Sofie Martins, University of Graz, 22.1.2026

FWF Österreichischer Wissenschaftsfonds

Computing cross-sections from the lattice

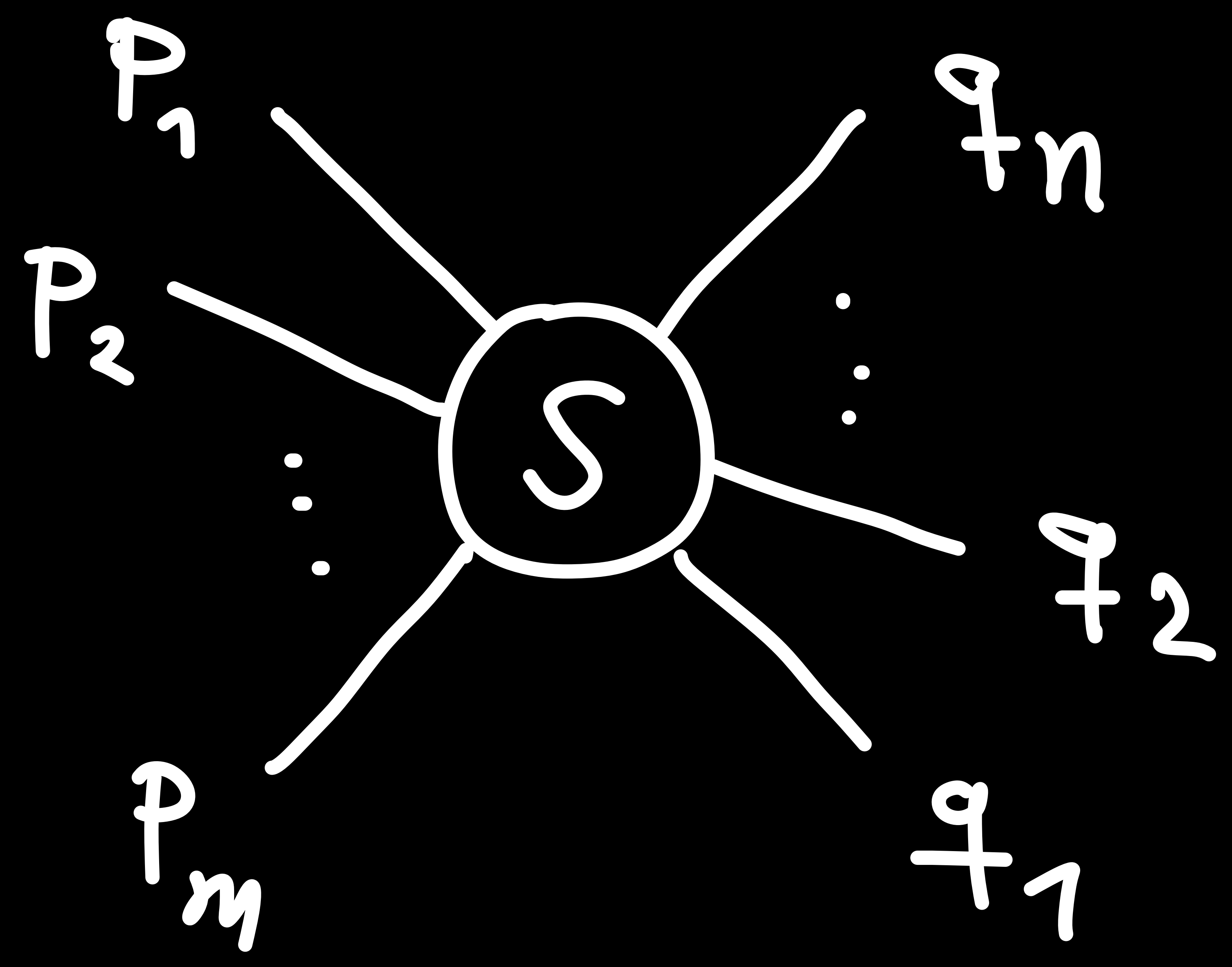
A recent shift in lattice thinking paradigms?

Outline

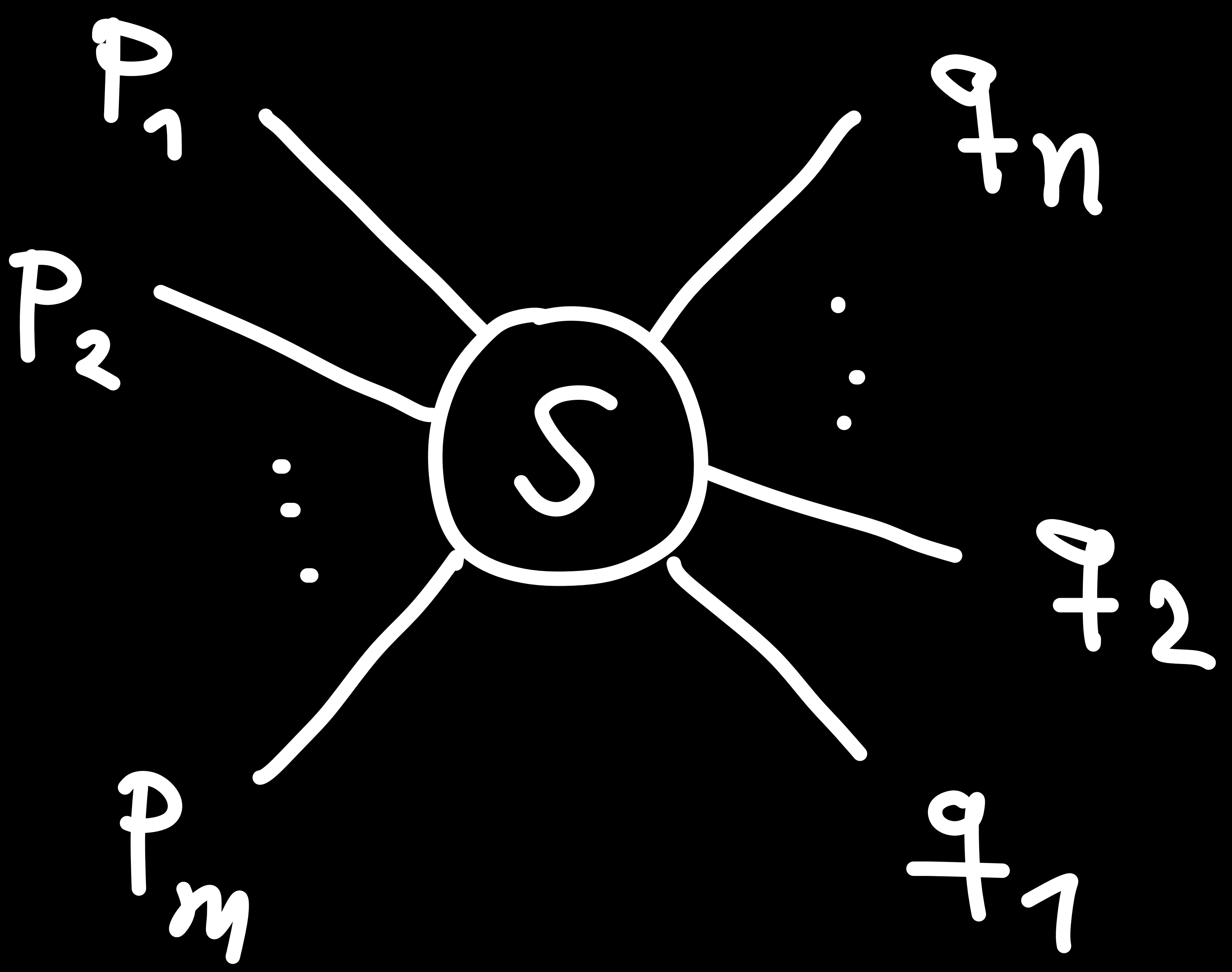
- What are scattering amplitudes?
- Our test case
- Computing spectral densities from lattice data
- Method by Hansen and Bulava
- Method by Patella and Tantalo
- Conclusions and Outlook

What are scattering amplitudes?

Computing a transition probability

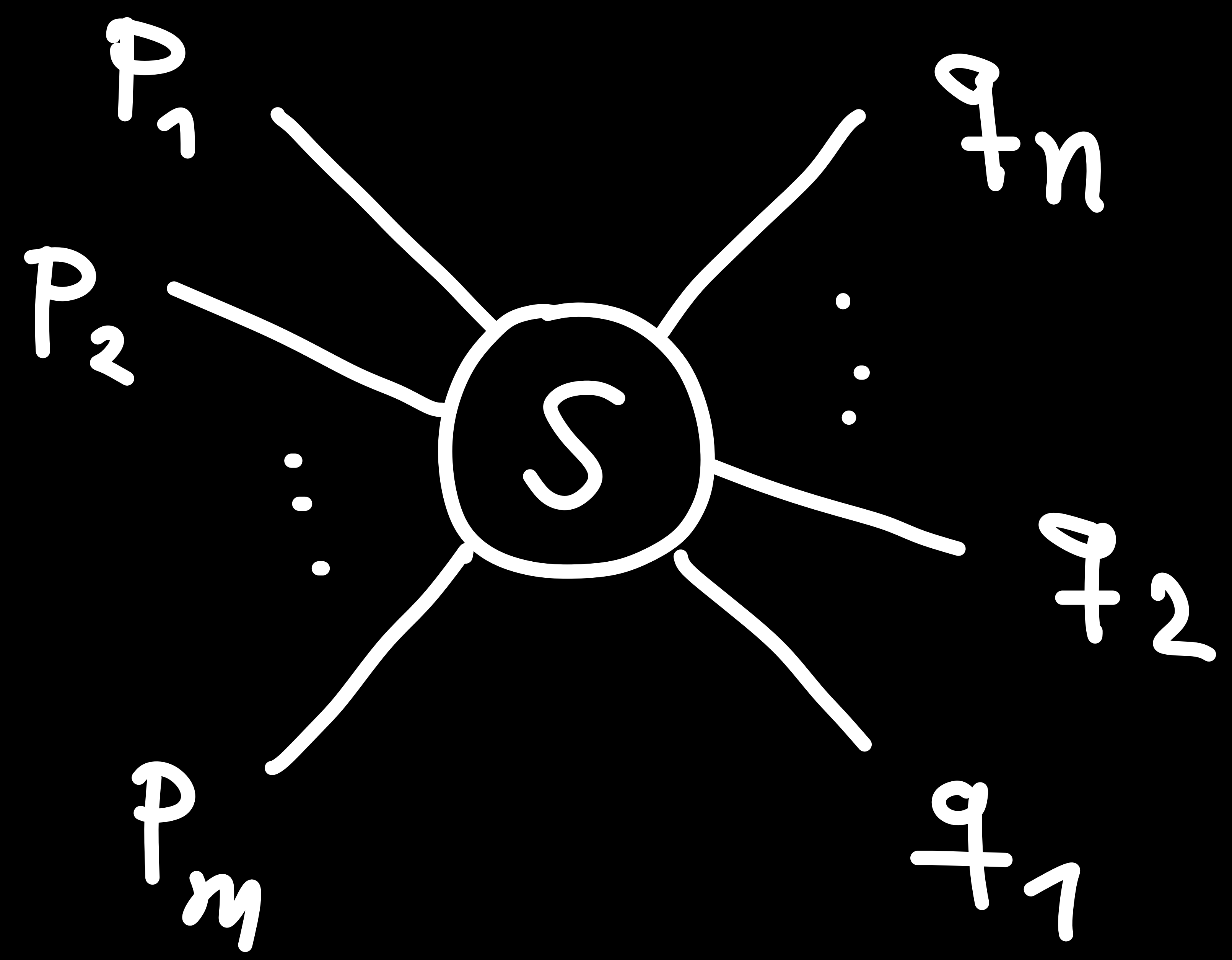


Computing a transition probability



$$\rightsquigarrow \quad P = \left| {}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in} \right|^2$$

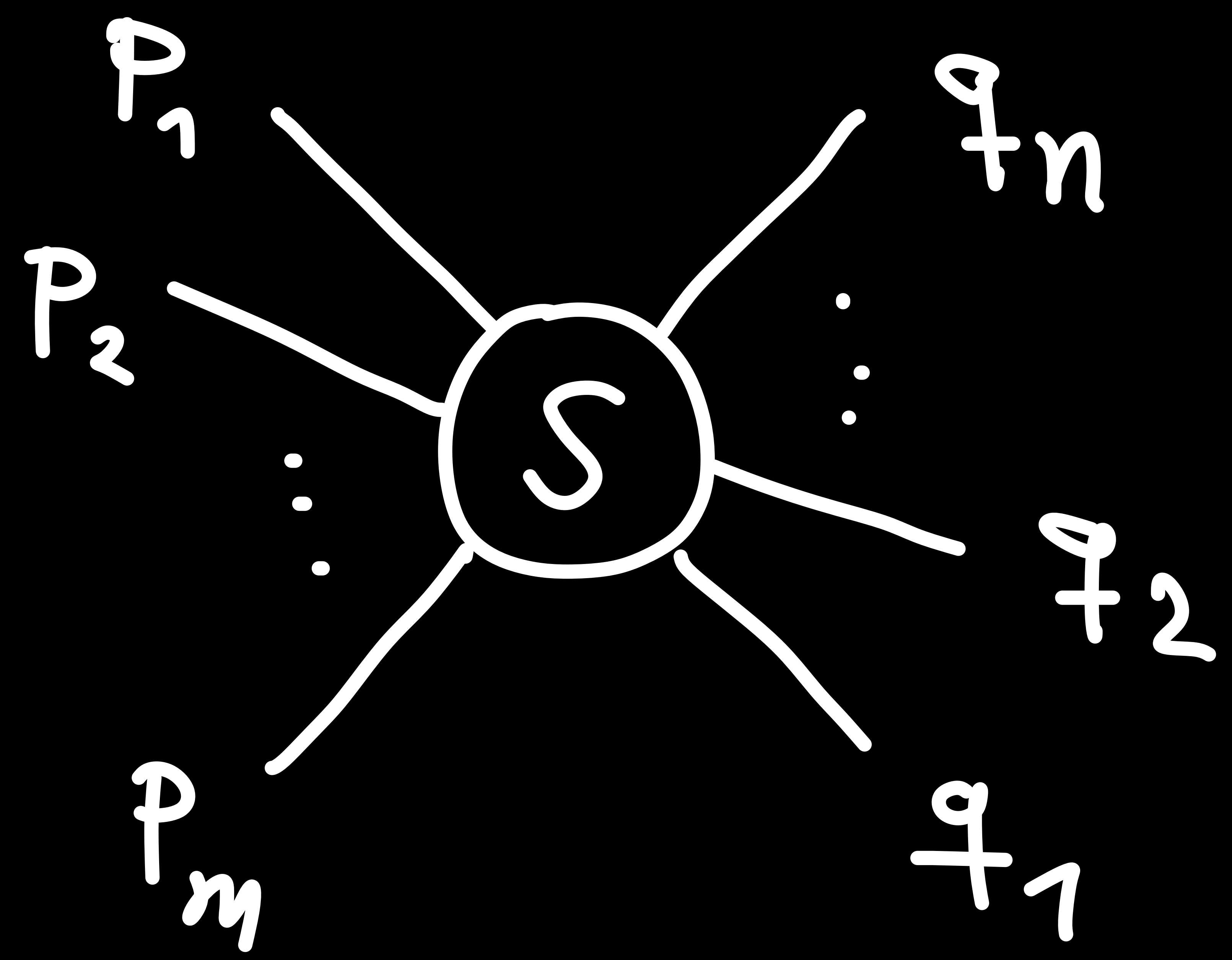
Computing a transition probability



$$\rightsquigarrow \quad P = |{}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in}|^2$$

With definite momenta p_f and q_i

Computing a transition probability



$$\rightsquigarrow \quad P = \left| {}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in} \right|^2$$

With definite momenta p_f and q_i

Physical theory encoded in S-Matrix

Cross sections conceptually

$$d\sigma(\cdots) = \int \left| {}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in} \right|^2 \times \text{KINEMATICS}$$

Cross sections conceptually

known $\qquad\qquad\qquad\qquad\qquad\qquad$ known

$$d\sigma(\ldots) = \int |_{out}\langle\{p_f\}|\{q_i\}\rangle_{in}|^2 \times \text{KINEMATICS}$$

Cross sections conceptually

Possibly non-perturbative physics

$$d\sigma(\cdots) = \int \left| {}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in} \right|^2 \times \text{KINEMATICS}$$

Perturbative approach LSZ reduction

Connect matrix element to time-ordered correlation functions

$$ {}_{out}\langle \{p_f\} | \{q_i\} \rangle_{in} \sim \langle 0 | T\{ \phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_m) \} | 0 \rangle $$

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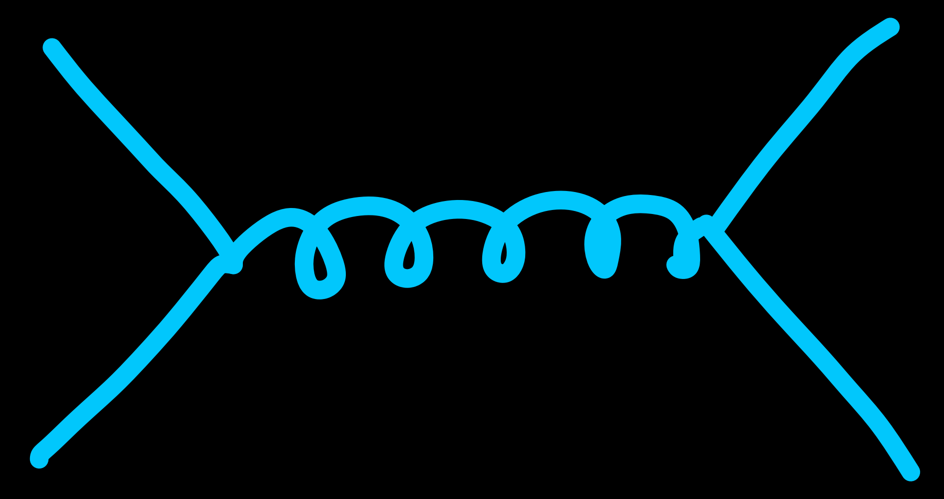
$\sim \sum$ Feynman Diagrams

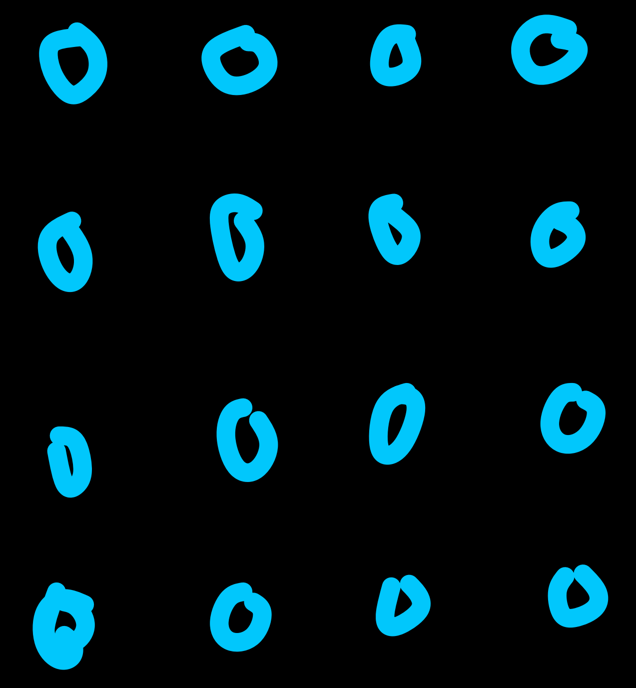
Perturbation theory vs the Lattice



What we compute in perturbation theory

$$\langle 0 | T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\} | 0 \rangle$$

Time-ordered correlation functions



What we compute on the lattice

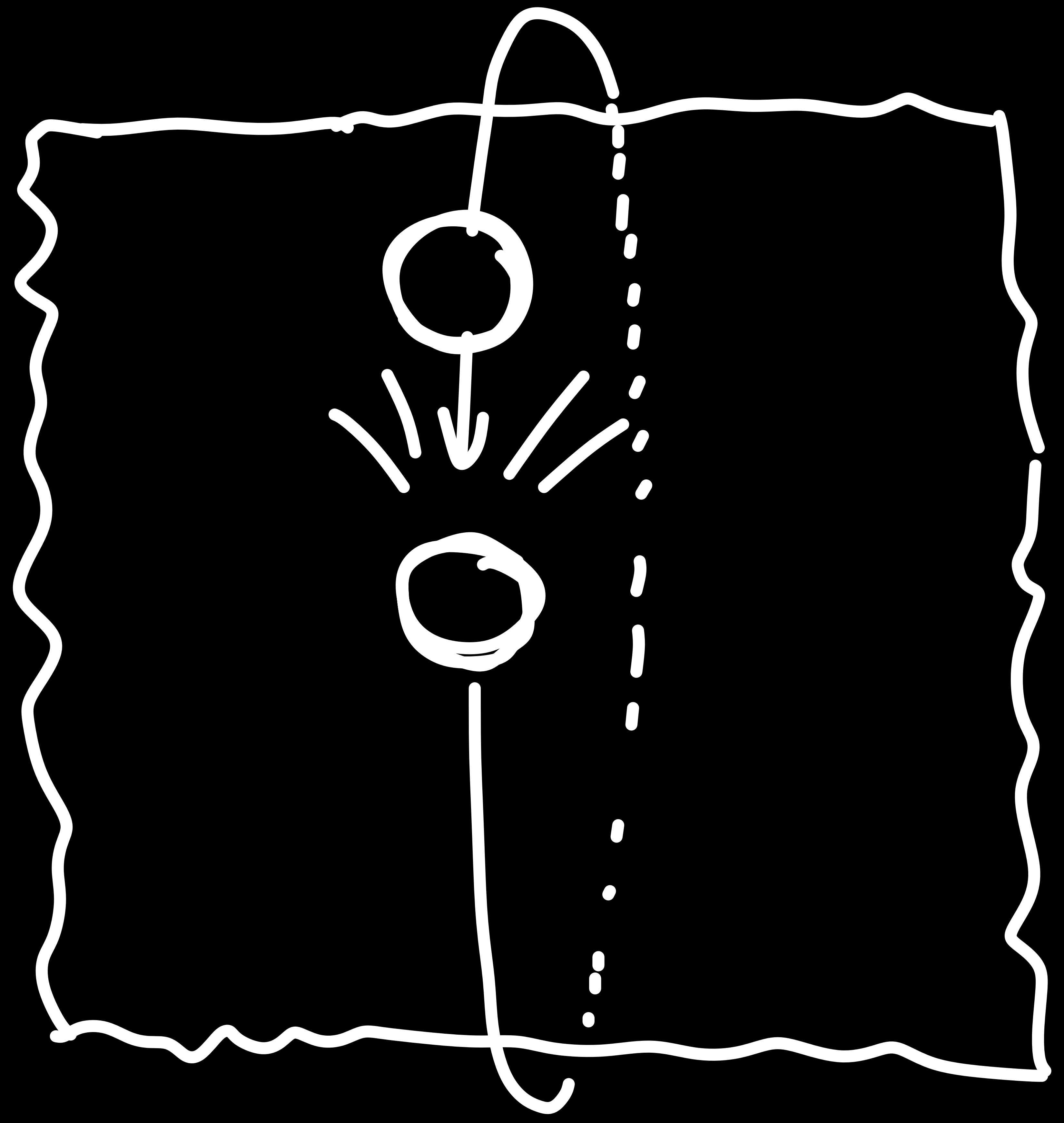
$$\langle 0 | \phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m) | 0 \rangle$$

No time ordering

Conversion: Lüscher Method

[Lüscher, 1986]

