# Category Theory in Analytical Mechanics

Application of the basics of category theory on Newtonian, Lagrangian, and Hamiltonian mechanics

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in partial fulfillment of the requirements for the degree of
Bachelor Physics – BSc

*Supervisor* Univ.-Prof. Dipl.-Phys. Dr.rer.nat. Axel Torsten Maas

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# Contents

**Abstract**

This thesis explores a possible application of category theory to analytical mechanics by focusing on structural similarities between Newtonian, Lagrangian, and Hamiltonian mechanics. These three formalisms provide equivalent descriptions of physical systems through distinct mathematical frameworks. Despite being defined in different spaces, the they exhibit similar features that suggest the existence of a shared underlying structure.
Category theory, a mathematical framework for abstracting mathematical objects and studying their relations, offers a way to formalize structural relationships between the three mechanics formulations. By interpreting the states and time evolutions as objects and morphisms, this thesis constructs categories corresponding to the Newtonian, Lagrangian, and Hamiltonian formulation. These categories showcase the fundamental structural similarities of the formalisms and enable the definition of functors between the frameworks. These functors serve as mappings between the three categories. An application of this categorical framework is explicitly demonstrated through the example of a mathematical pendulum. This example points out incompatibilities of the objects with constraints, and inconsistencies in the previously defined transformations. To recover a representation of the dynamics of a pendulum via categories, subcategories are defined that consider the constraints on the system.

# 1 Introduction

In physics, there are often multiple ways to approach the same problem. Different theories and formalisms can be used to describe the same physical system, providing complementary perspectives and offering different insights into its behaviour.

Analytical mechanics is a prominent example with its three main formulations: *Newtonian mechanics*, *Lagrangian mechanics*, and *Hamiltonian mechanics*. These formalisms lead to equivalent equations of motion for a given physical system, despite relying on different mathematical structures. While they are based on different mathematical and conceptual spaces, they describe the same dynamics of the studied system, yielding identical solutions to the equations of motion.
These formulations of analytical mechanics are often used interchangeably, usually chosen based on convenience when solving a specific problem. This interchangeability and the shared physical descriptions suggest a fundamental structural relation, and provide an opportunity to study the connections between the different frameworks.

The change from one mechanics formulation to another can highlight redundancies. Where one formalism requires multiple variables to describe the system, a different formalism might only require a reduced set of variables.
Applying Ockham's razor to analytical mechanics motivates the reduction of redundant information, and encourages the search for a simplified framework that generalizes Newtonian, Lagrangian, and Hamiltonian mechanics. While having multiple descriptions of the same phenomena can demonstrate the consistency and universality of analytical mechanics, it can also be seen as a redundancy, and suggests the existence of a shared fundamental structure.

In mathematics, *category theory* provides a framework to study general structures and relations between mathematical objects. By abstracting mathematical objects to focus on their relational properties, and by studying the mappings between them, category theory can help formalize connections between seemingly different systems. Category theory thus offers a way to explore shared structures behind the various formulations of analytical mechanics. It could help identify how Newtonian, Lagrangian, and Hamiltonian mechanics are related, and how the descriptions can be transformed into each other.
This perspective aligns with Ockham's razor: instead of viewing the three formulations of analytical mechanics as entirely independent, category theory could showcase a shared description that highlights their fundamental structures while abstracting redundant details.

The following aims to explore the three main formulations of analytical mechanics within the framework of category theory. By interpreting mechanics through this categorical perspective, more fundamental structural properties could arise, providing insight into the connections between the different frameworks.

In section 2, Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics are introduced. Based on standard mechanics textbooks, [2] [10] [17], the main concepts and governing equations for each formalism are developed. Through this, similarities in the variables and time evolutions between the three descriptions appear, as summarized in section 2.4. These shared structures suggest a comparison with category theory. The most basic concepts in category theory are developed in section 3 through the definition of categories and the definition of functors in section 3.2. It is then investigated whether Newtonian, Lagrangian, and Hamiltonian mechanics each fulfill the requirements of a category, to then formulate the corresponding categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ in section 3. After the overarching structures of these categories have been established, a more generalized category, representing the three formalisms and showcasing their similarities, is described in section 4.4. This generalized category is used to characterize transformations between categories of this kind, introducing functors in section 5. Finally, this generalized framework from section 5.1 is applied in reverse, aiming to develop explicit functors between the Newtonian, Lagrangian, and Hamiltonian formulations. These functors describe how the formulations relate to another, and how to transform one mechanics description to a different one. An application of the developed categorical description to a specific physical system is given by the example of a simple pendulum in section 6.

This aims to explore shared structures in analytical mechanics by interpreting Newtonian, Lagrangian, and Hamiltonian mechanics in categorical terms. Applying category theory to mechanics can not only deepen the understanding of the different formalisms, but could also lead to further investigations into its applicability in theoretical physics.

# 2 Introduction to Classical Mechanics

Historically, describing the motion of an object was among the first problems studied in physics [10]. Determining the dynamics of a set of objects is the central objective of the field called *analytical mechanics* [15]. Throughout the centuries, different approaches to determine the so-called *equations of motion* of a physical system have evolved. Among the most prominent are *Newtonian mechanics*, *Lagrangian mechanics*, and *Hamiltonian mechanics*. All three formulations yield the same equations of motion for a given system, thus describing the same dynamical behaviour [15]. However, they are based on different principles and rely on distinct mathematical frameworks [10].

This section shall provide an overview of the three primary formulations in analytical mechanics. For each, the fundamental principles and mathematical structures shall be introduced, in combination with the key equations used in later applications. The following discussion is not exhaustive, and shall only serve as a concise summary of the key concepts necessary for subsequent sections.
To provide an example of a dynamic physical system, the motion of a simple mathematical pendulum will be studied in section 6, to obtain its equations of motion from the three introduced formalisms.

Here and in the following, **bold** letters denote vectors in two- or three-dimensional space. The indication of time dependence, e.g. $x(t)$, is initially stated for clarity. However, in subsequent instances the explicit dependence may be omitted for brevity $x(t) = x$, though it is to be understood that the variables remain time-dependent unless stated otherwise.

## 2.1 Newtonian Mechanics

The motion of one or more objects is described by specifying their positions at all times within a chosen coordinate system. This motion is often visualized using a trajectory or a worldline, which represents the position of an object as a function of time. [2]

To simplify such descriptions, a set of physical objects can often be idealized as a system of point masses. Within the framework of Newtonian mechanics, these point masses reside in three-dimensional Euclidean space. [2] In most cases, and in the following, the position $\mathbf{r}(t)$ of a single point mass is expressed in Cartesian coordinates, $\mathbf{r}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$. The change in position over time, the velocity of the point mass, is given by the time derivative $\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$.

The totality of positions $\mathbf{r}(t_0)$ and velocities $\dot{\mathbf{r}}(t_0)$ of all point masses at a specific time $t_0$ are referred to as the system's *initial conditions*. From those initial conditions, further kinematic quantities can be derived. [2]
Moreover, according to *Newton's law of determinacy*, the knowledge of the initial conditions suffices to describe the dynamics of the whole system for all times $t$, given sufficient smoothness conditions[2]. This means, that every position in time $\mathbf{r}(t), t \neq t_0$, as well as the velocity $\dot{\mathbf{r}}(t)$ can be determined from the set of initial conditions $(\mathbf{r}(t_0), \dot{\mathbf{r}}(t_0))$ [2].

The acceleration is defined as the second derivative of the position with respect to time, $\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}(t) = (\ddot{x}(t), \ddot{y}(t), \ddot{z}(t))$. Notably, the acceleration $\ddot{\mathbf{r}}(t)$ can be expressed as a function $\mathbf{F}$ of the position, velocity, and time: [2]

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \tag{1}$$

Equation eq. (1) is commonly referred to as *Newton's Equation*, and forms the basis of *Newtonian mechanics*. The exact form of $\mathbf{F}$ depends on the specific mechanical system studied and defines the system's dynamics. [2]

Within Newtonian mechanics, $\mathbf{F}$ represents the force acting on the point mass.
By introducing the momentum $\mathbf{p}$ with $\mathbf{p} = m\frac{d\mathbf{r}}{dt} = m\dot{\mathbf{r}}$, eq. (1) can be rewritten to obtain Newton's second law and *Newton's equation of motion*: [17] [2]

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \qquad \Leftrightarrow \qquad m\,\frac{d^2\mathbf{r}}{dt^2} = m\ddot{\mathbf{r}} = \mathbf{F} \tag{2}$$

Here, $m$ is the constant mass of the point mass. Equation (2) describes that the change in momentum with respect to time is equal to the sum of external forces acting on the point masses in the studied system.

For completeness, it should be mentioned that Newtonian mechanics can also be written in terms of a reduced set of coordinates, when constraints are placed on a system. A coordinate system different from $\mathbf{r} = (x, y, t)$ might be used to simplify equations. For an example, see section 6. However, for a general study of the Newtonian formalism, $\mathbf{r}(t), \dot{\mathbf{r}}(t) \in \mathbb{R}^3$ are considered, as above.

### 2.1.1 Time Variation

The time dependence and explicit form of the position $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}}(t)$ are determined by a system's equations of motion eq. (2). By setting up the equations of motion and solving the derived differential equations in consideration of the initial conditions, $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, and from those the time evolution of the whole system can be obtained.

However, the time evolution of $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ at a future time $t + \Delta t$ can be approximated using an expansion of the original state around $t$. For infinitesimal time steps $dt$, higher-order terms are assumed negligible.
Thus, for small timesteps $dt$, only the first-order terms are retained to give the (approximated) time variation:

$$\begin{aligned}
\mathbf{r}(t + dt) &= \mathbf{r}(t) + d\mathbf{r}(t)\, dt = \mathbf{r}(t) + \dot{\mathbf{r}}(\mathbf{t})\, dt = \mathbf{r}(t) + \frac{1}{m}\, \mathbf{p}(t)\, dt \\
\dot{\mathbf{r}}(t + dt) &= \dot{\mathbf{r}}(t) + d\dot{\mathbf{r}}(t)\, dt = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, dt = \dot{\mathbf{r}}(t) + \frac{1}{m}\frac{d\mathbf{p}(t)}{dt}\, dt = \dot{\mathbf{r}}(t) + \frac{1}{m}\mathbf{F}\, dt
\end{aligned} \tag{3}$$

Where the momentum $\mathbf{p}$ and the force $\mathbf{F}$ have been used to relate the time evolution to eq. (2).

### 2.1.2 Kinetic and Potential Energy

Within the framework of Newtonian mechanics, another quantity closely linked to forces acting on a point mass is the *energy* of an object. By displacing a point mass by the infinitesimal distance $d\mathbf{r}$ within the vector field of the force $\mathbf{F}$, work is done at the point mass: [17]

$$\mathbf{F} d\mathbf{r} = m\ddot{\mathbf{r}}\dot{\mathbf{r}}\, dt = \frac{1}{2} m\, d\dot{\mathbf{r}}^2$$

From this, the kinetic energy $T$ is obtained:

$$T = \frac{1}{2} m\dot{\mathbf{r}}^2 = \frac{1}{2}\frac{\mathbf{p}^2}{m} \tag{4}$$

Conservative forces $\mathbf{F}$ appear as the gradient of a scalar potential. Here, the scalar potential is the time-independent potential energy $U$, such that:

$$\mathbf{F} = -\nabla U \tag{5}$$

The total energy $E$ of a system is the sum of the kinetic and potential energy:

$$E = T + U \tag{6}$$

The energy $E(\mathbf{r}(t), \dot{\mathbf{r}})(t)$ does not explicitly depend on the time $t$. Therefore, $E$ is conserved over time: [17] [2]

$$\frac{dE}{dt} = \frac{d}{dt}\left(T + U)\right) = m\ddot{\mathbf{r}}\, \dot{\mathbf{r}} + \frac{dU}{dt}\dot{\mathbf{r}} = \dot{\mathbf{r}}\left(m\ddot{\mathbf{r}} - \mathbf{F}\right) = 0 \tag{7}$$

**Time Variation with respect to Energy**

Considering the kinetic Energy $T$ and potential energy $U$ from eq. (4) and eq. (5), the time variation from section 2.1.1 can be rewritten as:

$$\begin{aligned}\mathbf{r}(t+dt) &= \mathbf{r}(t) + d\mathbf{r}(t)\,dt = \mathbf{r}(t) + \dot{\mathbf{r}}(t)\,dt = \mathbf{r}(t) + \frac{dT}{d\mathbf{p}}\,dt \\ \dot{\mathbf{r}}(t+dt) &= \dot{\mathbf{r}}(t) + d\dot{\mathbf{r}}(t)\,dt = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\,dt = \mathbf{r}(t) - \frac{1}{m}\frac{dU}{d\mathbf{r}}\,dt\end{aligned} \tag{8}$$

## 2.2 Lagrangian Mechanics

In Lagrangian mechanics, the state of a system is described in its *configuration space* $Q$ [2] by the generalized coordinates $\mathbf{q} = q_1, q_2, ..., q_n$, with $n$ the number of degrees of freedom [10]. The generalized coordinates $q_i$ uniquely specify the configuration of a system [10]. The time derivative of $\mathbf{q}(t)$ is the generalized velocity $\dot{\mathbf{q}}$.

The equations of motion are derived from *Hamilton's principle of stationary action*, which states that the trajectory of a system is an extremal of the action $S$: [2]

$$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt$$

Where $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the *Lagrange function*: [2]

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\dot{\mathbf{q}}, t) - U(\mathbf{q}, t) \tag{9}$$

with the kinetic energy $T$ from eq. (4) and the potential energy $U$ eq. (5).

By performing a variation of $S$ [10], the *Euler-Lagrange equation* is obtained: [2]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{10}$$

These equations eq. (10) form a set of $n$ coupled second-order differential equations, describing the time evolution of the generalized coordinate $\mathbf{q}(t)$ and velocity $\dot{\mathbf{q}}(t)$ [2].

The Lagrangian formalism is particularly useful for a system with constraints and independent coordinates [10]. Instead of having to directly calculate forces as in section 2.1, it relies on the scalar kinetic and potential energy from section 2.1.2. Therefore, the formulation and properties of the Lagrangian do not depend on the choice of coordinates for the system [10] [2].

### 2.2.1 Time Variation

To analyze the time evolution of a Lagrangian system, the time variation of $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ over a small timestep $dt$ shall be considered. As in section 2.1.1, the time evolution $\mathbf{q}(t+dt), \dot{\mathbf{q}}(t+dt)$ of the generalized coordinates and velocities can be approximated via a Taylor expansion to the first order. Substituting the Euler-Lagrange equations into the expansion gives an explicit expression for $\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{q}}(t)$:

$$\mathbf{q}(t+dt) = \mathbf{q}(t) + d\mathbf{q}(t)dt = \dot{\mathbf{q}}(t) + \dot{\mathbf{q}}(t)\,dt \tag{11}$$

$$\dot{\mathbf{q}}(t+dt) = \dot{\mathbf{q}}(t) + d\dot{\mathbf{q}}(t)dt = \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t)dt = \dot{\mathbf{q}}(t) + \frac{\partial L}{\partial \mathbf{q}}dt \tag{12}$$

## 2.3 Hamiltonian Mechanics

Hamiltonian mechanics provides yet a different framework for describing the dynamics of a physical system. In contrast to the second order Euler-Lagrange equation eq. (9), *Hamiltonian mechanics* aims to describe the dynamics of a system in terms of first order equations of motion. Instead of having $n$ generalized coordinates $q_i$ in a n-dimensional configuration space, the Hamiltonian formulation uses $2n$ independent variables to describe the system in its*phase space*. [10] The so-called *canonical variables* $(\mathbf{q}(t), \mathbf{p}(t))$ are given by the generalized coordinate $\mathbf{q}(t)$ and the *canonical momentum* $\mathbf{p}(t)$ with: [2]

$$p_i = \frac{\partial L(q_j, \dot{q_j}, t)}{\partial \dot{q_i}}$$

For all times $t$, the state of the system is specified by the set of canonical variables $(\mathbf{q}, \mathbf{p})$.
The change from the Lagrangian to the Hamiltonian formulation corresponds to the change in variables $q_i, \dot{q_i}, L(q_i, \dot{q_i}, t) \rightarrow q_i, p_i, H(q_i, p_i, t)$ [17], given by the *Legendre transformation*: [10]

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{n} p_i \dot{q_i} \; - \; L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{13}$$

Differentiating $H(\mathbf{q}, \mathbf{p}, t)$ with respect to each variable, and comparing $dH$ with the differential of the left hand-side of eq. (13), gives Hamilton's equations: [10]

$$\dot{q_i} = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p_i} = -\frac{\partial H}{\partial q_i} \tag{14}$$

For a conservative system with velocity-independent potentials and the kinetic energy in the form of eq. (4), the Hamiltonian $H$ can be re-written as eq. (15), and gives the total energy $E$ of the system [2].

$$H = T + U = E \tag{15}$$

### 2.3.1 Time Variation

As in section 2.1.1 and section 2.2.1, the change in the coordinates for a small time variation $dt$ shall be considered. By expanding the generalized coordinate $\mathbf{q}(t)$ and canonical momentum $\mathbf{p}(t)$ around $t$ up to the first order, the time evolution of $(\mathbf{q}(t), \mathbf{p}(t))$ for small time steps $dt$ can be obtained. With eq. (14) it reads as:

$$\begin{aligned} \mathbf{q}(t + dt) &= \mathbf{q} + d\mathbf{q}\, dt = \mathbf{q}(t) + \dot{\mathbf{q}}\, dt = \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} dt \\ \mathbf{p}(t + dt) &= \mathbf{p} + d\mathbf{p}\, dt = \mathbf{p}(t) + \dot{\mathbf{p}}\, dt = \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}} dt \end{aligned} \tag{16}$$

## 2.4 Comparison of Newtonian, Lagrangian, and Hamiltonian Mechanics

In practice, the formalism used to study a physical system is chosen based on convenience when solving a particular problem in mechanics. At first glance, Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics seem to be very different formulations. They use different coordinates, and different equations eq. (2), eq. (6), eq. (9), eq. (13) to obtain the equations of motion. However, despite these differences, a shared underlying structure can be observed.

