field, it has to occur in a combination that does not change under gauge transformations. The result of such a combination is a composite operator that can be considered as a bound state of elementary fields, similar as hadrons in QCD [12]. An example for an operator build from a gauged scalar field would be

$$O_2(x) = \left(\sum_a \psi_a \psi_a \right)(x) \tag{5}$$

Where the sum goes over the gauge index, e.g. all colors in QCD. This poses a physical object that is assembled by two elementary fields. The notion of gauge invariant observables has far reaching consequences in the context of the electroweak sector of the standard model, see e.g. [12, 22].
What if an operator should resemble a particle that carries a physical charge? Such charges are conserved quantities that originate in the existance of global symmetries [12]. For an operator to carry such a charge, it must be in the representation of the global symmetry group. For more details regarding this issue consider [12].

3.1.2 External symmetries

Since the fields we consider live on a dynamical spacetime, we need to take diffeomorphisms into account when considering physical observables [10]. For them, the same rules apply as for gauge transformations. If an observable is to be measureable, it has to be invariant under diffeomorphisms. As discussed in section 2.1.1, this implies that arguments need to be considered as event labels and that tensors with open indices are not suitable observables. The latter consequence also applies to the metric field. The implications of this statement regarding the metric are discussed in section 4.

[4]The following arguments hold for both flat and curved spacetimes.

3.2 Correlation functions

Now that the notion of physical observables is clarified, it is possible to discuss physical correlation functions. Using the same motivation as above, physically meaningful correlation functions must be build from gauge and diffeomorphism invariant operators [12, 10]. Additionally, each operator is evaluated at an argument that is given by an event label. The simplest example is the two point function, i.e. the propagator of an invariant operator

$$D(x, y) = \langle O(x)O(y)\rangle \tag{6}$$

If a scattering process of different physical observables $O_j(x_j)$ is to be considered, a n-point function is introduced. The general definition is

$$\Gamma^{j_1\dots j_n}(x_1, \dots, x_i) = \langle O_{j_1}(x_1)\dots O_{j_n}(x_n)\rangle \tag{7}$$

The definitions (6) and (7) are completely generalized and valid in any spacetime.
Usually, the next step would be to perform a Fourier transformation to operate in momentum space. This is not possible in curved spacetimes, since a global momentum space representation does not exist [23]. Therefore, we have to stay in position space. But, operating with correlation functions as functions of events is not useful. Hence, it will be necessary to consider another diffeomorphism invariant approach that sets the events in relation and quantifies this link [10].

3.3 Geodesic distance

A notion of distance that is invariant under diffeomorphisms is the geodesic distance. It is the distance along a unique geodesic line connecting two spacetime events[5]. Following [24], the definition of this invariant distance is

$$\sigma(x, y) = \frac{1}{2}(\lambda_1 - \lambda_0)\left\langle \min_{z(\lambda)}\left(\int_{\lambda_0}^{\lambda_1} d\lambda g_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right)\right\rangle \tag{8}$$

Where x and y are the endpoints, λ is an affine parameter, $g_{\mu\nu}$ the metric and $z^\mu(\lambda)$ are the coordinate functions that connect the endpoints. These functions are constrained by the boundary conditions $z^\mu(\lambda_0) = x^\mu$ and $z^\mu(\lambda_1) = y^\mu$, where x^μ and y^μ are coordinate representations of the events. The geodesic distance is symmetric in its arguments. The prefactor on the right hand side is a convention and since λ is defined to be an affine parameter, the parameter endpoints can be chosen to be $\lambda_0 = 0$ and $\lambda_1 = 1$. The minimum

[5]Note that uniqueness is not necessarily fulfilled, depending on the spacetime and the chosen points to connect.

occuring in the definition chooses the connecting line between x and y to be of geodesic nature. This means that $z^\mu(\lambda)$ is the solution to the geodesic equations provided by the full quantum metric. The integral is along this line and the derivative of z^μ is the tangent vector of the geodesic.
The most decisive feature of this definition is that it is an expectation value. This stems from the fact that we consider the metric to be a quantum field. Hence, distances themselves are subject to quantum fluctuations and are therefore observables [10]. At first, it seems that we are going round in circles and that also the notion of the geodesic distance is not useful. But with the tools we will introduce in section 4 it will become tractable.

Propagator
Going back to correlation functions and the example of the propagator, the geodesic distance enables us to rewrite (6) as

$$D(x,y) = D(\sigma_{xy}) = \langle O(x)O(y)\rangle \tag{9}$$

Where the shorthand notation $\sigma(x,y) = \sigma_{xy}$ was introduced. This transition seems rather ad hoc, since we didn't specify the structure of spacetime or the considered operator. Is this transition always possible, even if we don't know the underlying spacetime? In fact, this transition is only feasible, if spacetime is on average maximally symmetric[6] [25, 26, 14]. Otherwise, the propagator on the curved spacetime can depend partially on e.g. the spatial distance and the time distance independently [27]. The consequences of this statement are discussed in section 4.2.2.

3.3.1 Interpretation

Before continuing to a more detailed discussion about the scalar propagator, it is helpful to briefly discuss the interpretation of the geodesic distance. The definition (8) makes it a rather intransparent and generic quantity. In this sense, only a rather generic interpretation is possible [24]. Similar as in Minkowski spacetime, two points can be connected in three different ways: time-, space- and lightlike. The latter case is not important for the scalar propagator, since the field is considered massive. If the geodesic distance is timelike, the connection is considered to be causal and linked to the proper time interval of the two events by

$$\Delta\tau = \sqrt{-2\sigma_{xy}} \tag{10}$$

[6]For a short discussion about maximally symmetric spacetimes see appendix B.

If the geodesic distance is spacelike, the connection is considered to be non-causal and linked to the proper distance of two events by

$$\Delta s = \sqrt{2\sigma_{xy}} \tag{11}$$

This means that the geodesic distance is either the proper time interval or the proper distance for timelike or spacelike behavior, respectively. Also, this introduces a notion of causality. The geodesic distance is symmetric in its arguments, which means that any quantity that solely depends on the geodesic distance automatically fulfills time reversal symmetry[7]. This does not influence causality, but it says that one can't say which event occured first. The only statement possible is to say that they are either causally connected or not. Of course there is still an ambiguity left, since the square root actually requires to consider both signs in (10) and (11). But without considering the unphysical coordinates, there is no way to decide which one is the right choice.

3.4 Scalar propagator in flat position space

Since we operate in position space, it is required to consider the scalar propagator in the very same. The usual expression for the Feynman propagator in Minkowski spacetime is [13]

$$D_F(x,y) = \int \frac{d^4p}{(2\pi)^2} \frac{1}{p^2 + m^2 + i\varepsilon} e^{-ip\cdot(x-y)} \tag{12}$$

Because we use the opposite signature of the metric as usually used in quantum field theory, the sign of the mass term is changed. This integral can be carried out and we arrive at a propagator that is defined in position space [13]

$$D_F(x,y) = -\frac{\delta(\sigma_{xy})}{4\pi} + \mathrm{sgn}(\sigma_{xy}) \frac{im^2}{4\pi^2} \frac{K_1(m\sqrt{2\sigma_{xy} + i\varepsilon})}{m\sqrt{2\sigma_{xy} + i\varepsilon}} \tag{13}$$

The function $K_1(w)$ is the modified Bessel function of first order. This expression shows that the propagator is a function of the geodesic distance, therefore $D_F(x,y) = D_F(\sigma_{xy})$. As mentioned, the lightlike case will not be important and the Dirac delta will therefore be neglected.

Figure 1 shows a plot of the absolute value of the propagator (13) as a function of the geodesic distance for different masses. Note that the plot is divided into two parts. The negative values of the geodesic distance indicate a timelike connection and the positive values indicate a spacelike one.

[7] The same holds for space, i.e. it fulfills symmetry under parity transformation.

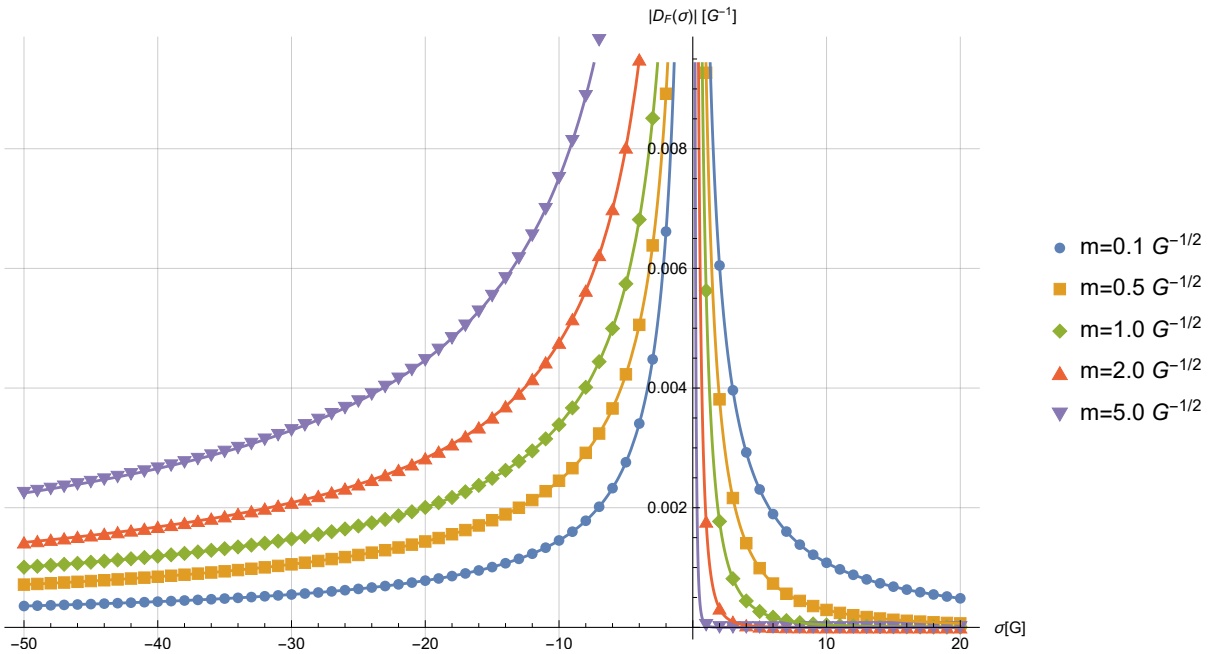


Figure 1: Absolute value of the scalar propagator in position space as a function of the geodesic distance for different masses. For the numerical implementation the mass was set to $m = 1G^{-1/2}$ and the pole parameter was set to $\varepsilon = 10^{-6}$.

3.4.1 Mass definition in position space

In standard quantum field theory the mass of a particle is defined through the pole structure of the propagator in momentum space [2]. Since we have no momentum space available, we need to find another way of determining a mass. Similar to the former case we examine the behavior of the propagator. At large spacelike distances, the propagator can be approximated by [13]

$$D_F(\sigma) = \frac{im}{4\pi^2}\frac{K_1(m\sqrt{2\sigma+i\varepsilon})}{\sqrt{2\sigma+i\varepsilon}} \overset{m\sqrt{2\sigma}>>1}{\approx} \frac{i}{8\pi^2\sigma}\frac{\sqrt{\pi m\sqrt{2\sigma+i\varepsilon}}}{\sqrt{2}}e^{-m\sqrt{2\sigma+i\varepsilon}} \tag{14}$$

The propagator is in the spacelike region solely imaginary, except for the ε prescription. From this approximation the mass can be extracted. A bit of reshuffling yields

$$\frac{1}{2}\ln m - m\sqrt{2\sigma+i\epsilon} = \ln\left[\frac{2^{7/2}\pi^{3/2}\sigma}{(2\sigma+i\varepsilon)^{1/4}}\,\mathrm{Im}(D_F(\sigma))\right] \tag{15}$$

Now, using that we assume $m\sqrt{2\sigma} >> 1$, the first term on the left hand side can be neglected. In the end the relation boils down to the approximation

$$m \approx \frac{-1}{\sqrt{2\sigma+i\varepsilon}}\ln\left[\frac{2^{7/2}\pi^{3/2}\sigma}{(2\sigma+i\varepsilon)^{1/4}}\,\mathrm{Im}(D_F(\sigma))\right] = m_{approx}(\sigma) \tag{16}$$

Which gives a mass approximation for a scalar particle that depends on the geodesic distance. Unfortunately, the function only asymptotically approaches the exact value in the limit $\sigma \to \infty$. Since this approximation is made in flat spacetime, the transition to a curved spacetime is intricate, as the form of the propagator will change.
To assess the approximation we put the exact scalar propagator (13) in the approximation (16). Different values for the exact mass are used and for the numerical calculation $\varepsilon = 10^{-6}$. Because of the non vanishing ε, the real part of the mass approximation has to be taken. Figure 2 shows the mass approximation as a function of the geodesic distance and figure 3 shows the relative error η in percent. Both for different exact masses that are given in the legends.
The error plot shows that the approximation works best if the mass is close to one, since then the negligence of the log term in (15) is exact. The v-shaped form in the error comes from an overshooting of the approximation for masses that are smaller than one.

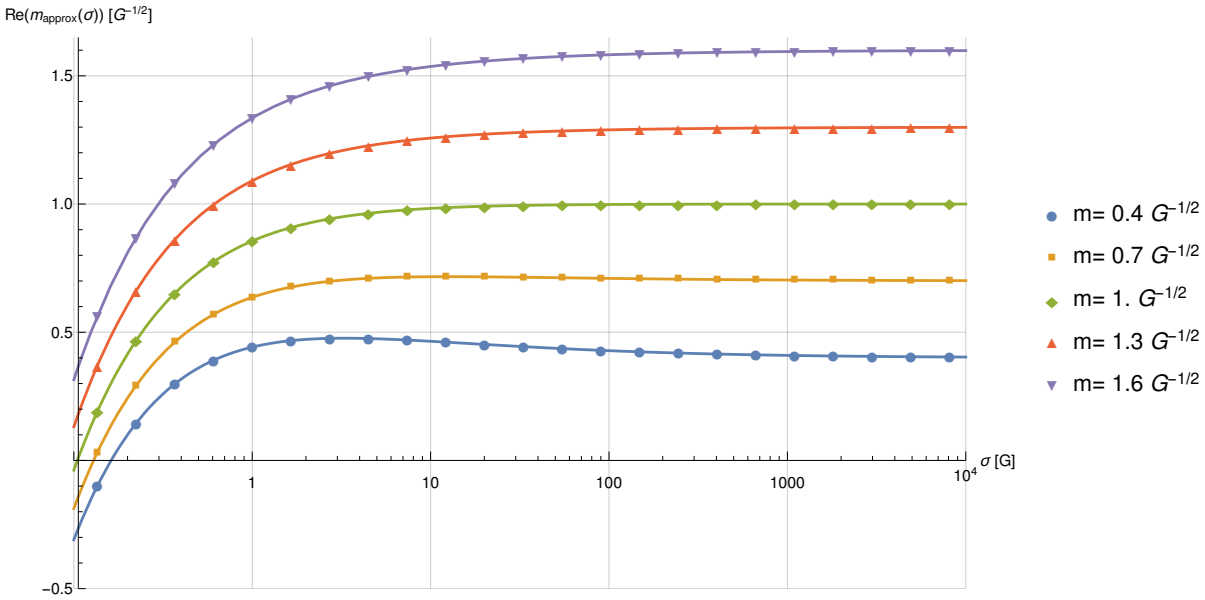


Figure 2: Mass approximation as a function of the geodesic distance. The exact mass is varied and the pole offset is $\varepsilon = 10^{-6}$.