- Due to the finite volume any particle interacts with itself over the boundary



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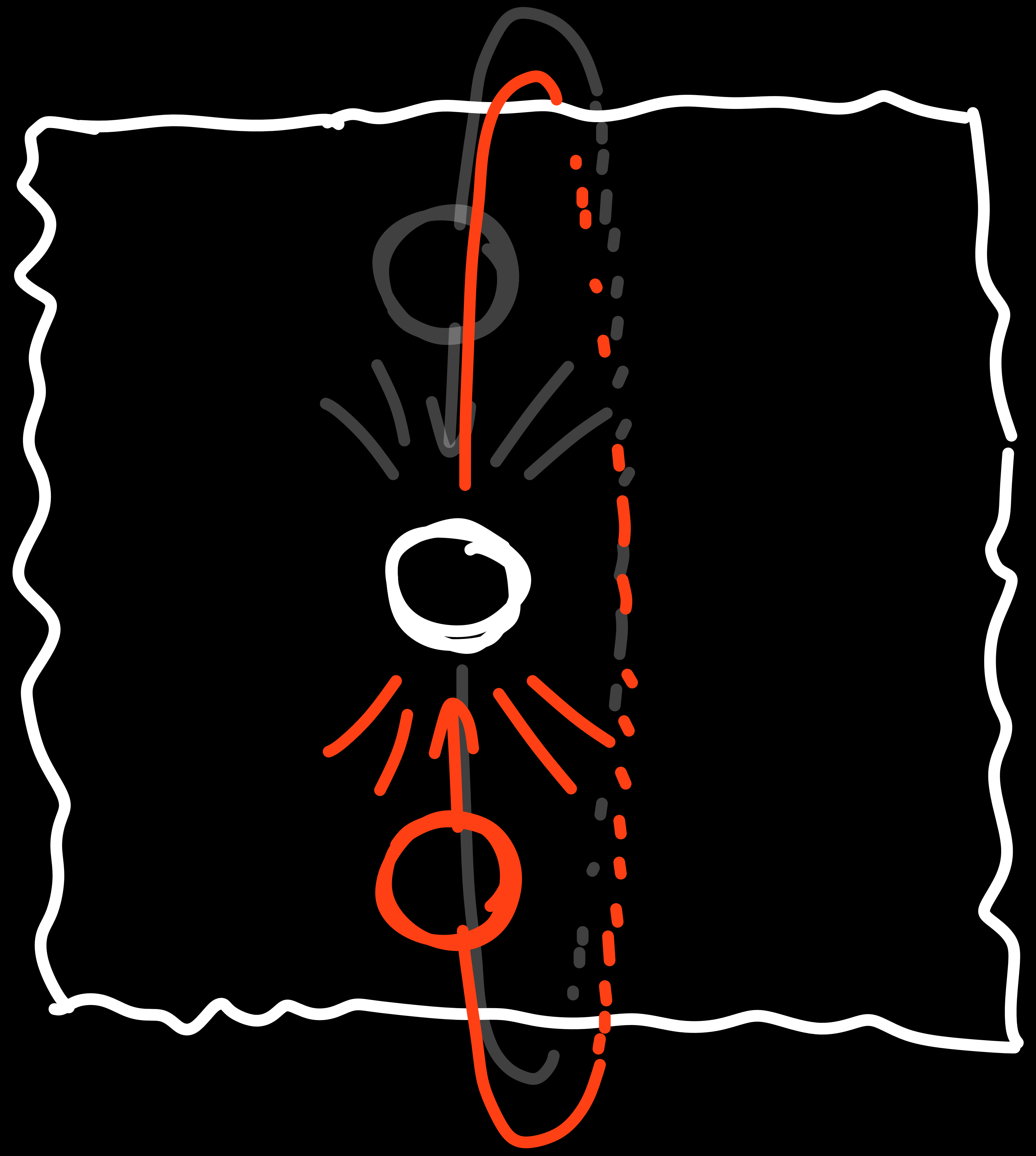

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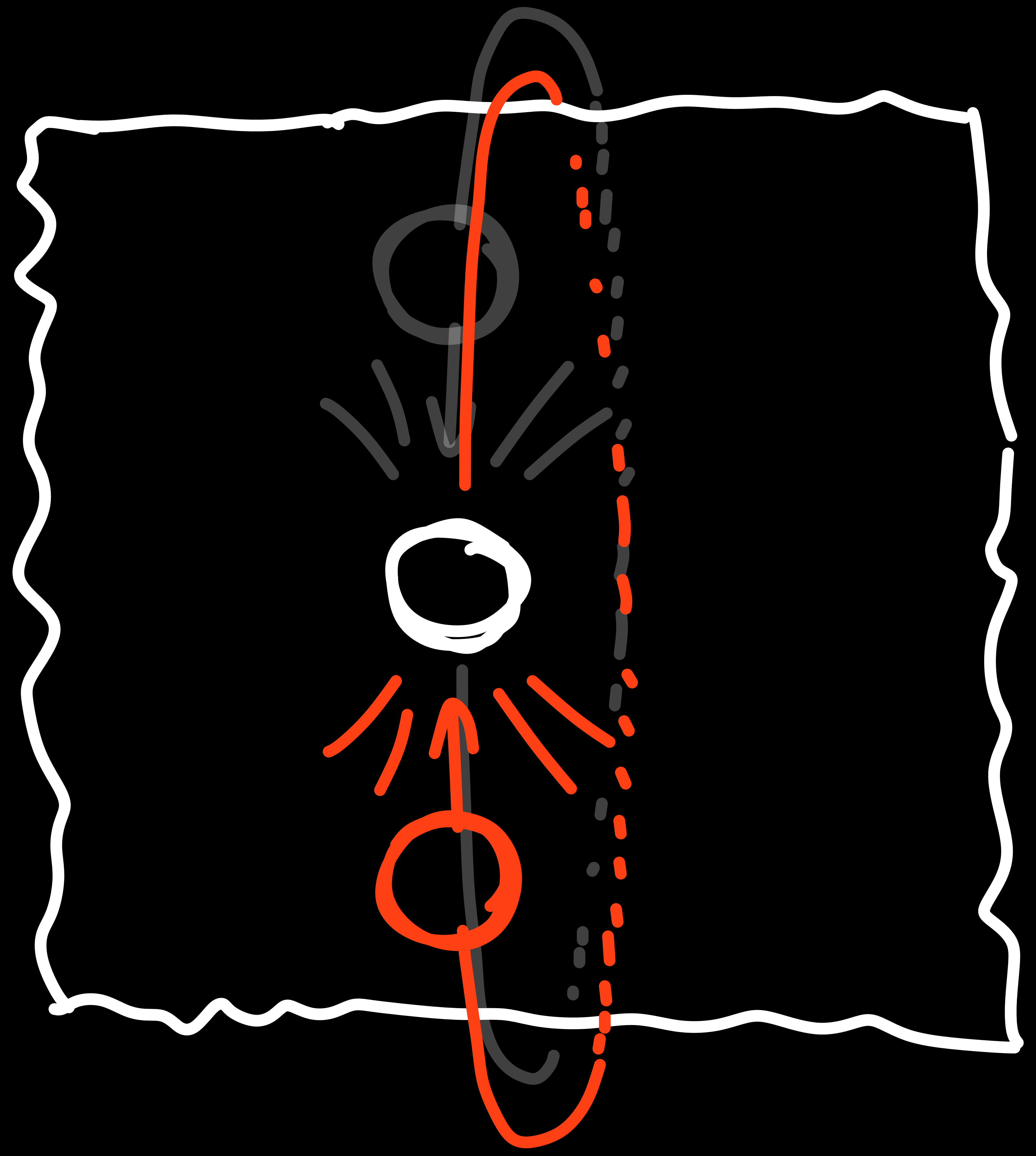
[Lüscher, 1986]

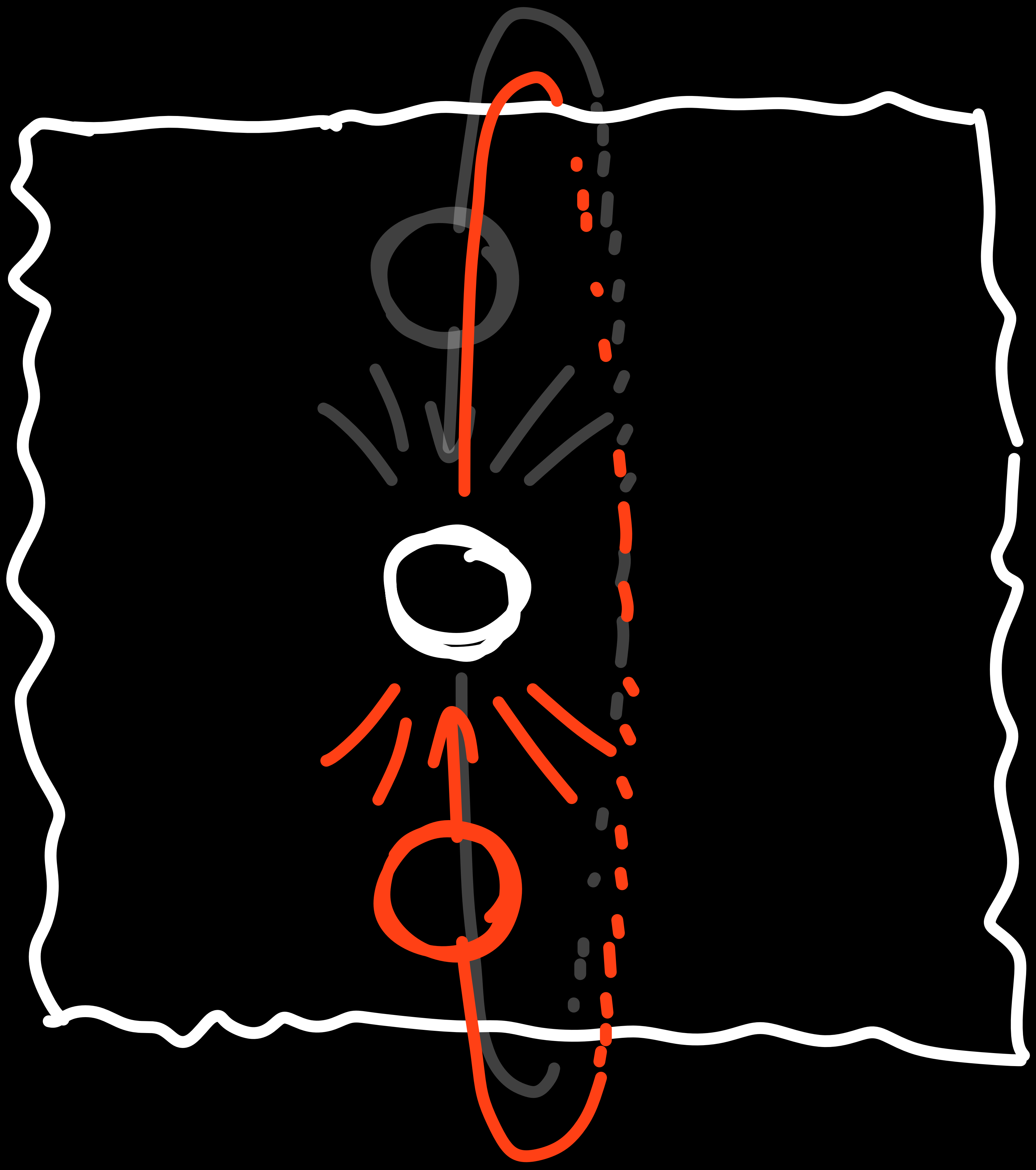
- Due to the finite volume any particle interacts with itself over the boundary
- This interaction is forward and back in time
- Cancellation reconstructs time ordering



Conversion: Lüscher Method

[Lüscher, 1986]

- Due to the finite volume any particle interacts with itself over the boundary
- This interaction is forward and back in time
- Cancellation reconstructs time ordering
- Relation between finite volume energy levels evaluated in Euclidean space to infinite-volume Minkowski space Scattering Amplitudes



A comment

Non-perturbative physics

(Matrix element)

Perturbation Theory

(Time-ordered correlation functions)

Non-perturbative physics

(Correlation function, spectroscopy)

Rethinking the lattice extraction method

Non-perturbative physics

(Matrix element)

→

~~Perturbation Theory~~

~~(Time-ordered correlation functions)~~

→

Non-perturbative physics

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Rethinking the lattice extraction method

Non-perturbative physics

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~~Perturbation Theory~~

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Non-perturbative physics

(Correlation function, spectroscopy)

Use spectral densities instead of correlation functions

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\phi(y)|0\rangle$$

Apply LSZ again

$$\langle 0 | \phi^\dagger(x) \underset{\mathbb{1}}{\wedge} \phi(y) | 0 \rangle$$

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\,\underset{\mathbb{1}}{\wedge}\,\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\,\mathbb{1}\,\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\, \rho(p^2)$$

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\underset{\mathbb{1}}{\wedge}\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

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$\Downarrow$

Extract this from the lattice

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\,\underset{\mathbb{1}}{\wedge}\,\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \, \rho(p^2)$$

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

$\mathbb{1}$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \rho(p^2)$$

Valid for general energies

Apply LSZ again

$$\langle 0|\phi^\dagger(x)\underset{\mathbb{1}}{\hat{}}\phi(y)|0\rangle = \sum_n \langle 0|\phi^\dagger(x)|n\rangle\langle n|\phi(y)|0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \rho(p^2)$$

😀 Valid for general energies

☹️ LSZ requires a mass gap, so not applicable for QCD+QED

Our testing theory

Testing scattering for weak physics

QCD

Strong coupling

Testing scattering for weak physics

QCD

Strong coupling

- Fine lattice spacing required

Testing scattering for weak physics

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- Line of constant physics interpolation necessary

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In practice: Master Field Simulations

Testing scattering for weak physics

QCD

Electroweak

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In practice: Master Field Simulations

[Francis et. al., 2020]

Weak coupling

Testing scattering for weak physics

QCD

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Weak coupling

- No continuum extrapolation

In practice: Master Field Simulations

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In practice: Master Field Simulations

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Testing scattering for weak physics

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- Easily controllable FV effects

In practice: Master Field Simulations

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Testing scattering for weak physics

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In practice: Master Field Simulations

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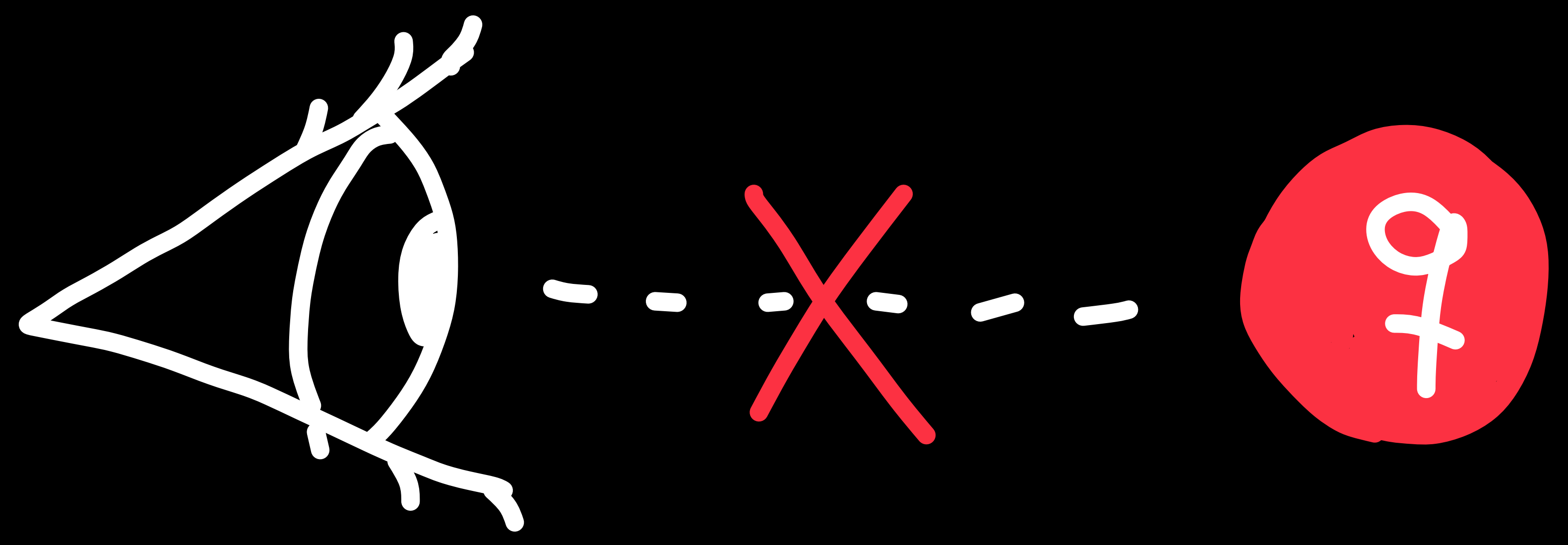
↳ How to simulate massive gauge bosons?
Spontaneous symmetry breaking?

FMS Framework

[Maas, 2019] & references therein

1. Observers are color blind



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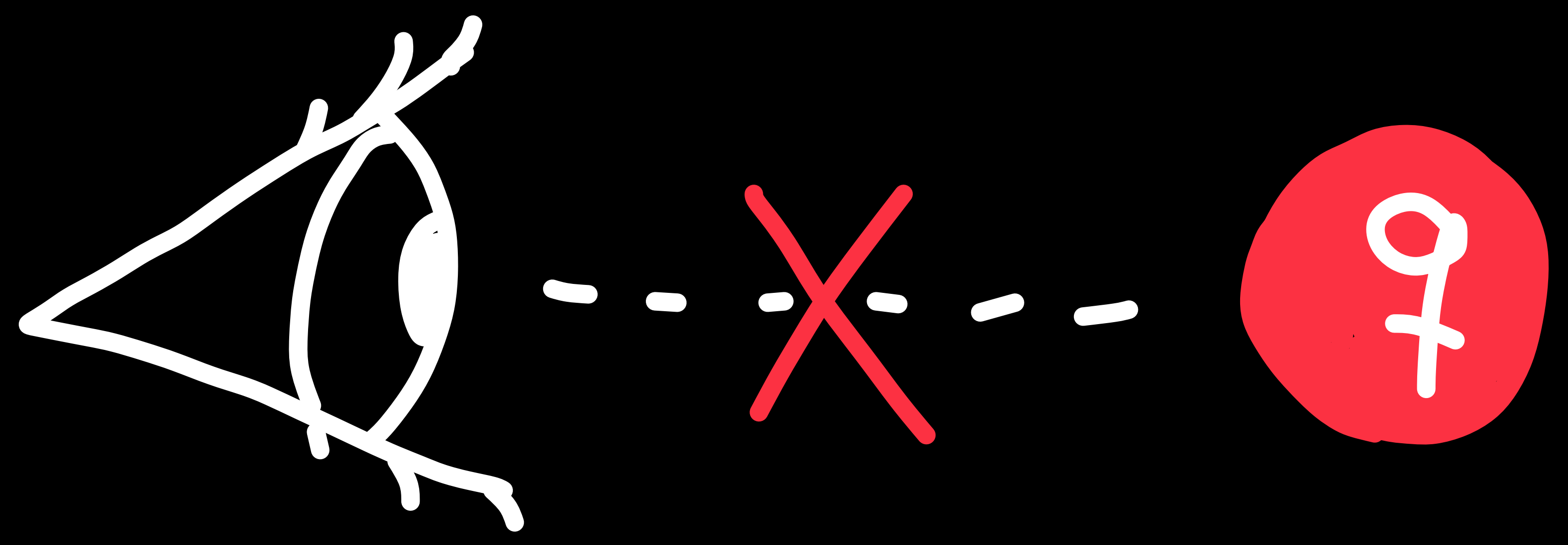

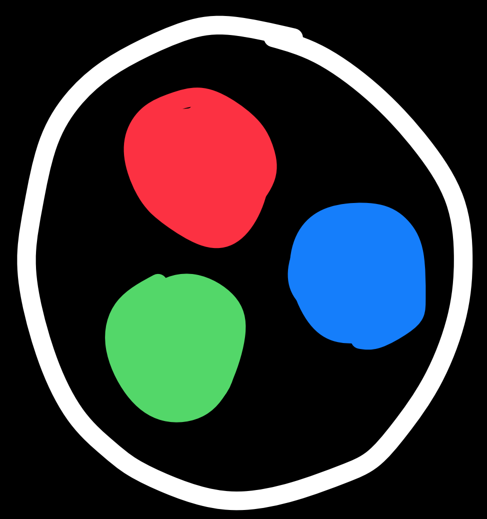

2. Physical objects are color singlets

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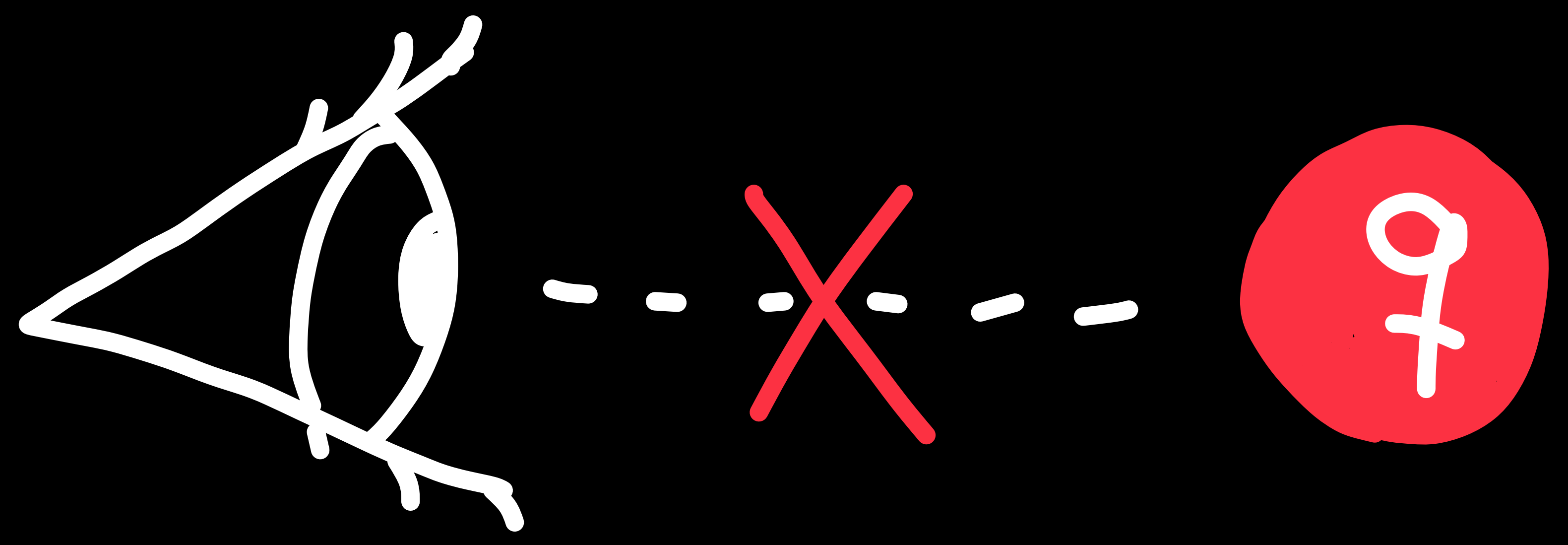

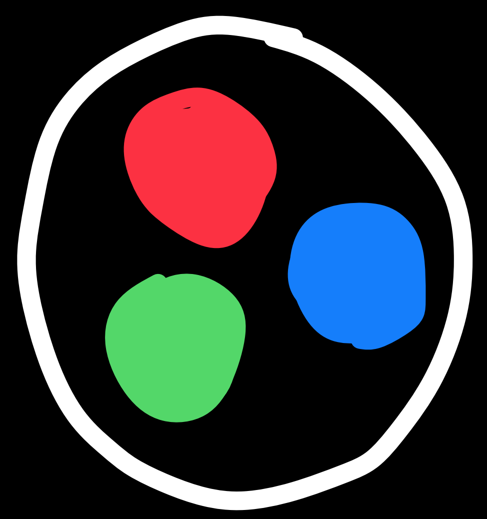

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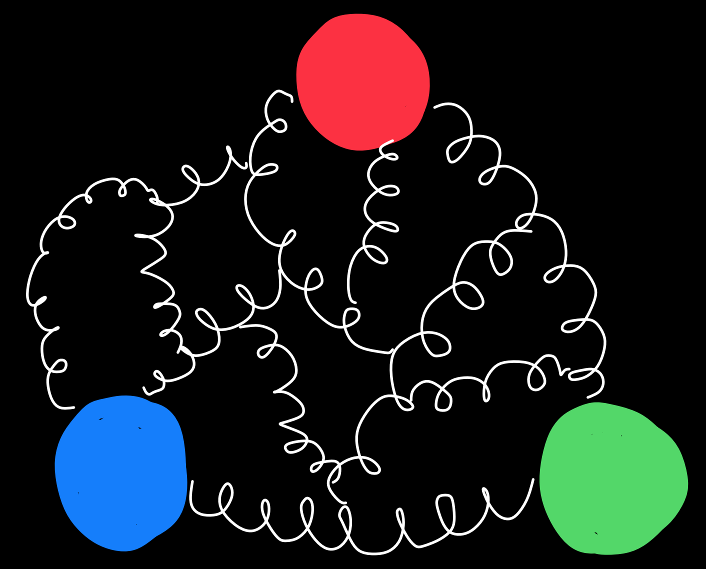