All three formalisms describe the physical system through time-dependent quantities in a state space. By generalizing the coordinates in eq. (3), eq. (11), and eq. (16), the state space variables for all formalisms can be written as $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$. In this notation, $\mathbf{Q}(t)$ describes a position (position coordinate $\mathbf{r}(t)$ or the generalized coordinate $\mathbf{q}(t)$). $\mathbf{P}(t)$ is a velocity-dependent entry in the state space, representing the velocity $\dot{\mathbf{r}}(t)$ in Newtonian, the generalized velocity $\dot{\mathbf{q}}(t)$ in Lagrangian, or the canonical momentum $\mathbf{p}(t)$ in Hamiltonian mechanics. According to the law of determinacy, see section 2.1, the information of the state of a system $\mathbf{V}(t_0) = (\mathbf{Q}(t_0), \mathbf{P}(t_0))$ at one time $t_0$ is enough

to determine all other states $\mathbf{V}(t \neq t_0)$ [2]. Thus, $\mathbf{V}(t_0)$ suffices to describe the whole dynamics of the studied system.

The time evolution of the system can be described by a variation of the time $t + dt$ based on a known time $t$. For small timesteps $dt$, this time variation can be approximated by a linear expansion. In Newtonian mechanics, this is given by eq. (3) if formulated via Newton's second law, or by eq. (8) in terms of the kinetic and potential energy. Comparing those equations with the Lagrangian time variation eq. (11) and the Hamiltonian equivalent eq. (16) showcases a similar structure. All time variations depend on the initial state $\mathbf{V}(t)$, and the derivative of the respective coordinates $d\mathbf{V}(t)\ dt$. A more general representation of the time evolution for all formalisms can thus be written as:

$$\mathbf{V}(t+dt) = \begin{pmatrix} \mathbf{Q}(t+dt) \\ \mathbf{P}(t+dt) \end{pmatrix} = \begin{pmatrix} \mathbf{Q}(t) + \dot{\mathbf{Q}}(t)dt \\ \mathbf{P}(t) + \dot{\mathbf{P}}(t)dt \end{pmatrix} \equiv \begin{pmatrix} \mathbf{Q}(t) \\ \mathbf{P}(t) \end{pmatrix} + \mathbf{d}_t(\mathbf{Q}, \mathbf{P})\ dt = \mathbf{V}(t) + \mathbf{d}_t\mathbf{V}(t)\ dt \tag{17}$$

Where $(\dot{\mathbf{Q}}(t)dt, \dot{\mathbf{P}}(t)dt) \equiv \mathbf{d}_t(\mathbf{Q}, \mathbf{P})\ dt) = \mathbf{d}_t\mathbf{V}(t)\ dt$ gives the time derivative of $\mathbf{V}(t)$. In this $\mathbf{d}_t$, the equations of motion eq. (2), eq. (10), and eq. (14) can be substituted into the time derivatives of the coordinates of $\mathbf{V}(t)$. This gives the time derivative in terms of the equations central to the chosen formalism (the force $\mathbf{F}$ or the energy $E$, the Lagrangian $L$, or the Hamiltonian $H$) and the known variables $(\mathbf{Q}(t), \mathbf{P}(t))$.

For comparison, the the respective state space variables $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$ and their derivatives $(\dot{\mathbf{Q}}(t), \dot{\mathbf{P}}(t))$ in Newtonian, Lagrangian, and Hamiltonian mechanics are summarized in table 1. For a better overview, the governing equations central to the formalisms are added to table 1. In the last column, the respective equations of motion, that determine the time derivative $\mathbf{d}_t\mathbf{V}(t)$, are given.

**Table 1:** Comparison of Newtonian, Lagrangian, and Hamiltonian mechanics, including the respective positions $\mathbf{Q}$, and velocity or momentum $\mathbf{P}$, and their time derivatives.

| **Formalism** | **Q** | $\dot{\mathbf{Q}}$ | **P** | $\dot{\mathbf{P}}$ | **Governing Equations** | **Equations of Motion** |
|---|---|---|---|---|---|---|
| Newtonian | $\mathbf{r}$ | $\dot{\mathbf{r}} = \frac{1}{m}\mathbf{p}$ | $\dot{\mathbf{r}}$ | $\ddot{\mathbf{r}} = \frac{1}{m}\frac{d\mathbf{p}}{dt}$ | Momentum $\mathbf{p}$ and Force $\mathbf{F}$ | $\mathbf{F} = m \cdot \frac{d\mathbf{p}}{dt}$ |
| | $\mathbf{r}$ | $\dot{\mathbf{r}} = \frac{dT}{d\mathbf{p}}$ | $\dot{\mathbf{r}}$ | $\ddot{\mathbf{r}} = \frac{1}{m}\frac{dU}{d\mathbf{r}}$ | Total Energy $E$ | $E = T + U$ |
| Lagrangian | $\mathbf{q}$ | $\dot{\mathbf{q}} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ | $\dot{\mathbf{q}}$ | $\ddot{\mathbf{q}} = \frac{\partial L}{\partial \mathbf{q}}$ | Lagrangian $L$ | $\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$ |
| Hamiltonian | $\mathbf{q}$ | $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$ | $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ | $\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$ | Hamiltonian $H$ | $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$, $\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$ |

Section 2.4 showcases a common underlying structure behind the dynamic descriptions of Newtonian, Lagrangian, and Hamiltonian mechanics. The variables and their time evolutions, derived from the respective equations of motion and summarized in table 1, share the same form of eq. (17). This generalized representation eq. (17) gives an abstraction of the variables used in each formalism.

In mathematics, the general structure of mathematical objects, their mappings, and the abstraction of different formalisms is studied within the framework of *Category Theory* [16]. Inspired by this approach, the following section 3 gives an introduction to the very basics of Category Theory, with the aim of using its framework to explore the shared structures and connections between Newtonian, Lagrangian, and Hamiltonian mechanics. This abstract perspective could provide further insights into the unifying principles underlying analytical mechanics.

# 3 Introduction to Category Theory

Category theory provides an abstract perspective on mathematical objects and the connections between them. When some objects are related, the relation can be represented by an arrow, which maps one object onto another [14]. A combination of multiple objects and arrows between them can be represented in a diagram. This diagram then gives a visualization of the relation between the objects, and mathematical properties of the construction can be studied based on the abstract form of the diagram, instead of explicitly looking at detailed formalisms. [13]

The arrows, representing the maps between objects, are called *morphisms* [16]. While in physics, such mappings are most often functions of variables, morphisms in category theory play a more general role and are not required to be functions [14]. A description in terms of the morphisms between the objects oftentimes showcases the structure of a system in a more direct way than when the focus in solely on the objects. To further investigate this mathematical structure, the relations between the morphisms can be studied. *Functors* are structure-preserving maps between categories. [16] They act as a morphism of categories [13], mapping objects from one category to objects from another, and the respective morphisms to each other [16]. On an additional level, the mappings between functors can be studied through *natural transformations* [14].

It was the study of those natural transformations that first motivated the development of category theory in the middle of the $20^{th}$ century. The aim to formally describe and define natural transformations required Samuel Eilenberg and Saunders McLane to work in reverse, defining functors and categories [14]. This created the initial framework of category theory.

Since then, category theory has developed into a broader field of study, extending beyond natural transformations. While the key notions of category theory include categories, functors, natural transformations, universality, and adjoints [16], many additional concepts have been established. In pure mathematics, the framework of category theory is often used to unify and clarify mathematical structures, reaching beyond its initial purpose in algebraic topology [16]. Additionally, category theory has found applications in a variety of different fields of study. The most prominent examples can be found in computer science, where the framework can be applied to functional programming and type theory [14] [3]. In physics, Bob Coecke introduced the framework of category theory to quantum mechanics [6] [7] and first with Samson Abramsky to quantum information theory and quantum computing [1]. Additionally, current studies in theoretical physics involve the application of category theory to superconductors [5] and condensed matter physics [12]. Higher category theory is used in the study of topological quantum field theory [11] [8] [9] [4]. Furthermore, category theory can also be used to relate different fields to each other, such as physics, topology, logic, and computation [3]. The list of examples is not exhaustive, and the growing number of applications across a wide range of disciplines highlights category theory as a framework for analyzing and unifying complex systems across diverse disciplines.

In the following, the two most basic concepts in category theory, categories and functors, are introduced through their formal definitions. The foundations of those provide a groundwork for applying category theory to analytical mechanics. Further details on categories and functors, as well as more advanced notions such as natural and adjoints, are not included as they are not required for the analysis in section 4 and section 5 and beyond the scope of this work.

## 3.1 Categories

In essence, categories consist of such *objects* and *arrows*, which relate the objects to each other. In principle, those objects can be anything, ranging from sets, to numbers and vector spaces, even vegetables in a recipe can be objects [7]. The exact nature of those objects is of lesser relevance, and they can be quite abstract.
Important for the definition of a category are the mappings from one object to another, represented by the arrows. The arrows from any object to any object of the category are also called the morphisms of the category, and need to fulfill a set of requirements regarding the way they map the objects. [13] [16]

In a more formal way a category can be constructed from the following definition definition 3.1: [16] [7]

**Definition 3.1 (Category)** *A category $\mathcal{C}$ consists of:*

1. ***Objects:*** *a collection of objects A, B, C, ..., $O \in \mathcal{C}$*
2. ***Morphisms:*** *a collection of arrows f, g, h, ...*
   *such that for any pair of objects $A, B \in \mathcal{C}$ the morphism $f \in \mathcal{C}(\mathrm{A}, \mathrm{B})$ with $f : \mathrm{A} \to \mathrm{B}$ represents the mapping from A to B*

   *This can also be denoted as $f_{A,B}$ or*

   $$A \xrightarrow{f} B$$

3. ***Composition:*** *For any $A, B, C \in \mathcal{C}$ and any $f \in \mathcal{C}(\mathrm{A}, \mathrm{B})$, $g \in \mathcal{C}(\mathrm{B}, \mathrm{C})$ a morphism of f with g such that*

   $$\mathcal{C}(\mathrm{A}, \mathrm{B}) \times \mathcal{C}(\mathrm{B}, \mathrm{C}) \to \mathcal{C}(\mathrm{A}, \mathrm{C})$$
   $$(g, f) \to g \circ f$$

   *The composition of morphisms is itself a morphism: $f \circ g \equiv h$ with $h \in \mathcal{C}(\mathrm{A}, \mathrm{C})$. This can be represented graphically by arrows:*

   $$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{g \circ f} & \downarrow{\scriptstyle g} \\ & & C \end{array} \tag{18}$$

   - ***Associativity:*** *The composition of morphisms is associative. For any $f \in \mathcal{C}(\mathrm{A}, \mathrm{B})$, $g \in \mathcal{C}(\mathrm{B}, \mathrm{C})$, $h \in \mathcal{C}(\mathrm{C}, \mathrm{D})$:*

     $$h \circ (g \circ f) = (h \circ g) \circ f$$

4. ***Identity morphism*** *For each object $A \in \mathcal{C}$, there exists a morphism $1_A \in \mathcal{C}(\mathrm{A}, \mathrm{A})$ which leaves the object unchanged: $1_A : \mathrm{A} \to \mathrm{A}$*

   *The identity can be applied right and left:*

   $$1_B \circ f_{AB} = f_{AB} \circ 1_A$$

   *The commutative diagram in eq. (18) can then be written as:*

   $$\begin{array}{ccccc} {\scriptstyle 1_A}\circlearrowright & A & \xrightarrow{f} & B & \circlearrowright{\scriptstyle 1_B} \\ & & \searrow^{g \circ f} & \downarrow{\scriptstyle g} & \\ & & & C & \circlearrowright{\scriptstyle 1_C} \end{array}$$

More precisely, the objects are parts of the class $\mathbf{Obj}(\mathcal{C})$, therefore $\mathrm{A} \in \mathbf{Obj}(\mathcal{C})$, and the morphisms are elements of the hom-set $f \in \hom_{\mathcal{C}}(\mathrm{A}, \mathrm{B})$ [16]. For simplicity, the above introduced short form $\mathrm{A} \in \mathcal{C}$ and $f \in \mathcal{C}(\mathrm{A}, \mathrm{B})$ shall be used whenever a more explicit notation is not necessary.

## 3.2 Functors

Functors are structure-preserving maps between categories [16]. A functor $F$ from one category to another maps the objects of the first category onto objects in the second, and the morphisms of the one category to morphisms in the other. When the arrow part is applied to morphisms, composition of morphisms and the identity morphisms from definition 3.1 are preserved [16].

A more detailed definition follows in definition 3.2: [16]

**Definition 3.2 (Functor)** *Consider two categories $\mathcal{C}$ and $\mathcal{D}$.*
*A functor $F : \mathcal{C} \to \mathcal{D}$ describes the mapping of the category $\mathcal{C}$ to the category $\mathcal{D}$.*
*$F$ consists of two mappings:*

1. *The **object part** maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$:*

$$F : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{D})$$

   *Or in the introduced short notation:*

$$F : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$$

2. *The **arrow part** maps morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$:*

$$F : hom_c(\mathrm{A}, \mathrm{B}) \to hom_D(F\mathrm{A}, F\mathcal{B})$$

   *for all $A$, $B \in \mathcal{C}$.*

*The same symbol $F$ is used for the object part and the arrow part.*
*Furthermore, a functor must fulfill the following conditions:*

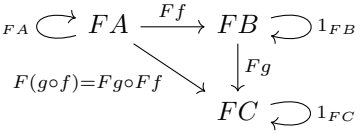
3. ***Composition**, whenever defined, is preserved.*
   *For a covariant functor:*

$$F(g \circ f) = Fg \circ Ff$$

4. ***Identity** is preserved:*

$$F1_A = 1_{FA}$$

$$FA \xrightarrow{Ff} FB$$




5. ***Composition of functors** is defined analogous to the composition of morphisms:*
   *For two functors $F : \mathcal{C} \Rightarrow \mathcal{D}$ and $G : \mathcal{D} \Rightarrow \mathcal{E}$ and an object $A \in \mathcal{C}$ the composition $G \circ F : \mathcal{C} \Rightarrow \mathcal{E}$ for the object part is given by*

$$(G \circ F)(A) = G(FA)$$

   *and for the morsphism part with $f \in \mathcal{C}(\mathrm{A}, \mathrm{B})$, or $f \in \mathrm{hom}_{\mathcal{C}}(\mathrm{A}, \mathrm{B})$:*

$$(G \circ F)(f) = G(Ff)$$

Comparing the above definition 3.1 with table 1 suggests that Newtonian, Lagrangian, and Hamiltonian mechanics may submit to the structure of a category. The states $V$ could represent objects within a category representing analytical mechanics. Time variations transform one state $V(t_0)$ into a different one $V(t \neq t_0)$, and thus they could be interpreted as a mapping from one object to another, representing a morphisms. Additionally, switching between the different formalisms could be achieved via functors according to definition 3.2. However, this hypothesis requires a more detailed examination.

In the following section 4, and attempt will be made to construct categories according to definition 3.1 from the mechanics formulations in section 2 and table 1, in order to verify a categorical structure on analytical mechanics formalisms. Provided section 4 successfully defines categories, similarities in their structure are studied to formalize functors between the categories according to definition 3.2 in section 5.

# 4 Categories in Analytical Mechanics

When introducing the three main formulations of analytical mechanics, namely Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics, particular emphasis is placed on the differences between them. The different formalisms operate within distinct mathematical spaces and take different approaches, utilizing their own basic equations, to obtain the equations of motion that describe the dynamics of a chosen object or system, see section 2.

These individual equations of motion all share the same solution, thus describing the same dynamics of a system [2]. Therefore, Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics are said to be physically equivalent. However, the three different formalisms of theoretical mechanics also have another commonality: they share a similar underlying structure, as developed in section 2.4.