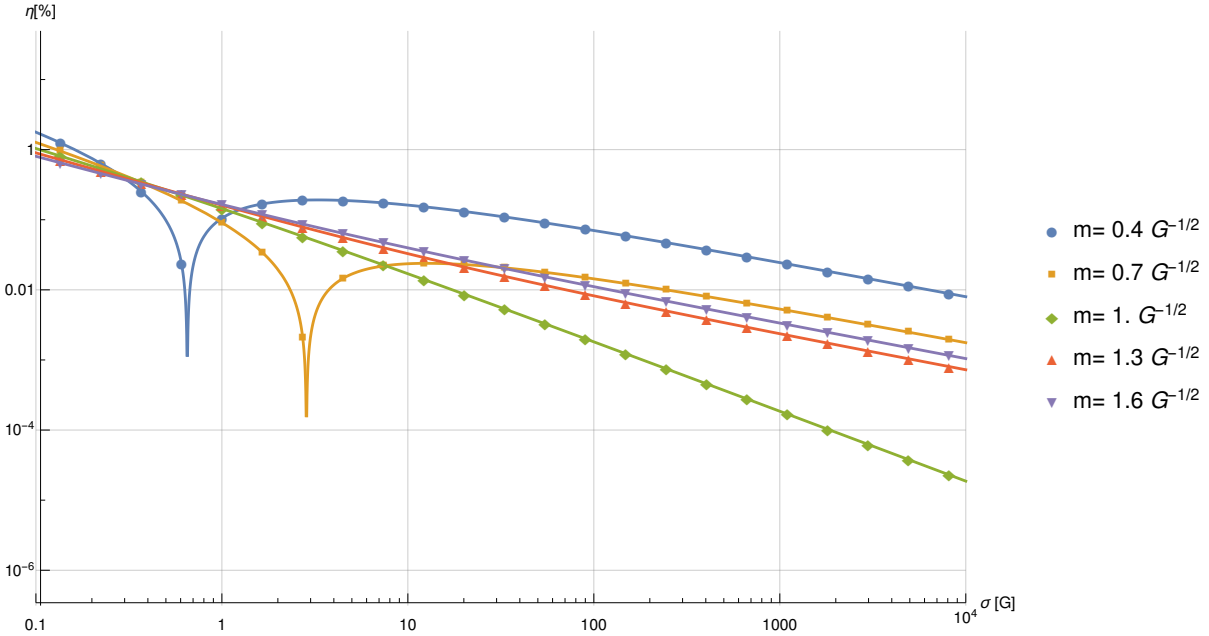


Figure 3: Relative error of the mass approximation in percent and as a function of the geodesic distance. The exact mass is varied and the pole offset is $\varepsilon = 10^{-6}$.

4 Metric split

The previous discussion provides statements about what valid physical observables of a quantum theory are and how they emerge from the field content of the theory. The next step would be to calculate expectation values of those observables. Such calculations are even in standard quantum field theory non-trivial endeavors [12]. Since we work with a dynamical spacetime, it gets even more intricate [11, 14]. Therefore, it would be advantageous to get a handle on the interplay of quantum field theory and the dynamical spacetime. This is provided by the metric split and the Fröhlich-Morchio-Strocchi mechanism (FMS) [12, 10].

4.1 Splitting fields

Splitting a field only makes sense in certain circumstances. A prominent example occurs in the Higgs sector of the standard model [2, 12]. There, the Higgs field is basically split in its (constant) vacuum expectation value (vev) and quantum fluctuations. The usefulness of this split comes from two distinct properties of the field. First of all, the field has a dominant contribution. In the Higgs case this is provided by the wine-bottle shaped potential and the non-vanishing minimum. The second premiss is that the quantum fluctuations are small compared to the vev. If these requirements are matched, a split in vev and flucutations can be undertaken. For this split to be a valid operation, a suitable gauge fixing procedure has to be employed [12]. In the standard model this leads to the Brout-Englert-Higgs mechanism and the dynamical production of mass [2, 12]. Also, usually not covered in standard textbooks, it has extensive consequences regarding the phenomenology of the standard model, see e.g. [12, 22].
If we take a closer look at quantum gravity in the form of the metric formalism, we observe that the two requirements are also matched [10]. The dominant part of the metric is provided by the description of our universe with a classical metric, e.g. Robertson-Walker spacetime in the ΛCDM model. And the quantum fluctuations of gravity are in the majority of cases small. Only in the most extreme situations they become large. These are for example the vicinity of the singularity of a black hole or at short times after the big bang. Hence, we can employ the metric split as a technical tool to make statements about observables that contain the metric [10].

4.1.1 Gauge fixing

It is important to note that in order to perform the split and for it to have any significant impact, it is necessary to fix a gauge[8] [12, 10]. This is done in a way that the expectation value of the field equals the vev, i.e. the expectation value of the fluctuation field vanishes. Since in the case of gravity the path integral ranges over all metrics and diffeomorphism transformations thereof and all diffeomorphism orbits are assumed to have the same size, the expectation value of the metric vanishes without gauge fixing (c.f. section 3.1.2). Hence

$$\langle g_{\mu\nu}(x)\rangle = 0 \tag{17}$$

In order to split the metric in a way similar to the Higgs case, the vev of the metric should not vanish, i.e. a metric configuration must be distinguished. This can be done by implementing a suitable gauge condition with the help of a gauge fixing procedure like the Faddeev-Popov procedure [28]. The details of the procedure are not mandatory for the following and are therefore not covered here. After gauge fixing the expectation value of the metric is

$$\langle g_{\mu\nu}(x)\rangle = g^c_{\mu\nu}(x) \tag{18}$$

Where $g^c_{\mu\nu}$ satisfies the gauge condition. Note that this means that spacetime is on average described by this classical metric. After the gauge fixing procedure the metric is split into

$$g_{\mu\nu} = g^c_{\mu\nu} + \gamma_{\mu\nu} \tag{19}$$

Where $g^c_{\mu\nu}$ is the mentioned classical part and $\gamma_{\mu\nu}$ is the quantum part. Because of (18), the classical part needs to be a metric, whereas the quantum part is in general not. Also, the classical part does not need to be a solution to Einstein's field equations. But such a choice is recommended, since otherwise the split has no real practical use.

4.2 Consequences

Whenever the metric is contained in an operator, it is affected by the split. This implies three major impacts on observables, the geodesic distance and correlation functions.

4.2.1 Observables

The metric split provides a means to expand expectation values of observables that contain the metric. As an example consider an operator $O(g_{\mu\nu})$ that comprises the metric, e.g. the curvature scalar $R(x)$. The split allows us to expand the expectation value of the

[8] Gauge fixing breaks the gauge symmetry explicitly. Also note that fixing a gauge does not influence gauge independent quantities.

operator as [10]

$$\langle O(g_{\mu\nu})\rangle = \langle O(g^c_{\mu\nu})\rangle + \sum_{i,\text{Perm. of } g^c \text{ and } \gamma} \langle O(g^c_{\mu\nu}, n-i; \gamma_{\mu\nu}, i)\rangle \tag{20}$$

The summand has arguments $n-i$ and i that show the number of times the previous argument occurs in the expression. This expansion is essentially the idea of the FMS mechanism that is applied in Brout-Englert-Higgs physics [12, 10].
If the term that depends only on the classical part of the metric then dominates, it provides an approximation for the total right hand side. Providing statements about terms with $\gamma_{\mu\nu}$ requires additional information about the very same. Note that the left hand side of the equation is gauge invariant, which means that only the entire sum on the right hand side is gauge invariant [12, 10]. Therefore, using the dominating term as an approximation means using a gauge variant term as an approximation for a gauge invariant quantity.
This series expansion may produce the impression that the application of the metric split breaks gauge invariance. But the actual breaking happens when the gauge is fixed.

4.2.2 Geodesic distance

Since the geodesic distance is defined as an expectation value in (8), implementing the split will eventually provide a possibility to calculate the classical part of the distance. Employing the split in the definition (8) yields

$$\sigma_{xy} = \frac{1}{2}\min_{z(\lambda)}\left(\int_0^1 d\lambda g^c_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right) + \frac{1}{2}\left\langle \min_{z(\lambda)}\left(\int_0^1 d\lambda \gamma_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right)\right\rangle \tag{21}$$

Since the occuring operations in the metric are linear, the geodesic distance splits in a classical part and quantum fluctuations. In short

$$\sigma_{xy} = \sigma^c_{xy} + \delta\sigma_{xy} \tag{22}$$

The classical part can then actually be calculated. The minimization chooses the geodesic connection, which means that the length of the tangent vector stays constant along the line [6]. Since the integrand is nothing more than the very same, the expression is simplified to

$$\sigma_{xy} = \frac{1}{2}g^c_{\mu\nu}\frac{d\bar{z}^\mu}{d\lambda}\frac{d\bar{z}^\nu}{d\lambda} + \frac{1}{2}\left\langle \min_{z(\lambda)}\left(\int_0^1 d\lambda \gamma_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right)\right\rangle \tag{23}$$

The barred z indicates that it is the solution to the geodesic equations[9]. The quantum fluctuations can not be simplified that way, because the quantum part of the metric is

[9] The geodesic equations are provided in appendix A.

not a genuine metric and thus the integrand not a vector norm.

Minkowski spacetime

For the case of Minkowski spacetime the geodesic distance is given by [7]

$$\sigma^c_{xy} = \frac{1}{2}\left[-(t_x - t_y)^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right] G \tag{24}$$

In this case the symmetry of the arguments is obvious. As mentioned in 3.3.1 there are three cases to distinguish. In this signature the spacelike and non-causal case is for positive values and the timelike and causal case is applicable for negative values.

As argument of propagators

This split of the geodesic distance affects also propagators. As mentioned in section 3.3 a propagator is a function of only the geodesic distance, if the spacetime is on average maximally symmetric. This means that the vev of the metric, i.e. the classical part of the split, must be maximally symmetric. There are only three spacetimes that fulfill the requirement of maximal symmetry [5]: Minkowski, de Sitter and anti de Sitter. If the classical part matches the requirement, the split can be employed to conduct an expansion of the propagator

$$D(\sigma_{xy}) = D(\sigma^c_{xy} + \delta\sigma) = D(\sigma^c_{xy}) + \sum_n \frac{1}{n!} \left.\frac{\partial^n D(\sigma_{xy})}{\partial {\sigma_{xy}}^n}\right|_{\sigma^c_{xy}} (\sigma_{xy} - \sigma^c_{xy})^n \tag{25}$$

which is essentially an expansion in the quantum fluctuations $\delta\sigma$. The leading term is then the propagator evaluated with the classical geodesic distance. This expansion will become handy later on. Note that again only the entire sum is necessarily gauge invariant. The operator considered here does not contain the metric. Of course it is possible that it does. In that case also an expansion in the metric occuring in the operator has to be performed.

4.2.3 Correlation functions

If we now consider n-point correlation functions that are defined as[10]

$$\Gamma(x_1, \ldots, x_n) = \langle O(x_1) \ldots O(x_n) \rangle \tag{26}$$

[10]Same idea applies to correlation functions that contain different observables.

we can consider it as a function of geodesic distances[11], i.e.

$$\Gamma(x_1, \dots, x_n) = \Gamma(\sigma_{x_1x_2}, \sigma_{x_1x_3}, \dots) \tag{27}$$

Where on the right hand side the geodesic distances with all possible combinations of the points occur. By implementing the split in the geodesic distance, this correlation function can be expanded in the same manner as (25). This leads to

$$\Gamma(\sigma_{x_1x_2}, \sigma_{x_1x_3}, \dots) = \Gamma(\sigma^c_{x_1x_2}, \sigma^c_{x_1x_3}, \dots) + \mathcal{O}(\delta\sigma) \tag{28}$$

Again the leading order is given by the correlation function evaluated at classical distances. This simplifies calculations significantly, since the quantum fluctuations of the metric and thus the corrections to the geodesic distance must not be known. Keep in mind that similar as for the previous expansions only the whole sum is gauge invariant.

4.2.4 Emergence of flat space quantum field theory

An interesting consequence is, that with the help of the metric split standard flat spacetime quantum field theory can be recovered [10]. If the classical part of the metric is Minkowski spacetime, the classical geodesic distance is the well known distance (24) that occurs in the flat spacetime theory. As an example, consider a scalar quantum field theory and its propagator

$$D(x, y) = D(\sigma_{xy}) = \langle \phi(x)\phi(y) \rangle \tag{29}$$

The explicit expression for the propagator is given in (13). If then the expansion (25) is applied, it simplifies to

$$D(\sigma_{xy}) = D(\sigma^c_{xy}) + \mathcal{O}(\delta\sigma) = \langle \phi(x)\phi(y) \rangle (\sigma^c_{xy}) + \mathcal{O}(\delta\sigma) \tag{30}$$

Where the argument of the expectation value is implicating that the propagator depends on the geodesic distance. In that sense, the standard flat spacetime quantum field theory is recovered, if the quantum fluctuations of the geodesic distance are small and the classical distance is given by the Minkowskian one.

This has an important implication. Since the right hand side is only gauge invariant as a whole, the leading term is actually not physical. But, this term is the one we use in flat spacetime quantum field theory. Why does then flat space quantum field theory work so well? This stems from the dominance of the leading term. Since the quantum fluctuations are small, the higher order terms in the sum are negligible. Meaning that the

[11] For a n-point function there are $n(n-1)/2$ geodesic distances to replace the events as arguments.

gauge invariant propagator is very well approximated by the non-invariant leading term.

4.3 Choice of $g^c_{\mu\nu}$

In order to continue and start building the scattering model we are looking for, we need to choose a classical part of the metric. As mentioned in section 3.3 and 4.2.2, for the propagator to be only a function of the geodesic distance, the vev of the metric must be one of the three maximally symmetric spacetimes. For a first endeavor it will suffice to use Minkowski spacetime as our classical part, i.e. $g^c_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, 1, 1)$. This has also the advantage that we can use the flat spacetime propagator (13) and that the geodesic distance is calculated straightforwardly by (24). Additionally, we can adopt the causal structure of Minkowski spacetime to our system. Although we now specified the classical part of the metric, we still continue rather general, until the point where actual calculations are done.