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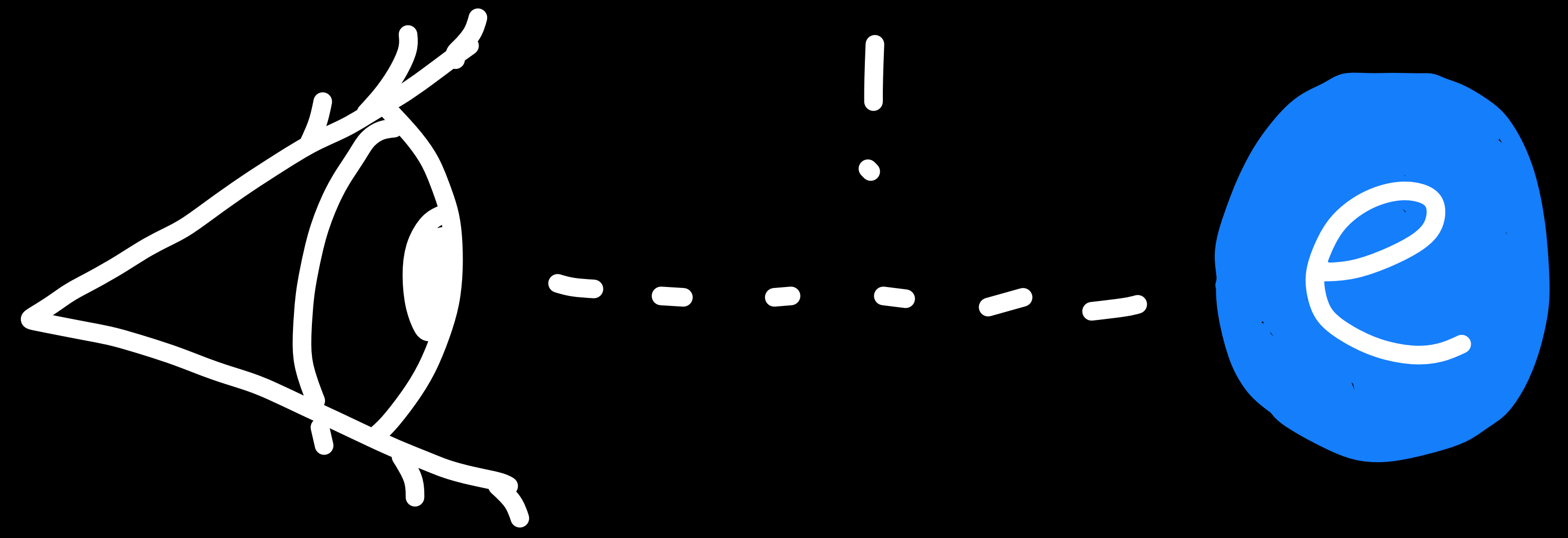

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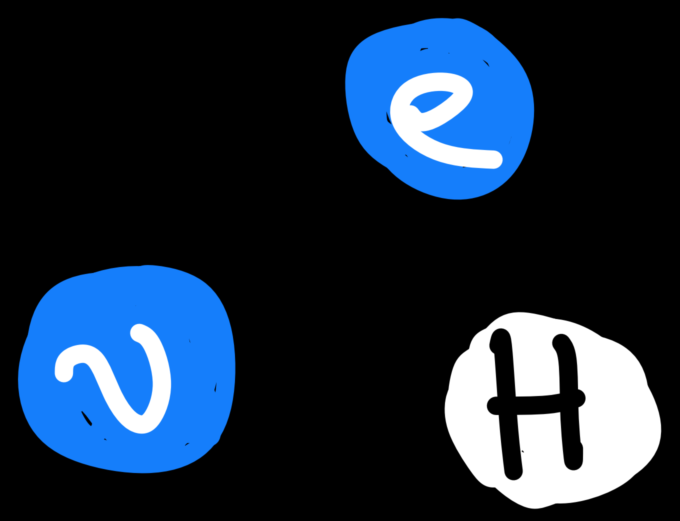

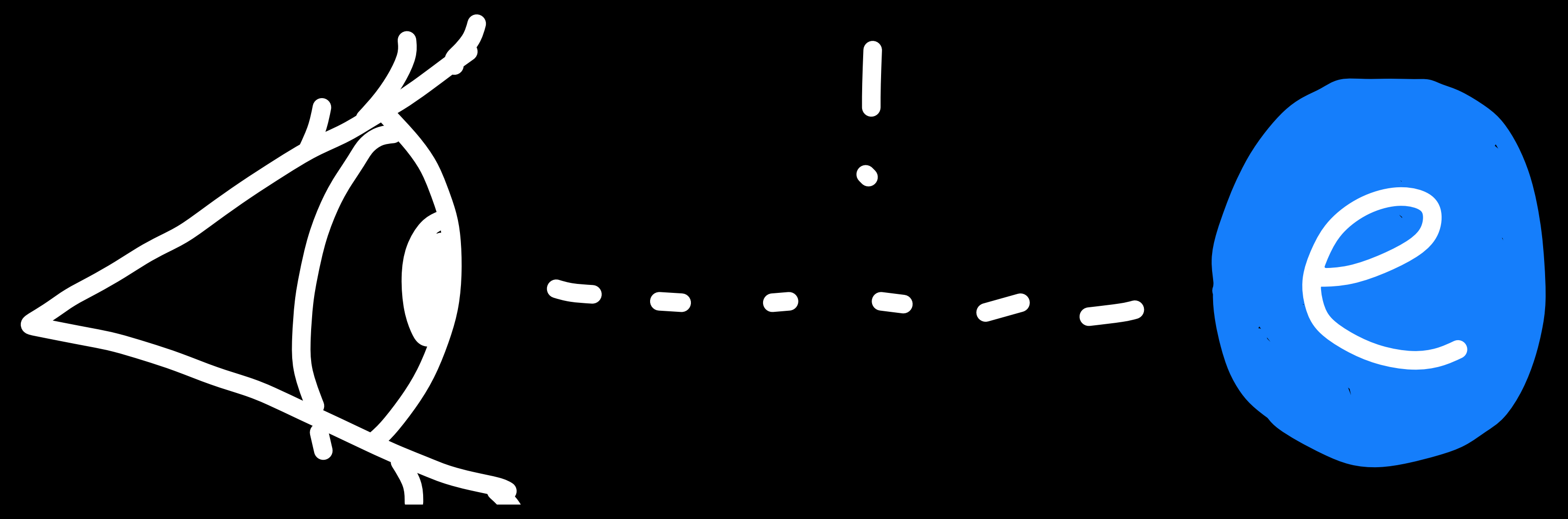

2. Leptons and neutrinos are individually observable

FMS Framework

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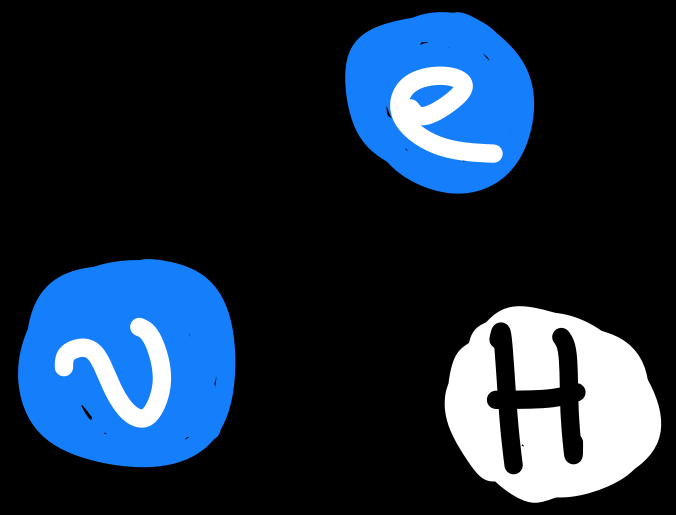

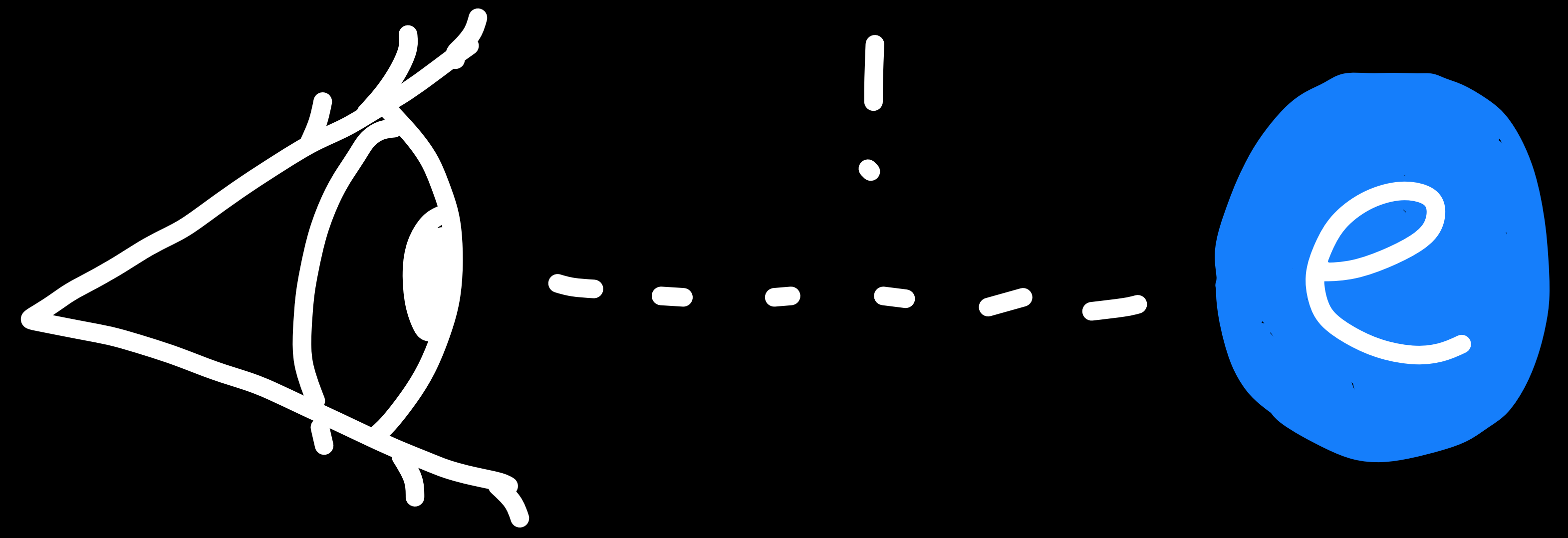

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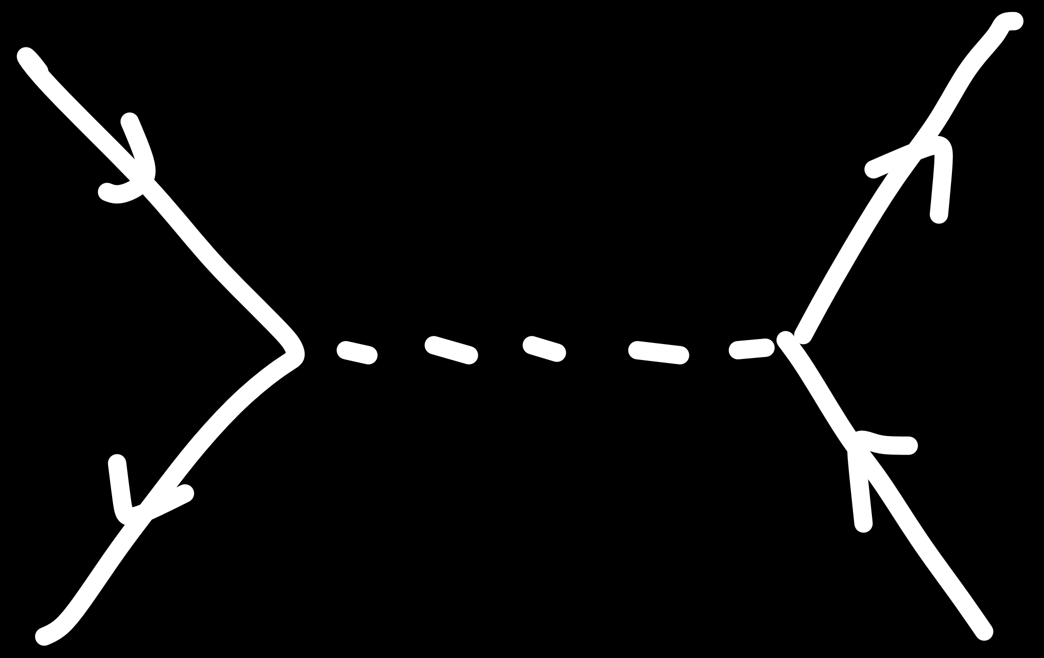
3. Perturbative treatment possible using the BRST construction



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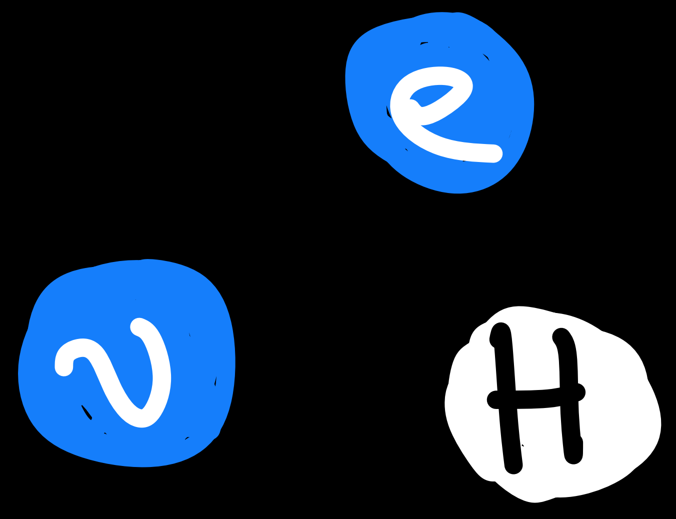

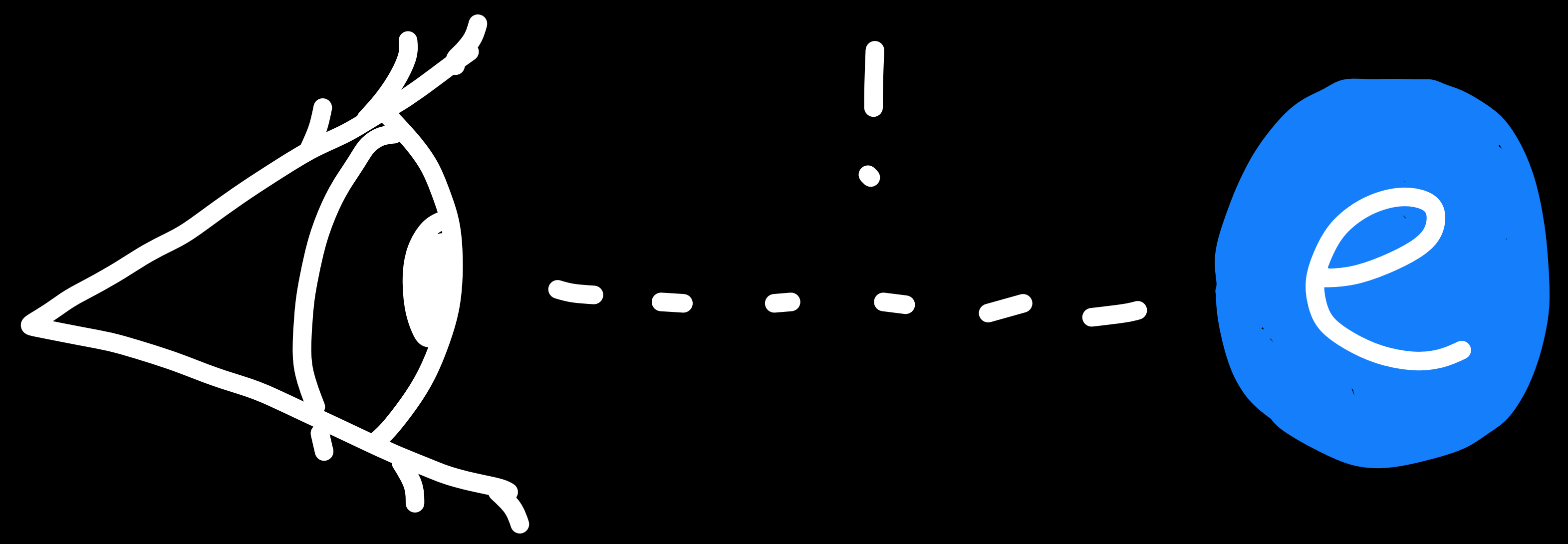

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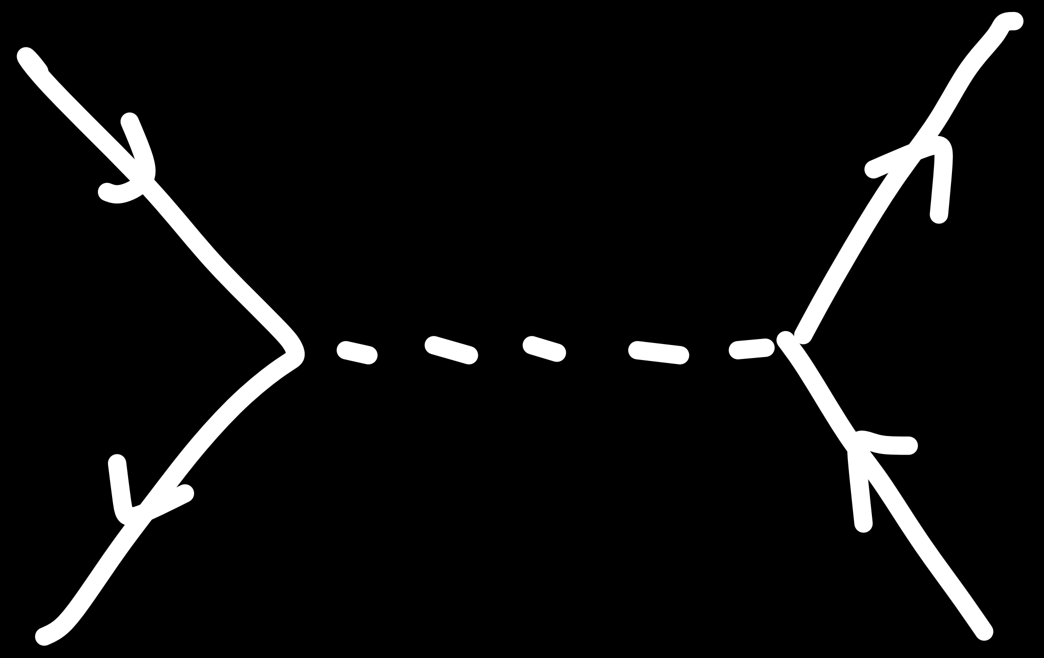

Elitzur's theorem: Gauge symmetries cannot be spontaneously broken!

FMS Framework

[Maas, 2019] & references therein

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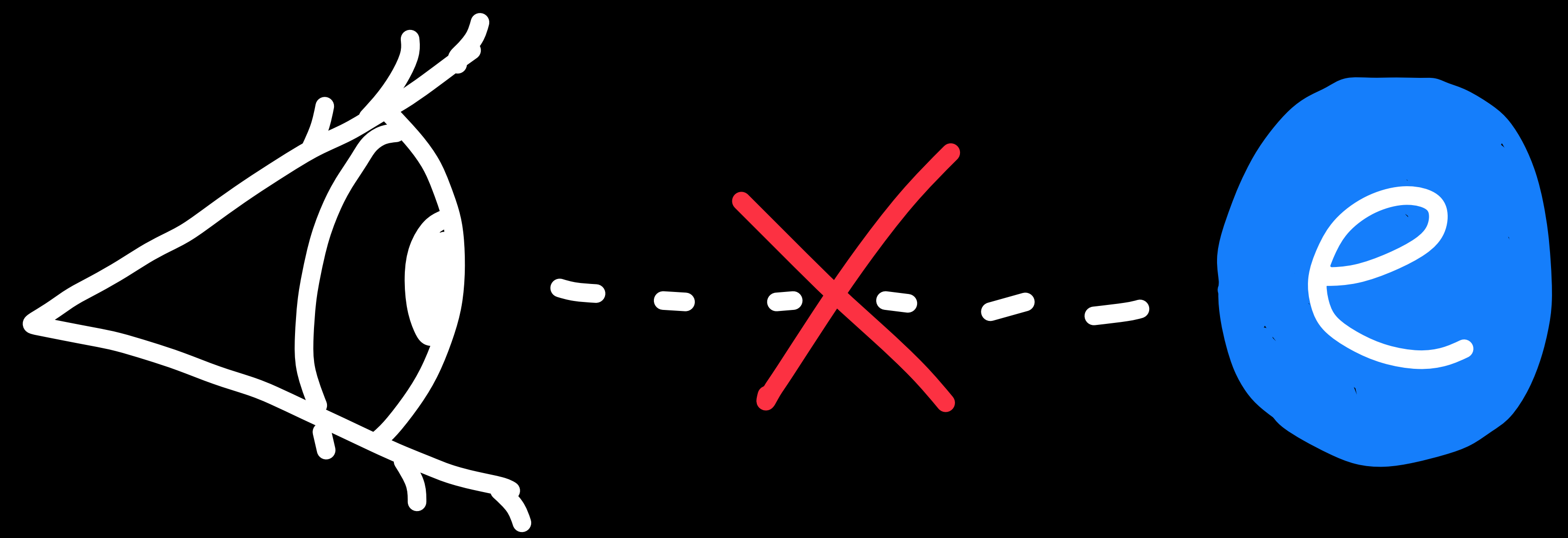

FMS Framework

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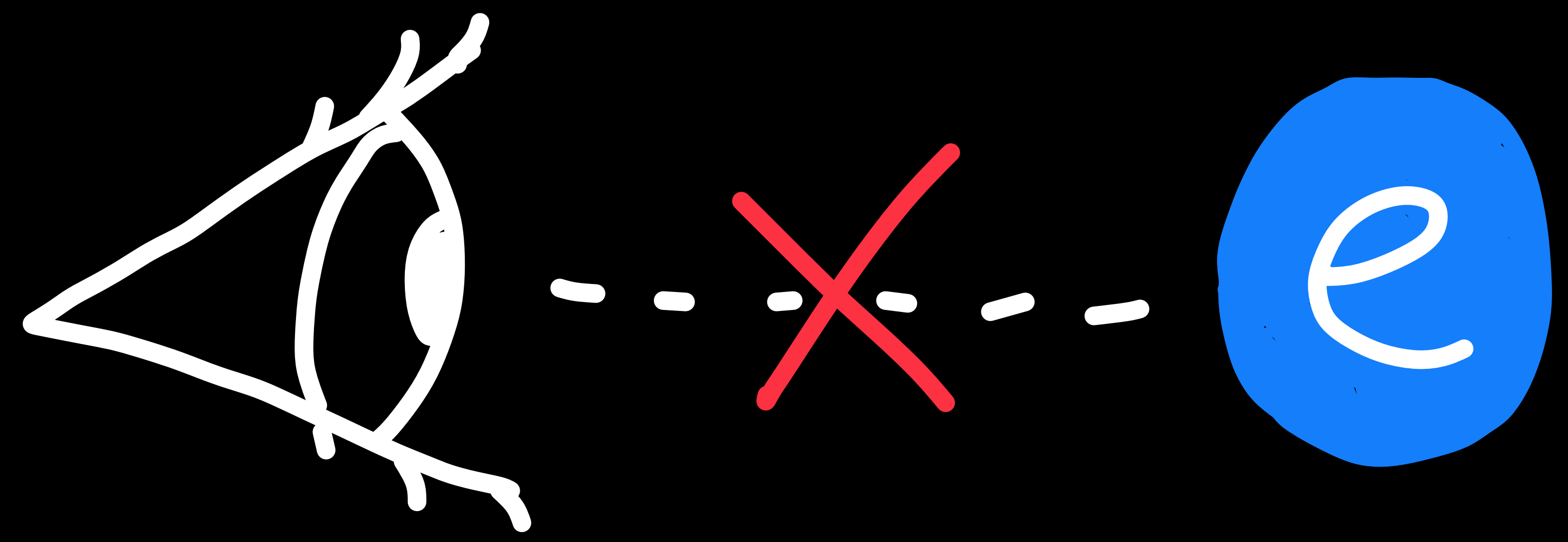