Each formulation is built upon a set of coordinates and a function describing the evolution of these coordinates over time. For all three, the coordinates are time-dependent states, given by the position $\mathbf{Q}(t)$ and a velocity-dependent quantity (velocity or momentum) $\mathbf{P}(t)$, as explicitly given in table 1. The time evolution of these coordinate states is determined by the equations of motion. Given one initial state and the equations of motion, all other states at different times can be determined [2].
This structure of coordinates and time evolution functions, describing the mapping from one set of coordinates to another, suggests a category-like structure. Specifically, the coordinate states $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$ can be interpreted as objects within a category, and the time evolution functions can be interpreted as morphisms (arrows) $f$.

To further investigate this structure, the three mechanics formalisms — Newtonian, Lagrangian, and Hamiltonian — will be studied within the framework of category theory. By comparing the requirements for a category, definition 3.1, with the structures and equations inherent to the three formulations, three categories $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ will be constructed corresponding to their respective formalisms. This approach aims to establish a categorical framework for the formalization of analytical mechanics.

The development of this approach will be done progressively. Setting up a Newtonian category $\mathcal{N}$ within the formulation of Newtonian mechanics will be described in greater detail, serving as an example for the later categories. The formulation of a Lagrangian category $\mathcal{L}$ and a Hamiltonian category $\mathcal{H}$ will be introduced more concisely, relying on the parallels to the Newtonian construction. Great detail will be omitted in the construction of $\mathcal{L}$ and especially $\mathcal{H}$ to maintain brevity and avoid redundancy, since additional repetition of a similar approach would add little clarity. However, these similarities in the construction of $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ bring attention to the shared structures of the categories. Based on the similarities between the categories, a more abstract category $\mathcal{A}$ will be formulated in section 4.4.

## 4.1 Newtonian Mechanics, $\mathcal{N}$

As developed in section 2.1 and summarized in table 1, Newtonian mechanics uses Cartesian coordinates in Euclidean space. Objects are described by their position $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}}(t)$. The change in $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ is determined by Newton's equations of motion eq. (2). In combination with a set of initial conditions, all other possible positions and velocities of an object or a system at different times $t$ can be obtained. The time evolution from one state $(\mathbf{r}, \dot{\mathbf{r}})$ can be approximated via eq. (3) or eq. (8).

Comparing section 2.1 and table 1 with the definition of a category definition 3.1 in section 3, Newtonian mechanics, with the position and velocity as objects and the time evolution of those as morphisms, suggests the structure of a category. To explore this idea, the following will go through definition 3.1 and identify the elements in Newtonian mechanics that correspond to the components of a category, to construct a category $\mathcal{N}$ that represents Newtonian mechanics.

**Newtonian Category $\mathcal{N}$**

Newtonian mechanics forms a category $\mathcal{N}$ with:

1. **Objects:** The vector $(x, y, z) \in \mathbb{R}^3$, representing the Euclidean coordinates of a point in three-dimensional space, is an object in $\mathcal{N}$. Similarly, the vector $(\dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^3$, describing the time derivative of $(x, y, z)$, is another object in $\mathcal{N}$. The space of all possible positions $(x, y, z)$ and velocities $(\dot{x}, \dot{y}, \dot{z})$ of a given system at different times $t$ is represented by the vectors $\mathbf{r} = (x, y, z)$ and $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ respectively.

   Thus, the full collection of objects in $\mathcal{N}$ is given by $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{R}^3$:

$$(\mathbf{r}, \dot{\mathbf{r}}) \;=\; (x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \mathbf{Obj}(\mathcal{N})$$

2. **Morphisms:** Morphisms in $\mathcal{N}$ describe the mapping from one set of objects in $\mathcal{N}$ to another. For the positions and velocities $(\mathbf{r}, \dot{\mathbf{r}})$ this corresponds to a function $f_{\mathcal{N}}$ representing the time evolution of the state $(\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) = (\mathbf{r}(t_0), \dot{\mathbf{r}}(t_0))$ at time $t_0$ to a different time $t_1$:

$$f_{\mathcal{N}} : (\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) = (\mathbf{r}(t), \dot{\mathbf{r}}(t)) \to (\mathbf{r_1}, \dot{\mathbf{r}}_\mathbf{1}) = (\mathbf{r}(t_1), \dot{\mathbf{r}}(t_1))$$

   From the equations of motion eq. (1), the time evolution $f_{\mathcal{N}}$, for small time-steps $dt$, where $t_0 = t$ is some initial time and $t_1 = t + dt$ a different time, is given by eq. (3) or eq. (8):

$$\mathbf{r}(t_1) = \mathbf{r}(t + dt) = \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, dt = \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, dt = \mathbf{r}(t) + \frac{dT}{d\mathbf{p}}\, dt$$
$$\dot{\mathbf{r}}(t_1) = \dot{\mathbf{r}}(t + dt) = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, dt = \dot{\mathbf{r}}(t) + \frac{1}{m}\frac{d\mathbf{p}(t)}{dt}\, dt = \mathbf{r}(t) - \frac{1}{m}\frac{dU}{d\mathbf{r}}\, dt$$

   This results in the action of a morphism $f_{\mathcal{N}}$ on a state $(\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0})$:

$$f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t + dt) \\ \dot{\mathbf{r}}(t + dt) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, dt \end{pmatrix} \tag{19}$$

   This fulfills the conditions of a morphism in section 3, since for any state $(\mathbf{r}(t_0), \dot{\mathbf{r}}(t_0))$ of a given system at a defined $t_0$, all other positions and velocities $(\mathbf{r}(t \neq t_0),\ \dot{\mathbf{r}}(t \neq t_0))$ can be obtained via the equations of motion.

   To ensure that for any two states in $\mathcal{N}$, the morphism $f_{\mathcal{N}}$ maps one state to another, all possible states of the system and thus all times $t$ need to be obtainable with $f_{\mathcal{N}}$. Consider an initial state $(\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) = (\mathbf{r}(t), \dot{\mathbf{r}}(t))$. The definition of a morphism in definition 3.1 requires that all other states $(\mathbf{r}(t \neq t),\ \dot{\mathbf{r}}(t \neq t))$ can be obtained by applying the morphism $f_{\mathcal{N}}$ to this initial state. To achieve this with eq. (19), the total stepsize in eq. (3) and eq. (19) can be varied. By leaving the infinitesimal stepsize $dt$ unchanged, this is achieved by a variation of $n\, dt$ with $n \in \mathbb{Z}$ However, the total stepsize $ndt$ needs to remain sufficiently small such that the linear approximation used in eq. (3) is still valid. From the initial state $(\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) = (\mathbf{r}(t), \dot{\mathbf{r}}(t))$, all other states are then obtained by applying the morphism $f_{\mathcal{N},n} := (\mathbf{r}(t), \dot{\mathbf{r}}(t)) \to (\mathbf{r}(t + ndt), \dot{\mathbf{r}}(t + ndt)) = (\mathbf{r_n}, \dot{\mathbf{r}}_\mathbf{n})$. Equation (19) then reads as eq. (20):

$$f_{\mathcal{N},n} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t + ndt) \\ \dot{\mathbf{r}}(t + ndt) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, ndt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, ndt \end{pmatrix} \tag{20}$$

3. **Composition:**

The composition requirement in definition 3.1 ensures that for two morphisms $f_{\mathcal{N},a} : (\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) \to (\mathbf{r_a}, \dot{\mathbf{r}}_\mathbf{a}) \in \mathcal{N}$ and $f_{\mathcal{N},b} : (\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) \to (\mathbf{r_b}, \dot{\mathbf{r}}_\mathbf{b}) \in \mathcal{N}$, their composition $f_{\mathcal{N},a} \circ f_{\mathcal{N},b} \equiv f_{\mathcal{N},c}$ with $f_{\mathcal{N},c} : (\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) \to (\mathbf{r_c}, \dot{\mathbf{r}}_\mathbf{c})$ is itself again a morphism in $\mathcal{N}$.

This is represented graphically as:

$$\begin{array}{ccc} (\mathbf{r_0}, \dot{\mathbf{r}}_\mathbf{0}) & \xrightarrow{f_{\mathcal{N},a}} & (\mathbf{r_a}, \dot{\mathbf{r}}_\mathbf{a}) \\ & \searrow^{f_{\mathcal{N},c} = f_{\mathcal{N},a} \circ f_{\mathcal{N},b}} & \downarrow f_{\mathcal{N},b} \\ & & (\mathbf{r_c}, \dot{\mathbf{r}}_\mathbf{c}) \end{array}$$

For the morphism eq. (20) in $\mathcal{N}$, the composition represents the successive application of time variation under Newton's equations of motion eq. (3). Assuming infinitesimal timesteps $dt$, the consecutive application of time evolution in eq. (3) is linear. Given by eq. (19):

$$f_\mathcal{N}\ \mathbf{r}(t + dt) = \mathbf{r}(t + dt + dt) = \mathbf{r}(t + 2dt)$$

This is true for any timestep $ndt$, with $n \in \mathbb{Z}$, and a constant stepsize $dt$, which is sufficiently small for the linear approximation in eq. (3) to be valid. The composition of the morphisms $f_{\mathcal{N},b} \circ f_{\mathcal{N},a}$ is then simply the addition of their respective timesteps:

$$f_{\mathcal{N},a} \circ f_{\mathcal{N},b} = f_{\mathcal{N},a+b} = f_{\mathcal{N},c} \quad \Leftrightarrow \quad a + b = c \qquad a, b \in \mathbb{Z}$$

Commutativity of the morphisms is here ensured by the associativity of integer numbers. Since $a+b = b+a = c$ for $a, b \in \mathbb{Z}$, the commutation $f_{\mathcal{N},a} \circ f_{\mathcal{N},b} = f_{\mathcal{N},b} \circ f_{\mathcal{N},a} = f_{\mathcal{N},c}$ is valid. When interpreting the morphisms as time evolutions, this means that, for sufficiently small timesteps, the order in which multiple timesteps are taken does not change the resulting state.

Applying the composition on a state $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$ is explicitly demonstrated as:

$$\begin{aligned} f_{\mathcal{N},b} f_{\mathcal{N},a} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} &= f_{\mathcal{N},b} \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, adt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, adt \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, adt + \dot{\mathbf{r}}(t)\, bdt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, adt + \ddot{\mathbf{r}}(t)\, bdt \end{pmatrix} = \\ &= \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, (a+b)dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, (a+b)dt \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, cdt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, cdt \end{pmatrix} = \begin{pmatrix} \mathbf{r_c}(t) \\ \dot{\mathbf{r}}_\mathbf{c}(t) \end{pmatrix} = f_{\mathcal{N},c} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \end{aligned}$$

For multiple morphisms $f_{\mathcal{N},a}, f_{\mathcal{N},b}, f_{\mathcal{N},c}$, associativity of the time evolutions eq. (20) is ensured via the associativity of integers:

$$(f_{\mathcal{N},a} \circ f_{\mathcal{N},b}) \circ f_{\mathcal{N},c} = f_{\mathcal{N},a} \circ (f_{\mathcal{N},b} \circ f_{\mathcal{N},c})$$

$$\begin{aligned} (f_{\mathcal{N},c} f_{\mathcal{N},b}) f_{\mathcal{N},a} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} &= \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, (a + (b+c))dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, (a + (b+c))dt \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, (a+b+c)dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, (a+b+c)dt \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, ((a+b)+c)dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\, ((a+b)+c)dt \end{pmatrix} = f_{\mathcal{N},c}(f_{\mathcal{N},b} f_{\mathcal{N},a}) \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \end{aligned}$$

4. **Identity:** For each object $(\mathbf{r}(t), \dot{\mathbf{r}}(t)) \in \mathcal{N}$, the identity morphism $1_{\mathcal{N}}$ leaves the object unchanged.

$$1_{\mathcal{N}} : (\mathbf{r}(t), \dot{\mathbf{r}}(t)) \to (\mathbf{r}(t), \dot{\mathbf{r}}(t))$$

Having a set of time-dependent coordinates as objects, the identity is preserved when no change in time given, $dt = 0$:

$$1_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t + (dt = 0)) \\ \dot{\mathbf{r}}(t + (dt = 0)) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) + \dot{\mathbf{r}}(t) \cdot 0 \; dt \\ \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t) \cdot 0 \; dt \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \tag{21}$$

## 4.2 Lagrangian Mechanics, $\mathcal{L}$

Within the framework of Lagrangian mechanics, a physical system is described by the generalized coordinate $\mathbf{q}$ and velocity $\dot{\mathbf{q}}$, see section 2.2. In combination with the time evolution of $(\mathbf{q}, \dot{\mathbf{q}})$ via the Euler-Lagrange equation eq. (10), this again suggests to admit to the basic structure of a category.

The formulation of a Lagrangian category $\mathcal{L}$ follows the same procedure as for the Newtonian category $\mathcal{N}$ in section 4.1. In an abstract sense, the difference lies primarily in the choice of objects, where $(\mathbf{q}, \dot{\mathbf{q}})$ replace $(\mathbf{r}, \dot{\mathbf{r}})$, and the time evolution changes from eq. (3) to eq. (11). The following argument is therefore shortened, given its similarity to the previous discussion. For a complete derivation, refer to section 4.1, with adjusted objects and morphisms.

**Lagrangian Category $\mathcal{L}$**

Lagrangian mechanics forms a category $\mathcal{L}$ with:

1. **Objects:** The objects in $\mathcal{L}$ are the generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$.

   The collection of all objects in $\mathcal{L}$ is therefore given by $(\mathbf{q}, \dot{\mathbf{q}}) \in Q \Rightarrow (\mathbf{q}, \dot{\mathbf{q}}) \in \mathbf{Obj}(\mathcal{L})$, where $Q$ is the system's velocity phase space.

2. **Morphisms:** A morphism $f_{\mathcal{L}}$ in the Lagrangian category $\mathcal{L}$ can be understood as the mapping of a state $(\mathbf{q_0}, \dot{\mathbf{q}}_\mathbf{0}) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))$ at time $t$ to another state $(\mathbf{q_1}, \dot{\mathbf{q}}_\mathbf{1}) = (\mathbf{q}(t + dt),\ \dot{\mathbf{q}}(t + dt))$ at a time $t + dt$ according to the equations of motion.

$$f_{\mathcal{L}} : (\mathbf{q_0}, \dot{\mathbf{q}}_\mathbf{0}) \to (\mathbf{q_1}, \dot{\mathbf{q}}_\mathbf{1})$$

   The equations of motion are obtained from the Euler-Lagrange equation eq. (10) and give the time evolution for small timesteps $dt$ according to eq. (11):

$$\begin{aligned} \mathbf{q}(t_1) &= \mathbf{q}(t + dt) = \dot{\mathbf{q}}(t) + \dot{\mathbf{q}}(t)\, dt \\ \dot{\mathbf{q}}(t_1) &= \dot{\mathbf{q}}(t + dt) = \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t) dt = \dot{\mathbf{q}}(t) + \frac{\partial L}{\partial \mathbf{q}} dt \end{aligned}$$

   Therefore, applying the functor $f_{\mathcal{L}}$ gives the time variation:

$$f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t + dt) \\ \dot{\mathbf{q}}(t + dt) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t) + \dot{\mathbf{q}}(t)\, dt \\ \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t)\, dt \end{pmatrix} \tag{22}$$

   By varying the stepsize $dt$, all possible states of a given Lagrangian system can be reached. However, in the following the infinitesimal $dt$ shall remain constant for all time evolutions. To account for timesteps of different magnitude, the variation of timesteps $dt$ is instead written as $n\ dt$ with $n \in \mathbb{N}$. The map from an initial state $(\mathbf{q_0}, \dot{\mathbf{q}}_\mathbf{0}) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))$ to the state $(\mathbf{q_n}, \dot{\mathbf{q}}_\mathbf{n}) = (\mathbf{q}(t + ndt),\ \dot{\mathbf{q}}(t + ndt))$ is denoted by $f_{\mathcal{L},n}$.

3. **Composition:** Composition in $\mathcal{L}$ describes the succession of time steps, where applying time evolution multiple times with a small step size $dt$ is equivalent to applying a single time evolution with a larger step size.