5 The physical setting

As written in the introduction, we want to consider the interaction of a black hole with a scalar particle. With the development of new methods to observe (the shadow of) black holes like the event horizon telescope [8, 29], they become an active area of investigations for gravitational and quantum gravitational phenomena. It would therefore be worthwhile to make statements about the possible corrections of quantum effects to the imaging of the shadow of black holes. With this in mind, we build a model of the scattering of a black hole with a scalar field in the form of a scattering function.
With the tools introduced in section 4 we can simplify the considered scattering function substantially. This simplification also means a truncation and approximation of the model. Nonetheless, as a first approach it provides some insights and also ground for further research.

5.1 Constructing the scattering function

To keep the effort manageable, a three point scattering function will suffice. Therefore, we need a physical operator for the scalar and one for the black hole. For the scalar field it is plainly the field operator $\phi(x)$. The black hole operator $B(x)$ needs more discussion, which is done in section 5.2. For now, the scattering function is

$$\Gamma(x, y, z) = \langle B(x)\phi(y)\phi(z)\rangle \tag{31}$$

Note, that the arguments of the scattering function are labels for spacetime events. Hence, the first thing we do is to transition from the event description to geodesic distances. The three events can be exchanged for three distances

$$\Gamma(\sigma_{xy}, \sigma_{xz}, \sigma_{yz}) = \langle B(x)\phi(y)\phi(z)\rangle\,(\sigma_{xy}, \sigma_{xz}, \sigma_{yz}) \tag{32}$$

The round bracket on the right hand side indicates that the expectation value depends on the three distances. Of course, in order to calculate the distances, coordinate representations for the events must be used.
The next thing that can be done is employing the split in the geodesic distance.

$$\Gamma(\sigma_{xy}, \sigma_{xz}, \sigma_{yz}) = \Gamma(\sigma^c_{xy} + \delta\sigma_{xy}, \sigma^c_{xz} + \delta\sigma_{xz}, \sigma^c_{yz} + \delta\sigma_{yz}) \tag{33}$$

Performing a series expansion in all three arguments provides the expression

$$\Gamma(\sigma_{xy}, \sigma_{xz}, \sigma_{yz}) = \Gamma(\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) + \mathcal{O}(\delta\sigma_{xy}, \delta\sigma_{xz}, \delta\sigma_{yz}) \tag{34}$$

This means that the scattering function is dominated by the very same, evaluated with classical geodesic distances, since we assume that the quantum fluctuations of the geodesic distance are small. Therefore, the scattering can be approximated by the leading order correlation function

$$\Gamma(\sigma_{xy}, \sigma_{xz}, \sigma_{yz}) \approx \Gamma(\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) = \langle B(x)\phi(y)\phi(z)\rangle \, (\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) \tag{35}$$

It is important to note that the coordinates that must be introduced to calculate the geodesic distances live on the classical spacetime described by the classical part of the metric $g^c_{\mu\nu}$.

5.2 Black hole operator

Describing a black hole with a quantum operator is a non-trivial task and most likely not possible until a full quantum theory of gravity is known. Therefore, we have to make assumptions and approximations. General relativity makes the existence of black holes necessary and provides also a way of describing their classical properties [7, 5]. The most characteristic feature of a black hole is that of a curvature singularity. This should manifest itself in the black hole operator. Hence, a straightforward way is to use a curvature scalar as the operator. But the curvature scalar is again subject to quantum fluctuations. Since the black hole is a macroscopic object and dominated by a classical part, we want to employ the same strategy as above and split off the classical from the quantum part. Which means, that we perform the split of the operator in a classical part $\rho(x)$ and quantum corrections $\delta\rho(x)$

$$B(x) = \rho(x) + \delta\rho(x) \tag{36}$$

Note that we required earlier that the full operator is manifestly invariant under gauge and diffeomorphism transformations. We want to adopt this property already for the classical part. Hence, $\rho(x)$ is identified with a scalar valued curvature invariant that is described by general relativity. Before we discuss such curvature invariants, we briefly discuss the spacetime that contains the black hole.

Schwarzschild spacetime

To keep it simple we use the most basic case of a spacetime that contains a singulartiy: the Schwarzschild solution of general relativity. The line element for a mass in this spherically symmetric spacetime is given by [7]

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{37}$$

Where $d\Omega^2$ is the usual shorthand for the angular part and $r_s = 2GM$ is the Schwarzschild radius that marks the event horizon. Striking about the line element is that there are two singularities. One at $r = r_s$ and one at $r = 0$. But to figure out if they are genuine singularities of spacetime or just bad coordinate choices, a diffeomorphism invariant way of characterizing them is needed [7].

Curvature scalars

Such invariant quantities are called curvature invariants or scalars [16]. Since curvature is described by the Riemann tensor $R_{\mu\nu\lambda\rho}$, invariants are build from contractions of the very same and the metric $g_{\mu\nu}$. In N spacetime dimensions there are $1/12N(N-1)(N-2)(N+3)$ different curvature invariants [16]. Hence, in $N = 4$ dimensions there are 14. As the Schwarzschild spacetime is a vacuum solution, 10 are given by the Ricci tensor $R_{\mu\nu} = 0$, since a vanishing tensor component is invariant. The four remaining are [16]

$$K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \tag{38}$$

$$I_1 = g^{-1/2}\varepsilon^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma\nu\kappa}R_{\lambda\mu\nu\kappa} \tag{39}$$

$$I_2 = R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}R_{\rho\sigma}{}^{\lambda\mu} \tag{40}$$

$$I_3 = g^{-1/2}R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}\varepsilon_{\rho\sigma}{}^{\tau\xi}R_{\tau\xi}{}^{\lambda\mu} \tag{41}$$

Where $\varepsilon_{\lambda\mu\rho\sigma}$ is the Levi-Civita tensor[12]. The scalar invariant K is called the Kretschmann scalar. For the Schwarzschild spacetime the quantities are

$$K^s = \frac{12r_s^2}{r^6} \tag{42}$$

$$I_1^s = I_3^s = 0 \tag{43}$$

$$I_2^s = -\frac{12r_s^3}{r^9} \tag{44}$$

They show singular behavior at $r = 0$, which means that this must be a genuine singularity. The other singularity at $r = r_s$ does not occur in the invariant expressions and is therefore a coordinate singularity. This type stems from the choice of the coordinate system and can be eliminated by a suitable transformation [7]. Although it is not a real singularity, it represents a bound that splits the spacetime in two parts, i.e. the interior and exterior of the black hole [7, 5]. This comes from the fact that the Schwarzschild coordinates do not cover the whole spacetime. For a detailed discussion about this division consider [7].

Since we now have an invariant characterization of the classical black hole, we can define the classical part of the black hole operator. We observe that the scalar invariants

[12]The definitions of the occurring tensors are given in appendix A.

K^s and I_2^s have dimensions of Gaussian curvature squared and cubed, respectively. But, we want the black hole description to have dimensions of a Gaussian curvature. Therefore, the classical part of the black hole operator is constructed from a combination of both invariants with according powers

$$\rho(r)|_x = a(K^s)^{1/2} + b(I_2^s)^{1/3} = \left(\sqrt{12}a - \sqrt[3]{12}b\right)\frac{r_s}{r^3} \tag{45}$$

Since we don't have any indication for what to set the variables, we make an a priori choice and set $a = 1$ and $b = 0$. Therefore, the classical part of the black hole operator is

$$\rho(r)|_x = \sqrt{12}\frac{r_s}{r^3} \tag{46}$$

The x indicates the still existing connection to the spacetime event that is represented by coordinates of the classical spacetime. Note that the black hole operator does now not represent the black hole itself, but rather the evaluation of the thereby generated curvature of spacetime. Now the question arises how we can connect this quantity to our classical spacetime, although it is defined in Schwarzschild spacetime.

5.2.1 Embedding of the black hole

The connection of (46) with the classical spacetime will be established using physical quantities. One, that can be determined in both spacetimes, is the geodesic distance. The idea is, to calculate it in each spacetime and set them equal. If the geodesic distance in Schwarzschild spacetime is determined as a function of the radial coordinate, the corresponding radial coordinate to a geodesic distance in the classical spacetime is found. Figure 4 shows a sketch of the two systems. The left hand side of the depiction lives in the classical spacetime, the right hand side in the Schwarzschild spacetime.

Figure 4: Sketch of the two spacetimes that contain the considered geodesic distances.

Event Ω in the classical spacetime represents the position of the black hole. Note that the resolution of the black hole is restricted to the outside of the event horizon, due to the use of Schwarzschild coordinates and their restricted covering of the spacetime[13]. In

[13]Further investigations are required, to tell if the choice of a different coordinate system, e.g. Kruskal-Szekeres coordinates, would improve the resolution.

mathematical terms the idea is

$$\sigma^c_{\Omega x} = \sigma^s(r) \Rightarrow r(\sigma^c_{\Omega x}) = (\sigma^s)^{-1}(\sigma^c_{\Omega x}) \tag{47}$$

where $(\sigma^s)^{-1}$ denotes the inverse function of the Schwarzschild distance. Since the distances individually depend on two points with corresponding coordinates in each spacetime, we need to figure out what points (coordinates) to use for both spacetimes. The geodesic distance $\sigma^c_{\Omega x}$ is between the location of the evaluation of the black hole operator x and the location of the black hole Ω.

The Schwarzschild case is more intricate. First of all, without loss of generality we choose the angular coordinates to be constant for the two points, i.e. $d\Omega = 0$. Secondly, we make the choice to set $dt = 0$, i.e. we stay on a spacelike hypersurface. This implies that we do not consider dynamics of the black hole. This actually restricts the value range of both geodesic distances to spacelike connections, i.e. with our choice of signature to positive values. Considering both choices, the line element can be written as

$$ds = \frac{1}{\sqrt{1 - \frac{r_s}{r}}} dr \tag{48}$$

This is convenient, because this quantity is plainly connected to the geodesic distance and can be integrated directly

$$\sqrt{2\sigma^s(r)} = s(r) = \int_{r_s}^{r} \frac{1}{\sqrt{1 - \frac{r_s}{r'}}} dr' = r\sqrt{1 - \frac{r_s}{r}} + \frac{1}{2} r_s \log\left(-1 + 2\frac{r}{r_s}(1 + \sqrt{1 - \frac{r_s}{r}})\right) \tag{49}$$

Note that the integration starts at r_s. Though the radial coordinate can not be extracted analytically, numerically it is possible. In order to do so, a value for the Schwarzschild radius has to be chosen, i.e. the mass of the black hole. It will be set to $r_s = 2G^{1/2}$, which implies that the black hole has a mass of one Planck mass $m_P = G^{-1/2}$.

Figure 5 shows the values for the Schwarzschild radial coordinate as a function of the geodesic distance. The plot is generated by calculating geodesic distance values for different radial coordinates. This produces a list of radial values with corresponding geodesic distances depicted by the blue dots in figure 5. By interpolating this list, we arrive at a function $r(\sigma)$ which is pictured as the continuous line.

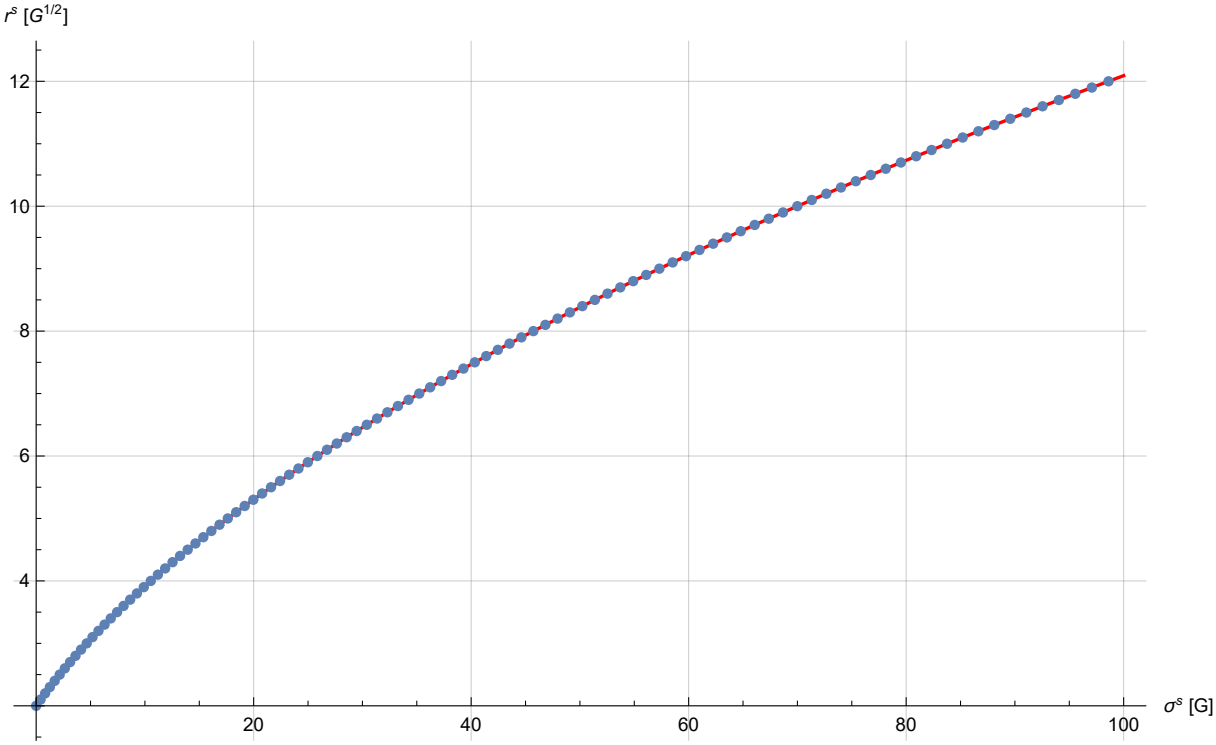


Figure 5: The Schwarzschild radial coordinate plotted against the geodesic distance with the Schwarzschild radius set to $r_s = 2G^{1/2}$. This marks the boundary of the data on the y-axis. The dots indicate the data points and the continuous line is the interpolation.

Now that the radial coordinate is determined as a function of the geodesic distance, it is possible to determine the curvature scalar. In mathematical terms

$$\sigma^s(r) = \sigma^c_{\Omega x} \quad \Rightarrow \quad r(\sigma^c_{\Omega x}) \quad \Rightarrow \rho(r)|_x = \sqrt{12}\frac{r_s}{(r(\sigma^c_{\Omega x}))^3} \tag{50}$$

The distance $\sigma^c_{\Omega x}$ is an input from the classical spacetime and depends on the position of the black hole and the point, where the curvature scalar should be evaluated.
Figure 6 shows the plot of the curvature evaluation as a function of the geodesic distance. The restriction of the resolution clearly passes on to the curvature. Thus, it does not show any singular behavior.