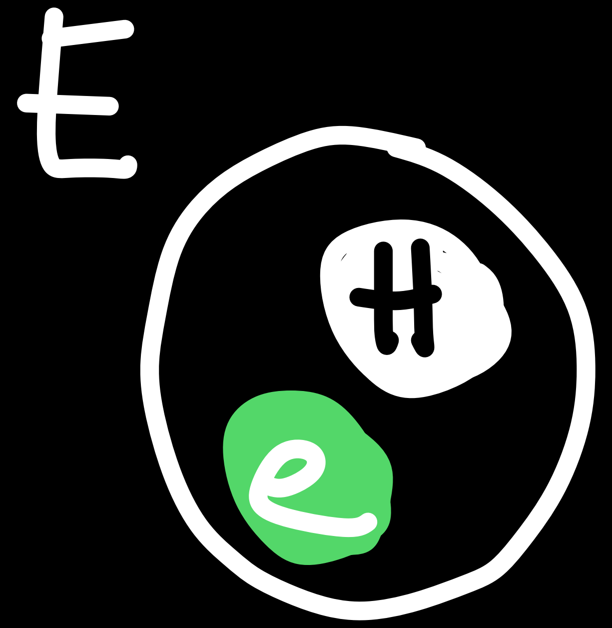

2. Physical fermions are composite

FMS Framework

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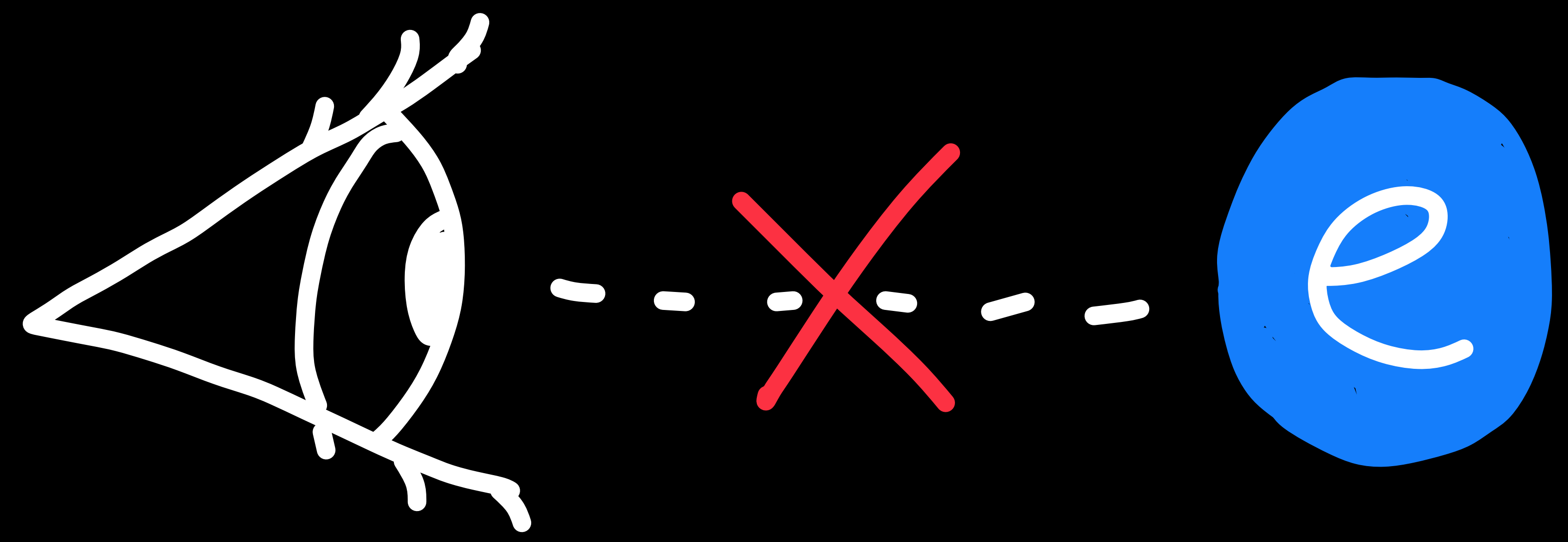

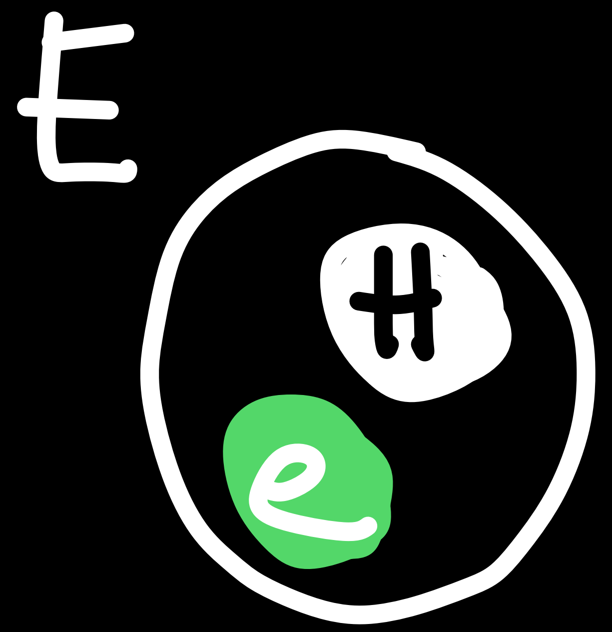

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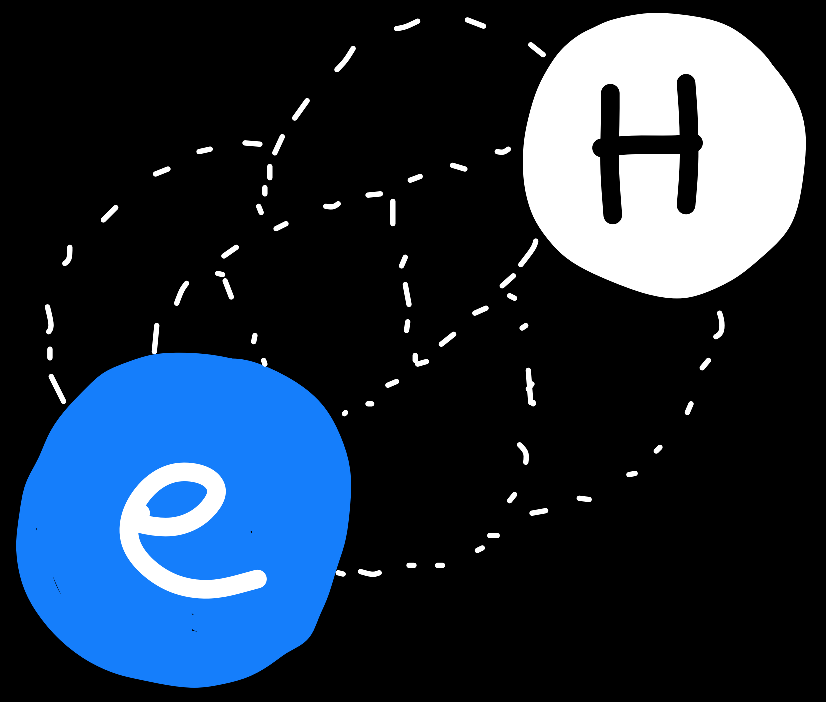
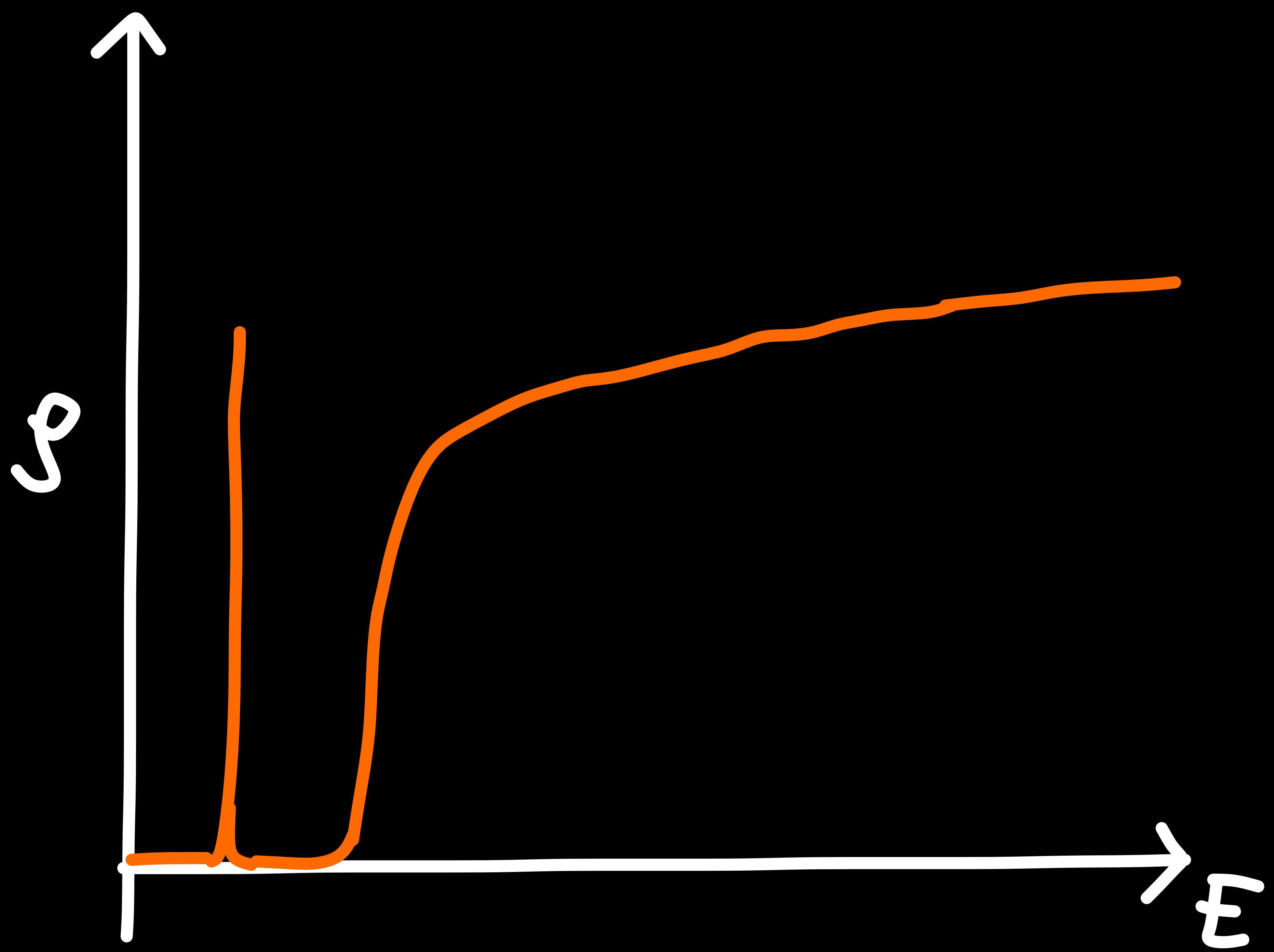


Spectral Densities from the Lattice

Backus Gilbert Approach

[Backus, Gilbert, 1968 & 1970]

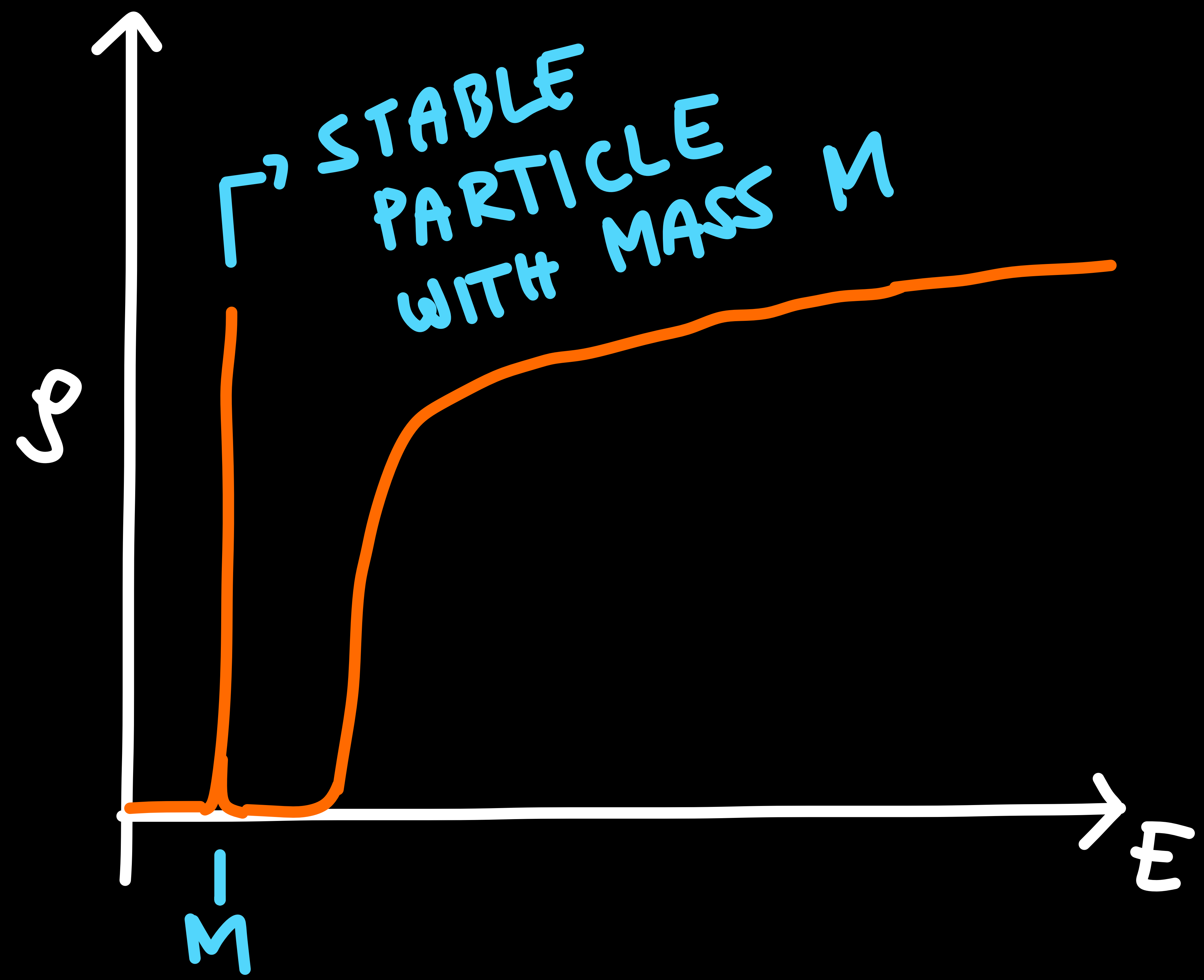


Continuum

Lattice / Finite Volume

Backus Gilbert Approach

[Backus, Gilbert, 1968 & 1970]



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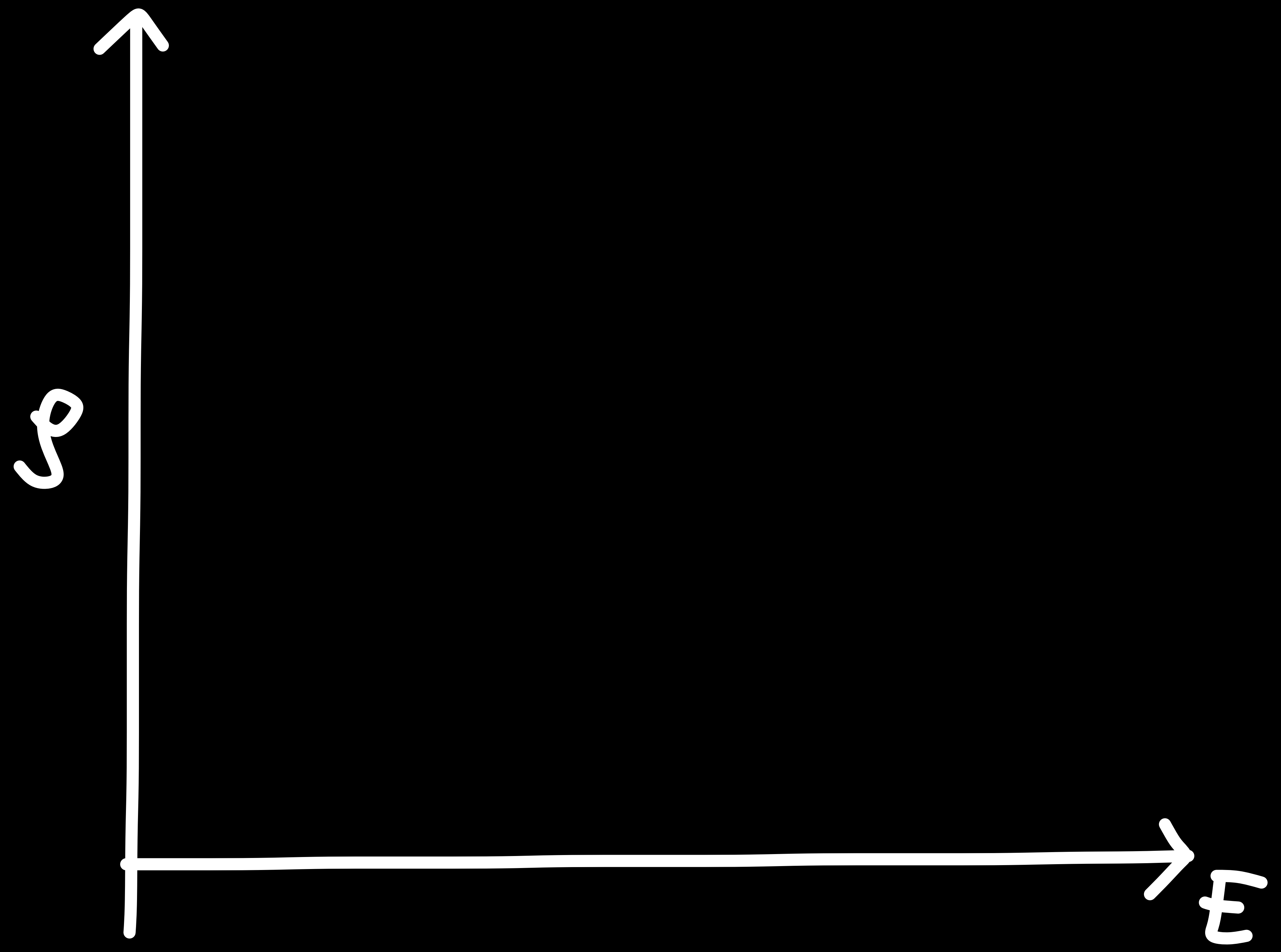

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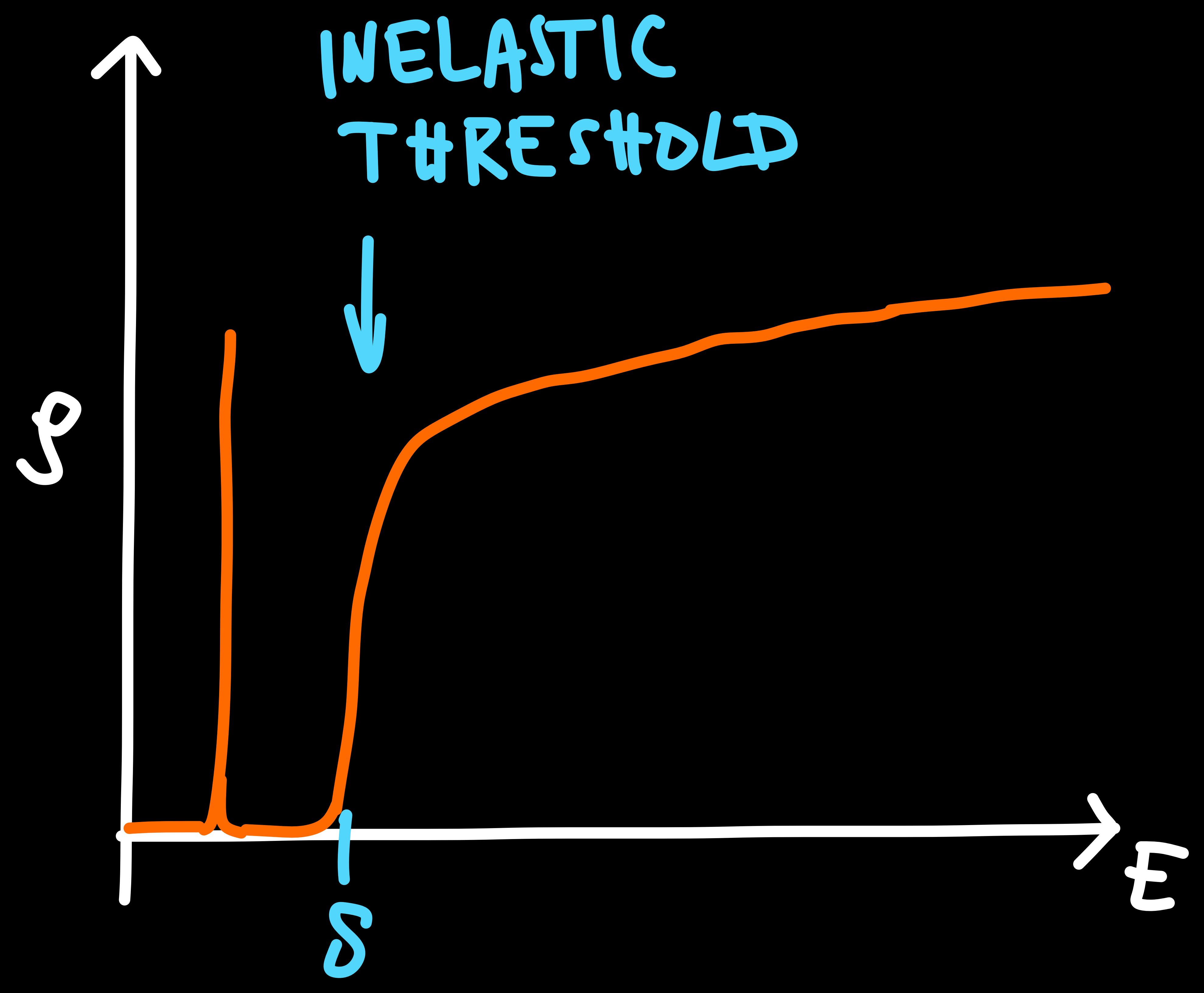