   For example, for the two morphisms $f_{\mathcal{L},1}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = (\mathbf{q}(t+dt),\ \dot{\mathbf{q}}(t+dt)) \in \mathcal{L}$ and $f_{\mathcal{L},2}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = (\mathbf{q}(t+2dt),\ \dot{\mathbf{q}}(t+2dt)) \in \mathcal{L}$ their composition is given by the commutative diagram:

$$\begin{array}{ccc} (\mathbf{q_0}, \dot{\mathbf{q}}_\mathbf{0}) & \xrightarrow{f_{\mathcal{L},1}} & (\mathbf{q_1}, \dot{\mathbf{q}}_\mathbf{1}) \\ & \searrow^{f_{\mathcal{L},3} = f_{\mathcal{L},2} \circ f_{\mathcal{L},1}} & \downarrow f_{\mathcal{L},2} \\ & & (\mathbf{q_3}, \dot{\mathbf{q}}_\mathbf{3}) \end{array}$$

   The morphism $f_{\mathcal{L},2} \circ f_{\mathcal{L},1} = f_{\mathcal{L},3}$ is equivalent to the combined time evolution over the entire interval, going from the initial state $(\mathbf{q_0}, \dot{\mathbf{q}}_\mathbf{0})$ directly to $(\mathbf{q_3}, \dot{\mathbf{q}}_\mathbf{3})$:

$$\begin{gathered} f_{\mathcal{L},2}\, f_{\mathcal{L},1}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = f_{\mathcal{L},2}\,(\mathbf{q}(t+dt), \dot{\mathbf{q}}(t+dt)) = (\mathbf{q}(t+dt+2dt), \dot{\mathbf{q}}(t+dt+2dt)) = \mathbf{q}(t+3dt), \dot{\mathbf{q}}(t+3dt) \\ \Rightarrow f_{\mathcal{L},3}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = (\mathbf{q}(t+3dt), \dot{\mathbf{q}}(t+3dt)) \end{gathered}$$

   Here it is again assumed that the time steps $n\ dt$ with $n \in \mathbb{Z}$ is sufficiently small such that the first-order approximation in section 2.2.1 holds. In this notation, the composition of morphisms with different time steps $a\ dt$ and $b\ dt$ resulting in a final time step $(a+b)\ dt$ is denoted as $f_{\mathcal{L},a} \circ f_{\mathcal{L},b} = f_{\mathcal{L},a+b}$ .

   This succession of states is associative, since the order of taking the step $a\ dt$ followed by $b\ dt$, or vice versa, does not affect the result. The same applies to multiple time steps:

$$(f_{\mathcal{L},a} \circ f_{\mathcal{L},b}) \circ f_{\mathcal{L},c} = f_{\mathcal{L},a} \circ (f_{\mathcal{L},b} \circ f_{\mathcal{L},c})$$

   For example:

$$(f_{\mathcal{L},1} \circ f_{\mathcal{L},2}) \circ f_{\mathcal{L},3} = f_{\mathcal{L},1} \circ (f_{\mathcal{L},2} \circ f_{\mathcal{L},3})$$

$$\begin{aligned} &(f_{\mathcal{L},1} f_{\mathcal{L},2}) f_{\mathcal{L},3}\ \mathbf{q_0} = f_{\mathcal{L},3} f_{\mathcal{L},3}\ \mathbf{q_0} = \mathbf{q}(t+3dt+3dt) = \mathbf{q}(t+6dt) = \mathbf{q_6} = \\ &f_{\mathcal{L},1}(f_{\mathcal{L},2} f_{\mathcal{L},3})\ \mathbf{q_0} = f_{\mathcal{L},1} f_{\mathcal{L},5}\ \mathbf{q_0} = \mathbf{q}(t+\ dt+5dt) = \mathbf{q}(t+6dt) = \mathbf{q_6} \end{aligned}$$

4. **Identity:** Applying the identity morphism $1_\mathcal{L}$ leaves the objects unchanged,$1_\mathcal{L} : (\mathbf{q}(t), \dot{\mathbf{q}}(t)) \to (\mathbf{q}(t), \dot{\mathbf{q}}(t))$. Therefore this corresponds to no time evolution in the state $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{L}$.

   This condition is met for $(\mathbf{q}(t+dt), \dot{\mathbf{q}}(t+dt)) = (\mathbf{q}(t), \dot{\mathbf{q}}(t)) \Leftrightarrow dt = 0$. Therefore, $1_\mathcal{L} = f_{\mathcal{L},0}$, and $1_\mathcal{L}$ is given by:

$$1_\mathcal{L} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t+(dt=0)) \\ \dot{\mathbf{q}}(t+(dt=0)) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} \tag{23}$$

## 4.3 Hamiltonian Mechanics, $\mathcal{H}$

Embedding Hamiltonian mechanics within the framework of category theory to formulate the Hamiltonian category $\mathcal{H}$ again follows the same approach as in section 4.1 and section 4.2. Due to the repetitive nature of setting up the category, this section will only provide a concise summary. However, as the following is fully analogous to section 4.1 and section 4.2, by replacing the coordinates in section 4.1 or section 4.2 with the canonical variables $(\mathbf{q}(t), \mathbf{p}(t))$, and the respective equations of motion by the corresponding Hamiltonian eq. (13) and Hamilton's equations of motion eq. (14), the definition of $\mathcal{H}$ can be explored in greater detail.

**Hamiltonian Category $\mathcal{H}$**

Hamiltonian mechanics forms a category $\mathcal{H}$ with:

1. **Objects:** Objects in $\mathcal{H}$ are the set of canonical coordinates $\mathbf{q}$ and canonical momenta $\mathbf{p}$, given by $(\mathbf{q}, \mathbf{p}) \in \mathbf{Obj}(\mathcal{H})$, where $\mathbf{q}$ and $\mathbf{q}$ are in the system's phase space.

2. **Morphisms:** The morphism $f_{\mathcal{H},n}$ maps one phase space point $(\mathbf{q_0}, \mathbf{p_0}) = (\mathbf{q}(t), \mathbf{p}(t))$ to another $(\mathbf{q_n}, \mathbf{p_n}) = (\mathbf{q}(t + ndt), \mathbf{p}(t + ndt))$:

$$f_{\mathcal{H}} : (\mathbf{q_0}, \mathbf{p_0}) \to (\mathbf{q_n}, \mathbf{p_n})$$

   The time evolution is governed by Hamilton's equations of motion eq. (14) and is given by eq. (16):

$$\mathbf{q}(t + dt) = \mathbf{q} + d\mathbf{q}\, dt = \mathbf{q}(t) + \dot{\mathbf{q}}\, dt = \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}(t)dt$$
$$\mathbf{p}(t + dt) = \mathbf{p} + d\mathbf{p}\, dt = \mathbf{p}(t) + \dot{\mathbf{p}}(t)\, dt = \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}dt$$

   This gives the morphism $f_{\mathcal{H},n}$ as:

$$f_{\mathcal{H},n} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t + ndt) \\ \dot{\mathbf{q}}(t + ndt) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t) + \dot{\mathbf{q}}(t)\, ndt \\ \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t)\, ndt \end{pmatrix} \tag{24}$$

   Again, it is assumed that the differential timesteps $n\ dt$ are sufficiently small for the linear approximation in eq. (16) to hold.

3. **Composition:** The composition of morphisms $f_{\mathcal{H},a} \in \mathcal{H}$ and $f_{\mathcal{H},b} \in \mathcal{H}$ is given by the addition of time steps:

$$f_{\mathcal{H},a} \circ f_{\mathcal{H},b} = f_{\mathcal{H},a+b}$$

$$\begin{array}{ccc} (\mathbf{q_0}, \mathbf{p_0}) & \xrightarrow{f_{\mathcal{H},a}} & (\mathbf{q_a}, \dot{\mathbf{q}}_\mathbf{a}) \\ & \searrow^{f_{\mathcal{H},a+b} = f_{\mathcal{H},b} \circ f_{\mathcal{H},a}} & \downarrow f_{\mathcal{H},b} \\ & & (\mathbf{q_{a+b}}, \dot{\mathbf{q}}_\mathbf{a+b}) \end{array}$$

   Associativity is given by the independence of the final state $(\mathbf{q_n}, \mathbf{p_n})$ on the order in which the time steps are taken. For three morphisms $f_{\mathcal{H},a},\ f_{\mathcal{H},b}\ f_{\mathcal{H},c}$ with $a + b + c = n$ :

$$(f_{\mathcal{H},a} \circ f_{\mathcal{H},b}) \circ f_{\mathcal{H},c} = (f_{\mathcal{H},a} \circ f_{\mathcal{H},b}) \circ f_{\mathcal{H},c} = f_{\mathcal{H},a+b+c} = f_{\mathcal{H},n}$$
$$(f_{\mathcal{H},a} \circ f_{\mathcal{H},b}) \circ f_{\mathcal{H},c}\ \mathbf{q_0} = (f_{\mathcal{H},a} \circ f_{\mathcal{H},b}) \circ f_{\mathcal{H},c}\ \mathbf{q_0} = \mathbf{q}(t + adt + bdt + cdt) = \mathbf{q}(t + ndt)$$

4. **Identity:** Again, the condition for the identity morphism $1_\mathcal{H}$ to leave objects unchanged is satisfied for morphisms $f_{\mathcal{H},0} = 1_\mathcal{H}$ with no change in time, $dt = 0$:

$$(\mathbf{q}(t + dt), \mathbf{p}(t + dt)) = (\mathbf{q}(t), \mathbf{p}(t)) \Leftrightarrow dt = 0$$

   The identity morphism, $1_\mathcal{H}$ is therefore given by:

$$1_\mathcal{H} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t + (dt = 0)) \\ \mathbf{p}(t + (dt = 0)) \end{pmatrix} = \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \tag{25}$$

## 4.4 Comparison of $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$

Formulating the three categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ followed the same methodology based on definition 3.1. Additionally, when comparing section 4.1, section 4.2, and section 4.3, all three categories share the same form of objects and morphisms, while the composition of morphisms and the identity morphisms are defined in the same manner.
The objects within these categories represent states, which are composed of position- and velocity-dependent quantities. Despite differences in formulation, the time evolution of these states adheres to a common structural framework.

This has parallels with section 2.4. The objects in $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ can be represented by the state $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$, as developed in section 2.4. All three morphisms $f_{\mathcal{N}}$, $f_{\mathcal{L}}$, and $f_{\mathcal{H}}$ share the same form as the time evolution of $\mathbf{V}(t)$ in eq. (17). This similarity suggests the formulation of a category $\mathcal{A}$, serving as a generalized representation and summary of $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$.

**Generalized Category $\mathcal{A}$**

Based on the similar structure of Newtonian, Lagrangian, and Hamiltonian mechanics from section 2.4, and the similarities between the categories $\mathcal{N}$ (section 4.1), $\mathcal{L}$ (section 4.2), and $\mathcal{H}$ (section 4.3), a general approach for defining a category $\mathcal{A}$ within analytical mechanics follows definition 3.1 with:

1. **Objects:** Objects in $\mathcal{A}$ are the set of coordinate states $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$, where $\mathbf{Q}(t)$ describes a position as a coordinate and $\mathbf{P}(t)$ represents a velocity-dependent quantity, a velocity or a momentum.

2. **Morphisms:** For any object $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$, the morphism $f_{\mathcal{C},n}$ maps the state $(\mathbf{Q}(t), \mathbf{P}(t))$ to another state $\mathbf{V}(t + ndt) = (\mathbf{Q}(t + ndt), \mathbf{P}(t + ndt))$. To be able to map any initial state $\mathbf{V}(t_0)$ at any time $t_0$ to all other possible states $\mathbf{V}(t \neq t_0)$, the timestep $ndt$ can be adjusted by a variation of $n \in \mathbb{Z}$.

   Assuming the stepsize $dt$ and the total time variation $ndt$ are sufficiently small, the time evolution of $(\mathbf{Q}(t), \mathbf{P}(t))$ can be approximated by a first-order expansion, see eq. (17).

   $f_{\mathcal{C},n}$ is then explicitly given by:

$$f_{\mathcal{C},n}\, \mathbf{V}(t) = \begin{pmatrix} \mathbf{Q}(t + ndt) \\ \mathbf{P}(t + ndt) \end{pmatrix} = \begin{pmatrix} \mathbf{Q}(t) + \dot{\mathbf{Q}}(t) ndt \\ \mathbf{P}(t) + \dot{\mathbf{P}}(t) ndt \end{pmatrix} \equiv \begin{pmatrix} \mathbf{Q}(t) \\ \mathbf{P}(t) \end{pmatrix} + \mathbf{d}_t(\mathbf{Q}, \mathbf{P})\, ndt = \mathbf{V}(t) + \mathbf{d}_t\mathbf{V}(t)\, ndt \tag{26}$$

   The explicit form of $\mathbf{d}_t(\mathbf{Q}, \mathbf{P})\, dt$ is dependent on the chosen physical framework, and can be derived from the respective equations of motion.

3. **Composition:** Composition of morphisms is given by the addition of time steps.

   For $N$ morphisms $f_{\mathcal{C},n_i},\ i = 1, 2, \ldots, N$, their composition is given by:

$$f_{\mathcal{C},n_1} \circ f_{\mathcal{C},n_2} \circ \cdots \circ f_{\mathcal{C},n_N} = f_{\mathcal{C},n_1+n_2+\cdots+N}$$

$$f_{\mathcal{C},n_1+n_2+\cdots N}\, \mathbf{V}(t) = \mathbf{V}(t) + \mathbf{d}_t\mathbf{V}(t)\ \cdot (n_1 + n_2 + \cdots + N) \cdot ndt$$

   For any two morphisms $f_{\mathcal{C},a} : (\mathbf{Q}(t), \mathbf{P}(t)) \to (\mathbf{Q}(t + adt), \mathbf{P}(t + adt))$, $f_{\mathcal{C},b} : (\mathbf{Q}(t), \mathbf{P}(t)) \to (\mathbf{Q}(t + bdt), \mathbf{P}(t + bdt))$, $\in \mathcal{C}$ and their composition $f_{\mathcal{C},a} \circ f_{\mathcal{C},b} \equiv f_{\mathcal{C},c}$, $f_{\mathcal{C},c} : (\mathbf{Q}(t), \mathbf{P}(t)) \to (\mathbf{Q}(t+cdt), \mathbf{P}(t+cdt))$ , the commutative diagram eq. (18) is then written as:

$$\begin{array}{ccc} (\mathbf{Q_0}, \mathbf{P_0}) & \xrightarrow{f_{\mathcal{C},a}} & (\mathbf{Q_a}, \mathbf{P_a}) \\ & \searrow^{f_{\mathcal{C},c} = f_{\mathcal{C},a} \circ f_{\mathcal{C},b}} & \downarrow f_{\mathcal{C},b} \\ & & (\mathbf{Q_c}, \mathbf{P_c}) \end{array}$$

   Where $(\mathbf{Q_a}, \mathbf{Q_a}) = (\mathbf{Q}(t + adt), \mathbf{P}(t + adt))$.

Associativity is ensured for all $n_i \in \mathbb{Z}$ via the associativity of the addition of integer numbers. For three exemplary morphisms $f_{\mathcal{C},n_1},\ f_{\mathcal{C},n_2}\ f_{\mathcal{C},n_3}$ with $n_1 + n_2 + n_3 = N$:

$$(f_{\mathcal{C},n_1} \circ f_{\mathcal{C},n_2}) \circ f_{\mathcal{C},n_3} = (f_{\mathcal{C},n_1} \circ f_{\mathcal{C},n_2}) \circ f_{\mathcal{C},n_3} = f_{\mathcal{C},n_1+n_2+n_3} = f_{\mathcal{C},N}$$
$$\text{since}\ \ (n_1 + n_2) + n_3 = n_1 + (n_2 + n_3) = n_1 + n_2 + n_3 \quad \forall n_1, n_2, n_3 \in \mathbb{Z}$$

4. **Identity:** The identity morphism $1_{\mathcal{C}}$ leaves objects in $\mathcal{A}$ unchanged. For any time-dependent formulations, the state $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$ necessarily remains unchanged if no change in time is given, $t + dt = t \Leftrightarrow dt = 0$. $1_{\mathcal{C}}$ therefore acts as a morphism $f_{\mathcal{C},0}$ where $dt = 0$ :

$$1_{\mathcal{C}}\, \mathbf{V}(t) = f_{\mathcal{C},0}\, \mathbf{V}(t) = \begin{pmatrix} \mathbf{Q}(t + (dt = 0)) \\ \mathbf{P}(t + (dt = 0)) \end{pmatrix} = \begin{pmatrix} \mathbf{Q}(t) \\ \mathbf{P}(t) \end{pmatrix} + \mathbf{d}_t(\mathbf{Q}, \mathbf{P}) \cdot\ 0 \cdot dt = \mathbf{V}(t) \tag{27}$$

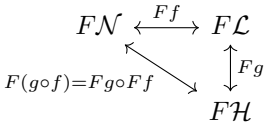
# 5 Functors in Classical Mechanics

When comparing the three categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$, they all show similar underlying structures, leading to a more generalized formulation of categories within theoretical mechanics as discussed in section 4.4.
The objects within these categories are the time-dependent states of the studied system. Despite differences in the exact form, the time evolutions of these states have a common structure, as shown in eq. (26). The primary differences between the categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ lie in the respective state spaces and the specific equations leading to the equations of motion.

This structural similarity suggests the existence of a transformation between the different categories, connecting these different formalisms. Returning to the basics of category theory as discussed in section 3, and combining them with the categories of section 4, this similarity transformation aligns with the concept of functors. In this context, functors could establish a link between the categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$.
The aim is to formulate functors $F$ between the categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ according to definition 3.2 provided in section 3.2. These mappings result in six transformations:

$$\mathcal{N} \leftrightarrow \mathcal{L}, \quad \mathcal{L} \leftrightarrow \mathcal{H}, \quad \mathcal{N} \leftrightarrow \mathcal{H}$$

with two functors for each pairing: one for the 'to' $\rightarrow$ direction and another for the reverse 'from' $\leftarrow$ direction.
Given that a physical system can be equivalently described in all three formalisms, these functors $F$ shall fulfill the following relations:




Before studying the detailed formulations of the functors, the underlying structure and effect of those functors can be studied through a more generalized framework, section 5.1, based on section 4.4.