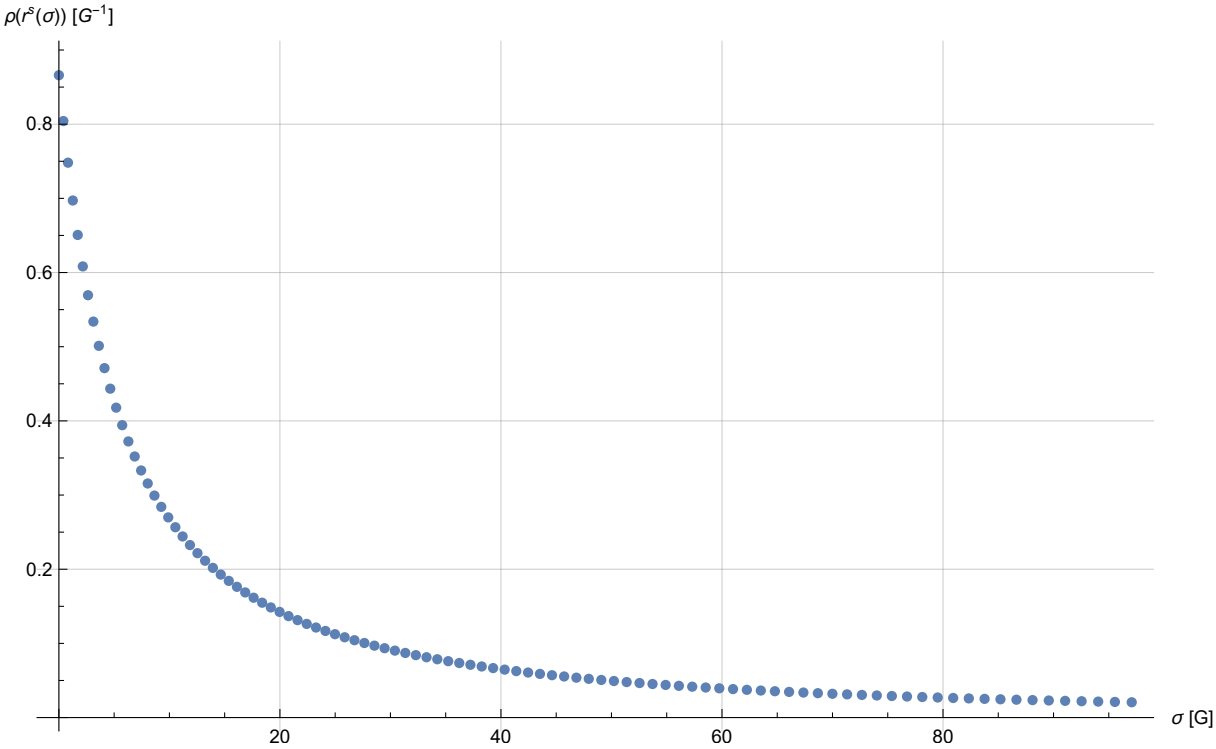


Figure 6: The evaluation of the curvature plotted against the geodesic distance. The Schwarzschild radius is set to $r_s = 2G^{1/2}$.

5.3 Final model

Since the black hole is now approximated by the evaluation of its curvature outside the event horizon, which is not an operator, the scattering function reduces further to

$$\Gamma(\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) = \langle \rho(x)\phi(y)\phi(z)\rangle + \langle \delta\rho(x)\phi(y)\phi(z)\rangle \approx \rho(r)|_x \langle \phi(y)\phi(z)\rangle \tag{51}$$

Where we assumed that the terms with the quantum fluctuations of the scalar curvature are negligible compared to the leading one. This means, that the expectation value only affects the scalar field and it can be written as

$$\Gamma(\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) = \rho(r)|_x D_F(\sigma^c_{yz}) \tag{52}$$

Where $D_F(\sigma)$ is the Feynman propagator of the scalar field in the classical spacetime. But since we use Minkowski spacetime as our classical part, the propagator is given by the equation (13). Hence,

$$\Gamma(\sigma^c_{xy}, \sigma^c_{xz}, \sigma^c_{yz}) = \left(\sqrt{12}\frac{r_s}{r^3}\right)\Big|_x \operatorname{sgn}(\sigma^c_{yz})\frac{im^2}{4\pi^2}\frac{K_1(m\sqrt{2\sigma^c_{yz} + i\varepsilon})}{m\sqrt{2\sigma^c_{yz} + i\varepsilon}} \tag{53}$$

Note that (53) is not gauge invariant, since it is only the first term of a sum that only as an entirety possesses gauge invariance.
In the end the scattering model got reduced to a product of the curvature evaluation of the black hole with the scalar propagator. In a numerical sense, this is rather straightforward to implement. Figure 7 shows a sketch of the final model. It contains four spacetime events represented by coordinates in the classical spacetime: the black hole position Ω, the curvature evaluation point x and the two scalar field points y and z. All events are connected via geodesic distances. Only the ones that will be considered in the numerical calculations are depicted in the sketch. The relevant distances for the model (53) are $\sigma^c_{\Omega x}$ and σ^c_{yz}. The former is solely spacelike and enters in the calculation of the curvature scalar. The latter is either timelike or spacelike and is the argument of the propagator. Note that the final scattering model is independent of the distances σ^c_{xy} and σ^c_{xz} and that it exhibits time reversal and parity inversion symmetry. They originate from the symmetry in the arguments of the geodesic distance.

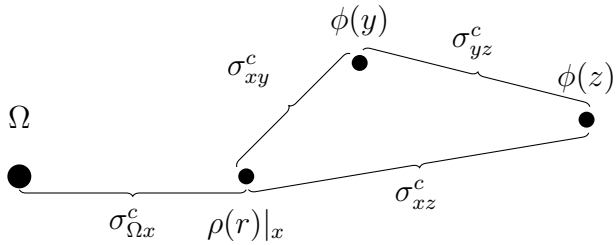


Figure 7: Depiction of the final model. The events are represented by coordinates in the classical spacetime.

6 Scattering scenarios and results

This section covers some possible scattering scenarios that are chosen to be as straightforward and transparent as possible, in order to acquire illuminating insights into the model. For the purpose of clear presentation, we want to extract two dimensional plots of the scattering function. Since the final model is not diffeomorphism invariant, we chose different scenarios that are motivated, to some extent, by coordinates. Nonetheless, considering the results in a manner of geodesic distances shows that the cases that originate from different coordinate scenarios are not distinguishable. This may be due to the simplicity, i.e. the symmetries of the spacetime. A more complex classical spacetime could possibly yield a more intricate result.
Since the final scattering model (53) is in position space, the actual interpretation of the results is non-trivial. The position space representation does not provide any kinematics, only correlations between the measurements of different quantities at a fixed coordinate constellation. As such, **the model is roughly interpreted as the influence of the black hole on the propagation of a scalar particle.** This is a general interpretation. More specific ones are given at the end of each scenario.

Parameters and scales
In order to do numerical calculations, the parameters of the model have to be fixed. There are two parameters: the mass of the black hole M and the mass of the scalar particle m. The black hole mass is set to one Planck mass, i.e. $M = 1G^{-1/2}$. For the scalar field it is reasonable to choose a mass that is comparable to that of the particles of the standard model. Since the standard model contains one particular scalar particle, the Higgs boson, we take it as a role model and choose a mass of $125GeV$ [1]. In units of the gravitational constant it is $\frac{125}{(1.2209\cdot 10^{19})}G^{-1/2} \approx 1.024 \cdot 10^{-17}G^{-1/2}$. These masses provide guiding values for typical scales of the geodesic distances. From dimensional considerations, the corresponding relation of the mass and the geodesic distance can be determined to be $\sigma \sim m^{-2}$. Therefore, the characteristic distance scale for the black hole is of $\mathcal{O}(1)$ in units of the gravitational constant G. And the length scale concerning the scalar propagator is of the order $10^{34}G$. Note that this scales are reference points for the evaluation to investigate the model at reasonable magnitudes.

Choosing the black hole to be more massive would lead to a decrease of the scattering amplitude. Consider the curvature scalar (46). The biggest value of this function is at $\rho(r_s) = \sqrt{12}\frac{1}{r_s^2}$. This means, that for bigger Schwarzschild radii the curvature value decreases. And, since the Schwarzschild radius is proportional to the mass, the scattering effect will diminish with increasing black hole mass. As a consequence, scenarios with

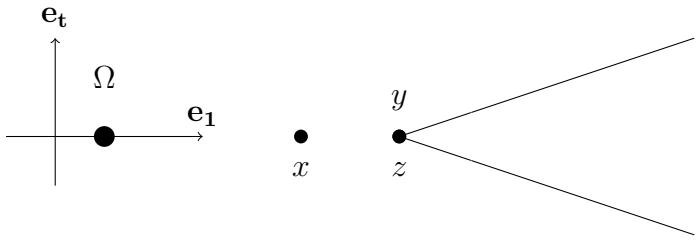
black holes that possess multiple solar masses will have a vanishingly small scattering effect, which makes actual observations dificult.
Although, the distances σ_{xy} and σ_{xz} do not enter in the final model explicitly, we still incorporate them in the calculations.

The superscript c indicating the classical part of the geodesic distance will be omitted in the following for reasons of readability. Keep in mind that all occurring distances ought to be the classical ones.

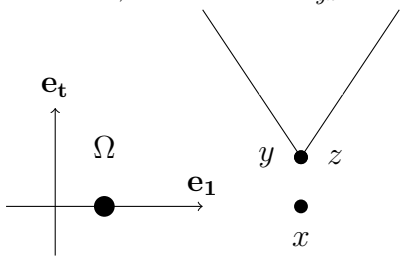
6.1 Symmetric modification

The approach via symmetric modification is done, as the name says, by symmetrically changing the coordinates for two of the four events. This is confined by the condition that the two geodesic distances σ_{xy} and σ_{xz} are chosen to be equal. It is reasonable to keep the distance between the black hole and the curvature evaluation point constant by fixating the points Ω and x. Note that if this distance is fixed, the curvature evaluation will yield a constant value. This results in a product of a constant and a scalar propagator for the scattering model. The distance σ_{yz} entering the propagator is then altered by changing the two points y and z. This provides a two dimensional plot of the scattering function depending on the distance σ_{yz}.

Coordinates
Without loss of generality we consider only a (1+1) dimensional system by setting the other spatial coordinates to zero (or constant). In these two dimensions, only two different ways to realize this scenario are possible. They are sketched in figure 8.



(a) As the points y and z always have equal coordinates in spatial direction, the distance σ_{yz} will always be timelike.



(b) The points y and z will always have equal time coordinates, this means that the distance σ_{yz} will always be spacelike.

Figure 8: Two dimensional system where y and z are symmetrically modified. The lines indicate the path that the coordinates take, where the distances σ_{xy} and σ_{xz} are set equal.

Both sketches show the position of the black hole Ω and the three points x, y and z. The lines indicate the coordinate paths that y and z follow. They are always updated in a way such that $\sigma_{xy} = \sigma_{xz}$. This implies that in the upper figure the distance σ_{yz} is timelike and in the lower figure it is spacelike. If the angle between those lines is big enough, the transition of $\sigma_{xy} = \sigma_{xz}$ from timelike to spacelike or vice versa is made. This seems, as if it should have an impact on the result. But as mentioned, these distances don't affect the process.
In the following, the two cases will be distinguished by the causal behavior of the distance σ_{yz}.

Numerical implementation σ_{yz} timelike
The modification recipe for the coordinates y and z in the sense of figure 8a is

$$y_n^{(j)} = y_0 + \delta_n(1, \kappa_j, 0, 0)^T \qquad z_n^{(j)} = z_0 + \delta_n(-1, \kappa_j, 0, 0)^T \tag{54}$$

The index $n \in \mathbb{N}$ indicates the step number and the index $j \in \mathbb{N}$ indicates that the procedure is done for different step sizes κ_j. The δ_n is the scale parameter, which ensures that the values of the geodesic distances lie in the range provided by the mass of the scalar particle. For the calculations it was set to $\delta_n = n10^{16}$. It depends on the step number to cover a wider range of distance values. For the implementation the relation $\kappa_j = \kappa_0 + j\kappa_k$ was used, where $\kappa_0 = 1$ and $\kappa_k = 0.5$ was chosen.

With this prescription the constraint $\sigma_{xy} = \sigma_{xz}$ is fulfilled. With increasing n the geodesic distance σ_{yz} increases in a timelike manner. Initial coordinates used for the calculations of this approach are

$$x = y_0 = z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{55}$$

With the help of (54) the coordinates are then updated and the geodesic distances and the scattering function are calculated. For each coordinate configuration, the distances and scattering function are related.

Numerical implementation σ_{yz} spacelike

The principle is the same as for the timelike case. The modification recipe for the coordinates y and z in the sense of figure 8b is

$$y_n^{(j)} = y_0 + \delta_n(\omega_j, 1, 0, 0)^T \qquad z_n^{(j)} = z_0 + \delta_n(\omega_j, -1, 0, 0)^T \tag{56}$$

The indices indicate the same as above and the scale parameter δ_n is chosen to be the same as before. The step size parameter ω_j is chosen to be a different one. It is not mandatory, but it introduces additional freedom in the implementation. Again, a linear dependency was chosen, i.e. $\omega_j = \omega_0 + j\omega_k$. The parameters were selected to be $\omega_0 = 1$ and $\omega_k = 0.5$. This prescription satisfies the condition $\sigma_{xy} = \sigma_{xz}$ and increases σ_{yz} with increasing n in a spacelike manner. Initial coordinates that were used for the calculations are

$$x = y_0 = z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{57}$$

Again, the coordinates are correspondingly updated, the geodesic distances and scattering function values are calculated and are set in relation for each coordinate configuration.

Also, regarding both the timelike and spacelike case, some calculations are done for different distances $\sigma_{\Omega x}$ (curvature values $\rho(r)|_x$), where it is still kept constant over one whole update procedure of y and z.

6.1.1 Results for σ_{yz} timelike

The absolute values of the geodesic distances are plotted in figure 9 for the control paramters κ_1, κ_2 and κ_3. The absolute values of the scattering function are plotted in figure 10 and 11. The former is with different parameters κ_j and the latter for κ_1 and different distances $\sigma_{\Omega x}$. The different distances are achieved by shifting the coordinates of the black hole in $-\mathbf{e}_1$ direction by one for each case. The according values for the distance, Schwarzschild radial coordinate and curvature scalar are given in the legend of the plot.

One property to notice is that the geodesic distance σ_{yz} does not change with different step size κ_j. This makes sense, since the step size is for both y and z in the $\mathbf{e_1}$ direction and therefore cancels. This implies also that the values for the scattering function are the same for different step sizes and therefore, as mentioned, not dependent on the distances σ_{xy} and σ_{xz}.