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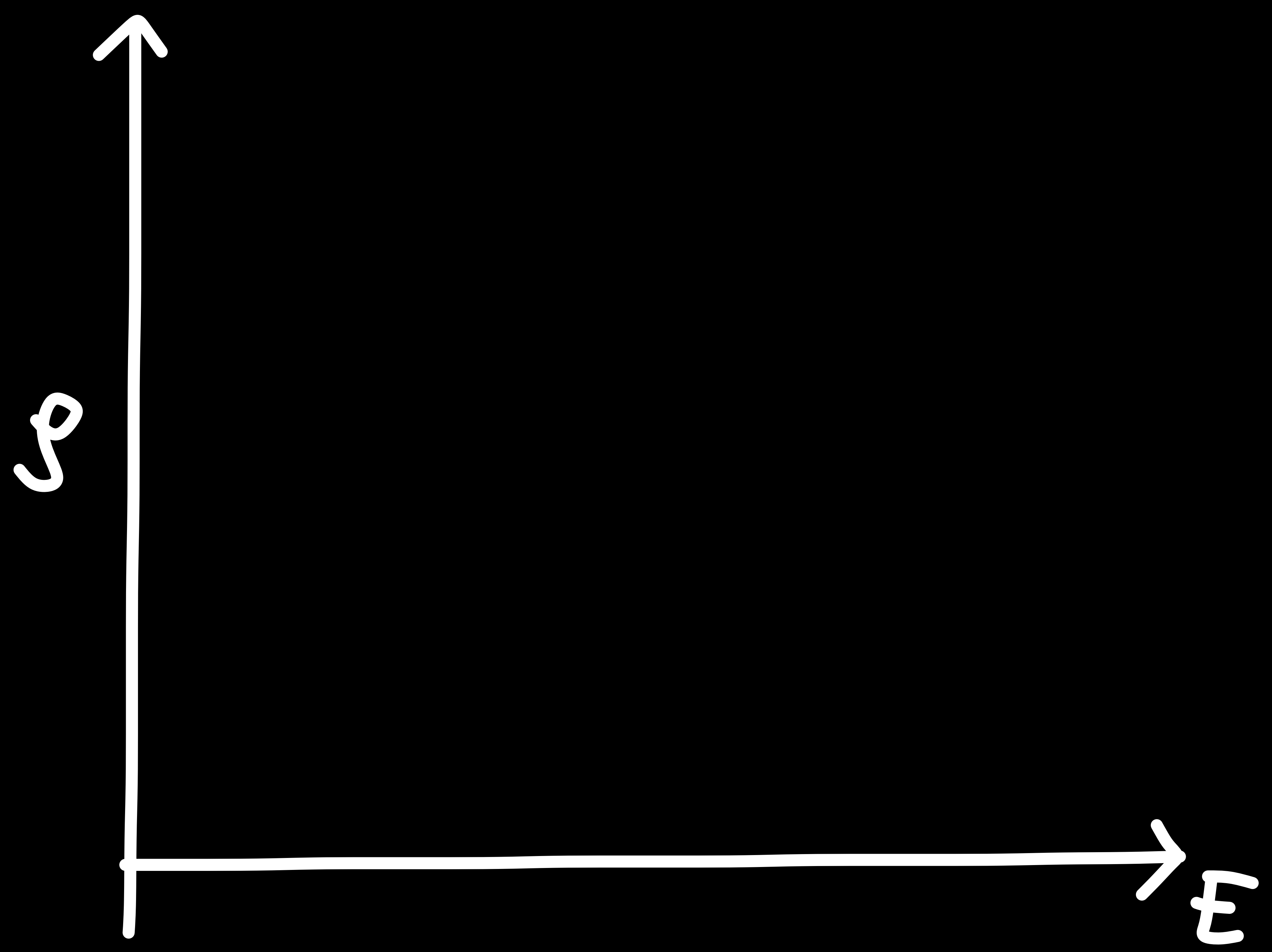

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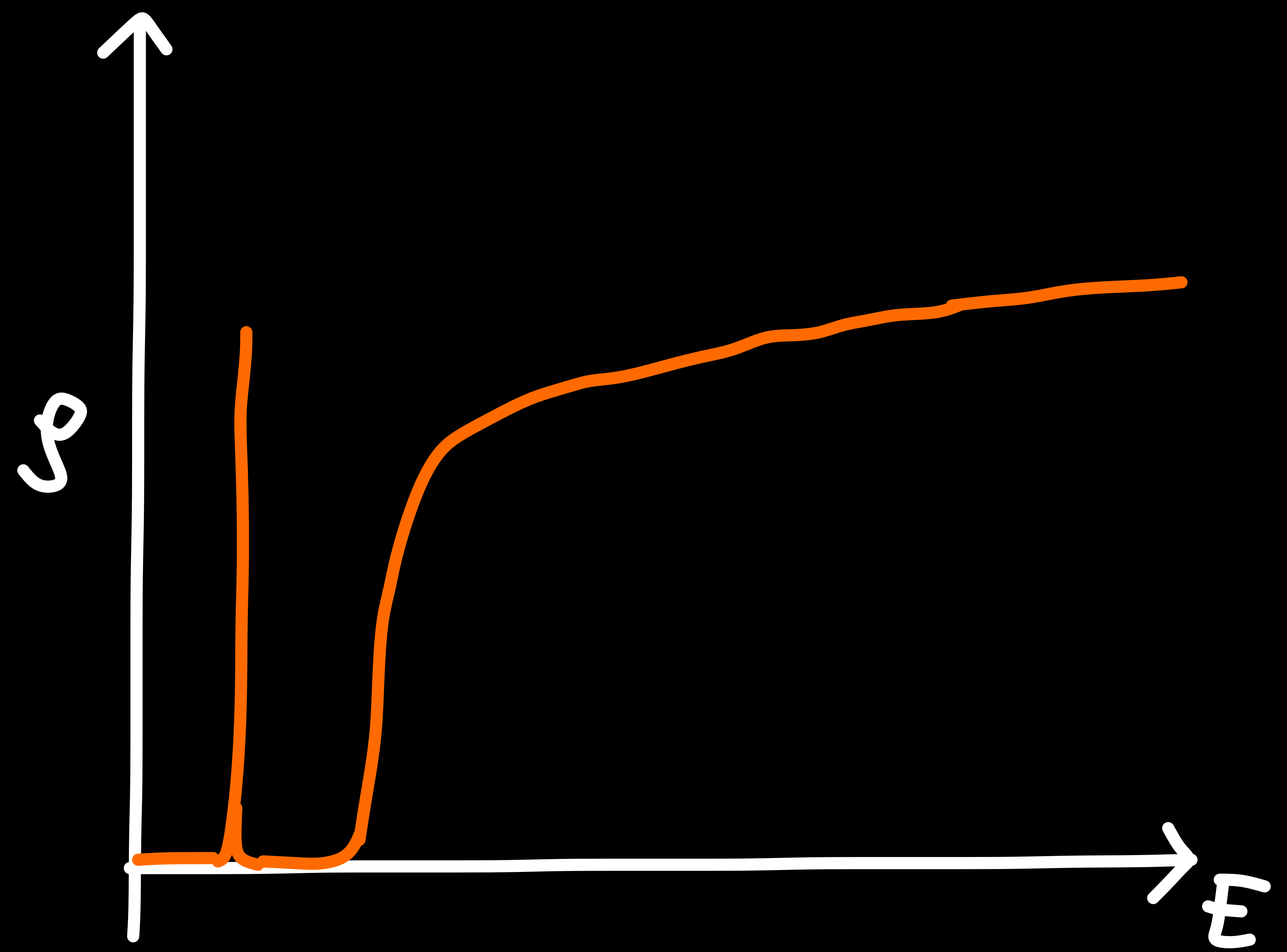

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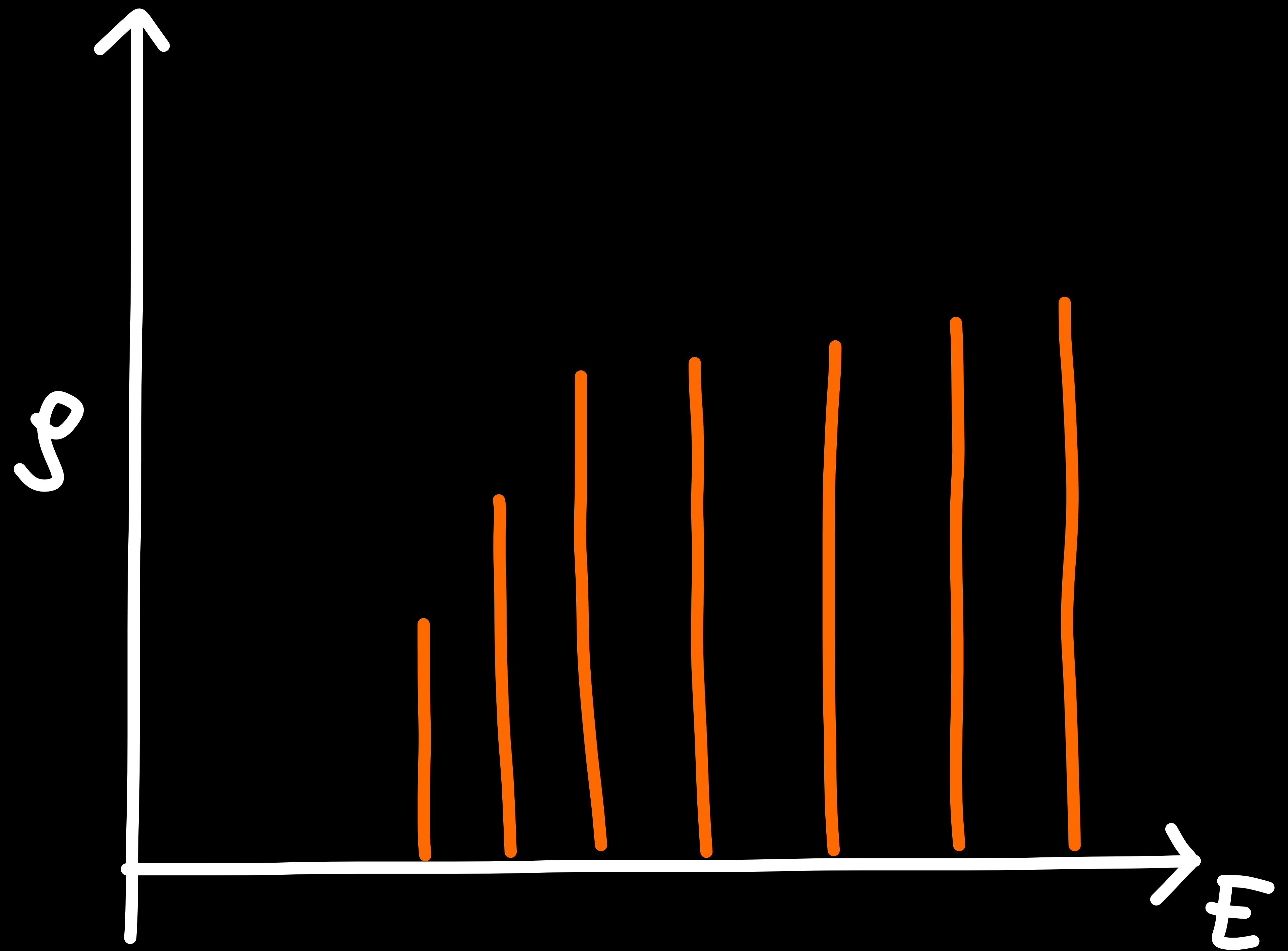

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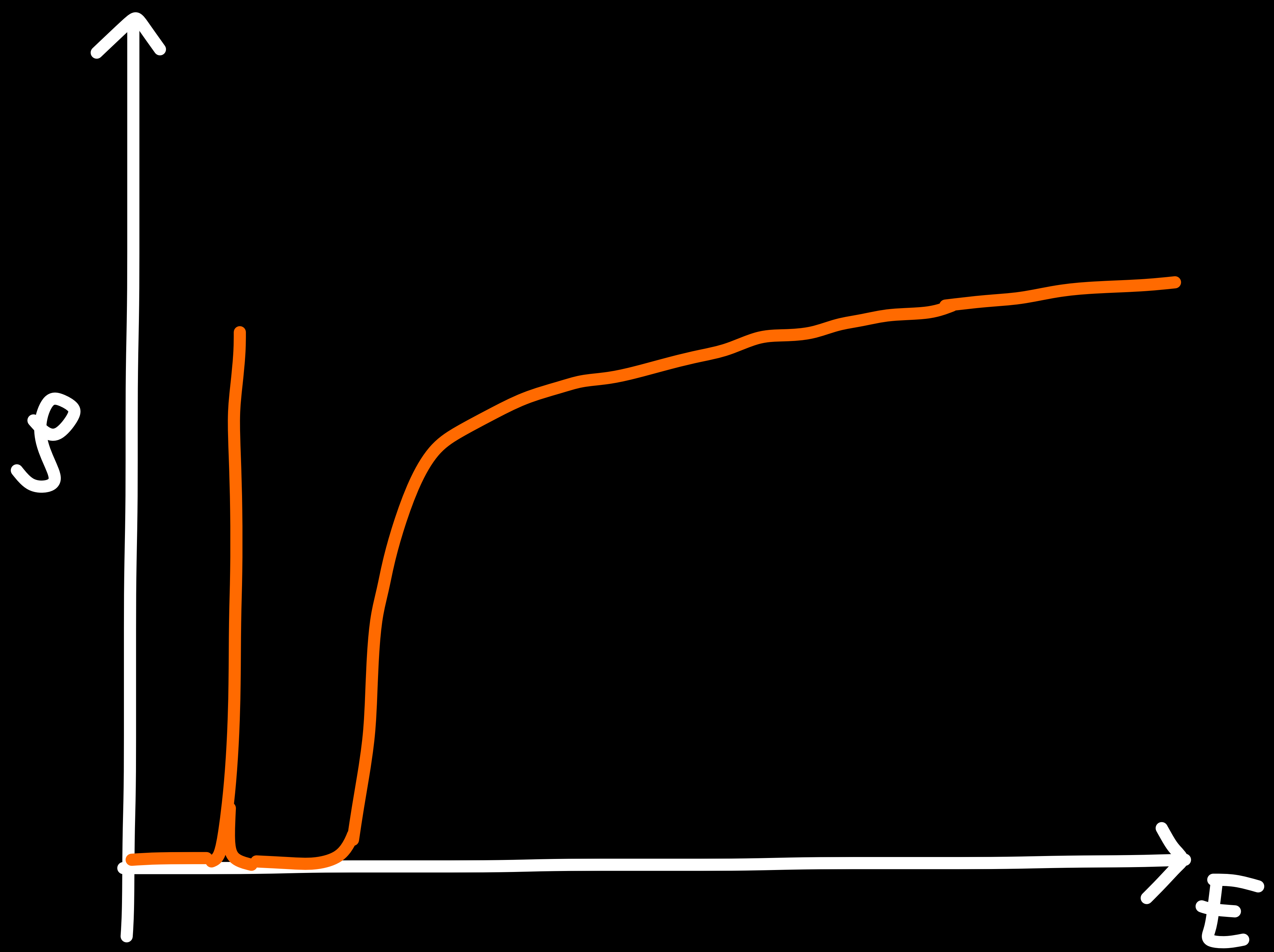

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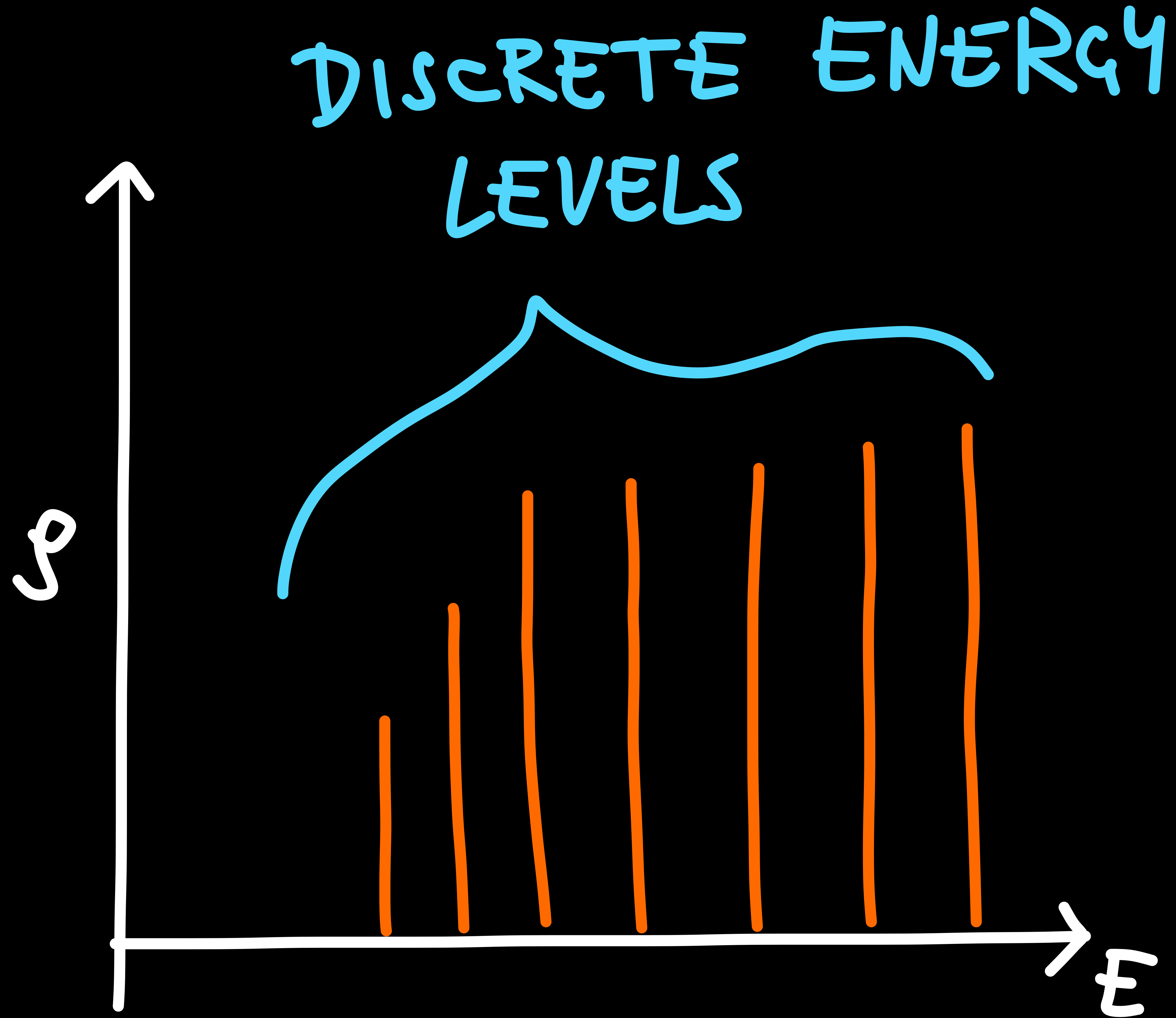
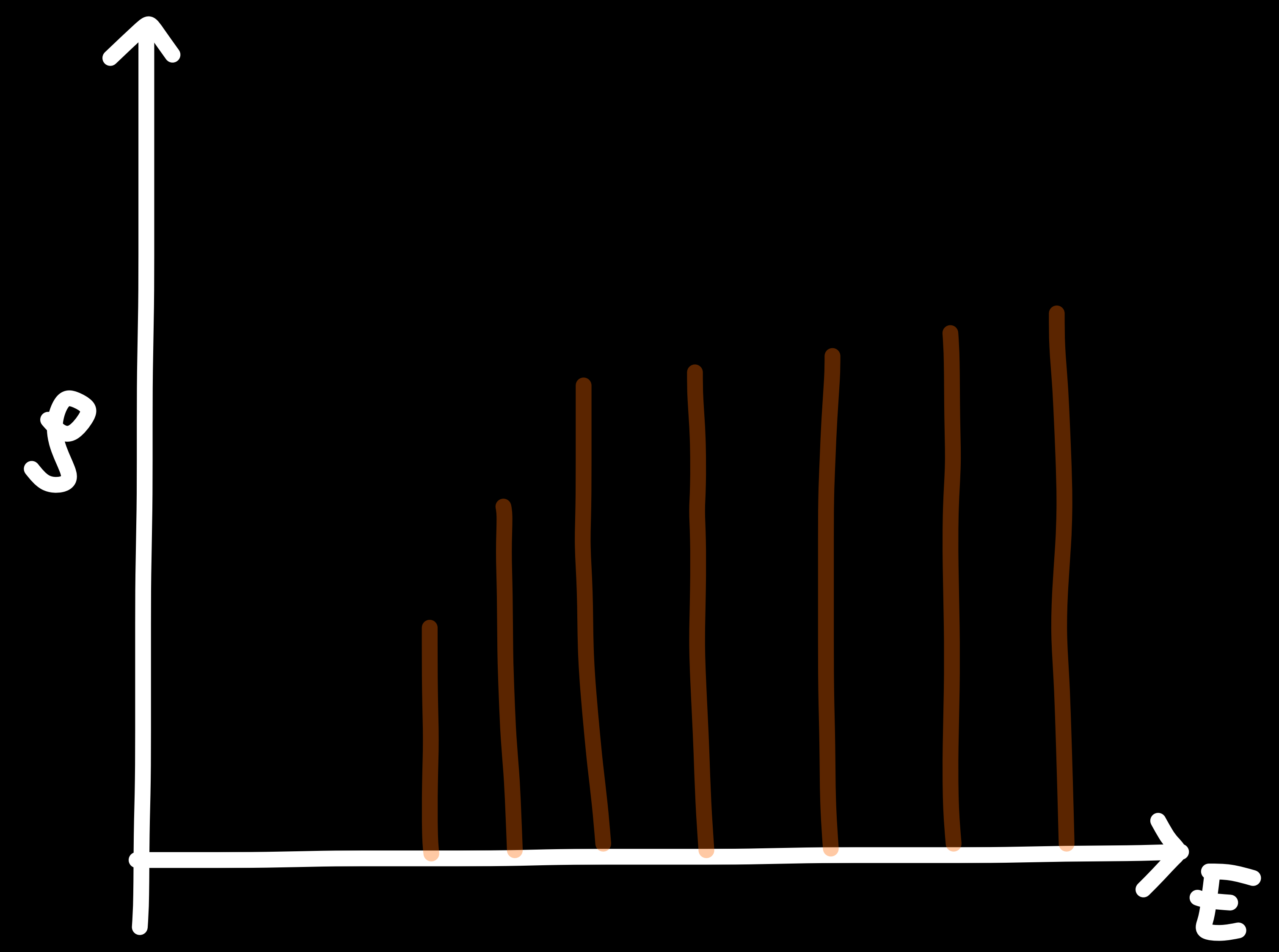


Lattice / Finite Volume

Backus Gilbert Approach

[Backus, Gilbert, 1968 & 1970]

Smearing



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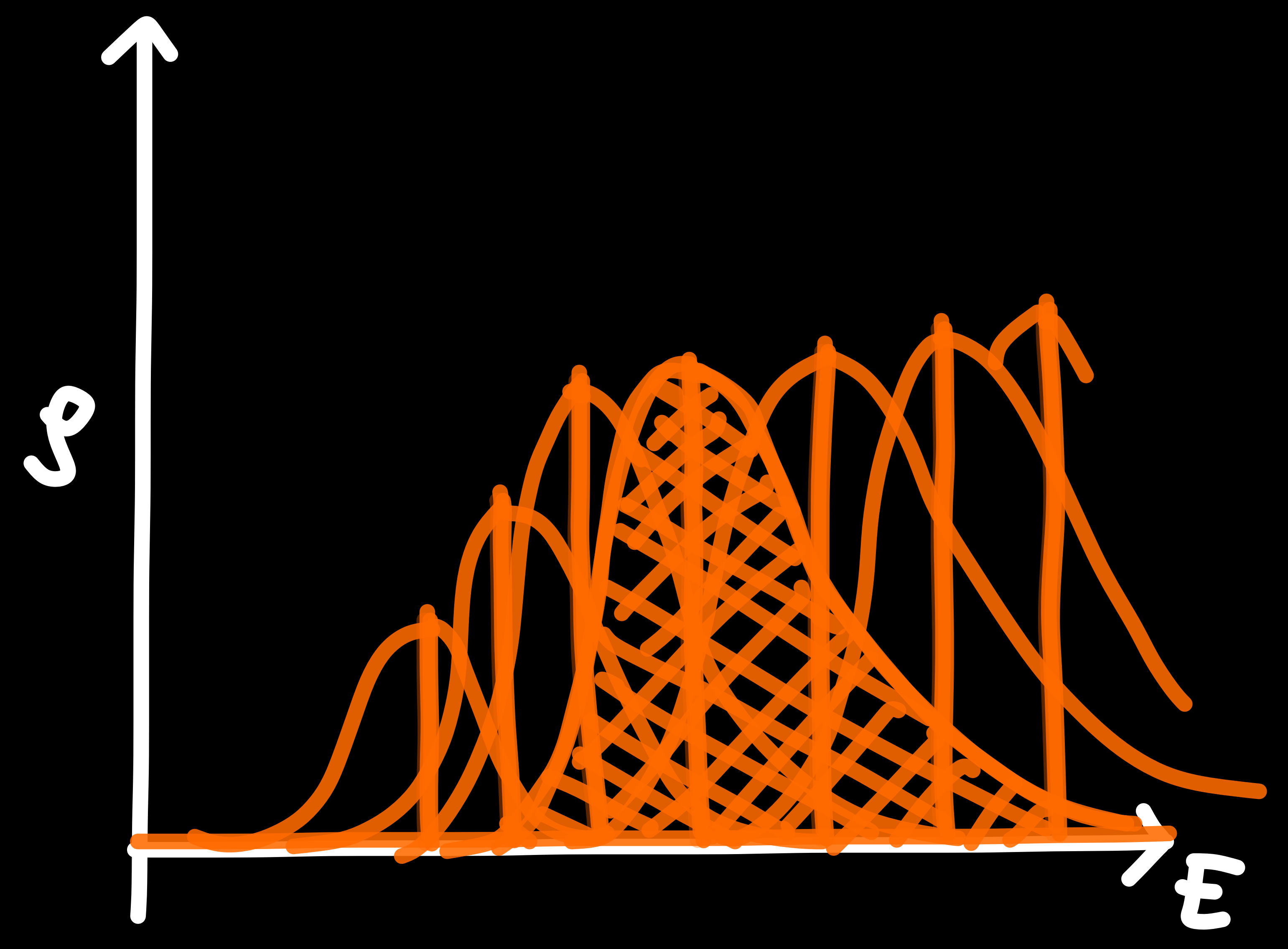