## 5.1 Generalized Functor $F_{\mathcal{A},\mathcal{B}}$

For all mappings between the here studied categories, the objects are the coordinate states $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$, as given in table table 1. Given two categories $\mathcal{A}$ and $\mathcal{B}$, with their objects $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t)) \in \mathbf{Obj}(\mathcal{A})$ and $\mathbf{V}'(t) = (\mathbf{Q}'(t), \mathbf{P}'(t)) \in \mathbf{Obj}(\mathcal{B})$, the object part $F^{obj}_{\mathcal{A},\mathcal{B}}$ of the functor $F_{\mathcal{A},\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$ therefore describes a coordinate transformation:

$$F^{obj}_{\mathcal{A},\mathcal{B}} : (\mathbf{Q}(t), \mathbf{P}(t)) \rightarrow (\mathbf{Q}'(t), \mathbf{P}'(t)) \tag{28}$$

All time variations $\mathbf{V}(t+dt) \in \mathcal{A}$ from table table 1, and therefore also all morphisms $f_{\mathcal{A}} : \mathbf{V}(t) \to \mathbf{V}(t+dt)$ in section 4, show the same structure as eq. (26):

$$f_{\mathcal{A}}\,\mathbf{V}(t) = \begin{pmatrix}\mathbf{Q}(t+dt)\\ \mathbf{P}(t+dt)\end{pmatrix} = \begin{pmatrix}\mathbf{Q}(t)+\dot{\mathbf{Q}}(t)dt\\ \mathbf{P}(t)+\dot{\mathbf{P}}(t)dt\end{pmatrix} \equiv \begin{pmatrix}\mathbf{Q}(t)\\ \mathbf{P}(t)\end{pmatrix} + \mathbf{d}_t \begin{pmatrix}\mathbf{Q}(t)\\ \mathbf{P}(t)\end{pmatrix} dt = \mathbf{V}(t) + \mathbf{d}_t\mathbf{V}(t)\,dt \tag{29}$$

Where $\mathbf{Q}$ and $\mathbf{P}$ are the coordinates, and $\mathbf{d}_t$ gives their time derivative. The exact form of $\mathbf{d}_t(\mathbf{Q},\mathbf{P})$ is determined by the system's equations of motion. For brevity, the multiplicative factor $n$ before the timestep $ndt$ as in eq. (26) is omitted.
The arrow part $F^{arr}_{\mathcal{A},\mathcal{B}}$ of the functor transforms morphisms in $\mathcal{A}$ to those in $\mathcal{B}$:

$$F^{arr}_{\mathcal{A},\mathcal{B}} : f_{\mathcal{A}} \to f_{\mathcal{B}}$$

With eq. (29), $F^{arr}_{\mathcal{A},\mathcal{B}}$ transforms as:

$$F^{arr}_{\mathcal{A},\mathcal{B}}\, f_{\mathcal{A}}\mathbf{V}(t) = f_{\mathcal{B}}\mathbf{V}'(t)$$

$$F^{arr}_{\mathcal{A},\mathcal{B}}\, f_{\mathcal{A}}\,\mathbf{V}(t) = F^{arr}_{\mathcal{A},\mathcal{B}}\,\mathbf{V}(t) + F^{arr}_{\mathcal{A},\mathcal{B}}\,\mathbf{d}_t\mathbf{V}(t)\,dt = \mathbf{V}'(\mathbf{V}(t)) + \mathbf{d}_t\mathbf{V}'(\mathbf{V}(t))\,dt \tag{30}$$

Examining the relation in eq. (30) in greater detail reveals that the application of the arrow part of the functor on the given state $F^{arr}_{\mathcal{A},\mathcal{B}}\,\mathbf{V}(t)$ is equivalent to the coordinate transformation given by $F^{obj}$ in eq. (28).

Additionally, for $F$ to be a functor according to definition 3.2, the following conditions need to be fulfilled:

1. **Preservation of composition**
   For a covariant functor $F$, and two morphisms $f_{\mathcal{A},a}$, $f_{\mathcal{A},b}$ the composition shall fulfill:
   $$F(f_{\mathcal{A},b} \circ f_{\mathcal{A},a}) = Ff_{\mathcal{A},b} \circ Ff_{\mathcal{A},a}$$
   With $f_{\mathcal{A},a}$, $f_{\mathcal{A},b}$ being of the form eq. (26), $f_b \circ f_a = f_{a+b} \equiv f_c$, and eq. (30), the composition is achieved by:
   $$\begin{aligned} F^{arr}_{\mathcal{A},\mathcal{B}}\,(f_{\mathcal{A},b} \circ f_{\mathcal{A},a})\,\mathbf{V}(t) = F^{arr}_{\mathcal{A},\mathcal{B}}\,f_{\mathcal{A},c}\,\mathbf{V}(t) = f_{\mathcal{B},c}\,\mathbf{V}'(t) = \mathbf{V}'(t+cdt) = \\ (F^{arr}_{\mathcal{A},\mathcal{B}}\,f_{\mathcal{A},b} \circ F^{arr}_{\mathcal{A},\mathcal{B}}\,f_{\mathcal{A},a})\,\mathbf{V}(t) = (f_{\mathcal{B},b} \circ f_{\mathcal{B},a})\,\mathbf{V}'(t) = f_{\mathcal{B},c}\,\mathbf{V}'(t) = \mathbf{V}'(t+cdt) \end{aligned} \tag{31}$$
2. **Preservation of identity** The preservation of the identity morphisms $1_{\mathcal{A}}$ requires:
   $$F1_{\mathcal{A}} = 1_{F\mathcal{A}}$$
   In the above introduced notation this is explicitly given by:
   $$F^{arr}_{\mathcal{A},\mathcal{B}}\,1_{\mathcal{A}}\,\mathbf{V}(t) = F^{arr}_{\mathcal{A},\mathcal{B}}\,f_{\mathcal{A},0}\,\mathbf{V}(t) = F^{arr}_{\mathcal{A},\mathcal{B}}\,\mathbf{V}(t) + F^{arr}_{\mathcal{A},\mathcal{B}}\,\mathbf{d}_t\mathbf{V}(t)\cdot 0\cdot dt = \mathbf{V}'(t) = 1_{\mathcal{B}}\,\mathbf{V}'(t) \tag{32}$$
3. **Composition of functors**
   For two functors $F_{\mathcal{A},\mathcal{B}} : \mathcal{A} \to \mathcal{B}$, $F_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}$, their objects $\mathbf{V}(t) \in \mathbf{Obj}(\mathcal{A})$, $\mathbf{V}'(t) \in \mathbf{Obj}(\mathcal{B})$, $\mathbf{V}''(t) \in \mathbf{Obj}(\mathcal{C})$, and the morphisms $f_{\mathcal{A}}$, $f_{\mathcal{B}}$, $f_{\mathcal{C}}$, the composition of functors is given by:
   $$(F^{arr}_{\mathcal{B},\mathcal{C}} \circ F^{arr}_{\mathcal{A},\mathcal{B}})\,f_{\mathcal{A}}\,\mathbf{V}(t) = F^{arr}_{\mathcal{A},\mathcal{C}}\,f_{\mathcal{A}}\,\mathbf{V}(t) = f_{\mathcal{C}}\,\mathbf{V}''(t) = F^{arr}_{\mathcal{B},\mathcal{C}}(F^{arr}_{\mathcal{A},\mathcal{B}})\,f_{\mathcal{A}}\,\mathbf{V}(t) = F^{arr}_{\mathcal{B},\mathcal{C}}\,f_{\mathcal{B}}\,\mathbf{V}'(t) \tag{33}$$

In the following, functors $F$ between the three categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ will be proposed. By constructing the functors as outlined above for the general case $F_{\mathcal{A},\mathcal{B}}$ it is ensured that the $F$ fulfill the definition of a functor according to definition 3.2. The specific form of the six functors $F_{\mathcal{N},\mathcal{L}}$, $F_{\mathcal{L},\mathcal{N}}$, $F_{\mathcal{L},\mathcal{H}}$, $F_{\mathcal{H},\mathcal{L}}$, $F_{\mathcal{N},\mathcal{H}}$, $F_{\mathcal{H},\mathcal{N}}$ will be derived explicitly whenever necessary and possible, to ensure a physically accurate transformation of the used coordinates and equations of motion. However, as the construction of each transformation follows the pattern for $F_{\mathcal{A},\mathcal{B}}$, detailed steps will

be omitted wherever repeating a similar demonstration does not give additional insights. Especially when a functor $F_{\mathcal{A},\mathcal{B}}$ has been defined, the definition of the reverse direction $F_{\mathcal{B},\mathcal{A}}$ is shortened due to shared properties.

As the transformation from the Lagrangian to the Hamiltonian formalism is known to be given by the Legendre Transformation eq. (13), the mapping from the Lagrangian category to the Hamiltonian and vice-versa, $\mathcal{L} \leftrightarrow \mathcal{H}$, will be studied first. This choice is made for clarity and to build upon familiar ground. Subsequently, the other two transformations, $\mathcal{N} \leftrightarrow \mathcal{L}$ and $\mathcal{N} \leftrightarrow \mathcal{H}$, will be explored in a similar manner.

## 5.2 Lagrangian $\mathcal{L} \leftrightarrow$ Hamiltonian $\mathcal{H}$

The Functor $F_{\mathcal{L},\mathcal{H}} : \mathcal{L} \to \mathcal{H}$ gives the transformation from the Lagrangian category $\mathcal{L}$ to the Hamiltonian category $\mathcal{H}$. Similarly, the Functior $F_{\mathcal{H},\mathcal{L}} : \mathcal{H} \to \mathcal{L}$ gives the back-transformation.
Traditionally, the transformation from the Lagrangian formalism to the Hamiltonian is given by the *Legendre Transformation* eq. (13), as mentioned in section 2.3.

### 5.2.1 Object Part

**Hamiltonian $\mathcal{H} \to \mathcal{L}$ Lagrangian**

The objects $(\mathbf{q}, \mathbf{p}) \in \mathcal{L}$ (canonical coordinate and momentum) transform into the generalized position and velocity $(\mathbf{q}, \dot{\mathbf{q}})$ via the Hamiltonian $H$ eq. (14):

$$F^{obj}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \in \mathcal{L} \tag{34}$$

**Lagrangian $\mathcal{L} \to \mathcal{H}$ Hamiltonian**

The back-transformation $F^{obj}_{\mathcal{L},\mathcal{H}} : \mathcal{L} \to \mathcal{H}$ is given by the definition of the canonical momentum $\mathbf{p}$:

$$F^{obj}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \frac{\partial L}{\partial \dot{\mathbf{q}}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \in \mathcal{H} \tag{35}$$

where $L$ is the Lagrange equation eq. (9) $L = T - V$, and $(\frac{\partial L}{\partial \dot{\mathbf{q}}})_i = \frac{\partial L}{\partial \dot{q}_i}$.

### 5.2.2 Arrow Part

The transformation from the Lagrangian function $L$ to the Hamiltonian function $H$ and vice versa is traditionally given by the Legendre Transformation eq. (13):

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}, t) &= \sum_{i=1}^{n} p_i \dot{q}_i \; - \; L(\mathbf{q}, \dot{\mathbf{q}}, t) \\ L(\mathbf{q}, \dot{\mathbf{q}}, t) &= \sum_{i=1}^{n} p_i \dot{q}_i \; - \; H(\mathbf{q}, \mathbf{p}, t) \end{aligned} \tag{36}$$

Therefore, the Legendre Transformation is expected to also appear in a categorized notation. When applying $F^{arr}_{\mathcal{H},\mathcal{L}}$ or $F^{arr}_{\mathcal{H},\mathcal{L}}$ on the morphisms in their respective categories, eq. (13) transforms the time derivatives of the objects, giving an explicit mapping of the morphisms.

**Hamiltonian $\mathcal{H} \to \mathcal{L}$ Lagrangian**

$F^{arr}_{\mathcal{H},\mathcal{L}} : \mathcal{H} \to \mathcal{L}$ transforms the morphisms in $\mathcal{H}$ to morphisms in $\mathcal{L}$: $F^{arr}_{\mathcal{A},\mathcal{B}} : f_{\mathcal{A}} \to f_{\mathcal{B}}$
Considering the actions of the morphisms $f$ on the corresponding objects, $F^{arr}_{\mathcal{H},\mathcal{L}}$ needs to fulfill:

$$F^{arr}_{\mathcal{H},\mathcal{L}}\ f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \mathbf{p} + \dot{\mathbf{p}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \dot{\mathbf{q}} + \ddot{\mathbf{q}}\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} \in \mathcal{L}$$

Adapting the notation of eq. (30), $F^{arr}_{\mathcal{H},\mathcal{L}}$ acts as:

$$F^{arr}_{\mathcal{H},\mathcal{L}}\ f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} + F^{arr}_{\mathcal{H},\mathcal{L}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} dt = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} + \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} dt = f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} \in \mathcal{L} \tag{37}$$

Comparing the above eq. (37) with the object part of the functor, $F^{obj}_{\mathcal{H},\mathcal{L}}$, shows that the action of $F^{arr}_{\mathcal{H},\mathcal{L}}$ on the term $(\mathbf{q}, \mathbf{p})$ in the time evolution is equivalent to the coordinate-transformation performed by $F^{obj}_{\mathcal{H},\mathcal{L}}$ in eq. (34). This has already been discussed for the general case in eq. (30).

The explicit form of the transformation $F^{arr}_{\mathcal{H},\mathcal{L}}\ \mathbf{d}_t(\mathbf{q}, \mathbf{p})$ is determined by the respective equations of motion. Based on eq. (16) and eq. (11), $F^{arr}_{\mathcal{H},\mathcal{L}}\ \mathbf{d}_t$ needs to transform as:

$$F^{arr}_{\mathcal{H},\mathcal{L}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} \overset{!}{=} \begin{pmatrix} \dot{\mathbf{q}} \\ \frac{\partial L}{\partial \mathbf{q}} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{pmatrix} = \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}$$

In the case of the transformation from the Hamiltonian to the Lagrangian formalisms, the change from the Hamiltonian $H$ to the Lagrangian $L$ is known to be the Legendre Transformation eq. (13). Therefore, applying $F^{arr}_{\mathcal{H},\mathcal{L}}$ on $H$ gives:

$$F^{arr}_{\mathcal{HL}}\ H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}\dot{\mathbf{q}}\ -\ H(\mathbf{q}, \mathbf{p}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

This means, in the time variation eq. (37), the arrow part of the functor $F^{arr}_{\mathcal{H},\mathcal{L}}$ acts on $\mathbf{d}_t$ as:

$$F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{p}} (\dot{\mathbf{q}}\, \mathbf{p} - L) \\ -\frac{\partial}{\partial \mathbf{q}} (\dot{\mathbf{q}}\, \mathbf{p} - L) \end{pmatrix} = \begin{pmatrix} \mathbf{p} \frac{\partial \mathbf{q}}{\partial \dot{\mathbf{p}}} + \dot{\mathbf{q}} \frac{\partial \mathbf{p}}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}} L(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}) \\ -\left( \dot{\mathbf{q}} \frac{\partial \mathbf{p}}{\partial \mathbf{q}} + \mathbf{p} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}) \right) \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{q}} \\ \frac{\partial L}{\partial \mathbf{q}} \end{pmatrix}$$

Combining these individual transformations, the explicit form of the mapping of the morphisms $f_{\mathcal{H}} \in \mathcal{H}$ in the Hamiltonian formulation to those in the Lagrangian $f_{\mathcal{L}} \in \mathcal{L}$ via the arrow part of the functor $F^{arr}_{\mathcal{H},\mathcal{L}}$ is given by:

$$F^{arr}_{\mathcal{H},\mathcal{L}}\ f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \frac{\partial}{\partial \mathbf{p}} (\dot{\mathbf{q}}\, \mathbf{p} - L)\ dt \\ \mathbf{q} - \frac{\partial}{\partial \mathbf{q}} (\dot{\mathbf{q}}\, \mathbf{p} - L)\ dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \mathbf{p} + \frac{\partial L}{\partial \mathbf{q}}\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} \tag{38}$$

For general $f_{\mathcal{H},n}$, $f_{\mathcal{L},n}$, the scaling of the time-derivative $dt$ is given by $ndt$, as in all previous instances.

Composition of morphisms is conserved under the application of $F^{arr}_{\mathcal{H},\mathcal{L}}$ in the same manner as for the general case studied in eq. (33). Similarly, the preservation of identity under $F^{arr}_{\mathcal{H},\mathcal{L}}$ follows eq. (32). Therefore, explicit demonstration is omitted for brevity.