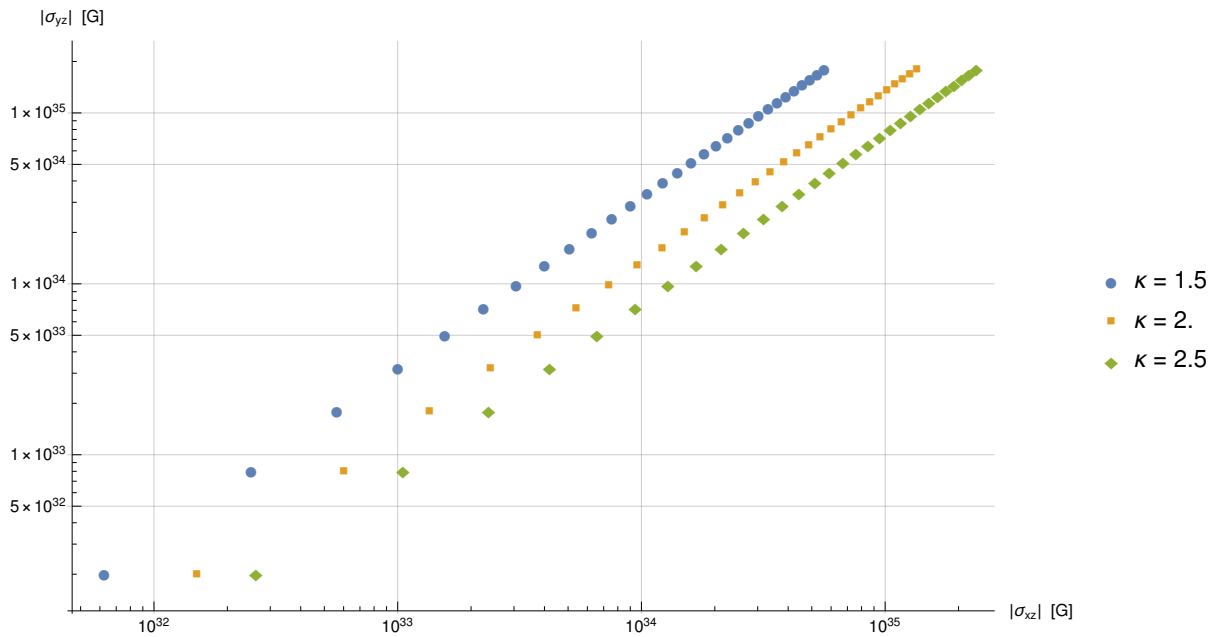


Figure 9: Absolute values of the geodesic distances for the symmetric modification scenario where the geodesic distance of the propagator is timelike. The absolute value was taken to have a logarithmic presentation. Note that $\sigma_{yz} < 0$.

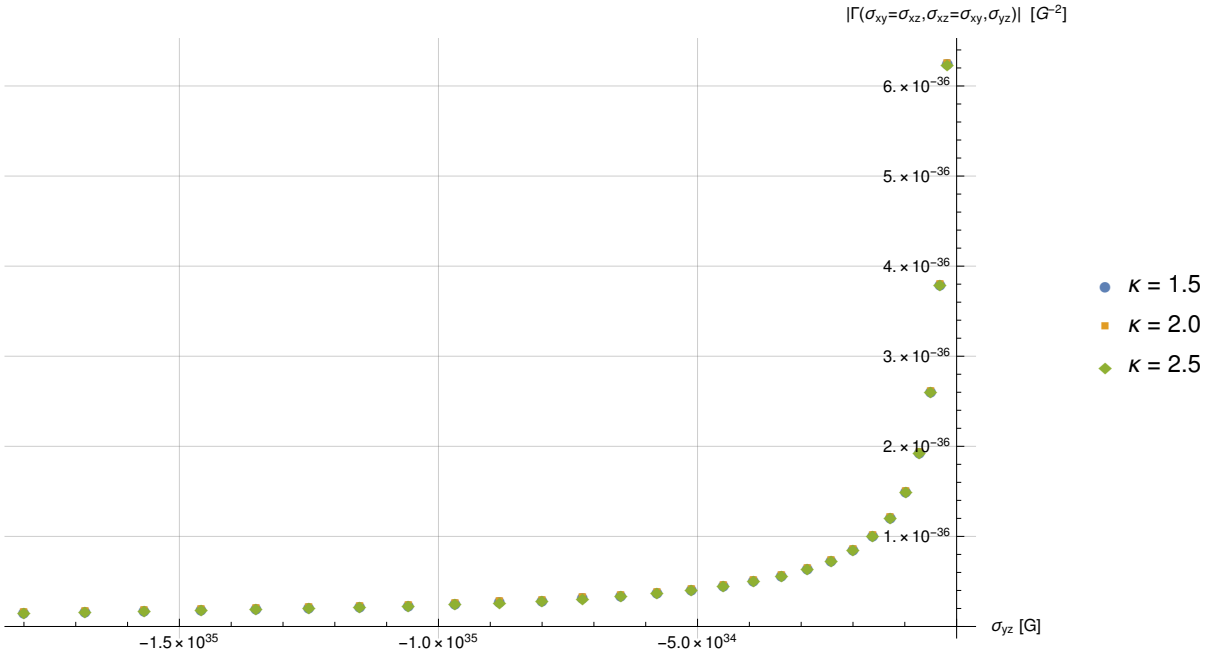


Figure 10: Absolute value of the scattering function for the symmetric modification scenario where the geodesic distance of the propagator is timelike and $\sigma_{\Omega x} = 0.5G$. The different weight factors correspond to different geodesic distances $\sigma_{xy} = \sigma_{xz}$. Their lack of influence is apparent.

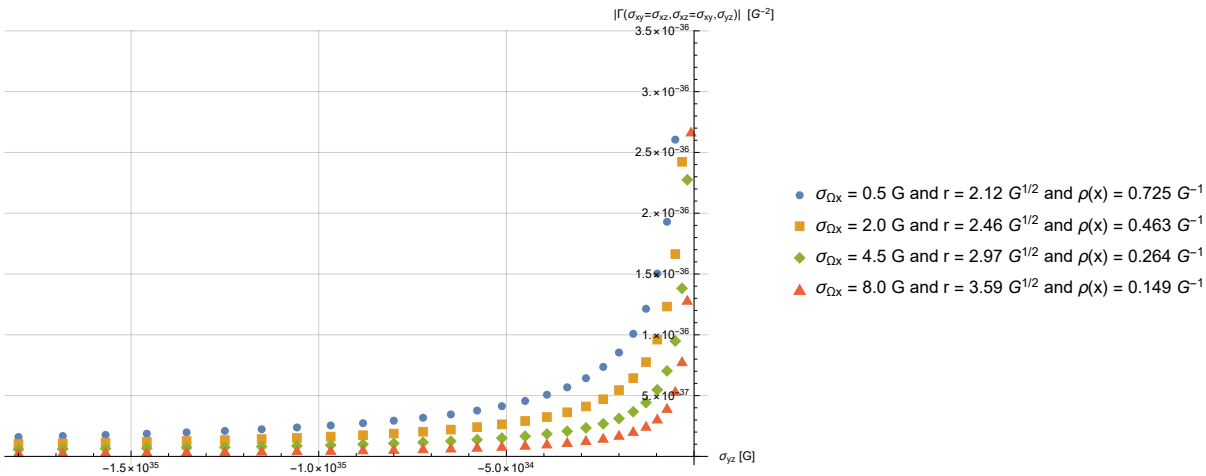


Figure 11: Absolute value of the scattering function for the symmetric modification scenario where the geodesic distance of the propagator is timelike. It shows the values for $\kappa_1 = 1.5$ and different distances between the black hole and the curvature evaluation. The legend provides the values for this distance, the according Schwarzschild radial coordinate and the curvature scalar.

6.1.2 Results for σ_{yz} spacelike

The second case, which corresponds to figure 8b, considers the distance σ_{yz} to be spacelike. The absolute values of the geodesic distances are shown in figure 12 and the absolute

values of the scattering function are shown in figure 13 for different step sizes and in 14 for different black hole positions.

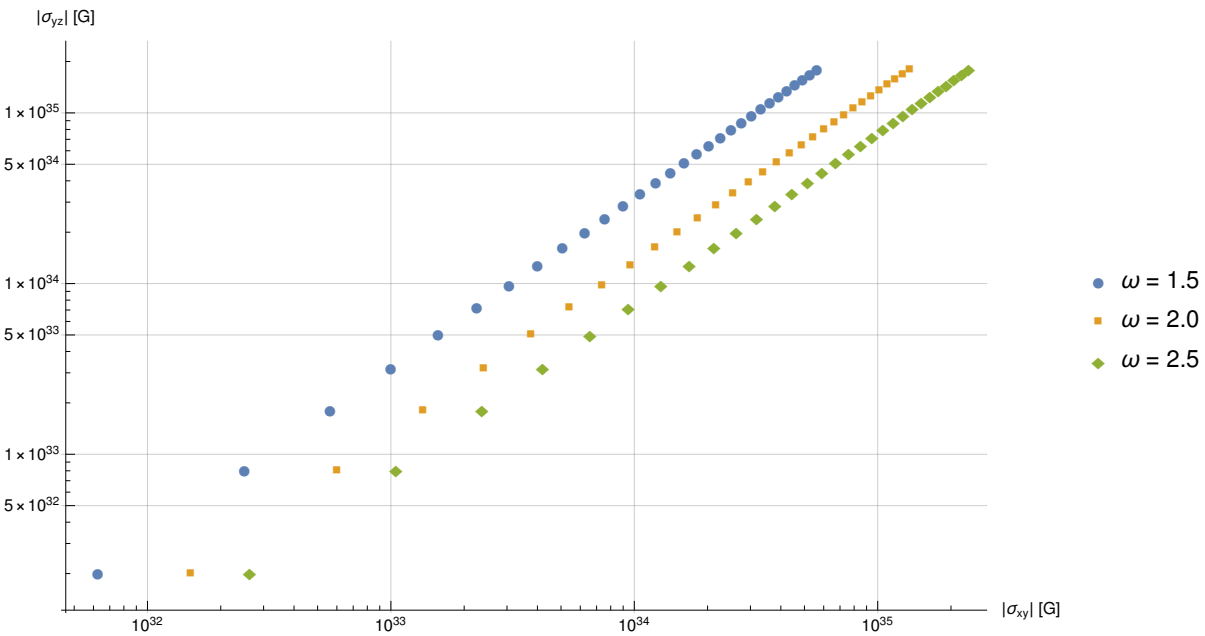


Figure 12: Absolute values of the geodesic distances for the symmetric modification scenario where the geodesic distance entering the propagator is spacelike.

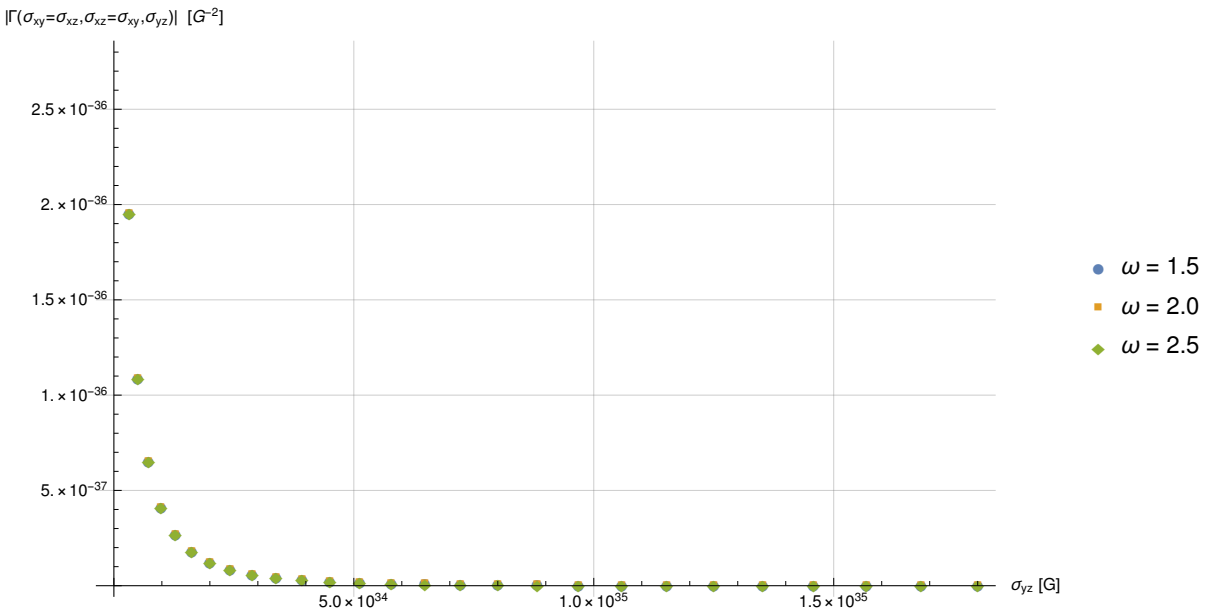


Figure 13: Absolute value of the scattering function for the symmetric modification scenario where the geodesic distance entering the propagator is spacelike and $\sigma_{\Omega x} = 0.5G$. The different weight factors correspond to different geodesic distances $\sigma_{xy} = \sigma_{xz}$.

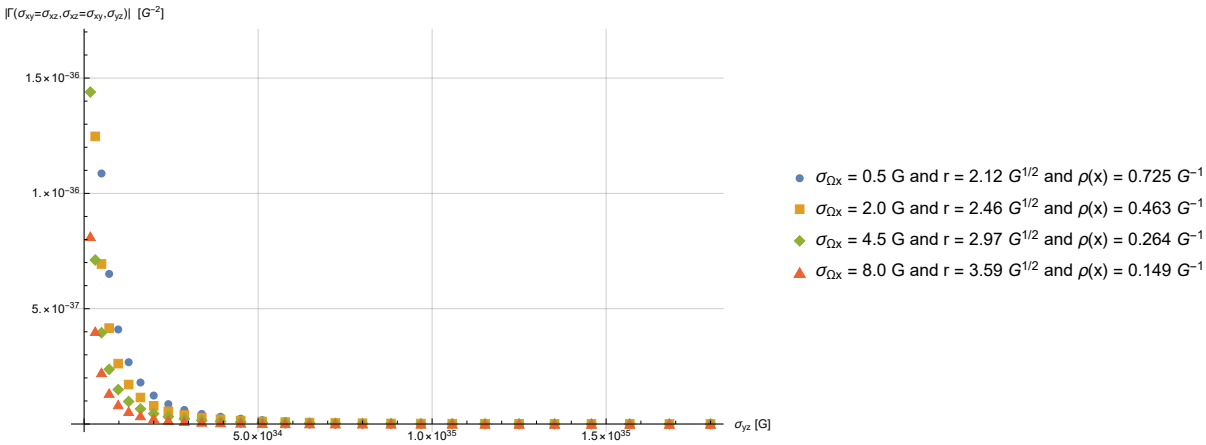


Figure 14: Absolute value of the scattering function for the symmetric modification scenario where the geodesic distance entering the propagator is spacelike. It shows the values for $\omega_1 = 1.5$ and different distances between the black hole and the curvature evaluation. The legend gives the values of this distance, the according Schwarzschild radial coordinate and the curvature scalar.