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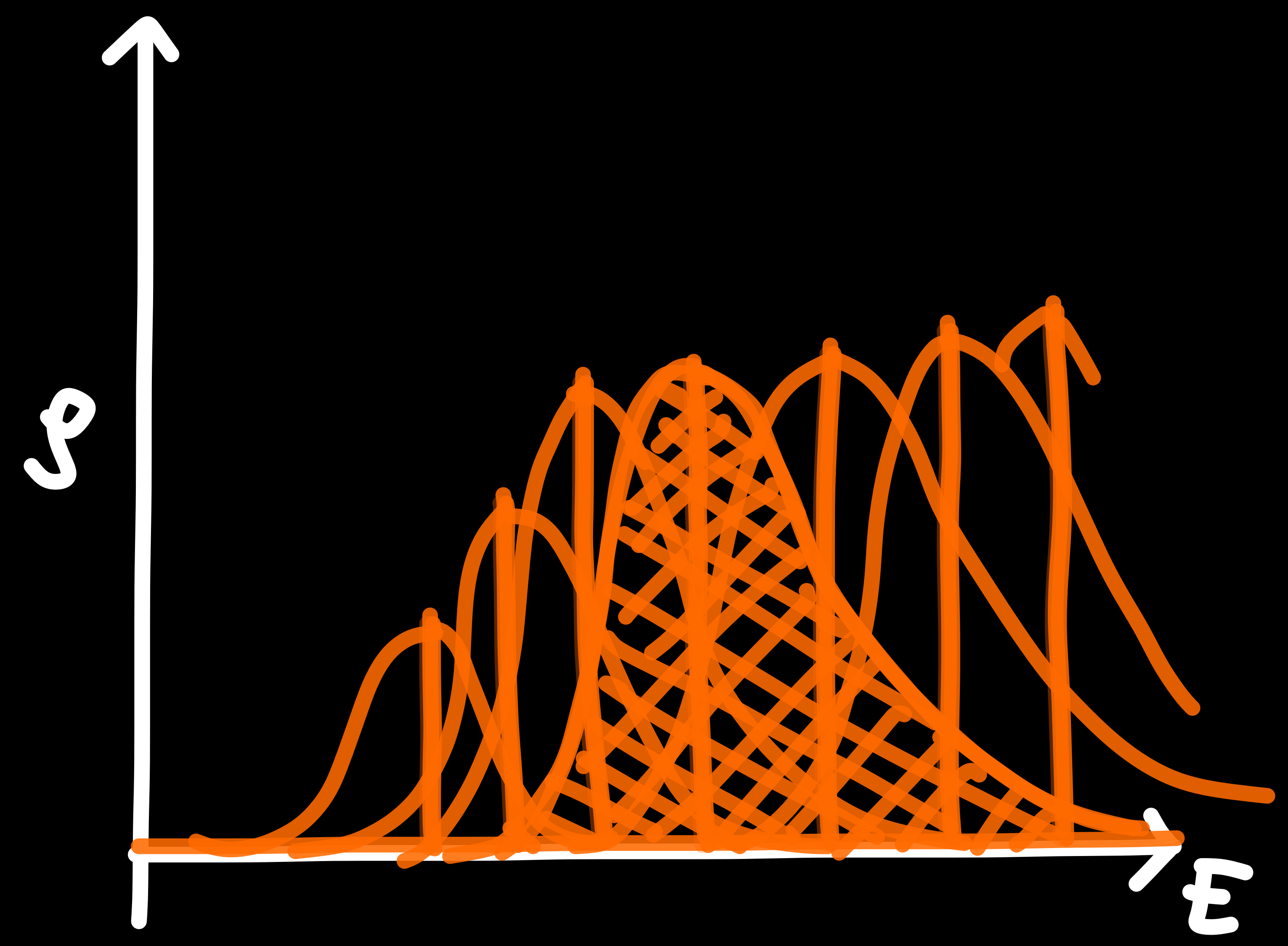
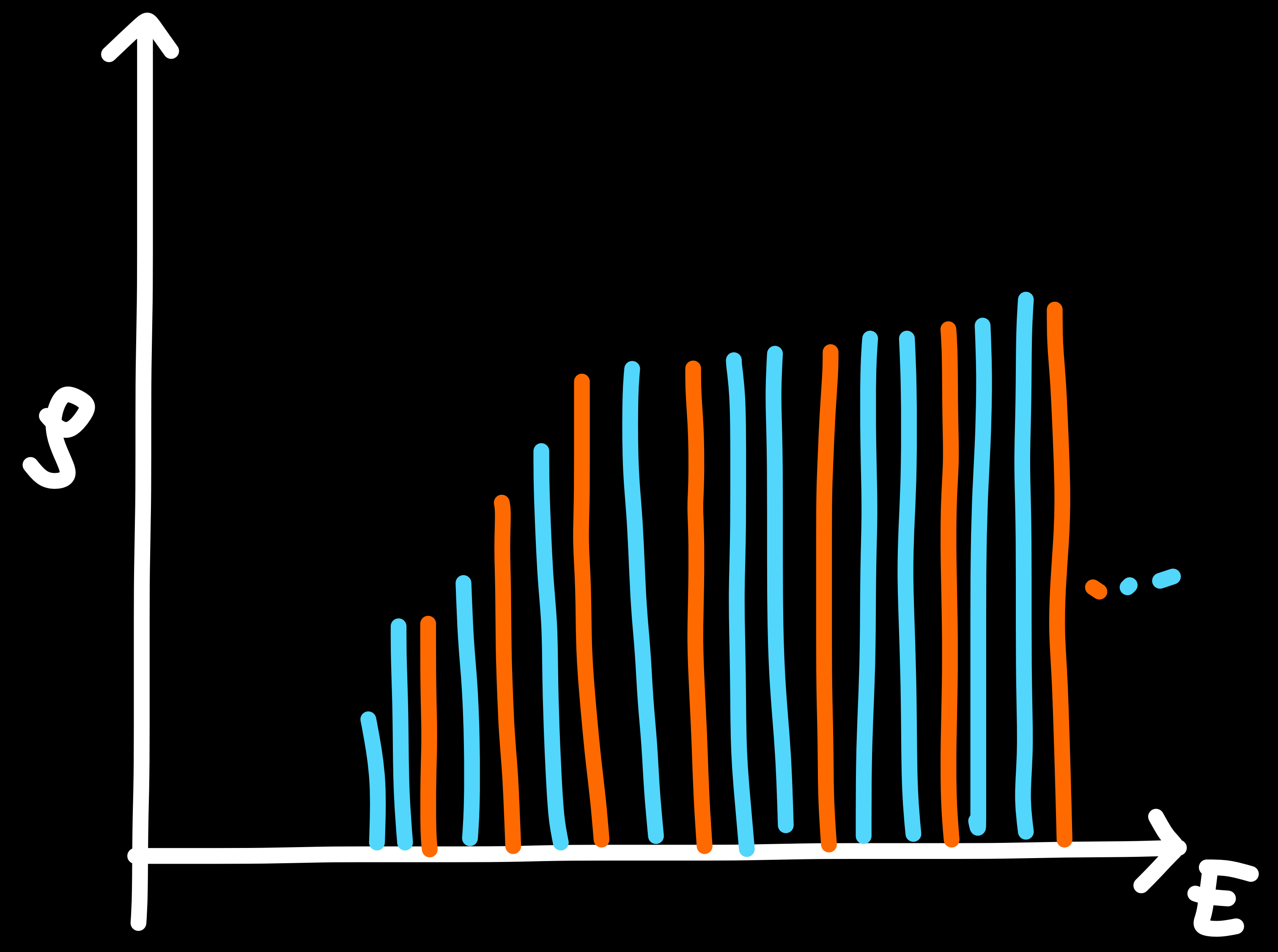


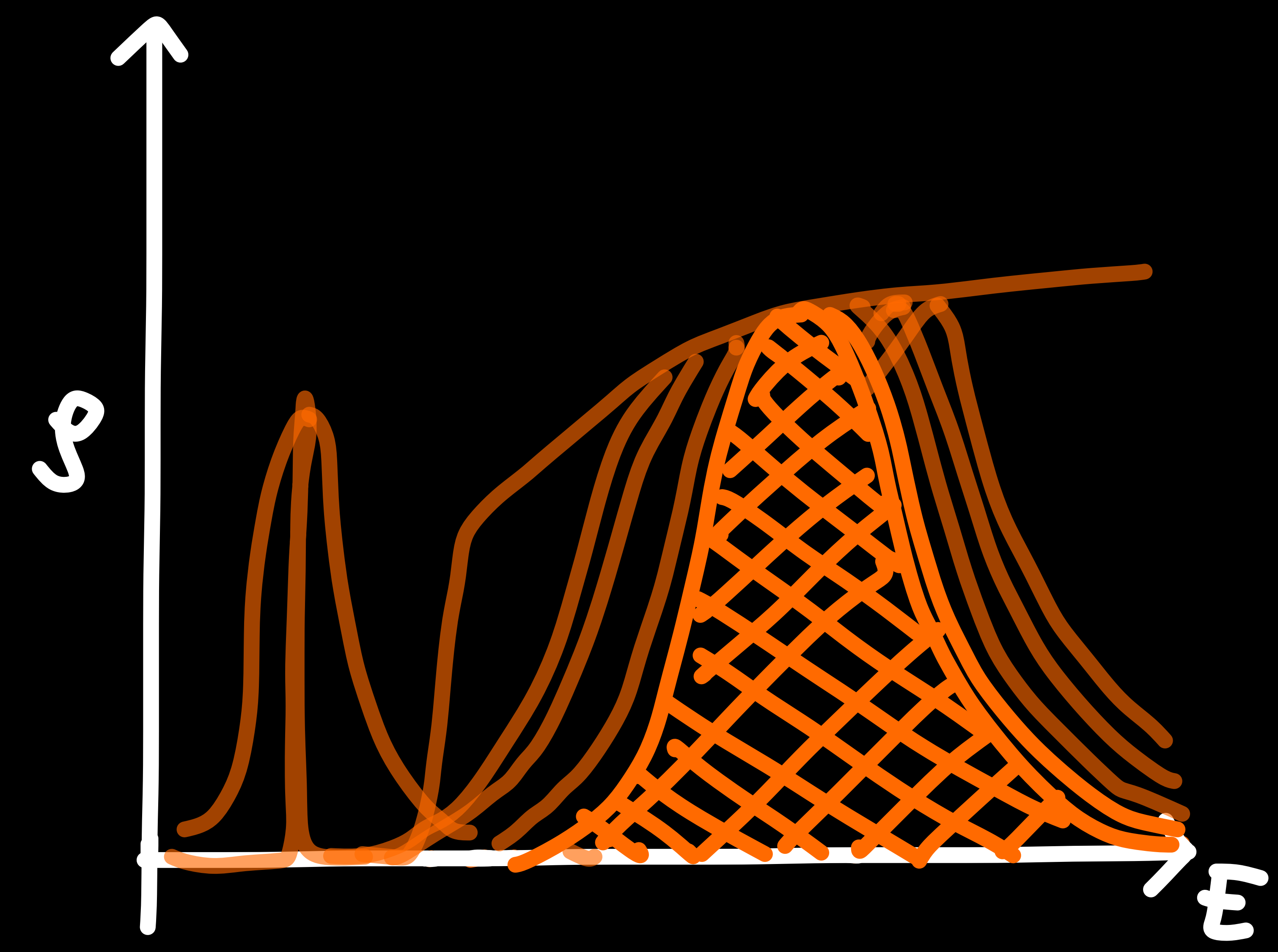


$$\hat{\rho}_L(\sigma, E_*) = \int_0^\infty dE\, \Delta_\sigma(E_*, E)\, \rho_L(E)$$

Backus Gilbert Approach

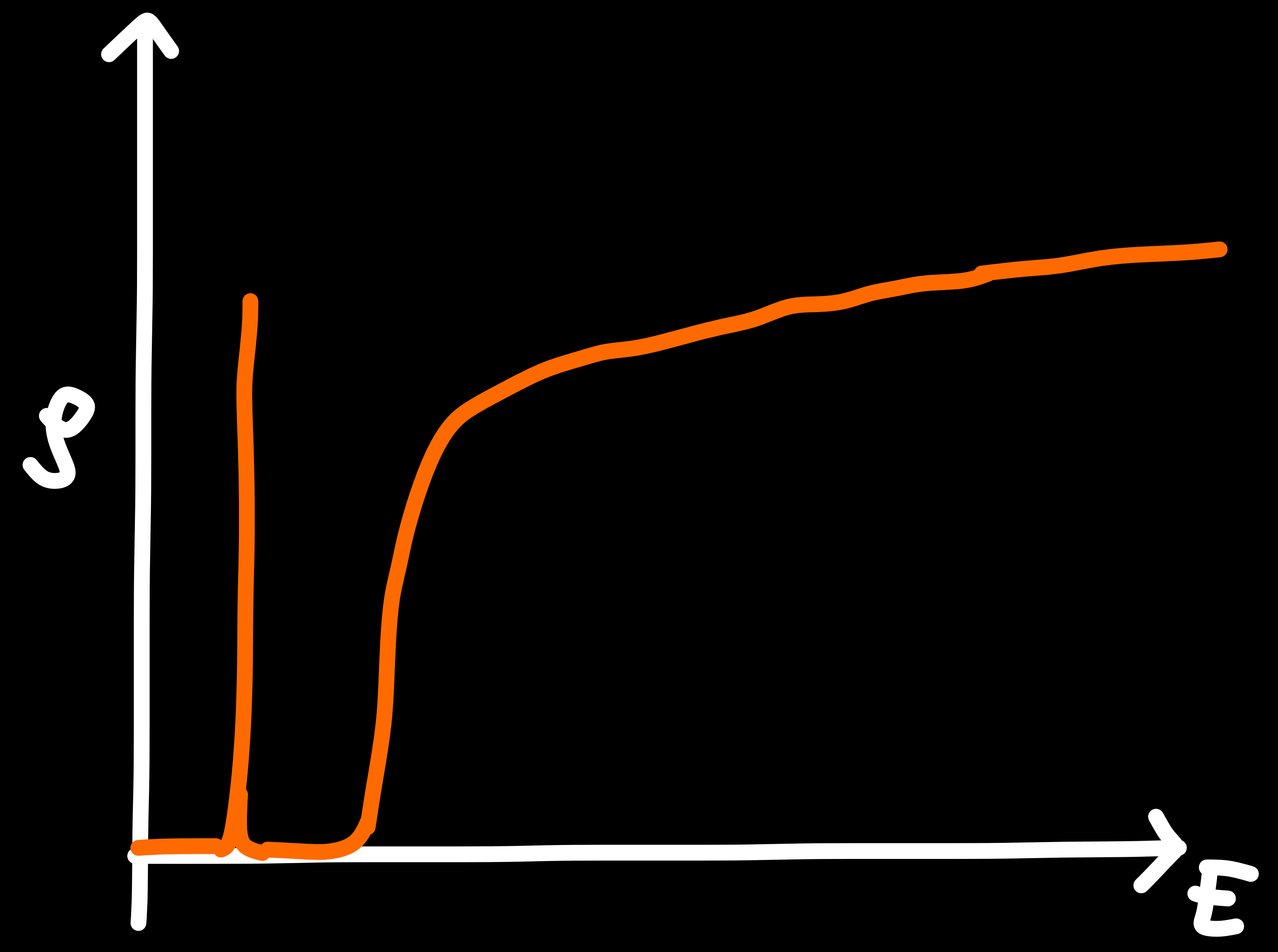
[Backus, Gilbert, 1968 & 1970]

Smearing



$$\lim_{L\to\infty} \hat{\rho}_L(\sigma, E_\star)$$

$\sigma \to 0$



$$\rho(E_\star) = \lim_{\sigma\to 0}\lim_{L\to\infty} \hat{\rho}_L(\sigma, E_\star)$$

Backus Gilbert Approach

[Backus, Gilbert, 1968 & 1970]

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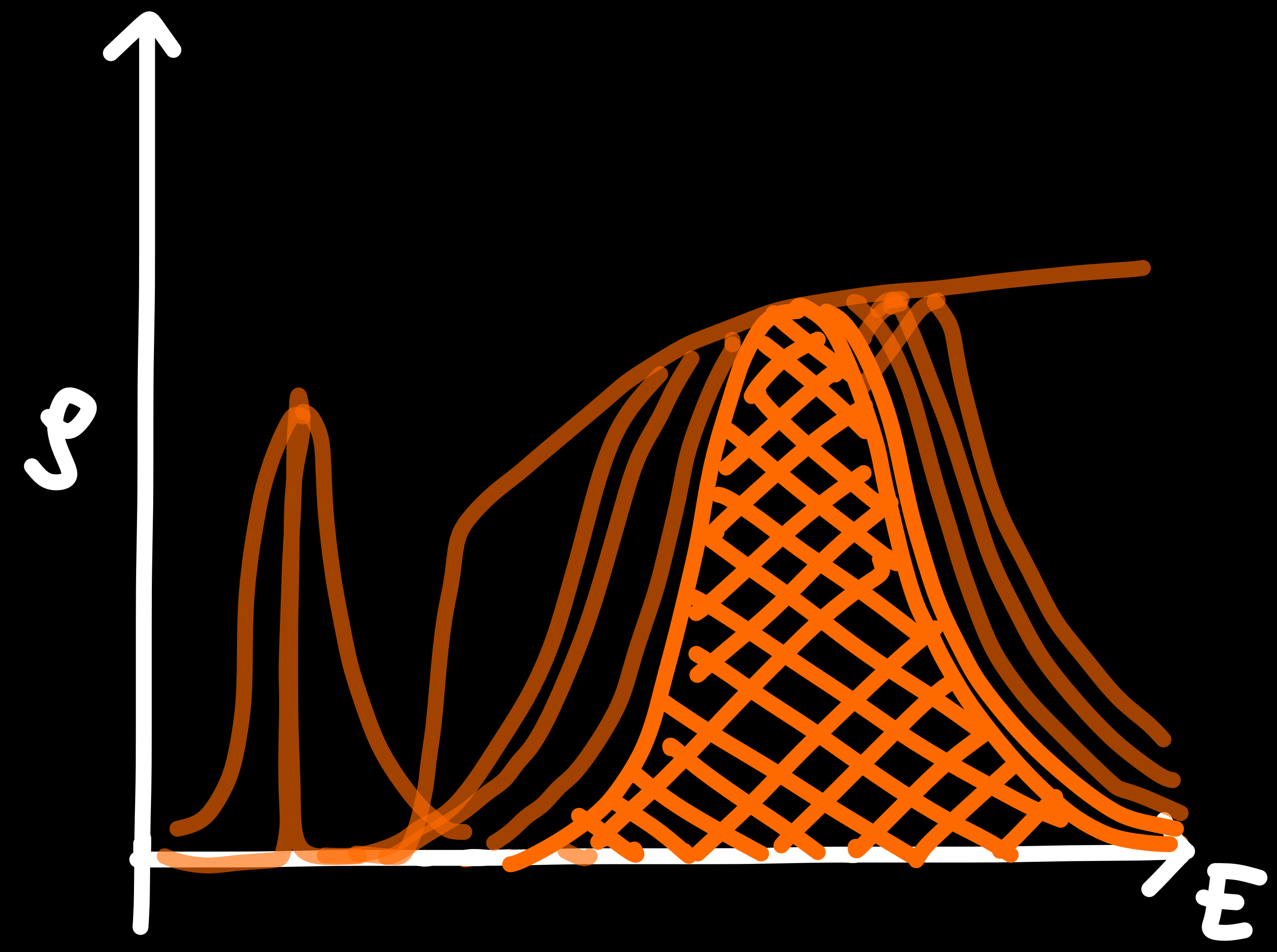
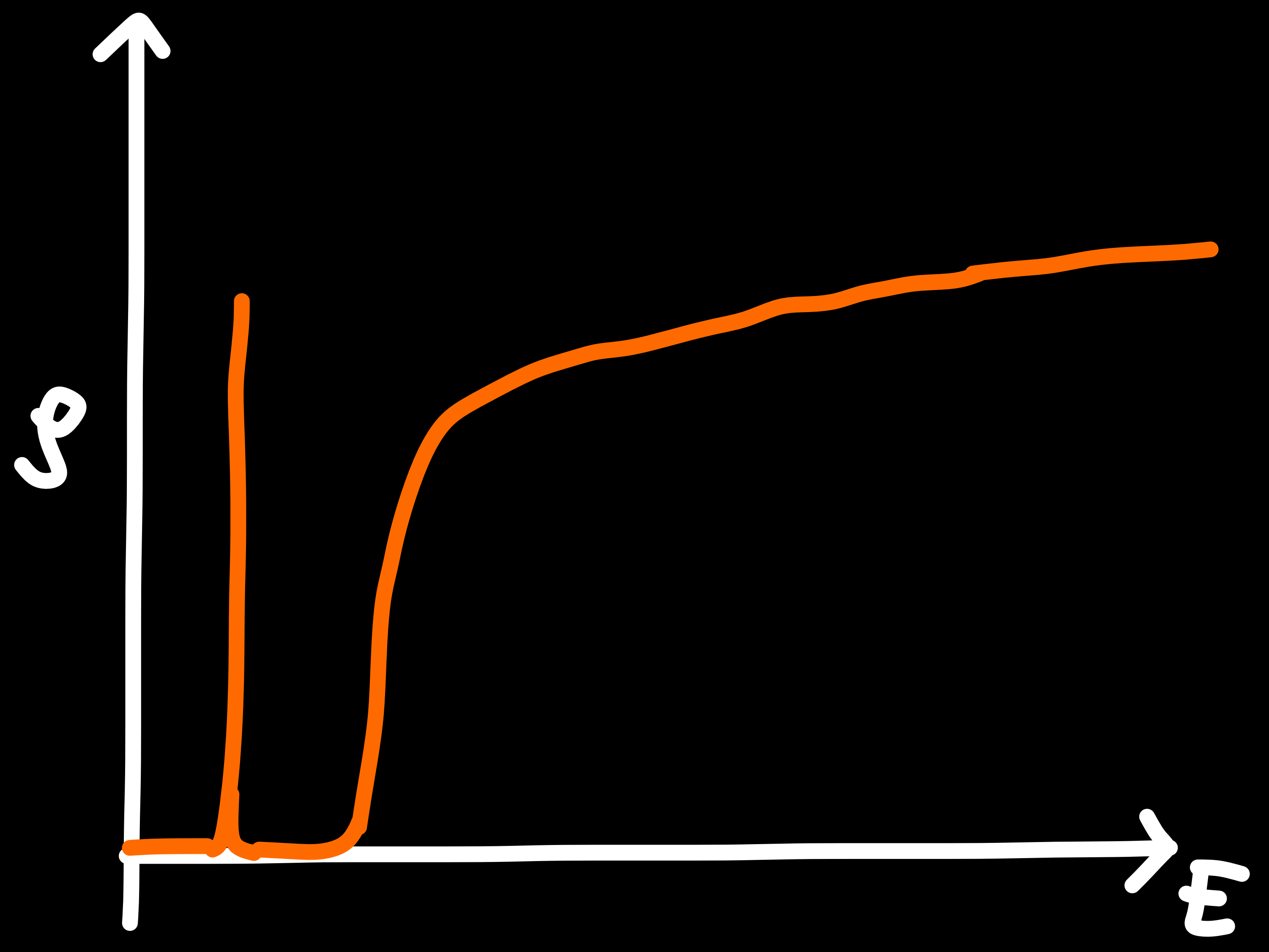


$$\xrightarrow{\sigma \to 0}$$



$$\lim_{L\to\infty} \hat{\rho}_L(\sigma, E_*)$$

$$\rho(E_*) = \lim_{\sigma\to 0} \lim_{L\to\infty} \hat{\rho}_L(\sigma, E_*)$$

!! IN THIS ORDER !!