**Lagrangian $\mathcal{L} \to \mathcal{H}$ Hamiltonian**

For the arrow part, the functor $F_{\mathcal{L},\mathcal{H}} : \mathcal{L} \to \mathcal{H}$ must fulfill:

$$F^{arr}_{\mathcal{L},\mathcal{H}}\ f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} +\ F^{arr}_{\mathcal{L},\mathcal{H}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} dt = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} +\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} dt = f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \in \mathcal{H}$$

Again, the transformation $F^{arr}_{\mathcal{L},\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}})$ is equivalent to the transformation of the coordinates $F^{obj}_{\mathcal{L},\mathcal{H}}$, given in eq. (35).

The explicit form of the transformation $F^{arr}_{\mathcal{L},\mathcal{H}}\ \mathbf{d}_t(\mathbf{q},\dot{\mathbf{q}})$ of the coordinate's time derivatives again depends on the equations of motion eq. (10) and eq. (14). $F^{arr}_{\mathcal{L},\mathcal{H}}$ acts on the Lagrangian $L$ according to the Legendre Transformation eq. (36):

$$F^{arr}_{\mathcal{LH}}\ L(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{p}\,\dot{\mathbf{q}}\ -\ L(\mathbf{q},\dot{\mathbf{q}},t) = H(\mathbf{q},\mathbf{p},t)$$

combining the Legendre Transformation with eq. (14) gives the time derivative as:

$$F^{arr}_{\mathcal{L},\mathcal{H}}\begin{pmatrix}\dot{\mathbf{q}}\\ \frac{\partial L}{\partial \mathbf{q}}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{q}}\\ \frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}\,\mathbf{p} - H\right)\end{pmatrix} = \begin{pmatrix}\frac{\partial H}{\partial \mathbf{p}}\\ \dot{\mathbf{q}}\frac{\partial \mathbf{p}}{\partial \mathbf{q}} + \mathbf{p}\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}H(\mathbf{q},\dot{\mathbf{q}},\mathbf{t})\end{pmatrix} = \begin{pmatrix}\frac{\partial H}{\partial \mathbf{p}}\\ -\frac{\partial H}{\partial \mathbf{q}}\end{pmatrix}$$

In total, the action of the arrow part of the functor $F^{arr}_{\mathcal{L},\mathcal{H}} : f_{\mathcal{L}} \to f_{\mathcal{H}}$ is explicitly given by:

$$F^{arr}_{\mathcal{L},\mathcal{H}}\ f_{\mathcal{L}}\begin{pmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{H}}\begin{pmatrix}\mathbf{q}+\dot{\mathbf{q}}\,dt\\ \dot{\mathbf{q}}+\frac{\partial L}{\partial \mathbf{q}}\,dt\end{pmatrix} = \begin{pmatrix}\mathbf{q}+\frac{\partial H}{\partial \mathbf{p}}\,dt\\ \mathbf{p}+\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}\,\mathbf{p}-H\right)\,dt\end{pmatrix} = \begin{pmatrix}\mathbf{q}+\frac{\partial H}{\partial \mathbf{p}}\,dt\\ \mathbf{p}-\frac{\partial H}{\partial \mathbf{q}}\,dt\end{pmatrix} = f_{\mathcal{H}}\begin{pmatrix}\mathbf{q}(t)\\ \mathbf{p}(t)\end{pmatrix} \tag{39}$$

Similar to $F^{arr}_{\mathcal{H},\mathcal{L}}$, the composition of morphisms and preservation of identity are ensured through the explicit demonstration of the general $F^{arr}_{\mathcal{A},\mathcal{B}}$ in eq. (31) and eq. (32).

## 5.3 Newtonian $\mathcal{N}$ $\leftrightarrow$ Lagrangian $\mathcal{L}$

The transformation from the Newtonian category $\mathcal{N}$ to the Lagrangian category $\mathcal{L}$ is performed by the functor $F_{\mathcal{N},\mathcal{L}} : \mathcal{N} \to \mathcal{L}$ The opposite direction is given by $F_{\mathcal{L},\mathcal{N}} : \mathcal{L} \to \mathcal{N}$.

In the following, the mappings between the objects $(\mathbf{r},\dot{\mathbf{r}}) \in \mathcal{N}$ and $(\mathbf{q},\dot{\mathbf{q}}) \in \mathcal{L}$, as well as the functors $f_{\mathcal{N}}$ and $f_{\mathcal{L}}$ via $F^{\,obj}$ and $F^{\,arr}$ will be developed. Through those transformations, the Newtonian formulation appears as a special case of the Lagrangian, where the configuration space $Q$ is given by the Euclidean space $\mathbb{R}^3$, and the Lagrangian $L$ is given by $L = T - V$ as in section 2.2 [2].

### 5.3.1 Object Part

In Newtonian mechanics, the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ are first introduced in a three-dimensional Euclidean space, $\mathbb{R}^3$. In Lagrangian mechanics, the system is described by the generalized coordinates $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$ in the configuration space $Q$. [2] Therefore, the general notation of the functors $F^{obj}_{\mathcal{N},\mathcal{L}}$, $F^{obj}_{\mathcal{L},\mathcal{N}}$, transforming the objects is:

$$F^{obj}_{\mathcal{N},\mathcal{L}} : \mathcal{N}(\mathbf{r},\dot{\mathbf{r}}) \in \mathbb{R}^3 \to \mathcal{L}(\mathbf{q}(\mathbf{r}),\dot{\mathbf{q}}(\dot{\mathbf{r}})) \in Q$$
$$F^{obj}_{\mathcal{L},\mathcal{N}} : \mathcal{L}(\mathbf{q}(\mathbf{r}),\dot{\mathbf{q}}(\dot{\mathbf{r}})) \in Q \to \mathcal{N}(\mathbf{r},\dot{\mathbf{r}}) \in \mathbb{R}^3$$

The configuration space $Q$ is defined through the generalized coordinates $\mathbf{q}$, and coincides with the Cartesian coordinates in absence of constraints. When constraints are placed on a system, $\mathbf{q}$ adapt to these constraints, reducing [17] and transforming the Cartesian coordinates $\mathbf{r}$ to points $\mathbf{q}$ in a Cartesian hyperspace [10]. Based on the given constraints, the transformation from the Euclidean $(\mathbf{r},\dot{\mathbf{r}})$ to the generalized coordinates and velocities $(\mathbf{q},\dot{\mathbf{q}})$, and vice-versa, is given by a coordinate transformation. For an example, see section 6.

As the explicit form of this transformation is dependent on the studied system and its constraints, the functors $F^{obj}_{\mathcal{N},\mathcal{L}}$, $F^{obj}_{\mathcal{L},\mathcal{N}}$ can only be expressed in this general way at this point.

**Newtonian $\mathcal{N} \to \mathcal{L}$ Lagrangian**

The transformation of the objects $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$ (position and velocity) to the objects $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{L}$ (generalized position and velocity) is given by a coordinate transformation $\mathbf{R}$:

$$F_{\mathcal{N},\mathcal{L}}^{obj} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \mathbf{q}(\mathbf{r}) \\ \dot{\mathbf{q}}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \in \mathcal{L} \tag{40}$$

The explicit form of $\mathbf{R}$ depends on the studied system and its constraints. For systems without any constraints, $(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{q}, \dot{\mathbf{q}})$. With constraints, $(\mathbf{q}, \dot{\mathbf{q}})$ lie in a lower-dimensional configuration space $Q$.

**Lagrangian $\mathcal{L} \to \mathcal{N}$ Newtonian**

The reverse transformation $F_{\mathcal{L},\mathcal{N}}^{obj}$, mapping the generalized coordinate and generalized velocity $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{L}$ to the position and velocity $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$ in Cartesian coordinates is given by:

$$F_{\mathcal{L},\mathcal{N}}^{obj} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \mathbf{R}' \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{r}(\mathbf{q}) \\ \dot{\mathbf{r}}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} \in \mathcal{N} \tag{41}$$

Here, $\mathbf{R}'$ symbolizes the reverse coordinate transformation from eq. (40).

Again, when there are no constraints on a system and the coordinates $\mathbf{r}$ cannot be reduced to lower-dimensional generalized coordinates $\mathbf{q}$, $\mathbf{R}'$ is an identity operator such that $\mathbf{R}'(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{q}, \dot{\mathbf{q}})$. When the coordinate transformation is invertible, $\mathbf{R}'$ is the inverse of $\mathbf{R}$. Examples include the transformation from polar coordinates or spherical coordinates to Cartesian coordinates. When constraints are given, the inversion of $\mathbf{R}$ might not be possible, and $\mathbf{R}'$ needs to be calculated via parametric equations.

### 5.3.2 Arrow Part

Two approaches can be employed to formulate the mapping between the time evolutions $f_{\mathcal{N}}$ and $f_{\mathcal{L}}$ by $F_{\mathcal{N},\mathcal{L}}^{arr}$. One approach takes the time variation as given in eq. (8), utilizing the total energy $E = T + V$ in the Newtonian system. Since the kinetic energy $T$ and potential energy $U$ replicate the structure of the Euler-Lagrange equations [2], this approach will be studied first. The second approach uses the momentum - force formulation of Newtonian mechanics, as first developed in section 2.1 and used in the time evolution in eq. (3). However, since this is different from the other arrow parts in its structure and the used equations, this second approach is not developed here but briefly discussed in section 7.

**Newtonian $\mathcal{N} \to \mathcal{L}$ Lagrangian**

The arrow part of the functor $F_{\mathcal{N},\mathcal{L}}$ maps the morphism $f_{\mathcal{N}}$ to the morphism $f_{\mathcal{L}}$, therefore, $F_{\mathcal{N},\mathcal{L}}^{arr}$ acts as:

$$F_{\mathcal{N},\mathcal{L}}^{arr}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F_{\mathcal{N},\mathcal{L}}^{arr} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} + F_{\mathcal{N},\mathcal{L}}^{arr}\ \mathbf{d}_t \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} dt = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} + \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} dt = f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} \in \mathcal{L} \tag{42}$$

The mapping $F_{\mathcal{N}\mathcal{L}}^{arr}\ (\mathbf{r}, \dot{\mathbf{r}}) \to (\mathbf{q}, \dot{\mathbf{q}})$ in eq. (42) is a coordinate transformation $\mathbf{R}$, equivalent to $F_{\mathcal{N}\mathcal{L}}^{obj}\ (\mathbf{r}, \dot{\mathbf{r}})$ in eq. (40). To map the time derivatives $F_{\mathcal{N},\mathcal{L}}^{arr}\ \mathbf{d}_t\ (\mathbf{r}, \dot{\mathbf{r}}) \to \mathbf{d}_t\ (\mathbf{q}, \dot{\mathbf{q}})$, compare the definition of the total energy $E$ eq. (6) with the Lagrangian $L$ eq. (9). The transformation from the Newtonian total Energy $E$ to the Lagrangian $L$ is then performed via:

$$F_{\mathcal{N}\mathcal{L}}^{arr}\ E = F_{\mathcal{N}\mathcal{L}}^{arr}\ (T + U) = T - U = L$$

with eq. (4) and a potential energy $U = U(\mathbf{r})$ [2].

Inserting $L$ into eq. (7) recovers the Euler-Lagrange equations eq. (10) in Cartesian coordinates:

$$\frac{dL}{dt} = \frac{d}{dt}(T-U) = m\ddot{\mathbf{r}}\,\dot{\mathbf{r}} - \frac{dU}{dt}\dot{\mathbf{r}} = \dot{\mathbf{r}}\,(m\ddot{\mathbf{r}} - \mathbf{F}) = 0$$
$$0 = m\ddot{\mathbf{r}} - \mathbf{F} = \frac{d}{dt}(m\dot{\mathbf{r}}) + \frac{\partial U}{\partial \mathbf{r}} = 0$$

Compare:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{r}}}\right) + \frac{\partial U}{\partial \mathbf{r}} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{43}$$

For systems without constraints, where the generalized variables $(\mathbf{q}, \dot{\mathbf{q}})$ in the configuration space $Q$ are equal to $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$, this relation suffices for the transformation:

$$\frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{\partial T}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} \qquad\qquad \frac{\partial L}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}}$$

However, for systems with constraints, additional coordinate transformations according to eq. (40) are required to obtain the Lagrangian $L$ in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$.

Using the time evolution as defined via the energies, eq. (8), $F^{arr}_{\mathcal{N},\mathcal{L}}$ maps the time derivatives as:

$$F^{arr}_{\mathcal{N},\mathcal{L}}\ \mathbf{d}_t \begin{pmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix}\dot{\mathbf{r}}\\ \ddot{\mathbf{r}}\end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix}\frac{1}{m}\frac{\partial T}{\partial \dot{\mathbf{r}}}\\ -\frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix}\dot{\mathbf{r}}\\ -\frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{q}}\\ \frac{\partial L}{\partial \mathbf{q}}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{pmatrix} = \mathbf{d}_t \begin{pmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{pmatrix}$$

The full transformation performed by $F^{arr}_{\mathcal{N},\mathcal{L}}$ is then given by:

$$F^{arr}_{\mathcal{N},\mathcal{L}}\ f_{\mathcal{N}} \begin{pmatrix}\mathbf{r}(t)\\ \dot{\mathbf{r}}(t)\end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix}\mathbf{r} + \dot{\mathbf{r}}\,dt\\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\,dt\end{pmatrix} = \begin{pmatrix}\mathbf{q} + \dot{\mathbf{q}}\,dt\\ \mathbf{q} + \frac{\partial}{\partial \mathbf{q}}(T-U)\,dt\end{pmatrix} = \begin{pmatrix}\mathbf{q} + \dot{\mathbf{q}}\,dt\\ \mathbf{q} + \frac{\partial L}{\partial \mathbf{q}}\,dt\end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{pmatrix} \tag{44}$$

Under $F^{arr}_{\mathcal{N},\mathcal{L}}$, composition of morphisms is preserved via eq. (31). The identity morphisms $1_{\mathcal{N}}$, $1_{\mathcal{L}}$ are preserved under $F^{arr}_{\mathcal{N},\mathcal{L}}$ as for the general case in eq. (32).

**Lagrangian $\mathcal{L}$ ↔ Newtonian $\mathcal{N}$**

The functor $F^{arr}_{\mathcal{L},\mathcal{N}}$ performs the mapping $F^{arr}_{\mathcal{L},\mathcal{N}} : f_{\mathcal{L}} \to f_{\mathcal{N}}$ according to eq. (30):

$$F^{arr}_{\mathcal{L},\mathcal{N}}\ f_{\mathcal{L}} \begin{pmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{pmatrix} + F^{arr}_{\mathcal{L},\mathcal{N}}\ \mathbf{d}_t \begin{pmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{pmatrix} dt = \begin{pmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{pmatrix} + \mathbf{d}_t \begin{pmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{pmatrix} dt = f_{\mathcal{N}} \begin{pmatrix}\mathbf{r}(t)\\ \dot{\mathbf{r}}(t)\end{pmatrix} \in \mathcal{N}$$

As in previous morphisms transformations, the first term $F^{arr}_{\mathcal{N},\mathcal{L}}(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{r}, \dot{\mathbf{r}})$ is equivalent a coordinate transformation, in this case as given by eq. (41). In the second term, $F^{arr}_{\mathcal{L},\mathcal{N}}$ acts on the time derivative of the objects as:

$$F^{arr}_{\mathcal{L},\mathcal{N}}\ \mathbf{d}_t \begin{pmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix}\dot{\mathbf{q}}\\ \frac{\partial L}{\partial \mathbf{q}}\end{pmatrix} \overset{!}{=} \begin{pmatrix}\dot{\mathbf{r}}\\ -\frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{r}}\\ \ddot{\mathbf{r}}\end{pmatrix} = \mathbf{d}_t \begin{pmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{pmatrix}$$

When considering a Lagrangian $L$ of the form eq. (9), the transformation from the Lagrangian formulation to an energy consideration as in section 2.1.2 appears naturally by inserting $L = T - U$ into the Euler-Lagrange equations:

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{r}}}(T-U)\right) - \frac{\partial}{\partial \mathbf{r}}(T-U) = \frac{d}{dt}\,(m\dot{\mathbf{r}}) + \frac{\partial U}{\partial \mathbf{r}} = m\ddot{\mathbf{r}} + \frac{\partial U}{\partial \mathbf{r}} = 0\end{aligned} \tag{45}$$

The last equivalence is ensured through eq. (2) and eq. (5). The coordinate transformation $(\mathbf{q}, \dot{\mathbf{q}}) \to (\mathbf{r}, \dot{\mathbf{r}})$ is performed by the coordinate transformation as given by the functor $F^{obj}_{\mathcal{L},\mathcal{N}}$.

With eq. (45), the total action of $F^{arr}_{\mathcal{L},\mathcal{N}}$ is given by:

$$F^{arr}_{\mathcal{L},\mathcal{N}}\, f_{\mathcal{L}} \begin{pmatrix}\mathbf{q}(t)\\ \dot{\mathbf{q}}(t)\end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix}\mathbf{q} + \dot{\mathbf{q}}\,dt\\ \dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}}\,dt\end{pmatrix} = \begin{pmatrix}\mathbf{r} + \frac{1}{m}\frac{\partial T}{\partial \dot{\mathbf{r}}}\,dt\\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\,dt\end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix}\mathbf{r}(t)\\ \dot{\mathbf{r}}(t)\end{pmatrix} \tag{46}$$

Composition of functors is preserved under eq. (46) as given in the more general case eq. (31). Similarly, preservation of identity is achieved through eq. (32).