6.1.3 Combined plot

Additionally, the plots 11 and 14 are combined in a single plot. As can be seen in figure 15, the form of the scattering function resembles that of a scalar propagator, c.f. figure 1. The timelike side decays in a polynomial way and the spacelike side decays exponentially. This is to be expected, since the model is just a product of the curvature evaluation (in this case a constant) with the scalar propagator. Hence, the scattering model poses a simple dressing of the scalar propagator.

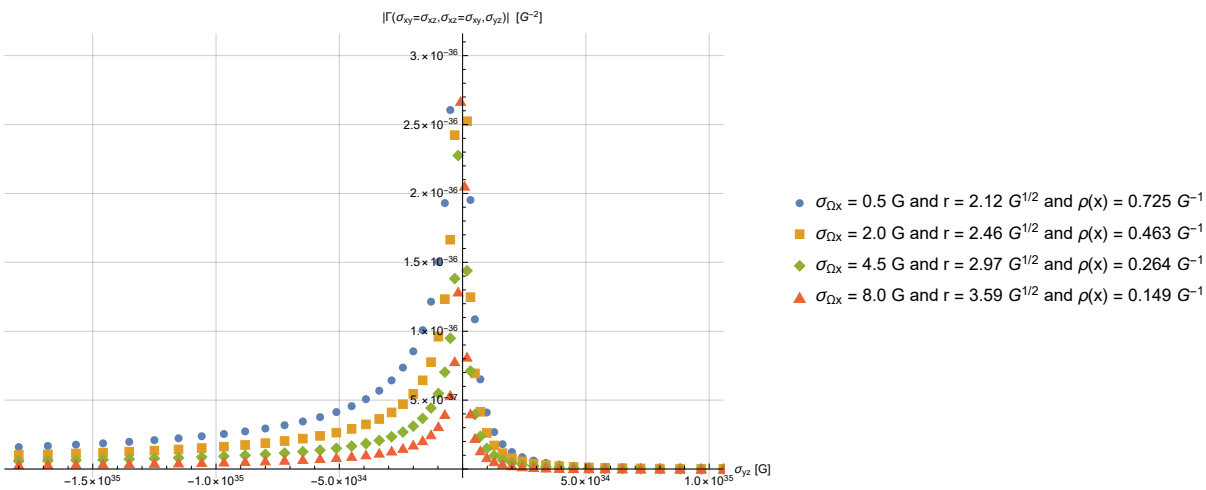


Figure 15: Combined plot of the symmetric modification scenario.

Interpretation

As mentioned earlier, this scattering model basically resembles the influence of a black hole on the propagation of a scalar particle.

Note that the curvature value decreases with the distance of the curvature evaluation point to the position of the black hole. This means, that the influence of the black hole on the propagation decreases, if the curvature is measured further away from the black hole. This is an expected feature, since the asymptotic behavior of the Schwarzschild spacetime is such that the curvature is decreasing for increasing distance.

In 3.3.1 we interpreted the geodesic distance for timelike behavior to be the proper time interval between two points. Adapting this to the results in figure 15 would mean that the scattering effect is bigger, if the proper time interval between y and z is small. I.e. the influence of the black hole on a briefly propagating scalar is bigger than on a longer propagating one.

Note that the actual values of the scattering function are ambiguously. This stems from the arbitrary choice of the prefactor in the classical part of the black hole operator $\rho(x)$. Without any experimental comparison there is no way to state, if this was the right one. So far, only relative statements are meaningful.

6.2 Equality of absolute values

Another scenario is provided by equating the absolute values of the geodesic distances between x, y and z and set σ_{xy} and σ_{xz} equally timelike (spacelike) and the σ_{yz} spacelike (timelike). In mathematical terms

$$|\sigma_{xy}| = |\sigma_{xz}| = |\sigma_{yz}| = \chi \qquad \mathrm{sgn}(\sigma_{xy}) = \mathrm{sgn}(\sigma_{xz}) \neq \mathrm{sgn}(\sigma_{yz}) \tag{58}$$

Where χ is considered a parameter. Since the distance σ_{yz} is the decisive one in the model, a distinction between its timelike and spacelike behavior will be made again. This scenario is in a way the link between the former and the following scenario.

6.2.1 σ_{yz} timelike

This is the situation where the scalar field events are connected causally and are in turn non-causally connected to the curvature evaluation point. To construct an update procedure for the coordinates that fulfills the condition (58), we need to consider the explicit expressions for the geodesic distances. Generally, we can set x to the origin, y in the $\mathbf{e}_t$ and $\mathbf{e}_1$ plane and z in the $\mathbf{e}_t$, $\mathbf{e}_1$ and $\mathbf{e}_2$ volume. The according equations for the

geodesic distances are then

$$2\sigma_{xy} = 2\chi = (-t_y^2 + y_1^2)G \tag{59}$$

$$2\sigma_{xz} = 2\chi = (-t_z^2 + z_1^2 + z_2^2)G \tag{60}$$

$$2\sigma_{yz} = -2\chi = (-(t_y - t_z)^2 + (y_1 - z_1)^2 + z_2^2)G \tag{61}$$

where $\chi > 0$ is implied. This system contains three equations but five variables, the coordinates for y and z. The quantity χ is considered a parameter at this stage. In order to proceed, the coordinates for y and the coordinate z_2 will be chosen and the coordinates t_z and z_1 will be determined. Solving the system yields

$$t_z^{(\pm)} = \frac{5t_y}{2} \pm \frac{\sqrt{\chi'(t_y^2 + \chi')(4z_2^2 + 17\chi')}}{2\chi'} \tag{62}$$

$$z_1^{(\pm)} = \frac{5t_y^2\chi' + 5(\chi')^2 \pm t_y\sqrt{\chi'(t_y^2 + \chi')(4z_2^2 + 17\chi')}}{2\chi'\sqrt{t_y^2 + \chi'}} \tag{63}$$

Where $\chi' = \chi/G$ is the dimensionless parameter. For the procedure the coordinates y are altered and z is adjusted according to the equations above. Without loss of generality the coordinate z_2 is also set to zero, since it provides no decisive difference in (62) and (63). Note that the coordinate y_1 does not appear explicitly but implicitly in $\chi = 2\sigma_{xy}$.

Numerical implementation σ_{yz} timelike

We begin by defining initial coordinates

$$x = z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad y_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{64}$$

The next step is actually to increment the y coordinates by

$$y_n = y_0 + \delta_n(1, 2, 0, 0)^T \tag{65}$$

Where the scale factor $\delta_n = n10^{16}$ ensures that we hit the scale we are interested in. The increment is chosen this way, such that the distance σ_{xy} is spacelike. The next step is to calculate the geodesic distance $\sigma_{xy}^{(n)} = \chi^{(n)}$, where the index n indicates the step in the procedure. Inserting this and the y_n coordinates into (62) and (63) yields the coordinates for z_n, where the subscript again indicates the step number. From this we can calculate

the remaining quantities $\sigma_{xz}^{(n)}$, $\sigma_{yz}^{(n)}$ and $\Gamma^{(n)}$. Then we start again by updating y and repeat the steps. The results for the distances and the scattering function are then combined in order to create plots.

The geodesic distances for the solution $t_z^{(-)}$ and $z_1^{(-)}$ are plotted in figure 16 and the absolute values for the scattering function in figure 17. The different values for $\sigma_{\Omega x}$ are created by shifting the coordinates Ω in negative $\mathbf{e}_1$ direction by one for each case.

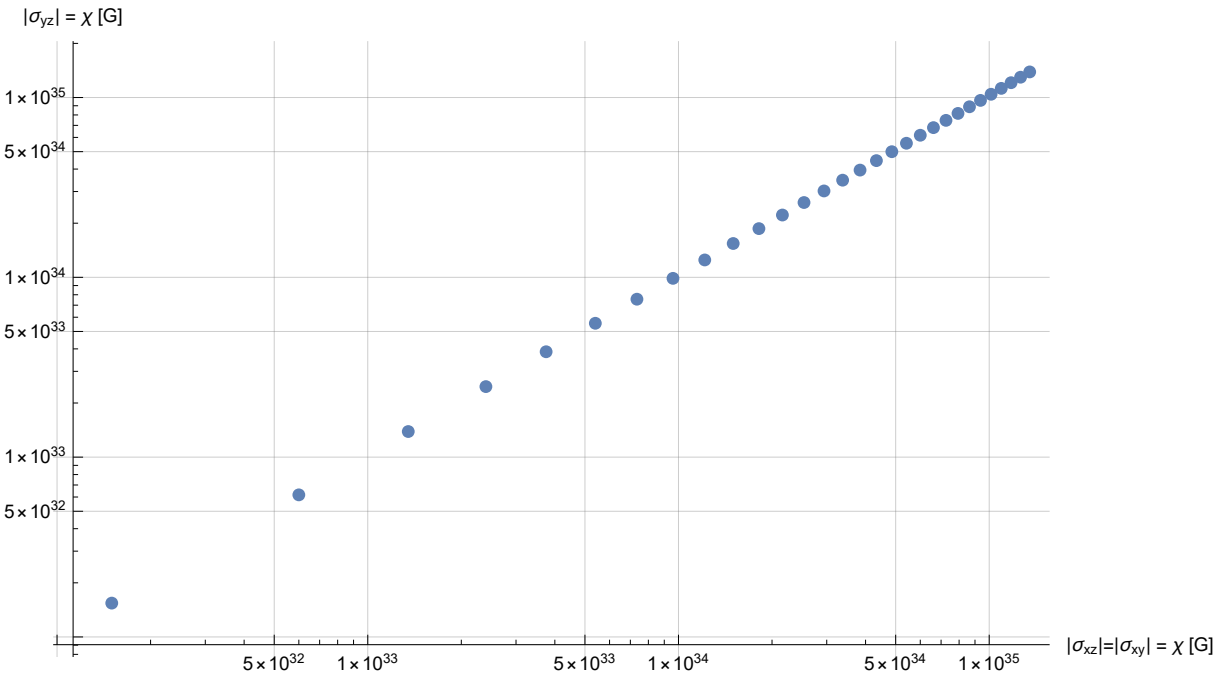


Figure 16: Absolute values of the geodesic distances where y and z are connected in a timelike manner. To get a logarithmic presentation, the absolute values of the distances are taken.

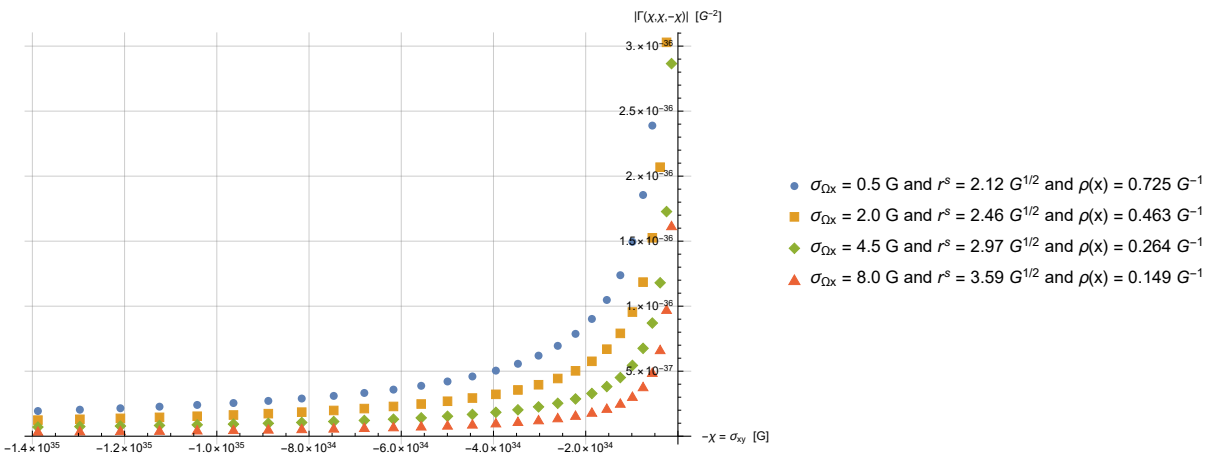


Figure 17: Absolute values of the scattering function where y and z are connected in a timelike manner. Also, different distances $\sigma_{\Omega x}$, i.e. curvature values are considered.

6.2.2 σ_{yz} spacelike

The procedure is basically the same as for the timelike case. The difference is in the calculation of the coordinate values of z, since the conditional equations are different. The equations are

$$2\sigma_{xy} = -2\chi = (-t_y^2 + y_1^2)G \tag{66}$$

$$2\sigma_{xz} = -2\chi = (-t_z^2 + z_1^2 + z_2^2)G \tag{67}$$

$$2\sigma_{yz} = 2\chi = (-(t_y - t_z)^2 + (y_1 - z_1)^2 + z_2^2)G \tag{68}$$

where $\chi > 0$ is implied again. The system of equations is handled in the same way as above. Solving for the z coordinates yields

$$t_z^{(\pm)} = \frac{5t_y}{2} \mp \frac{\sqrt{\chi(t_y^2 - \chi)(-4z_2^2 + 17\chi)}}{2\chi} \tag{69}$$

$$z_1^{(\pm)} = \frac{5t_y^2\chi - 5\chi^2 \pm t_y\sqrt{\chi(t_y^2 - \chi)(-4z_2^2 + 17\chi)}}{2\chi\sqrt{t_y^2 - \chi}} \tag{70}$$

This equations will be used to adjust the coordinates for z when y is changed. Again, z_2 is set to zero and again y_1 does only occur implicitly. The incrementation of the y coordinates is given by

$$y_i = y_0 + \delta_i(4, 0, 0, 0)^T \tag{71}$$

The increment is chosen this way, to always have a timelike distance for σ_{xy} and to cover the corresponding range of values.
The geodesic distances for the solution $t_z^{(-)}$ and z_1^- are plotted in figure 18 and the values for the absolute scattering function in 19.