[Hansen et. al, 2019]

Hansen-Lupo-Tantalo Method

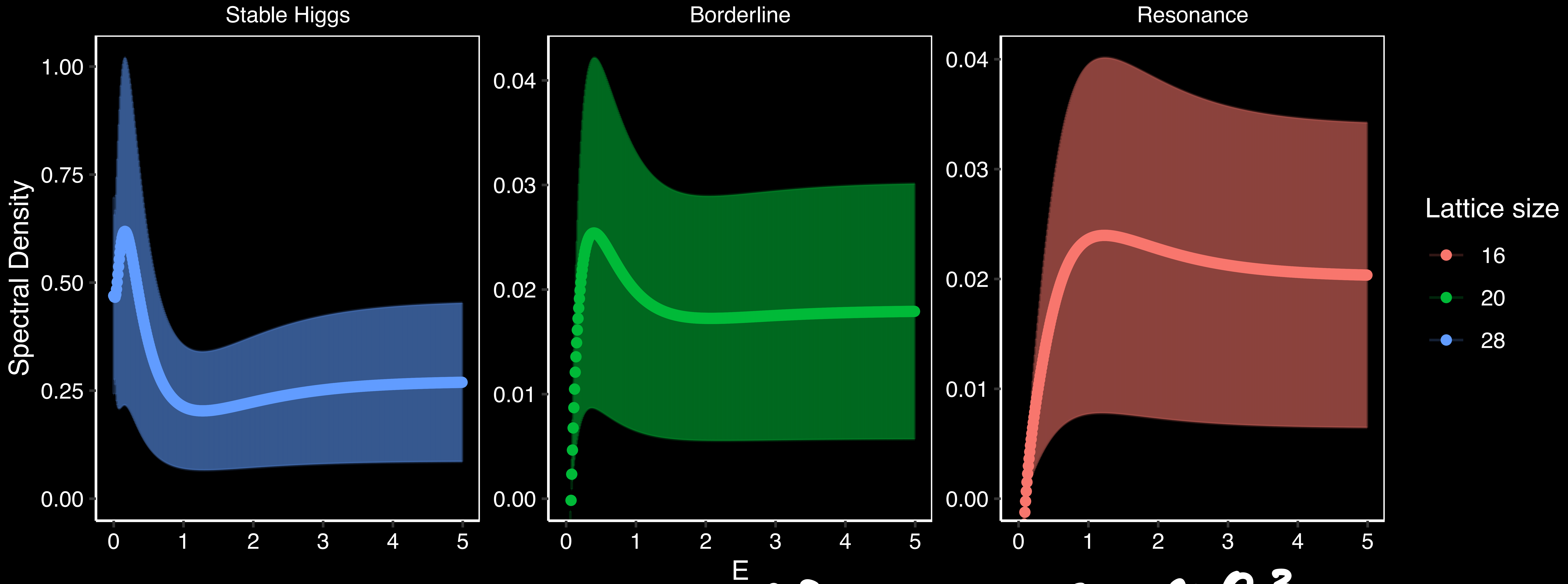
Expand kernel in functional basis preserving backpropagation

$$\Delta(E_*, E) = \sum_{t=0}^{T} g_t(E_*) \left[e^{-tE} + e^{-(T-t)E} \right]$$

Balance systematic and statistical uncertainties

$$A[g] = \int_{E_0}^{\infty} dE \left| \bar{\Delta}_\sigma(E_*, E) - \Delta_\sigma(E_*, E) \right|^2 \qquad \frac{\mathrm{Var}(C(t))}{C^2(0)}$$

Higgs stability



$m_W \approx 0.2$
$m_H \approx 0.3$
$H \not\rightarrow WW$

$m_W \approx 0.3$
$m_H \approx 0.6$
$H \xrightarrow{?} WW$

$m_W \approx 0.3$
$m_H \approx 1.2$
$H \longrightarrow WW$ ✓

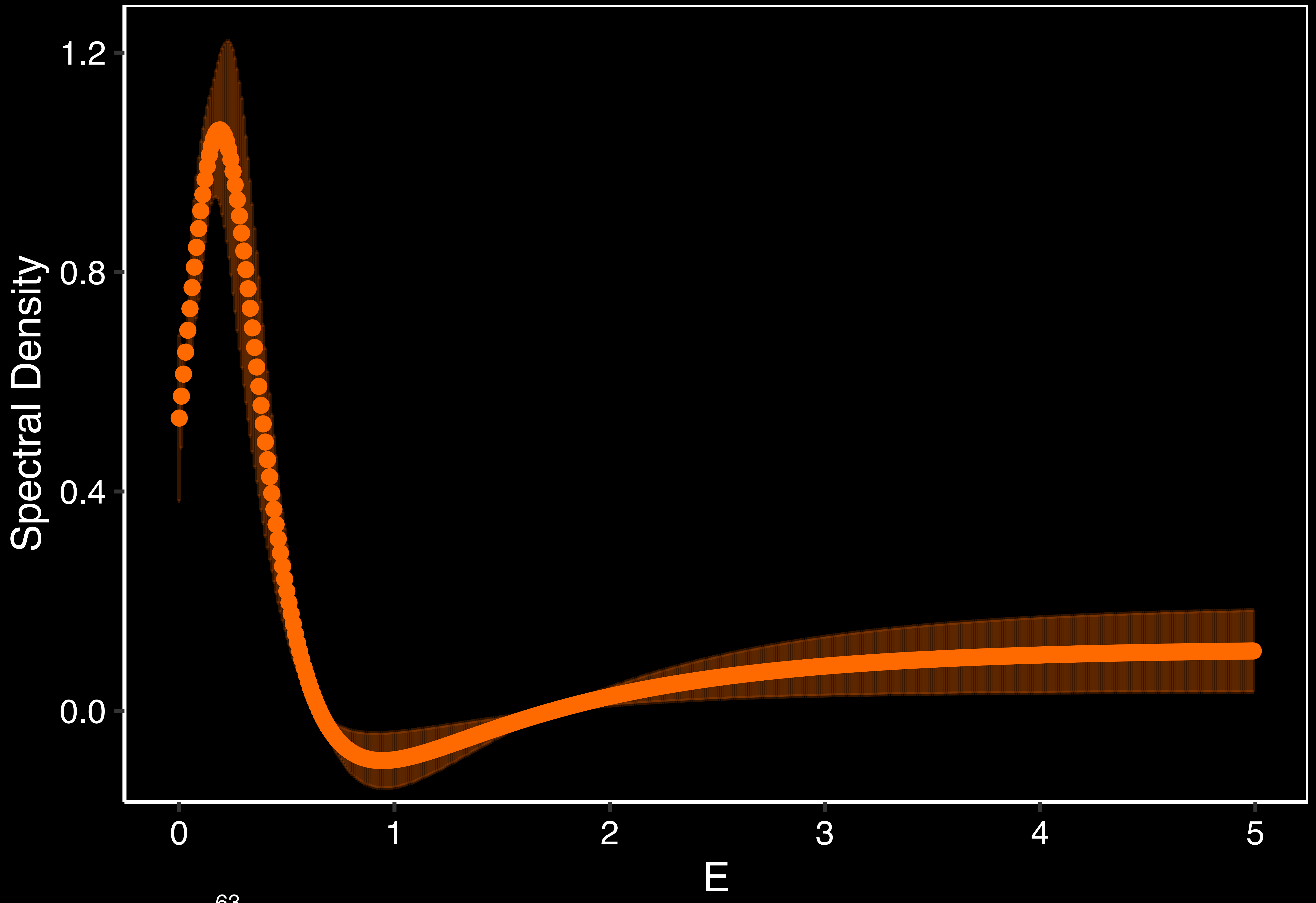
Example ensemble

The vector channel is much more precise

$m_W \approx 0.2$

$m_H \approx 0.3$

$T \times L^3 = 32 \times 32^3$



Summary

Spectral densities

- We can extend scattering amplitudes above the **inelastic threshold** by defining them in terms of **spectral densities**

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- Backus & Gilbert suggested to **smear** this discrete data into continuous data and perform limits to infinite volume and vanishing smearing

Summary

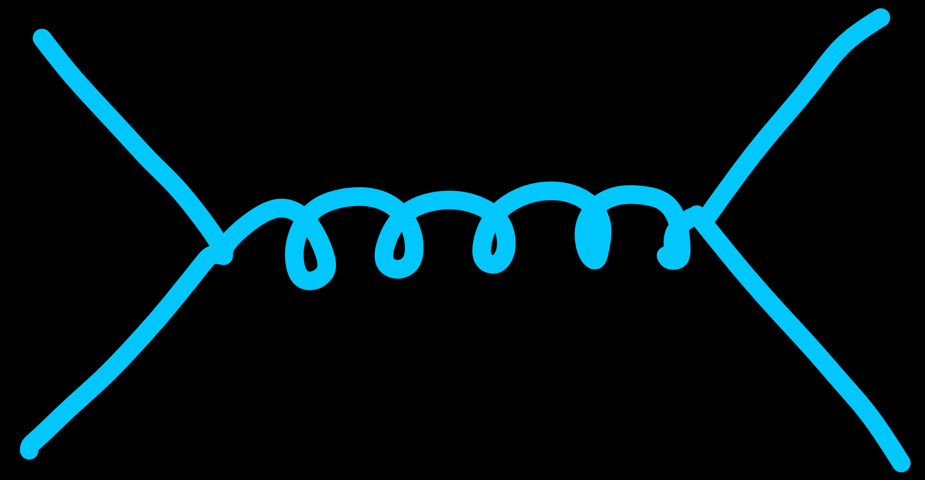
Spectral densities

- We can extend scattering amplitudes above the **inelastic threshold** by defining them in terms of **spectral densities**
- Computation of spectral densities on the lattice is an **ill-posed problem** because the finite volume enforces discrete energy levels
- Backus & Gilbert suggested to **smear** this discrete data into continuous data and perform limits to infinite volume and vanishing smearing
- HLT improved this by **optimization** of the functional basis w.r.t. **systematic** and **statistical** errors

Method by Hansen and Bulava

[Bulava, Hansen, 2019]

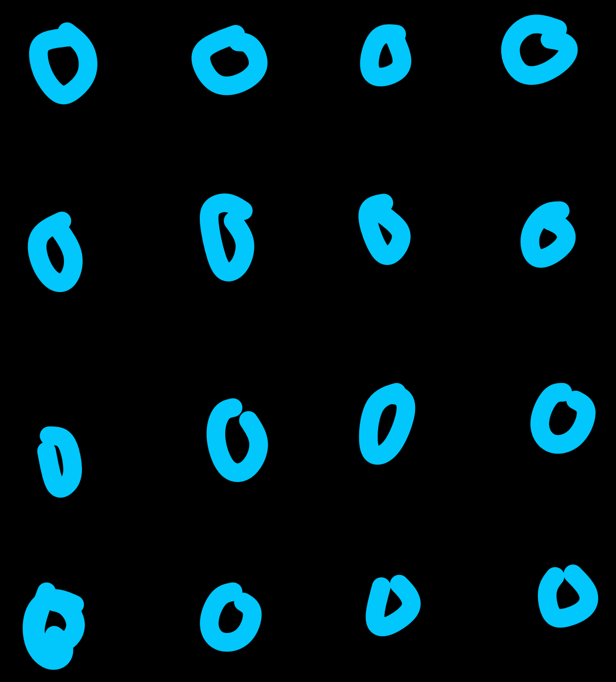
Idea



What we compute in perturbation theory

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|0\rangle$$

Time-ordered correlation functions

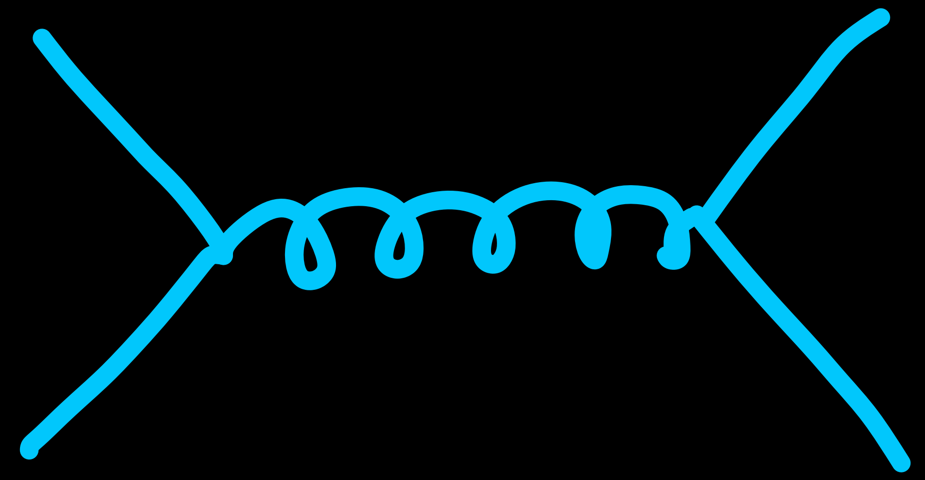


What we compute on the lattice

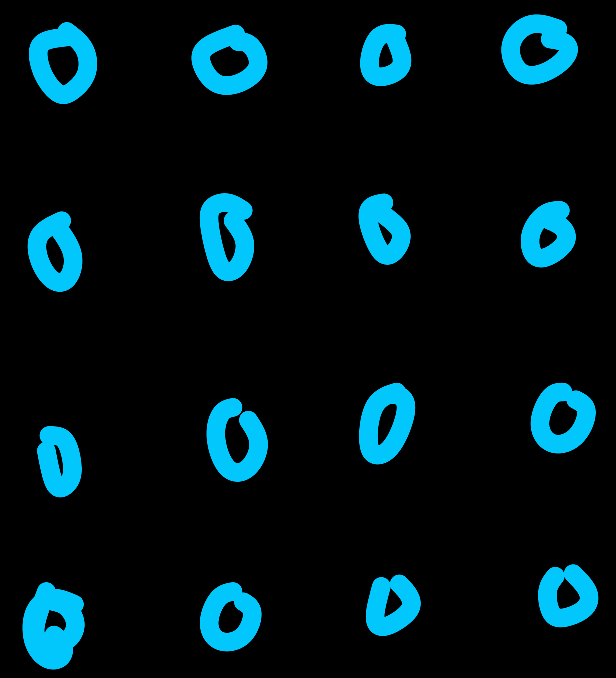
$$\langle 0|\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|0\rangle$$

No time ordering

Idea



What we compute in perturbation theory



What we compute on the lattice

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|0\rangle$$

$$\langle 0|\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|0\rangle$$

Identical on-shell pole structure

Express amplitude as spectral density

m particles out, n particles in

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- Disassemble amplitude spectrally

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$$i\mathcal{M}_c(\{p_f\}|\{q_i\}) \propto \rho_{p_m q_1}(\{p_f\}\backslash p_m, \{q_i\}\backslash q_1)$$

Express amplitude as spectral density

m particles out, n particles in

- Disassemble amplitude spectrally

$$i\mathcal{M}_c(\{p_f\}|\{q_i\}) \propto \rho_{p_m q_1}(\{p_f\}\backslash p_m, \{q_i\}\backslash q_1)$$

- Compute this spectral function by solving inverse problem

$$C_{p_m q_1}(\{p_f\}\backslash p_m, \{q_i\}\backslash q_1) = \int \frac{d^r E}{\pi^r} e^{-\sum E \Delta t} \rho_{p_m q_1}(\{p_f\}\backslash p_m, \{q_i\}\backslash q_1)$$

Compute spectral density

m particles out, n particles in

- The previously mentioned correlation function is constructed by "amputating" an m+n-point correlation function

$$C_{p_m q_1}(\{p_f\} \setminus p_m, \{q_i\} \setminus q_1) = 2\sqrt{E(p_m)E(q_1)}\, L^3 Z^{1/2}(p_m) Z^{1/2}(q_1)$$

$$\times \lim_{\Delta t_i \to \infty} \frac{C_{m+n}(\{p_f\}, \{q_i\}, \{\Delta t_i\})}{C_{2pt}(p_m, \Delta t_{m+n}) C_{2pt}(q_1, \Delta t_1)}$$

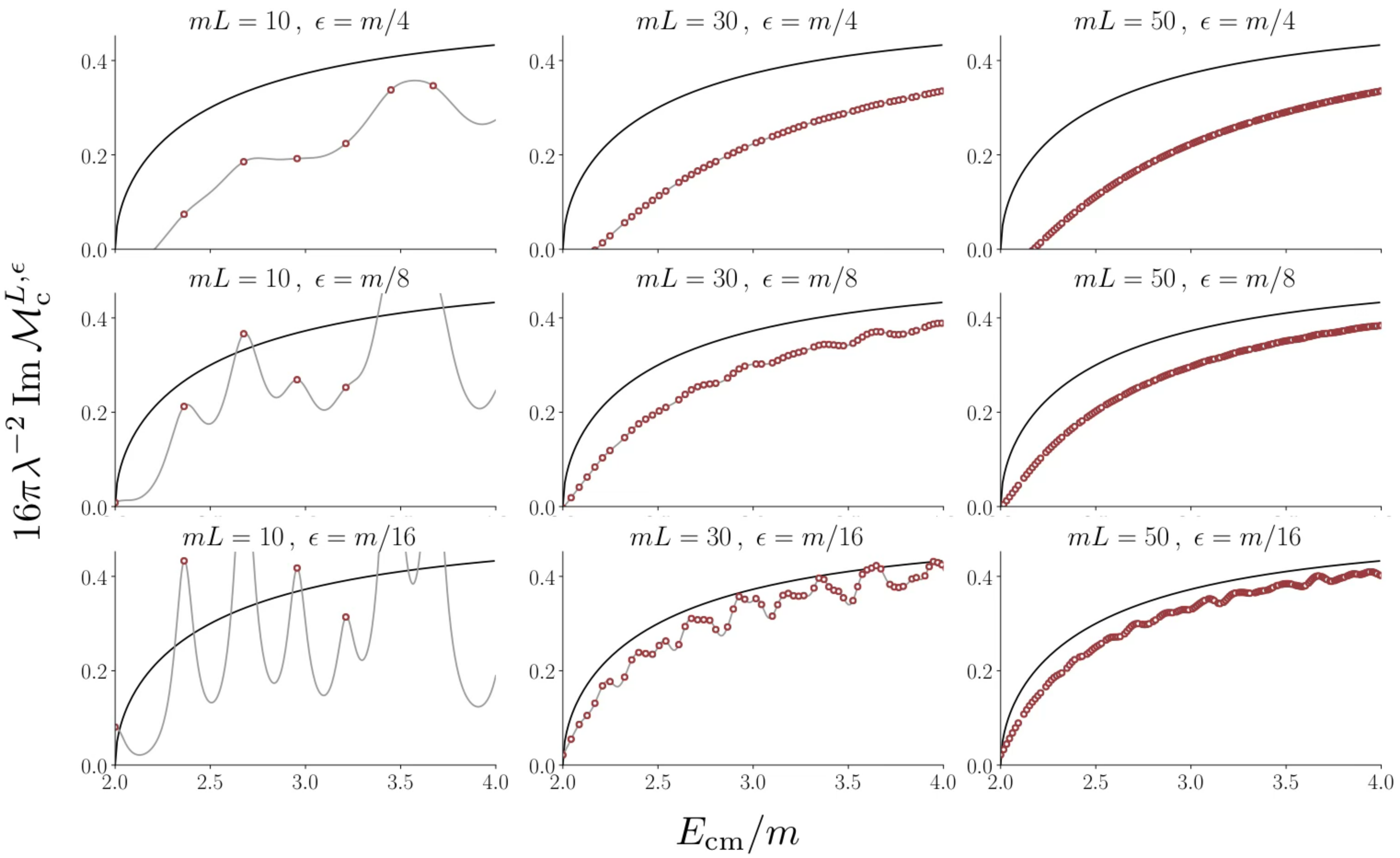
Examples for $2 \to 2$ exclusive amplitudes

Test of the method

[Bulava, Hansen, 2019]

ϕ^4-theory NLO

- Reconstructing next-to-leading order 2→2 process
- Analytical result available (solid line)
- Large finite volume effects

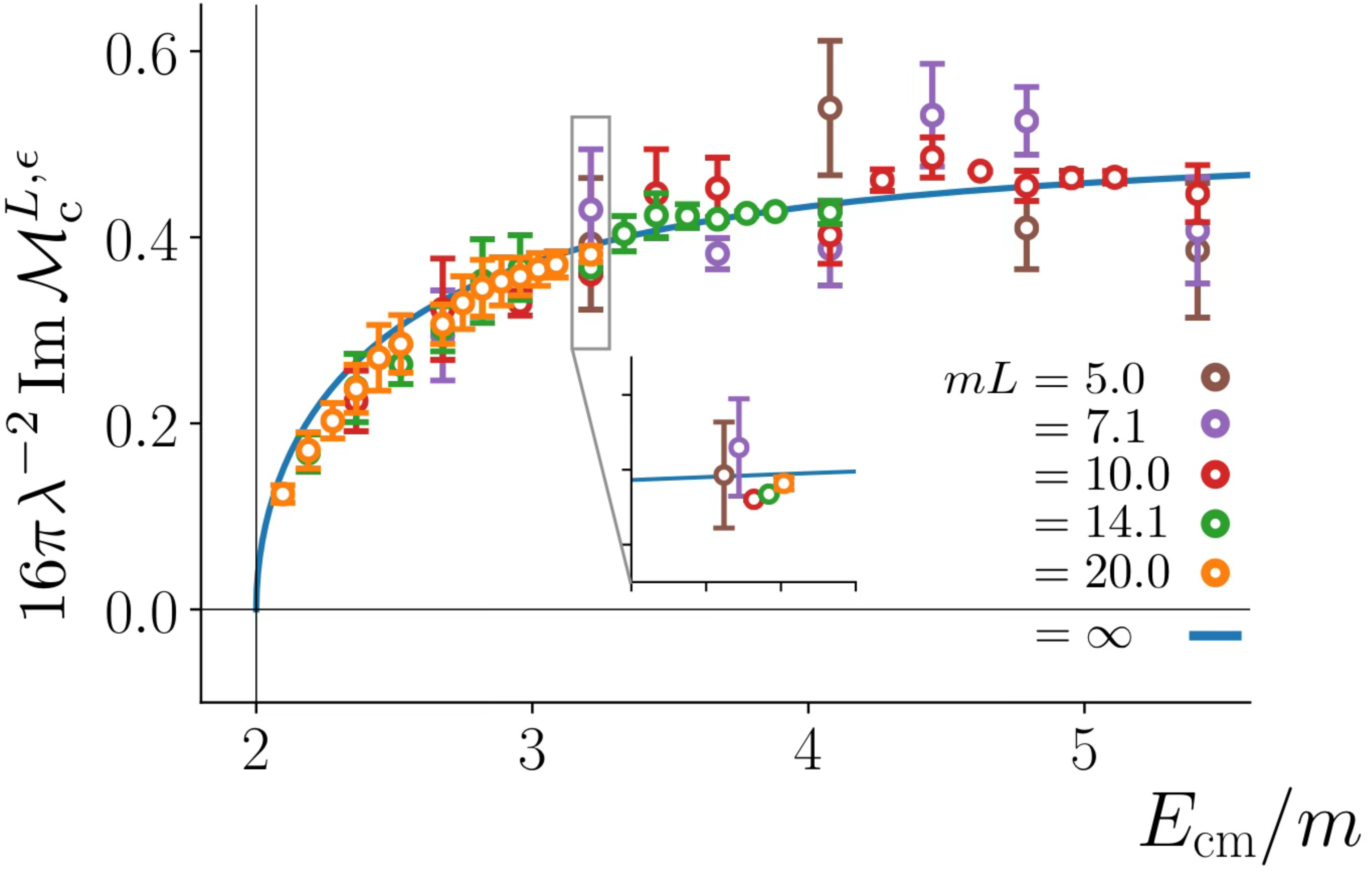


Test of the method

ϕ^4-theory NLO

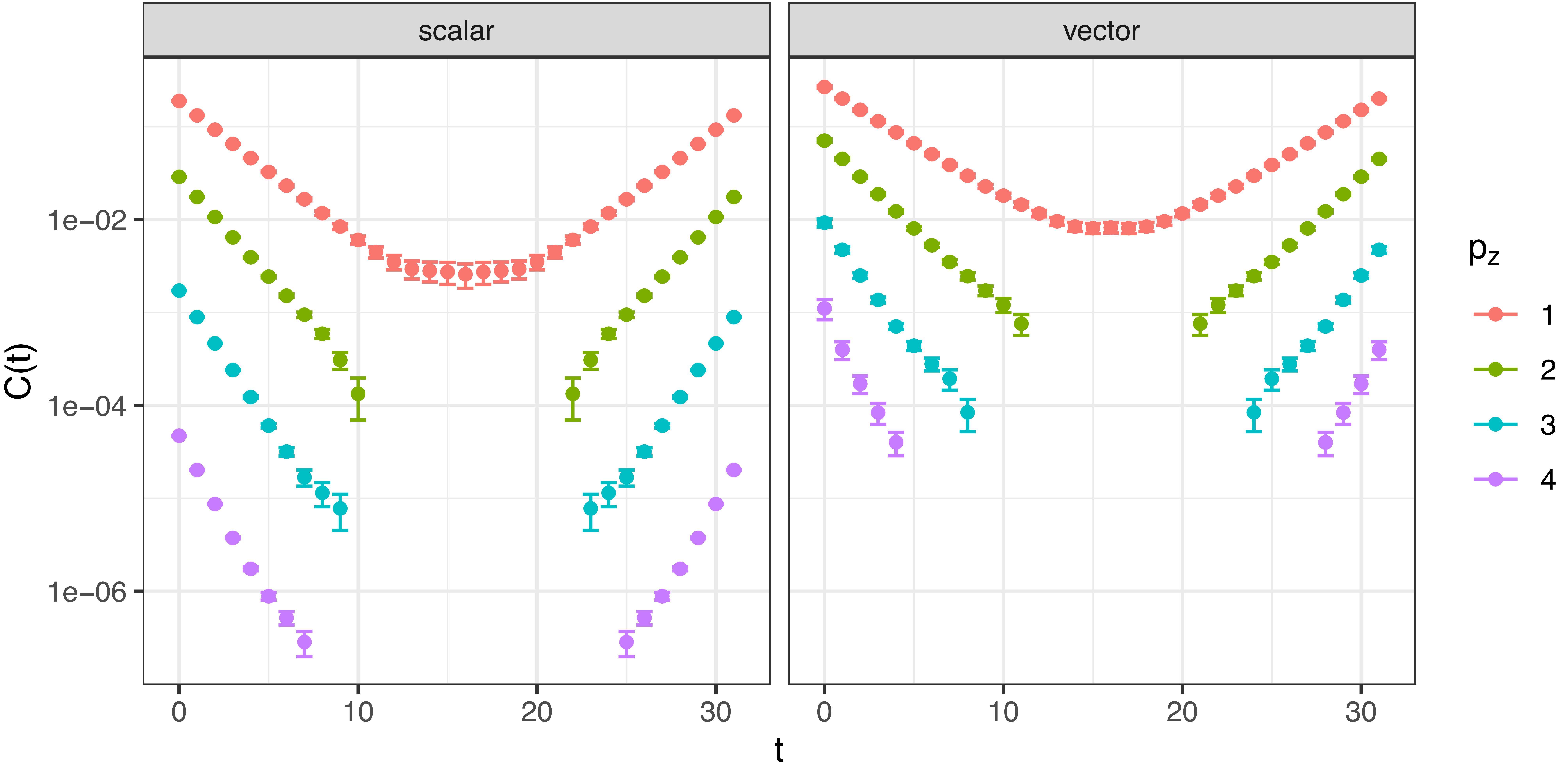
[Bulava, Hansen, 2019]