## 5.4 Hamiltonian $\mathcal{H}$ ↔ Newtonian $\mathcal{N}$

To transform the Newtonian variables into Hamiltonian ones, the extra step of first transforming Newtonian coordinates into Lagrangian generalized coordinates and then performing a Legendre-Transformation ($\mathcal{N} \to \mathcal{L} \to \mathcal{H}$) can be done, and vice versa. However, in the following the direct transformations $\mathcal{N} \leftrightarrow \mathcal{H}$ shall be developed.

Two approaches can be employed to formulate the mapping $\mathcal{N} \leftrightarrow \mathcal{H}$. One approach utilizes the Hamiltonian $H$, interpreted as the total energy $E$ in eq. (15) to obtain a relation between the velocity and momentum of a system. Since this more closely replicates the structures found in section 5.2 and section 5.3, this approach is followed in the following. The second approach overlooks the Hamiltonian $H$ and obtains the mapping from a canonical transformation. Since the latter is less straight forward and has a structure different from the other two transformations, $\mathcal{L} \leftrightarrow \mathcal{H}$, $\mathcal{N} \leftrightarrow \mathcal{H}$, this is only briefly considered in section 7.

### 5.4.1 Object Part

In contrast to the Newtonian variables $(\mathbf{r}, \dot{\mathbf{r}})$, the aim of the canonical variables $(\mathbf{q}, \mathbf{p})$ is to be independent, with no constraints among the coordinates themselves [10]. Instead of a full set of coordinates, as in the Newtonian description prior to taking constraints into account, the Hamiltonian formulation uses a reduced set of coordinates $(\mathbf{q}, \mathbf{p})$ [10].

Considering the mapping from the Newtonian coordinate $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ to the Hamiltonian canonical coordinates $\mathbf{q}$ and the canonical momentum $\mathbf{p}$, the transformation $\mathbf{Obj}(\mathcal{N}) \leftrightarrow \mathbf{Obj}(\mathcal{H})$ is similar to the transformations $\mathcal{N} \to \mathcal{L}$ and $\mathcal{L} \to \mathcal{H}$. The transformation of the coordinates $\mathbf{r} \to \mathbf{q}$ can again be simplified to a 'renaming' of the coordinates in $\mathcal{N}$ for the case of no constraints, since then the positions $\mathbf{r} \in \mathcal{N}$ and the canonical positions $\mathbf{q}$ coincide.
In case of constraints, the transformation $\mathbf{r} \leftrightarrow \mathbf{q}$ is given in the same way as for the transformation between the Lagrangian and the Hamiltonian, since the generalized position and canonical coordinate coincide. However, the transformation $\dot{\mathbf{r}} \leftrightarrow \mathbf{p}$ is less straightforward in general cases.

#### Newtonian $\mathcal{N} \to \mathcal{H}$ Hamiltonian

The mapping of the objects $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$ (position and velocity in Cartesian coordinates) to the canonical position and canonical momentum in phase space $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{H}$ is performed by $F^{obj}_{\mathcal{N},\mathcal{H}}$:

$$F^{obj}_{\mathcal{N},\mathcal{H}} \begin{pmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{pmatrix} = \begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} \in \mathcal{H} \tag{47}$$

The transformation of the positions $\mathbf{r} \to \mathbf{q}$ is a coordinate transformation from the Cartesian coordinates $\mathbf{r}$ in $\mathbb{R}^3$ to the generalized position $\mathbf{q} \in Q$. It is equivalent to the mapping $F^{obj}_{\mathcal{N},\mathcal{L}} : \mathbf{r} \to \mathbf{q}$ in section 5.3, since $\mathbf{q} \in \mathcal{L} = \mathbf{q} \in \mathcal{H}$.

The transformation of the velocity to the canonical momentum $\dot{\mathbf{r}} \to \mathbf{p}$ can be performed achieved using two different approaches. One method uses the definition of the Hamiltonian $H$, while the other involves a canonical transformation from Cartesian coordinates to phase-space coordinates. The latter approach is not described here, since its structure differs greatly from the other transformations.

#### $F^{obj}_{\mathcal{N},\mathcal{H}}$ via the kinetic Energy T

The transformation $(\mathbf{r}, \dot{\mathbf{r}}) \to (\mathbf{q}, \mathbf{p})$ can be performed more explicitly than eq. (47) by considering the energies of the system. First, the mapping $(\mathbf{r}, \dot{\mathbf{r}}) \to (\mathbf{q}, \dot{\mathbf{q}})$ is done by a coordinate transformation $\mathbf{R}$ in the same way as $F^{obj}_{\mathcal{N},\mathcal{L}} : (\mathbf{r}, \dot{\mathbf{r}}) \to (\mathbf{q}, \dot{\mathbf{q}})$. Then, the transformation $\dot{\mathbf{q}} \to \mathbf{p}$ is performed via the definition of the kinetic Energy eq. (4) and the Hamiltonian $H$ eq. (15) by:

$$\begin{pmatrix} \mathbf{q}(\mathbf{r}) \\ \dot{\mathbf{q}}(\mathbf{r}) \end{pmatrix} \to \begin{pmatrix} \mathbf{q} \\ \frac{\partial T}{\partial \dot{\mathbf{q}}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \frac{\partial H}{\partial \dot{\mathbf{q}}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \in \mathcal{H}$$

The above only applies for velocity independent potential energies $U$, where the Hamiltonian $H$ can be written as $H = T + U$ eq. (15). This approach is similar to the transformation by $F^{obj}_{\mathcal{L},\mathcal{H}}$ in eq. (35).

In general, the linear momentum in Newtonian mechanics, $\mathbf{p}_{\mathcal{N}} = m\dot{\mathbf{r}}$ differs from the canonical momentum $\mathbf{p} \in \mathcal{H}$. As the mapping $F^{obj}_{\mathcal{N},\mathcal{H}} : \mathbf{p}_{\mathcal{N}} \in \mathcal{N} \to \mathbf{p} \in \mathcal{H}$ depends on the specific system studied, a mapping $\dot{\mathbf{r}} \to \mathbf{p}_{\mathcal{N}} \to \mathbf{p}$ is not further developed here.

The full mapping of the objects is then given by:

$$F^{obj}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} \overset{\mathbf{R}}{\Rightarrow} \begin{pmatrix} \mathbf{q}(\mathbf{r}) \\ \dot{\mathbf{q}}(\mathbf{r}) \end{pmatrix} \to \begin{pmatrix} \mathbf{q} \\ \frac{\partial H}{\partial \dot{\mathbf{q}}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \in \mathcal{H} \tag{48}$$

### Hamiltonian $\mathcal{H} \leftrightarrow$ Newtonian $\mathcal{N}$

In the reverse direction, $F^{obj}_{\mathcal{H},\mathcal{N}}$ maps the canonical variables $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{H}$ to the position and velocity $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$:

$$F^{obj}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} \in \mathcal{N} \tag{49}$$

As before, the transformation of the positions $\mathbf{q} \to \mathbf{r}$ is a coordinate transformation, equivalent to the coordinate transformation $F^{obj}_{\mathcal{L},\mathcal{N}} : \mathbf{q} \to \mathbf{r}$ in section 5.3.2. The transformation from the canonical momentum to the velocity $\mathbf{p} \to \dot{\mathbf{r}}$ can again be transformed in multiple days. In the following, the mapping $F^{obj}_{\mathcal{H},\mathcal{N}}$ via the energies of the system is considered.

#### $F^{obj}_{\mathcal{H},\mathcal{N}}$ via the Hamiltonian

When using Hamilton's equations eq. (14) to perform the transformation $\mathbf{p} \to \dot{\mathbf{r}}$, the first step is a transformation of the canonical momentum to the generalized velocity $\mathbf{p} \to \dot{\mathbf{q}}$ by:

$$\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{q}(\mathbf{r}) \\ \dot{\mathbf{q}}(\mathbf{r}) \end{pmatrix}$$

Then, the mapping $\dot{\mathbf{q}} \to \dot{\mathbf{r}}$ is performed by the coordinate transformation $\mathbf{R}'$. For invertible coordinate transformations $\mathbf{R}$ in section 5.4.1, $\mathbf{R}'$ is the inversion of $\mathbf{R}$. The coordinate transformation $\mathbf{R}'$ is equivalent to section 5.2.1.

The full mapping $(\mathbf{q}, \mathbf{p}) \to (\mathbf{r}, \dot{\mathbf{r}})$ is performed by $F^{obj}_{\mathcal{H},\mathcal{N}}$ via:

$$F^{obj}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \overset{\mathbf{R}'}{\Rightarrow} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} \in \mathcal{N} \tag{50}$$

### 5.4.2 Arrow Part

The arrow parts $F^{arr}_{\mathcal{N},\mathcal{H}}$ and $F^{arr}_{\mathcal{H},\mathcal{N}}$ perform the mappings between the morphisms $f_{\mathcal{N}} \in \mathcal{N}$ and $f_{\mathcal{H}} \in \mathcal{H}$.

In the special case of a conservative system with velocity-independent potential energy $U$, the Hamiltonian $H$ can be written as $H = T + U$, see eq. (15) [2]. This equivalence of the Hamiltonian $H$ with the total energy $E$, eq. (6), allows the transformation between the Newtonian and Hamiltonian formalisms to be reduced to a change in variables.

For systems where $H$ is not equal to the total energy $E$, an alternative approach, considering the forces, can be chosen. However, this is less explicit if no concrete example is given.

**Newtonian $\mathcal{N} \to \mathcal{H}$ Hamiltonian**

$F^{arr}_{\mathcal{N},\mathcal{H}} : f_{\mathcal{N}} \to f_{\mathcal{H}}$ maps morphisms in $\mathcal{N}$ to morphisms in $\mathcal{H}$, such that:

$$F^{arr}_{\mathcal{N},\mathcal{H}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} + \ F^{arr}_{\mathcal{N},\mathcal{H}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} dt = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} + \ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} dt = f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \in \mathcal{H}$$

The mapping $F^{arr}_{\mathcal{N},\mathcal{H}}(\mathbf{r}, \dot{\mathbf{r}}) \to (\mathbf{q}, \mathbf{p})$ is performed in the same manner as eq. (48), where $\mathbf{r} \to \mathbf{q}$ is a coordinate transformation, and $\dot{\mathbf{r}} \to \frac{\partial T}{\partial \dot{\mathbf{q}}} = \frac{\partial H}{\partial \dot{\mathbf{q}}} = \mathbf{p}$.

For the transformation of the time evolution from eq. (3) and eq. (8) to eq. (16), consider the momentum $\mathbf{p}_{\mathcal{N}}$ in Newtonian mechanics (the subscript differentiates the Newtonian momentum $\mathbf{p}_{\mathcal{N}} = m\dot{\mathbf{r}}$ from the canonical momentum $\mathbf{p}$ in Hamiltonian mechanics). With a kinetic energy of the form eq. (4), $\dot{\mathbf{r}}$ can be written as $\dot{\mathbf{r}} = \frac{\partial T}{\partial \mathbf{p}_{\mathcal{N}}}$. Together with a potential energy from eq. (5), this resembles Hamilton's equations eq. (14):

$$\dot{\mathbf{r}} = \frac{\partial T}{\partial \mathbf{p}_{\mathcal{N}}} \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}}_{\mathcal{N}} = -\frac{\partial U}{\partial \mathbf{r}} \qquad \dot{\mathbf{p}} = \frac{\partial H}{\partial \mathbf{q}}$$

It is important to note, that for most cases the linear momentum $\mathbf{p}_{\mathcal{N}} = m\dot{\mathbf{r}} \in \mathcal{N}$ and the generalized momentum $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ are not the same. The explicit form of the transformation $F^{arr}_{\mathcal{N},\mathcal{H}}\ \mathbf{p}_{\mathcal{N}} = \mathbf{p}$ depends on the specific system and the used coordinate system.

For conservative systems where the total energy $E$ is of the form $E = T + U$ and eq. (15) $H = E$ applies, the transformation from the energies in Newtonian mechanics to Hamilton's equations is given by:

$$F^{arr}_{\mathcal{N},\mathcal{H}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \frac{\partial T}{\partial \mathbf{p}_{\mathcal{N}}} \\ \frac{1}{m} \frac{\partial U}{\partial \mathbf{r}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \frac{\partial}{\partial \mathbf{p}} \left( T(\dot{\mathbf{q}}) + U(\mathbf{r}) \right) \\ \mathbf{p} - \frac{\partial}{\partial \mathbf{q}} \left( T(\dot{\mathbf{q}}) + U(\mathbf{r}) \right) \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} = \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

Altogether, the transformation $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N} \to (\mathbf{q}, \mathbf{p}) \in \mathcal{H}$ is performed by $F^{arr}_{\mathcal{N},\mathcal{H}}$ via:

$$F^{arr}_{\mathcal{N},\mathcal{H}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \frac{1}{m} \frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \frac{\partial T}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial U}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \frac{\partial}{\partial \mathbf{p}} \left( T + U \right)\, dt \\ \mathbf{p} - \frac{\partial}{\partial \mathbf{q}} \left( T + U \right)\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \tag{51}$$

Again, composition of functors and the identity functor are preserved according to eq. (33) and eq. (32).

**Hamiltonian $\mathcal{H} \leftrightarrow$ Newtonian $\mathcal{N}$**

The transformation from morphisms $f_\mathcal{H} \in \mathcal{H}$ to $f_\mathcal{N} \in \mathcal{N}$ is performed by $F^{arr}_{\mathcal{H},\mathcal{N}}$:

$$F^{arr}_{\mathcal{H},\mathcal{N}}\ f_\mathcal{H} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} + F^{arr}_{\mathcal{H},\mathcal{N}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} dt = \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} + \mathbf{d}_t \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} dt = f_\mathcal{N} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \in \mathcal{N}$$

The mapping $F^{arr}_{\mathcal{H},\mathcal{N}}(\mathbf{q},\mathbf{p}) \to (\mathbf{r},\dot{\mathbf{r}})$ corresponds to the mapping $\mathbf{Obj}(\mathcal{H}) \to \mathbf{Obj}(\mathcal{N})$ in eq. (49). The mapping of the time derivatives must fulfill the following:

$$F^{arr}_{\mathcal{H},\mathcal{N}}\ \mathbf{d}_t \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} \overset{!}{=} \begin{pmatrix} \frac{\partial T}{\partial \mathbf{p}} \\ -\frac{\partial U}{\partial \mathbf{q}} \end{pmatrix} \overset{\mathbf{R}'}{\Rightarrow} \begin{pmatrix} \frac{\partial T}{\partial \mathbf{p}} \\ -\frac{1}{m}\frac{\partial U}{\partial \mathbf{r}} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{r}} \\ \ddot{\mathbf{r}} \end{pmatrix} = \mathbf{d}_t \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix}$$

For conservative systems, where the potential energy is independent of the velocity $U = U(\mathbf{q})$, and the kinetic energy is given by eq. (4), the Hamiltonian $H$ is equal to the total energy $E$, see eq. (15): $H = T + U = E$. The transformation of $H$ to $E$ to obtain the time variation based on eq. (8) is then done by inserting $H = T + U = E$ and performing a coordinate transformation from the canonical variables to the Cartesian position and velocity $(\mathbf{r},\dot{\mathbf{r}})$:

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{p}} &= \frac{\partial}{\partial \mathbf{p}}(T+U) = \frac{\partial T}{\partial \mathbf{p}} = \dot{\mathbf{r}} \\ \frac{\partial H}{\partial \mathbf{q}} &= \frac{\partial}{\partial \mathbf{q}}(T-U) = -\frac{\partial U}{\partial \mathbf{r}} = m\ddot{\mathbf{r}} \end{aligned}$$

The complete transformation performed by $F^{arr}_{\mathcal{H},\mathcal{N}}$ is then given as:

$$F^{arr}_{\mathcal{H},\mathcal{N}}\ f_\mathcal{H} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \mathbf{r} + \frac{\partial}{\partial \mathbf{p}}(T+U)\, dt \\ \mathbf{p} - \frac{\partial}{\partial \mathbf{q}}(T+U)\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \frac{\partial T}{\partial \mathbf{p}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \dot{\mathbf{r}}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = f_\mathcal{N} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \tag{52}$$

## 5.5 Composition of functors

Composition of the above functors is ensured in the same manner as for the general case developed in eq. (31). The composition of the functors $F_{\mathcal{N},\mathcal{L}}$, $F_{\mathcal{L},\mathcal{H}}$ and $F_{\mathcal{N},\mathcal{H}}$ can be represented in a diagram as:

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{F_{\mathcal{N},\mathcal{L}}} & \mathcal{L} \\ & \searrow^{F_{\mathcal{N},\mathcal{H}}} & \downarrow{\scriptstyle F_{\mathcal{L},\mathcal{H}}} \\ & & \mathcal{H} \end{array} \tag{53}$$

The composition of $F_{\mathcal{N},\mathcal{L}}$ with $F_{\mathcal{L},\mathcal{H}}$ should give $F_{\mathcal{N},\mathcal{H}}$, and should transform the Newtonian category $\mathcal{N}$ into the Hamiltonian $\mathcal{H}$. The functors $F_{\mathcal{L},\mathcal{N}}$, $F_{\mathcal{H},\mathcal{L}}$ and $F_{\mathcal{H},\mathcal{N}}$ would give diagram eq. (53) with reversed arrows. Additionally, the compositions $F_{\mathcal{N},\mathcal{L}} \circ F_{\mathcal{L},\mathcal{N}}$ , $F_{\mathcal{L},\mathcal{H}} \circ F_{\mathcal{H},\mathcal{L}}$ and $F_{\mathcal{N},\mathcal{H}} \circ F_{\mathcal{H},\mathcal{N}}$ should leave the original categories unchanged.