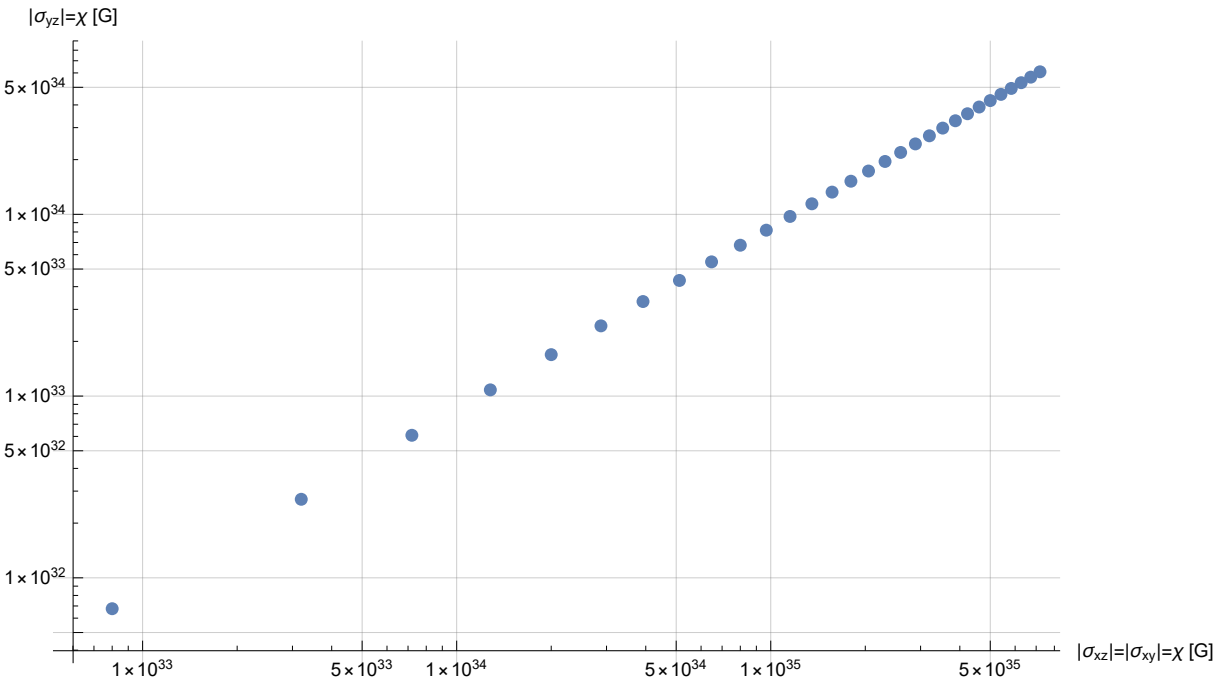


Figure 18: Absolute values of the geodesic distances where y and z are connected in a spacelike manner. For the sake of a logarithmic plot the absolute values of the distances were used.

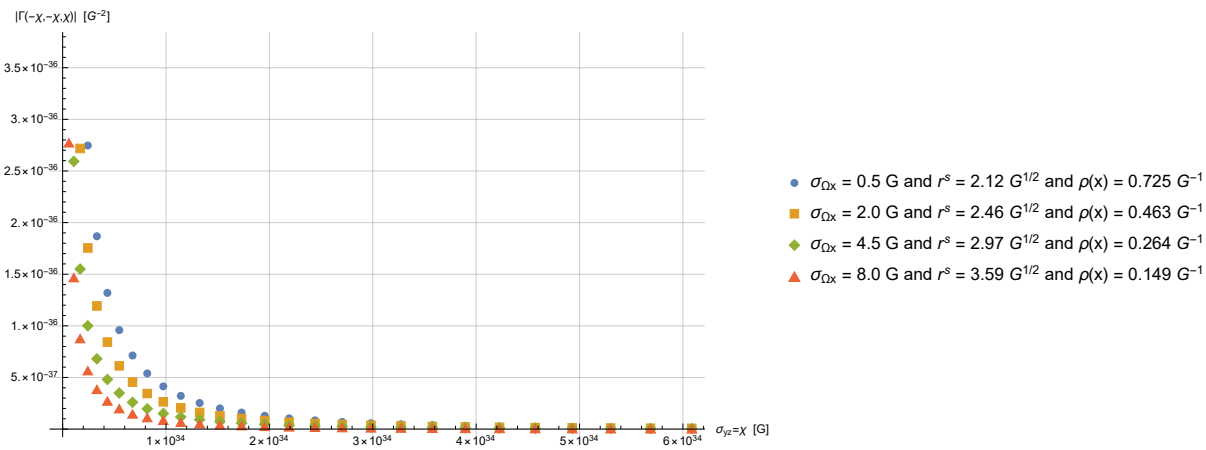


Figure 19: Absolute values of the scattering function where y and z are connected in a spacelike manner. Also, different distances $\sigma_{\Omega x}$ are considered.

6.2.3 Combined plot

It is again useful to combine the spacelike and timelike figures into one plot. This is given in figure 20.

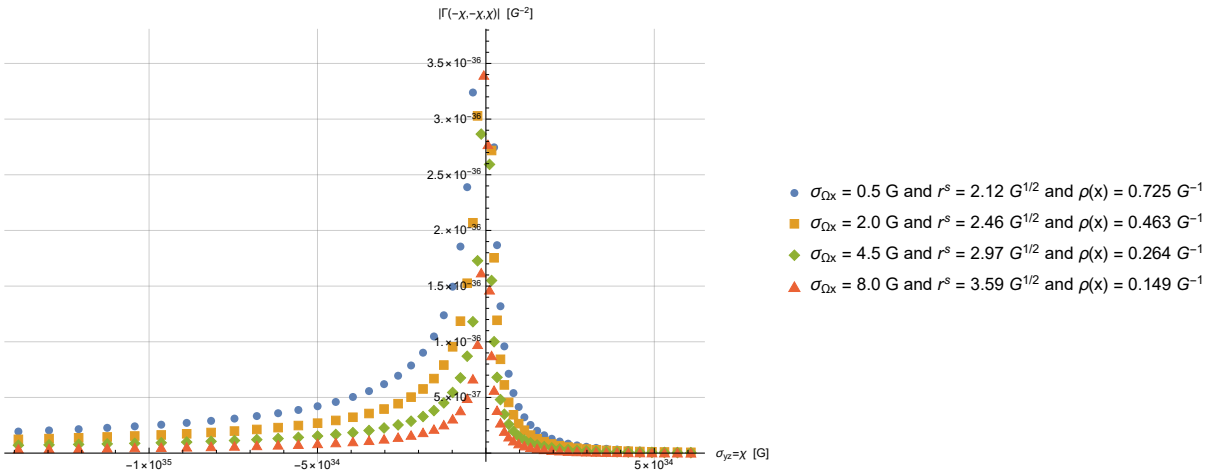


Figure 20: Combined plot of the scenario of equal absolute values.

Interpretation

The interpretation of this scenario is essentially the same as for the previous case in section 6.1.3. One thing to mention is that, albeit we chose a different coordinate configuration, the general form of the scattering function has not changed. This is also an expected feature, since we consider geodesic distances and not coordinates.

6.3 Full symmetric configuration/Coupling

In quantum field theory running couplings of n-point vertices can be calculated through n-point scattering functions. For example in QED the coupling of the fermion-antifermion-photon vertex is given by

$$\Gamma^{A\Psi\bar{\Psi}}(\mu^2, \mu^2, \mu^2) = ig(\mu^2) \tag{72}$$

Where μ is a scale parameter with dimension of mass. This is done in momentum space. Unfortunately, we do not have the possibility to operate in momentum space. Nevertheless, we want to implement the same idea of a symmetric configuration for the distances, i.e.

$$\Gamma(\chi, \chi, \chi) = ig(\chi) \tag{73}$$

Where the three arguments of the scattering function are the geodesic distances between the three events. The parameter χ plays a similar role as μ above. As the geodesic distance can take on spacelike and timelike values, χ could do the same. Since the three geodesic distances are set equal, the construction resembles an equilateral triangle. If we consider it to be spacelike, we have no problem, since we only need two spatial directions to build that triangle. But, for timelike distances it is not possible to build such a triangle, since we only have one time direction. An outline of the problem for the timelike case is

given in appendix C.

6.3.1 Results for χ spacelike

As mentioned, this scenario can be seen as an equilateral triangle living in two space dimensions. A sketch to emphasize this is drawn in figure 21.

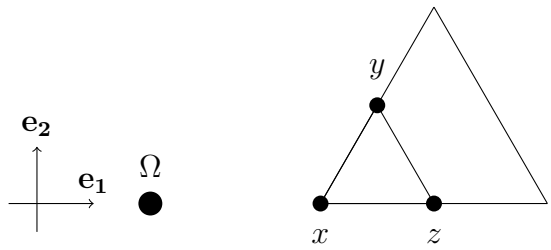


Figure 21: The equilateral triangle with spacelike distances in two space dimensions.

The event x is fixed and y and z are shifted, which is indicated by the bigger triangle in the sketch. The calculations are again repeated for different distances $\sigma_{\Omega x}$.

Numerical implementation

First of all, we choose a set of initial coordinates for x, y, z and Ω

$$x_0 = y_0 = z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Omega = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{74}$$

In order to build the equilateral triangle, we choose normalized increments for the coordinates

$$\Delta y = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} \qquad \Delta z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{75}$$

The angle between Δy and Δz is $\pi/3$ which ensures the equilateral property of the triangle. Of course, the scale of the distances has to be adjusted by a scale parameter δ_n, which we chose to be $\delta_n = 2 \cdot 10^{16} n$. The coordinates are then changed discretely by

$$y_n = y_0 + \delta_n \Delta y \qquad z_n = z_0 + \delta_n \Delta z \tag{76}$$

The update procedure is similar as that for the symmetric modification: calculate distances and the scattering function, update coordinates and repeat.

The geodesic distances for this scenario are plotted in figure 22 and the absolute values of the scattering function are plotted in figure 23.

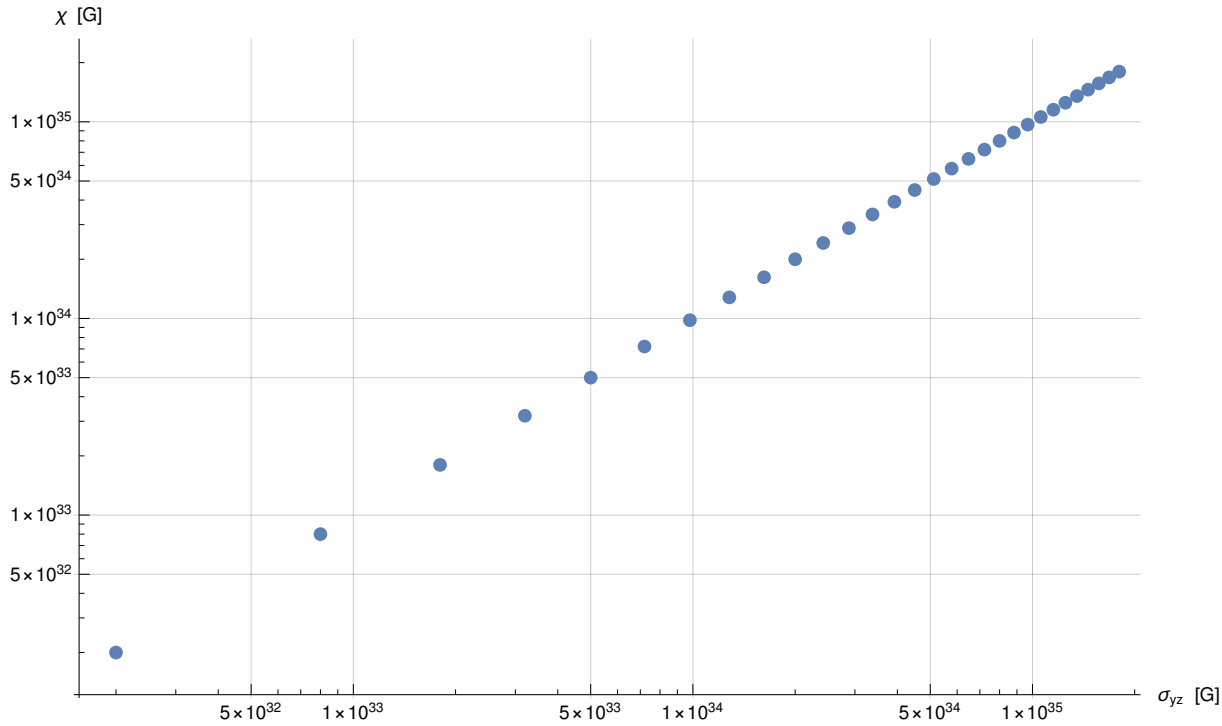


Figure 22: Geodesic distances for the coupling scenario. The distances $\sigma_{xy} = \sigma_{xz} = \sigma_{yz}$ are equal to χ, which shows in the plot.

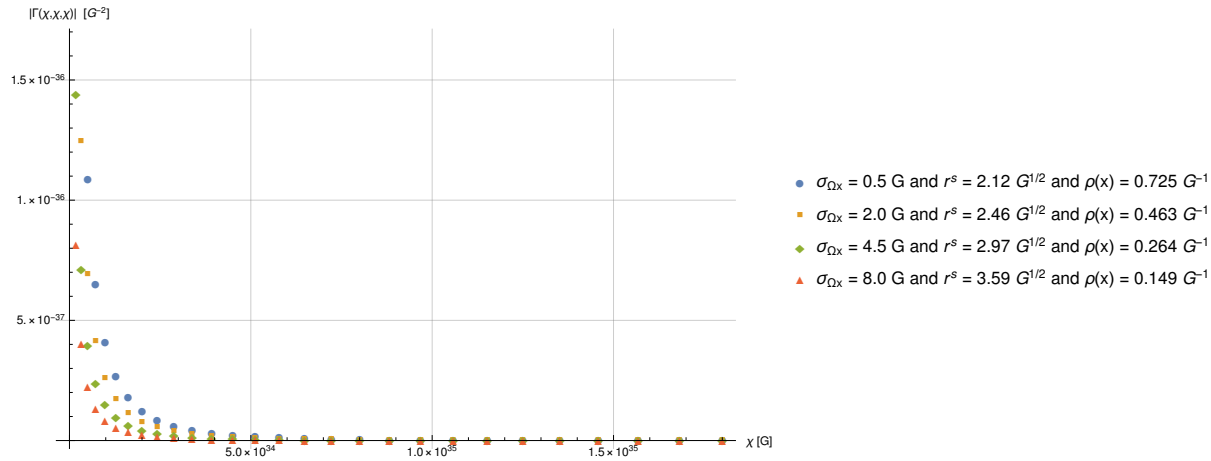


Figure 23: Absolute values of the scattering function for different distances between the black hole and the curvature evaluation point.

Interpretation

The solution to this scenario could provide a junction from our theoretical considerations to experiment. Unfortunately, the experimental obstacles seem rather huge. Since the magnitude of the scattering will decrease with increasing black hole mass, it will be very difficult to make observations that suggest properties of the coupling.

7 Conclusion and Outlook

7.1 Conclusion

We used the principle of gauge and diffeomorphism invariance to build a scattering model of a black hole and a scalar field. In that sense we discussed the important properties that observables must possess to be physical. In order to set physical observables in an invariant relation, we introduced the geodesic distance. Since this quantity contains in its definition the metric field, it is itself an observable. Employing the metric split provided the possibility to simplify and, at the same time, approximate observables. Due to gauge fixing of the metric field, which is necessary in order for the metric split to have meaning, we broke diffeomorphism symmetry explicitly. This reflects itself on the non-vanishing expectation values of diffeomorphism variant quantities, e.g. the metric. The FMS mechanism was adopted from the Higgs sector of the standard model to the case of gravity, in order to approximate correlation functions by a gauge variant dominant term. By using the FMS mechanism we could calculate the classical part of the geodesic distance and build series expansions of observables in orders of the quantum part of the metric. We applied these ideas to a three point scattering function of a black hole operator and a scalar field. The black hole operator was again expanded about a classical part, which we chose to be a curvature scalar with dimensions of a Gaussian curvature. Since this is an ordinary function, the scattering model reduced to a product of this curvature value and a scalar propagator. The final model is not diffeomorphism invariant, due to the employed metric split.