- Extrapolation to vanishing smearing width



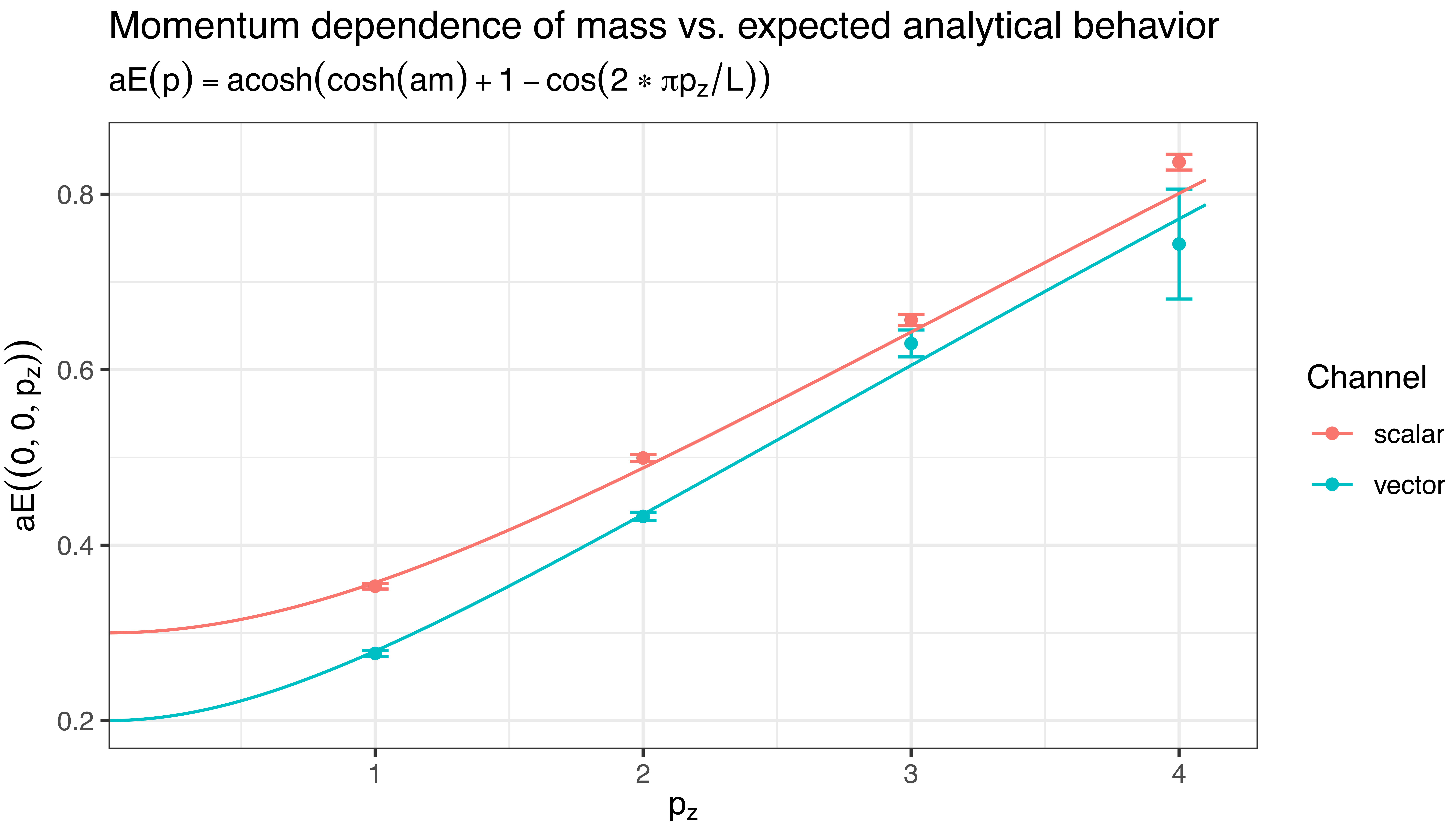
Electroweak theory

Boosted correlation functions



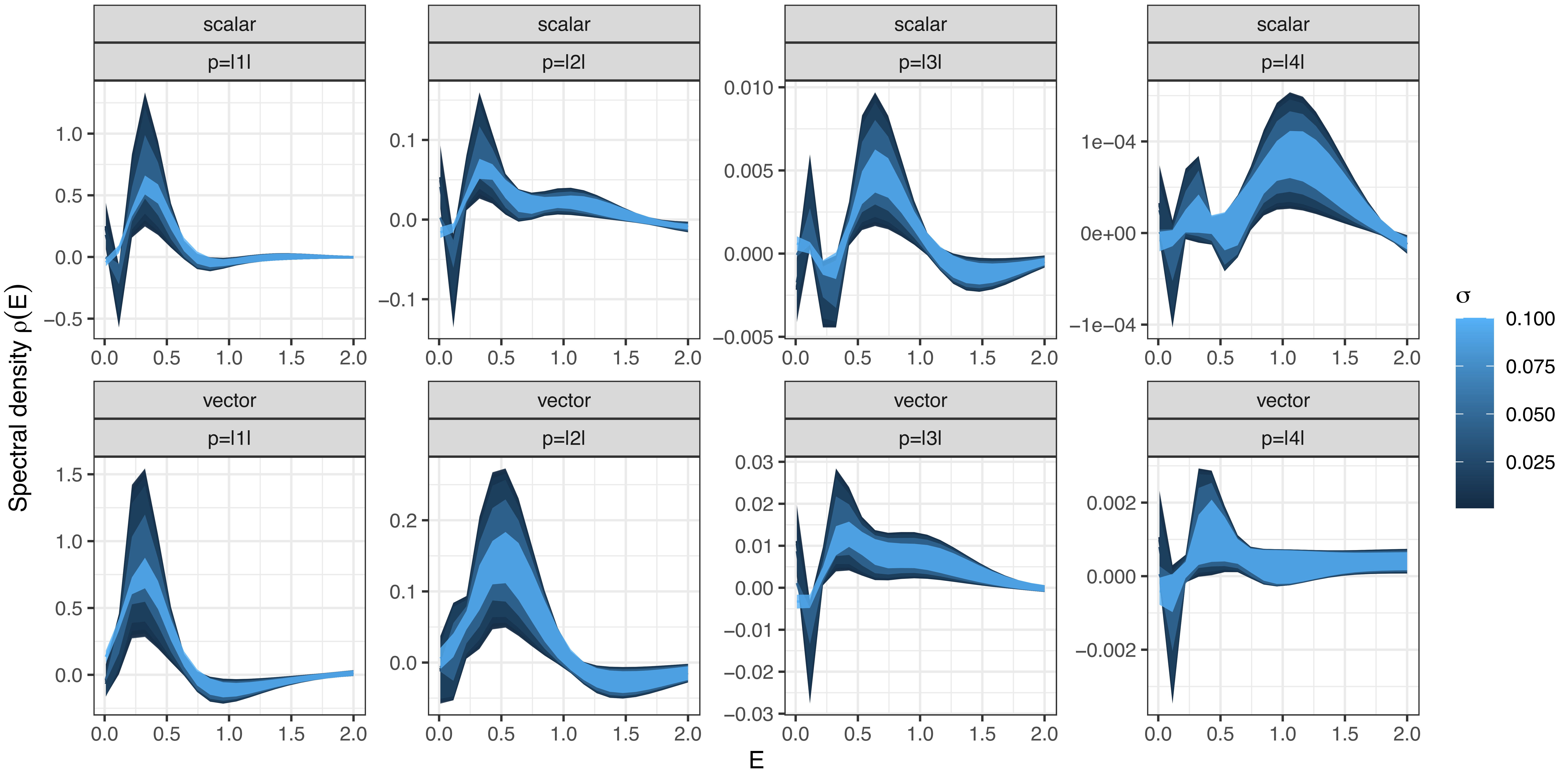
Electroweak theory

Naive energies through dispersion relation



Electroweak theory

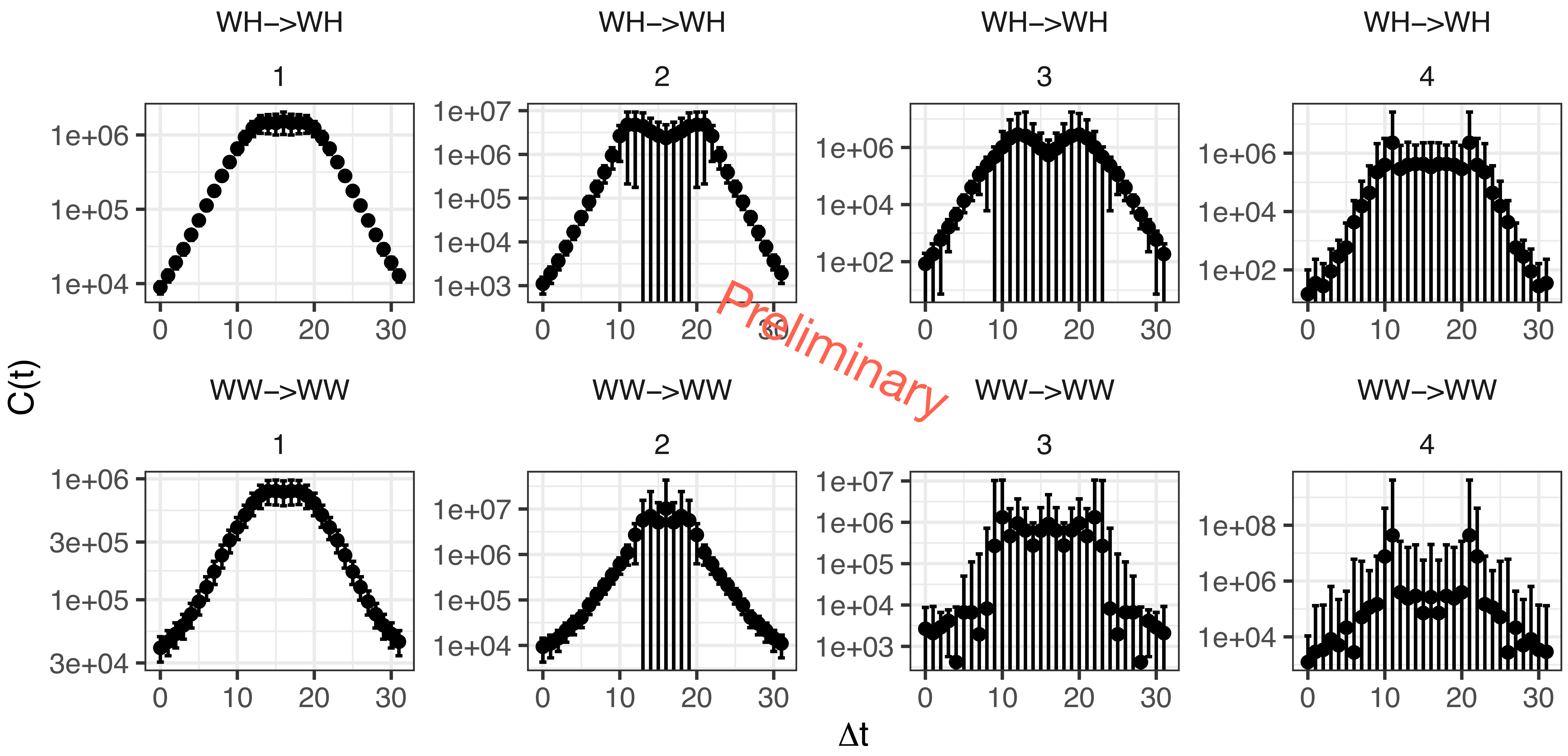
Boosted spectral densities (two-point functions)



Electroweak theory

$$C_{p_m q_1}(\{p_f\}\setminus p_m, \{q_i\}\setminus q_1, \Delta t)$$

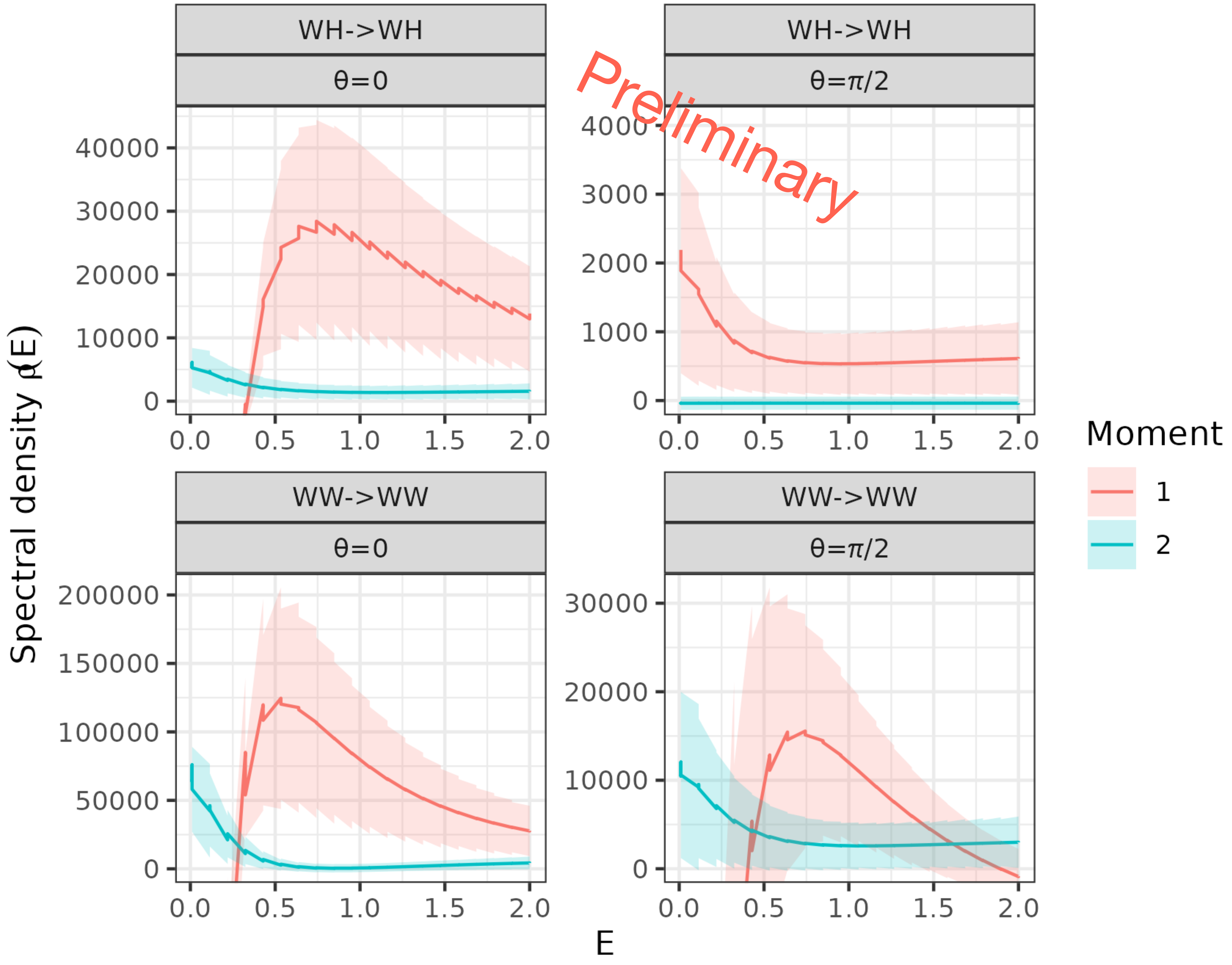
Ratio functions

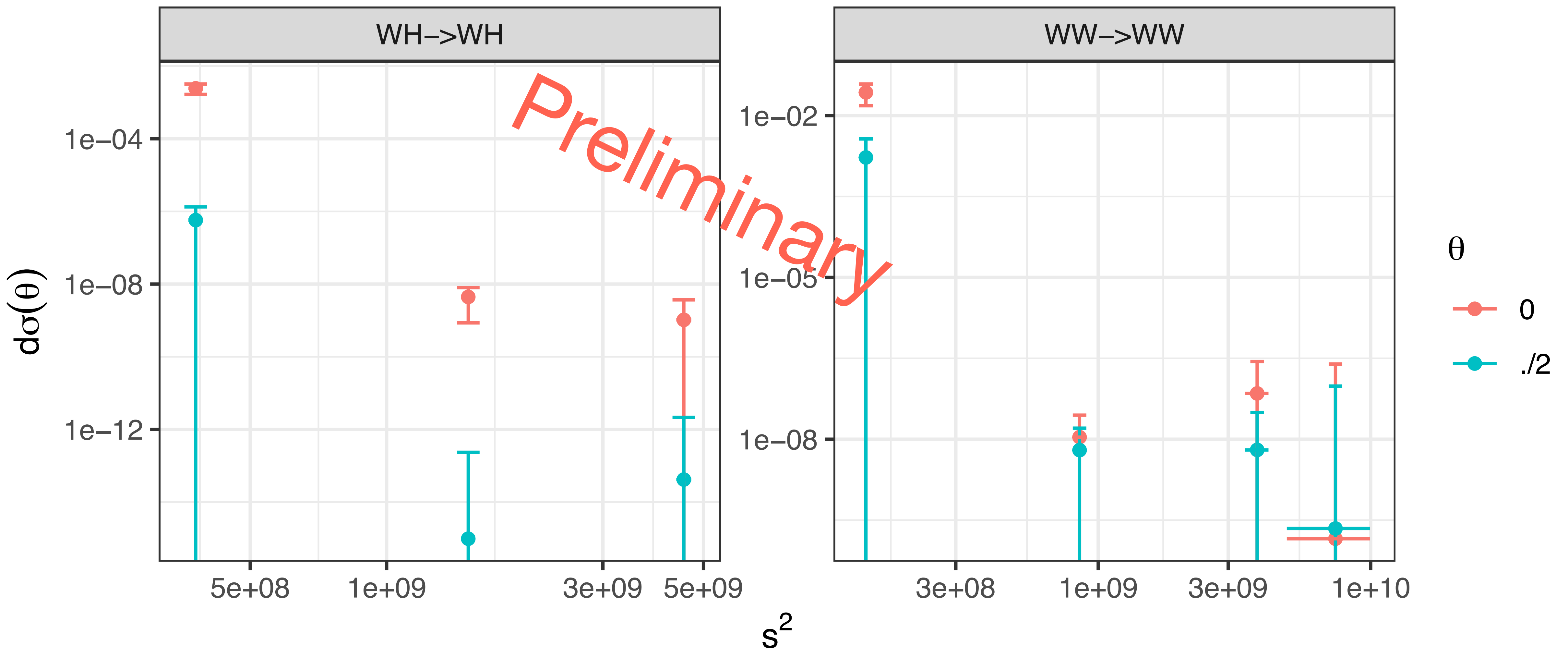


Electroweak theory

Spectral density of ratio

$$\varrho_{p_m q_1}(\{p_f\}\backslash p_m, \{q_i\}\backslash q_1)$$





Patella-Tantalo Approach

[Patella, Tantalo, 2024]

Skipping the spectral density

Expressing the scattering amplitudes as linear combinations of correlation functions

$$S_c(\sigma,\epsilon) = \sum_{n_1,\ldots,n_{n+m-1}\geq 1} \sum_{b\geq 0} \omega_{n,b}^{\sigma,\epsilon}$$

$$\times \int \left[\prod_{A=1}^{m+n} \frac{d^3 p_A}{(2\pi)^2} f_A^{(*)}(p_A) \right] \tilde{h}(\Delta(p))\, \Upsilon(\tau n; p) \left[\Delta(p)\right]^b \hat{C}_c(\tau n; P)$$

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LATTICE CORRELATORS

$$\times \int \left[\prod_{A=1}^{m+n} \frac{d^3 p_A}{(2\pi)^2} f_A^{(*)}(p_A) \right] \tilde{h}(\Delta(p)) \Upsilon(\tau n; p) [\Delta(p)]^b \hat{C}_c(\tau n; p)$$

Skipping the spectral density

Expressing the scattering amplitudes as linear combinations of correlation functions

NUMERICAL OPTIMIZATION

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KNOWN FUNCTIONS / KERNELS / CONSERVATION

Skipping the spectral density

Expressing the scattering amplitudes as linear combinations of correlation functions

Smearing

Precision

$$S_c(\sigma,\epsilon) = \sum_{n_1,\ldots,n_{n+m-1}\geq 1} \sum_{b\geq 0} \omega^{\sigma,\epsilon}_{n,b}$$

$$\times \int \left[\prod_{A=1}^{m+n} \frac{d^3p_A}{(2\pi)^3} \tilde{f}^{(*)}_A(p_A)\right] \tilde{h}(\Delta(p))\,\Upsilon(\tau n; p)\,[\Delta(p)]^b\,\hat{C}_c(\tau n; p)$$

No numerical results available (yet)

Summary and Outlook

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- Hansen-Bulava Method allows computation of general scattering amplitudes from spectral densities: The bottleneck is the precision that the data can achieve in the n+m point correlation functions

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- Hansen-Bulava Method allows computation of general scattering amplitudes from spectral densities: The bottleneck is the precision that the data can achieve in the n+m point correlation functions
- Patella-Tantalo Method could skip the ill-posed inversion and directly use the correlation functions. However, no numerical results exist at this point.

Outlook

- Improve precision of spectral density computation

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Nevanlinna-Pick-Interpolation [Fields, Christ, 2025]

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→ Nevanlinna-Pick-Interpolation [Fields, Christ, 2025]

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→ Nevanlinna-Pick-Interpolation [Fields, Christ, 2025]

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→ Add results for resonance Higgs

- Scan physical parameter space

Thank you!