In the course of the interpretation of the scattering scenarios, we observed that the approximated model resembles the simple dressing of a scalar propagator and that only relative statements about the scattering can be made.

All things considered, this work contains approximations and choices that need further investigations, in order to refine the model. But, the possibility to make significant statements is strongly intertwined with the pending experimental data.

7.2 Outlook

Since this is a rather crude first approach, plenty of extensions are possible in order to get the model closer to real physical systems. The following is a suggestion of continuations.

7.2.1 Curved classical spacetime

Choosing the classical part of the metric to be Minkowski spacetime was done mainly for simplicity, since this is the first approach with this method. Also, the scalar propagator

is known in flat spacetime and a sole function of the geodesic distance. The next step could be to consider a curved classical spacetime that still emits maximal symmetry, i.e. de Sitter and anti de Sitter spacetime. Two things would change in the procedure. The first is that all geodesic distances have to be calculated in that spacetime. The other change would be that an according scalar propagator has to be used. Both are rather formal changes and no big obstacles, since the numerical calculation of geodesic distances is possible and the form of a scalar propagator in (anti) de Sitter spacetime is known, e.g. [30, 26].
Going to a more general spacetime, e.g. Robertson-Walker for a cosmological model [7], would imply that the propagator is not a sole function of the geodesic distance. In this case the temporal and spatial distance would enter in different ways. This would change the diffeomorphism symmetry, since the set of diffeomorphisms would be truncated.

7.2.2 Dynamics

Since the connection between the black hole and the curvature evaluation point was chosen to be spacelike, it would be interesting to investigate a timelike connection. This means that the two points are connected in a causal manner and dynamics of the black hole could be considered. In order to do so, the time coordinate in Schwarzschild spacetime must not be constant. But, we run across a problem. Figure 24 shows a plot of the radial coordinate, if the two points are not considered spacelike connected.

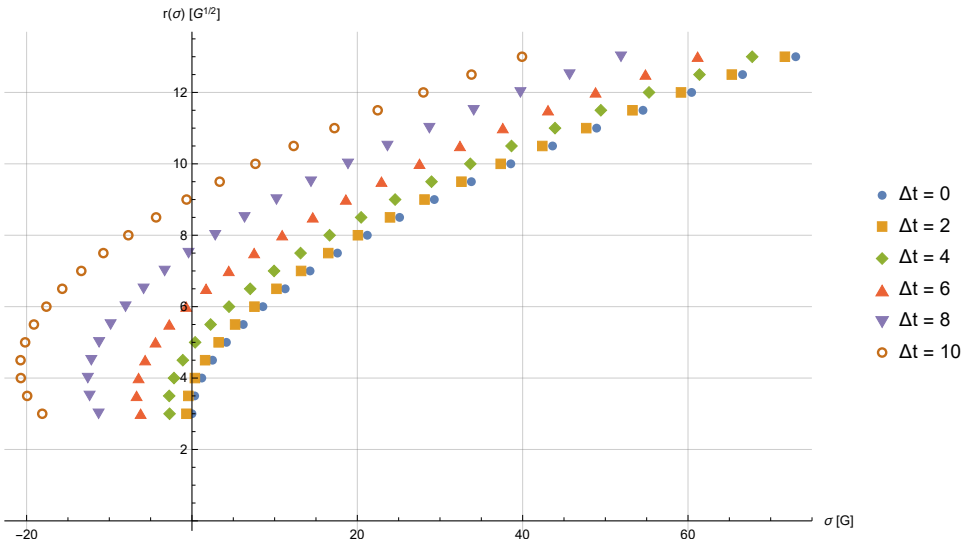


Figure 24: Radial coordiante as a function of the geodesic distance for different distances of the time coordinates.

Note that the Δt, given in the legend of the plot, is not the time distance in Schwarzschild spacetime. It is defined as $\Delta t = t_1 - t_2$, i.e. it is the time coordinate distance between

two points. This plot makes the problem apparent: the radial coordinate is not uniquely given by a timelike distance. It is also not clear which time coordinates to take in the Schwarzschild spacetime, since different temporal values yield different relations. In order to continue with a dynamical scenario, this problem needs to be investigated.

7.2.3 Four point function

An interesting extension would be to consider a four point scattering function, where two points correspond to (the same) black hole and two points again to a scalar field.

$$\Gamma(w, x, y, z) = \langle B(w)B(x)\phi(y)\phi(z)\rangle \tag{77}$$

These four points are connected via six geodesic distances, which makes the scenario more complex. Using the same tools as above, the leading order term would be

$$\Gamma^{LO}(\sigma_{wx}, \sigma_{wy}, \sigma_{wz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}) = \rho(r)|_w \rho(r)|_x D_F(\sigma_{yz}) \tag{78}$$

This would also be interesting in connection with the consideration of dynamics.
An interpretation of this case could be the description of either the scattering of a scalar on a black hole or the creation/annihilation of two black hole particles via the annihilation/creation of scalar particles. The creation of matter particles via annihilation of black hole particles could be considered Hawking radiation.

7.2.4 Kerr black hole

Black holes that occur in nature are not as simple as Schwarzschild black holes, i.e. they rotate. The strongest evidence for this statement comes from the Event Horizon Telescope collaboration. For comparison between the theory of general relativity and the observation they used the model of a Kerr black hole [31]. They also explain that non-spinning black holes can't explain the observations, since the produced jets in this case are not powerful enough.
Hence, in order to bring our model closer to observations, it would be beneficial to consider spinning black holes. The procedure discussed in section 5.2 is still applicable. But instead of using the Schwarzschild spacetime, the Kerr spacetime has to be used. The Kerr metric is given by [5]

$$ds^2 = -\left(1 - \frac{r_s r}{\Sigma^2}\right) dt^2 - \frac{a r_s r \sin^2\theta}{\Sigma^2}(dt d\phi + d\phi dt) + \frac{\Sigma^2}{\Delta} dr^2 \tag{79}$$

$$+\Sigma^2 d\theta^2 + \frac{\sin^2\theta}{\Sigma^2}\left((r^2 + a^2)^2 - a^2 \Delta \sin^2\theta\right) d\phi^2 \tag{80}$$

Where

$$\Delta(r) = r^2 - r_s r + a^2 \qquad \Sigma^2(r,\theta) = r^2 + a^2 \cos^2\theta \tag{81}$$

The parameter a is the angular momentum per unit mass. This makes it apparent that the calculation of the geodesic distance in this spacetime is way more complex than in Schwarzschild spacetime. Also the curvature scalar will be different, see e.g. [32] for a characterization of the Kerr spacetime with scalar invariants.

Appendices

A General relativistic definitions

The definition of the necessary equations and quantities follow [7]. The Einstein sum convention is used unless stated otherwise.

A.1 Christoffel symbols

In order to determine geodesic equations in curved spacetime and calculate the Riemann tensor, we need the Christoffel symbols. They are defined via the metric by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \tag{82}$$

Where $g^{\rho\sigma}$ is the inverse metric defined by $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$ and the abbreviation of the partial derivative signifies $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

A.2 Geodesic equations

For the calculation of the geodesic distance, the geodesic equations have to be determined and solved. They are given by

$$\frac{d^2 z^\mu}{d\lambda^2} + \Gamma^{\mu}{}_{\rho\sigma} \frac{dz^\rho}{d\lambda} \frac{dz^\nu}{d\lambda} = 0 \tag{83}$$

Where $z^\mu(\lambda)$ is the coordinate function that defines the geodesic line and λ is an affine parameter. We need boundary conditions to solve the equations, i.e.

$$z^\mu(\lambda_0 = 0) = x^\mu \quad \text{and} \quad z^\mu(\lambda_1 = 1) = y^\mu \tag{84}$$

This assumes that the affine parameter $\lambda \in [0, 1]$. The solution $z^\mu(\lambda)$ can then be used to calculate the (classical) geodesic distance (23).
This may seem straightforward, but the actual calculation for other spacetimes than the Minkowski case get complex really quick. Analytical calculations are in the majority of cases not possible. Numerical calculations are feasible, but since (83) poses four (coupled) differential equations of second order, they can be connected with computational effort.

A.3 Riemann tensor

From the Christoffel symbols the Riemann tensor can be calculated

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^\sigma{}_{\mu\rho} - \partial_\mu \Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\alpha\mu} \tag{85}$$

The Riemann tensor is necessary to calculate the curvature invariants in section 5.2.

A.4 Epsilon tensor

The Levi-Civita tensor, which is also needed to calculate the curvature invariants in section 5.2, is defined by [6]

$$\varepsilon_{\alpha\beta\gamma\delta} = g^{1/2}[\alpha\beta\gamma\delta] \tag{86}$$

$$\varepsilon^{\alpha\beta\gamma\delta} = -g^{-1/2}[\alpha\beta\gamma\delta] \tag{87}$$

Where $g = |det(g_{\mu\nu})|$ and $[\alpha\beta\gamma\delta]$ is the usual Levi-Civita symbol in four dimensions.

B Maximally symmetric spacetimes

A spacetime that exhibits the maximum amount of isometries is called maximally symmetric [5]. A n-dimensional manifold can have at most $1/2n(n+1)$ isometries. Considering $\mathbb{R}^n$, the isometry transformations are translations and rotations (or boosts, depending on the metric signature). A straightforward way of identifying if a spacetime is maximally symmetric is by checking if the Riemann tensor satisfies

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{88}$$

Where R is the Ricci scalar, which is constant for maximally symmetric spaces. It can have positive, negative or zero value. The vanishing case is Minkowski spacetime. The positive and negative cases are de Sitter and anti de Sitter spacetime, respectively. These are the only three spacetimes in four dimensions that fulfill the requirements to be maximally symmetric.

C Timelike coupling scenario

As written in the main text, the coupling scenario with three equal timelike distances is not possible. The following outlines the reason why.
The coordinates for x are set to the origin. For y the coordinates are set in the $\mathbf{e}_t$ and $\mathbf{e}_1$

plane. Without loss of generality, the coordinates for z live in $\mathbf{e}_t$, $\mathbf{e}_1$ and $\mathbf{e}_2$. Hence, the geodesic distances deliver three conditions

$$\chi = -t_y^2 + y_1^2 \tag{89}$$
$$\chi = -t_z^2 + z_1^2 + z_2^2 \tag{90}$$
$$\chi = -(t_y - t_z)^2 + (y_1 - z_1)^2 + z_2^2 \tag{91}$$

where $\chi < 0$[14]. Eliminating y_1 in the third equation by the first condition yields

$$\chi = -(t_y - t_z)^2 + (\sqrt{\chi + t_y^2} - z_1)^2 + z_2^2 \tag{92}$$

The coordinates are real numbers and as such we acquire a condition on t_y

$$\chi + t_y^2 > 0 \tag{93}$$

Therefore, we get two equations for the variables t_y, t_z, z_1 and z_2 (χ is considered a parameter).

$$\chi = -t_z^2 + z_1^2 + z_2^2 \tag{94}$$
$$\chi = -(t_y - t_z)^2 + (\sqrt{\chi + t_y^2} - z_1)^2 + z_2^2 \tag{95}$$

With the help of Mathematica the equations are solved for z_1 and z_2. This yields two solutions

$$z_1^{(\pm)} = \frac{2t_y t_z + \chi}{2\sqrt{t_y^2 + \chi}} \qquad z_2^{(\pm)} = \pm\frac{\sqrt{\chi}\sqrt{4t_y^2 - 4t_y t_z + 4t_z^2 + 3\chi}}{2\sqrt{t_y^2 + \chi}} \tag{96}$$

Keeping in mind that the coordinates must be real and that $\chi < 0$, we can elicit two conditions. The first one is the same that was already stated above

$$\chi + t_y^2 > 0 \tag{97}$$

The second one concerns the denominator of $z_2^{(\pm)}$, namely

$$\chi\left(4t_y^2 - 4t_y t_z + 4t_z^2 + 3\chi\right) > 0 \tag{98}$$

Which further yields the condition

$$4t_y^2 - 4t_y t_z + 4t_z^2 + 3\chi < 0 \tag{99}$$

[14]The factor $(2G)^{-1}$ is absorbed in χ, since it does not influence the thought process.

Now, condition (97) is used to eliminate χ

$$0 > 4t_y^2 - 4t_y t_z + 4t_z^2 + 3\chi > 4t_y^2 - 4t_y t_z + 4t_z^2 - 3t_y^2 \tag{100}$$

Finally, we get the inequality

$$0 > t_y^2 + 4t_z^2 - 4t_y t_z \tag{101}$$

Figure 25 is a plot of the right hand side of (101) as a function of t_z for different t_y, which are given in the legend. It shows, that the inequality is not fulfilled.

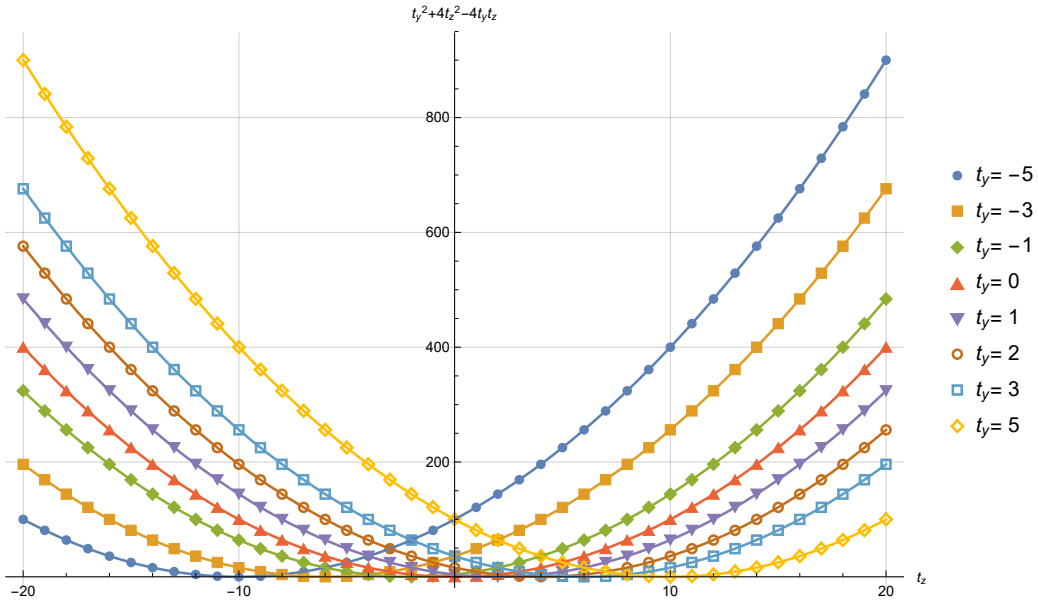


Figure 25: Demonstration that the inequality (101) is not fulfilled.

Although, this line of reasoning is not a rigorous proof, it outlines the conceptual problem. In order to build a timelike equilateral triangle, an additional time dimension is needed, i.e. the metric signature would need another negative entry.

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