In the following, the compositions resulting in eq. (53) and its reverse are demonstrated via the arrow parts of the functors $F^{arr}$, since the arrow part incorporates a transformation of the objects, as explained in section 5.1.

**Composition of Functors: $F^{\mathbf{arr}}_{\mathcal{N},\mathcal{L}}$, $F^{\mathbf{arr}}_{\mathcal{L},\mathcal{H}}$, and $F^{\mathbf{arr}}_{\mathcal{N},\mathcal{H}}$**

First, it is verified that the composition of $F^{\text{arr}}_{\mathcal{N},\mathcal{L}}$ and $F^{\text{arr}}_{\mathcal{L},\mathcal{H}}$ equals $F^{\text{arr}}_{\mathcal{N},\mathcal{H}}$, $\quad F^{\text{arr}}_{\mathcal{L},\mathcal{H}} \circ F^{\text{arr}}_{\mathcal{N},\mathcal{L}} = F^{\text{arr}}_{\mathcal{N},\mathcal{H}}$ by first applying $F^{\text{arr}}_{\mathcal{N},\mathcal{L}}$ to $f_\mathcal{N}$ and then applying $F^{\text{arr}}_{\mathcal{L},\mathcal{H}}$ on the result with:

$$F^{\text{arr}}_{\mathcal{L},\mathcal{H}}\ F^{\text{arr}}_{\mathcal{N},\mathcal{L}}\ f_\mathcal{N} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{\text{arr}}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = f_\mathcal{H} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix}$$

The whole mapping is equivalent to eq. (51).

**Composition of Functors: $F^{\mathbf{arr}}_{\mathcal{H},\mathcal{L}}$, $F^{\mathbf{arr}}_{\mathcal{L},\mathcal{N}}$, and $F^{\mathbf{arr}}_{\mathcal{H},\mathcal{N}}$**

In the reverse direction to diagram eq. (53), the composition $F^{\text{arr}}_{\mathcal{H},\mathcal{L}} \circ F^{\text{arr}}_{\mathcal{L},\mathcal{N}} = F^{\text{arr}}_{\mathcal{N},\mathcal{H}}$ is demonstrated via:

$$F^{\text{arr}}_{\mathcal{L},\mathcal{N}}\ F^{\text{arr}}_{\mathcal{H},\mathcal{L}}\ f_{\mathcal{H}} \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} = F^{\text{arr}}_{\mathcal{L},\mathcal{N}}\ F^{\text{arr}}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = F^{\text{arr}}_{\mathcal{L},} \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \ddot{\mathbf{r}}\, dt \end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}$$

This gives the same transformation as eq. (52).

**Composition of Functors: $F^{\mathbf{arr}}_{\mathcal{L},\mathcal{H}} \circ F^{\mathbf{arr}}_{\mathcal{H},\mathcal{L}}$, $F^{\mathbf{arr}}_{\mathcal{N},\mathcal{L}} \circ F^{\mathbf{arr}}_{\mathcal{L},\mathcal{N}}$, $F^{\mathbf{arr}}_{\mathcal{N},\mathcal{H}} \circ F^{\mathbf{arr}}_{\mathcal{H},\mathcal{N}}$**

The composition of the functors with their reverse functors should leave the morphisms and state to which they are applied to unchanged. As examples, the compositions $F^{\text{arr}}_{\mathcal{L},\mathcal{H}} \circ = F^{\text{arr}}_{\mathcal{H},\mathcal{L}}$, $F^{\text{arr}}_{\mathcal{N},\mathcal{L}} \circ = F^{\text{arr}}_{\mathcal{L},\mathcal{N}}$, and $F^{\text{arr}}_{\mathcal{N},\mathcal{H}} \circ = F^{\text{arr}}_{\mathcal{H},\mathcal{N}}$ are executed.

More concretely, the composition of $F^{\text{arr}}_{\mathcal{L},\mathcal{H}}$ and $F^{\text{arr}}_{\mathcal{H},\mathcal{L}}$ is given by:

$$F^{\text{arr}}_{\mathcal{H},\mathcal{L}}\ F^{\text{arr}}_{\mathcal{L},\mathcal{H}}\ f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix} = F^{\text{arr}}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}}\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix}$$

The composition of $F^{\text{arr}}_{\mathcal{N},\mathcal{L}}$ and $F^{\text{arr}}_{\mathcal{L},\mathcal{N}}$ is given by:

$$F^{\text{arr}}_{\mathcal{L},\mathcal{N}}\ F^{\text{arr}}_{\mathcal{N},\mathcal{L}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{\text{arr}}_{\mathcal{L},\mathcal{N}} \begin{pmatrix} \mathbf{q} + \dot{\mathbf{q}}\, dt \\ \dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \frac{1}{m} \frac{\partial T}{\partial \dot{\mathbf{r}}}\, dt \\ \dot{\mathbf{r}} - \frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}$$

The composition of $F^{\text{arr}}_{\mathcal{N},\mathcal{H}}$ and $F^{\text{arr}}_{\mathcal{H},\mathcal{N}}$ is given by:

$$F^{\text{arr}}_{\mathcal{H},\mathcal{N}}\ F^{\text{arr}}_{\mathcal{N},\mathcal{H}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{\text{arr}}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}}\, dt \\ \mathbf{p} - \frac{\partial H}{\partial \mathbf{q}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}$$

As different categories in physics operate in different mathematical spaces, the transformation from one space into another is not always a bijective mapping or solely a reduction of dimensions. Due to the distinct geometric structures, the transformations are not always fully reversible without introducing additional constraints or information. Inverse mappings might not always recover the full original space, and the mapping itself is not always a direct one-to-one correspondence between points or a simple projection, but involves more complex transformations.

Solely basing the mappings between the spaces, or in this context the functors between the categories, on mathematical formulations can also lead to situations where information about the physical system is lost, or where mapped states are not physically possible or represent a state and time evolution that cannot be interpreted for the specific system at hand. Therefore, the functors need to be examined and constructed in more rigor then presented in section 5 when considering concrete physical systems.
An example where additional constraints on the category need to be implemented to recover meaningful interpretations of the objects within the context of a simple pendulum is discussed in section 6.4 ongoing.

# 6 Mathematical Pendulum

In the following, the application of category theory to a physical system shall be showcased by an example. To apply the developed categorical framework of theoretical mechanics, a simple mathematical pendulum shall be considered. The pendulum is chosen since it is a standard textbook example, with well-developed formulations in Newtonian, Lagrangian, and Hamiltonian mechanics. By constructing the categories $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ for the pendulum, comparing them to known derivations of the equations of motion, and performing transformations between the formulations via the functors from section 5, the applicability of the categorical framework will be demonstrated.

A simple mathematical pendulum consists of a mass $m$, connected to a rigid rod of constant length $l$. The rod is fixed to the origin $O$, and the only force acting on the mass $m$ is the gravitational force $\mathbf{F_g} = -mg = const.$ When deflected from the resting position, the pendulum swings in a two-dimensional plane around $O$, with an angle $\theta$ with respect to the rod and the resting position, as sketched in fig. 1. The change in $\theta$ is denoted by its time derivative $\frac{d\theta}{dt} = \dot{\theta}$.
The rod is assumed to be incompressible and massless. Air resistance and similar phenomena will be neglected.

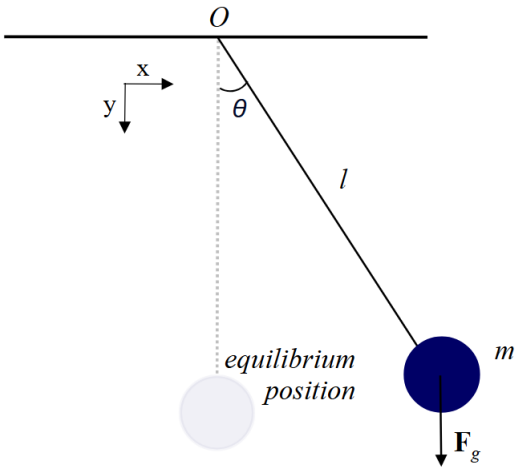



**Figure 1:** Symbolic image of a simple mathematical pendulum. The mass $m$ is connected to the origin $O$ through a rigid rod of length $l$. The displacement of $m$ from the equilibrium position (straight dotted line beneath $O$) is given by the angle $\theta$. The gravitational force $\mathbf{F_g} = -mg$ acts on $m$ and leads to a swinging motion around the equilibrium position.

At first, the dynamics of the pendulum shall be developed within the framework of theoretical mechanics as in section 2. Specifically, Newtonian mechanics section 6.1 is developed through the forces acting on the mass $m$ and the total energy $E$ of the system. Lagrangian mechanics studies the pendulum via the Euler-Lagrange equations in section 6.2. In section 6.3, Hamilton's equations of motion for the pendulum are determined.

These foundational equations are then used to construct the categories introduced in section 4. Applying the functors of section 5 to $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ shows that the mappings between the systems are consistent with the categorical structure.

## 6.1 Newtonian Mechanics

Within the framework of Newtonian mechanics, the simple pendulum is described by its position $\mathbf{r}(t) = (x(t), y(t))$ and velocity $\dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t))$ in Cartesian coordinates, and the forces $\mathbf{F}$ acting on the mass $m$. Due to the constraint imposed on the system by the rigid rod, the motion of the pendulum is confined to a harmonic motion around $O$. This suggests a coordinate transformation from Cartesian coordinates to polar coordinates to simplify the equations of motion.

As in section 2.1.2, the equations of motion can also be calculated via the kinetic energy $T$ and potential energy $U$, see section 6.1.3.

### 6.1.1 Forces in Cartesian Coordinates

Here, the only force considered is the constant gravitational force $\mathbf{F_g} = -m\mathbf{g}$, which acts on the mass $m$ in the negative y-direction as shown in figure 1. The deflection of $m$ from the equilibrium position, combined with the action of the force $\mathbf{F_g}$, causes the pendulum to oscillate around $O$ with velocity $\dot{\mathbf{r}}$. To better describe the effect $\mathbf{F_g}$ has on the pendulum, $\mathbf{F_g}$ can be decomposed into a parallel and perpendicular component, $\mathbf{F}_\perp$ and $\mathbf{F}_\parallel$ respectively. Based on the choice of axes in 1, $\mathbf{F}_\perp$ and $\mathbf{F}_\parallel$ are given by:

$$F_\perp = mg\frac{x}{l} \qquad\qquad F_\parallel = mg\frac{y}{l} \tag{54}$$

The motion of the pendulum is further constrained by the assumption that the rod is inextensible $l = const.$.As a consequence, the position of the pendulum must satisfy the constraint equation:

$$x^2 + y^2 = l^2 \tag{55}$$

Differentiating eq. (55) twice with respect to time yields the coupling between the acceleration $\ddot{\mathbf{r}}$ and the position $\mathbf{r}$:

$$\begin{aligned} 2x\dot{x} + 2y\dot{y} &= 0 \\ \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} &= 0 \end{aligned} \tag{56}$$

The first row in eq. (56) ensures that the velocities are perpendicular to the positions. From eq. (56), the equations of motion for the pendulum in Cartesian coordinates can be obtained. However, as the derivation is tedious and required the introduction of the tension in the rod as an additional quantity, it is omitted here.

### 6.1.2 Forces in Polar Coordinates

As the motion of the pendulum is confined to a circle around $O$, the position of $m$ can also be expressed in polar coordinates by the radius $R = l$ and the angle $\theta$. Since the rod is inextensible, $l = const.$, the radius remains constant, and only $\theta$ changes with time due to $\mathbf{F_g}$. Therefore, instead of describing the pendulum dynamics in a Cartesian coordinate system via $\mathbf{r}$ and $\dot{\mathbf{r}}$, it is sufficient to view $\theta$ as a *generalized coordinate*.

From the constraint 55, and the geometry of the system in fig. 1, the transformation of the Cartesian coordinates to polar coordinates can be expressed as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l\,cos(\theta) \\ l\,sin(\theta) \end{pmatrix} \tag{57}$$

In polar coordinates, the perpendicular and parallel components, eq. (54), of the gravitational force $\mathbf{F_g}$ are:

$$F_\perp = mgsin(\theta) \qquad\qquad F_\parallel = mgcos(\theta) \tag{58}$$

Since the motion is constrained to a circular trajectory, the acceleration $\ddot{r}$ can be expressed as the second derivative of the arc length with respect to time, $\ddot{r} = l\ddot{\theta}$. The perpendicular force component $F_\perp$ is then given by:

$$F_\perp = -mgsin(\theta) = ml\ddot{\theta} \tag{59}$$

Simplifying this leads to the equation of motion in polar coordinates:

$$\ddot{\theta} + \frac{g}{l}\,sin(\theta) = 0 \tag{60}$$

Instead of two coupled second-order differential equations, eq. (60) describes the same dynamical situation with one second-order differential equations in terms of a single generalized coordinate $\theta$.

### 6.1.3 Conservation of Energy

The pendulum can also be analyzed in terms of its mechanical energy $E$, given by the sum of the kinetic energy $T$ and the potential energy $U$
From eq. (4) with eq. (57), the kinetic energy $T$ is:

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

The potential energy $U$ is determined by the height of the pendulum relative to its minimal height at the resting position, and is given by:

$$U = mgh = mgl(1 - cos(\theta))$$

The total energy $E$ of the system is therefore given by:

$$E = T + U = \frac{1}{2} m l^2 \dot{\theta}^2 \; + \; mgl(1 - cos(\theta)) \tag{61}$$

The total energy $E$ is conserved over time, see eq. (7), yielding the equation of motion:

$$\frac{dE}{dt} = ml^2 \ddot{\theta} \; \dot{\theta} + mgl(sin(\theta)) \; \dot{\theta} = 0$$
$$\ddot{\theta} + \frac{g}{l} \; sin(\theta) = 0$$

## 6.2 Lagrangian Mechanics

In Lagrangian mechanics, the motion of the pendulum is described in the system's *configuration space* $Q$ [2] by the generalized coordinates $\mathbf{q}$, see section 2.2. Since the pendulum is attached to the origin $O$, and its motion is constrained to two dimensions, $m$ is confined to move along a circular path around $O$. The configuration space $Q$ is therefore embedded in a two-dimensional sphere: $Q = S^2 \subset \mathbb{R}^2$
However, since the rod is inextensible, $l = const.$, the mass $m$ can only move around the circumference of $S^2$, reducing the degrees of freedom to $dim(Q) = 1$. The generalized coordinate $\mathbf{q} = q_1 = q$ corresponds to the angular displacement $q = \theta$, and the velocity is given by $\dot{q} = \dot{\theta}$. The constraint in eq. (55) ensures the motion of $m$ is confined to the configuration space $Q$.

With the kinetic energy $T$ and the potential energy $U$ from eq. (57), the Lagrangian $L$ and is given by:

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 \; - \; mgl(1 - cos(\theta)) \tag{62}$$

Equation (62) leads to the Euler-Lagrange equation eq. (63) and results in the equation of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = ml^2 \ddot{\theta} + mgl sin(\theta) = 0 \tag{63}$$

$$\ddot{\theta} + \frac{g}{l} \; sin(\theta) = 0$$

## 6.3 Hamiltonian Mechanics

In Hamiltonian mechanics, the motion of the pendulum is described in the system's phase space via the canonical variables $(\mathbf{q}, \mathbf{p})$. For the pendulum, the generalized coordinate is $q = \theta$. The canonical momentum $p_\theta$ is given by section 2.3 $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$.
The Legendre transformation eq. (13) gives the Hamiltonian $H$:

$$H = p_\theta \dot{\theta} - L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - cos(\theta)) \\ = \frac{1}{2} \frac{p_\theta^2}{ml^2} + mgl(1 - cos(\theta)) \tag{64}$$

From the Hamiltonian *eq.* (64), the equations of motions are derived:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \qquad\qquad \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = mglsin(\theta) \tag{65}$$

Equation (65) describe the dynamics of the pendulum via a system of coupled first-order differential equations, instead of a single second-order differential equation in section 6.1 and section 6.2.

## 6.4 $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ for the Pendulum

Combining the coordinates and equations developed in the previous section 6.1 - section 6.3 with the categories from section 4, $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ can be constructed for the pendulum.
Based on section 4.4, the description of the categories only requires the states $\mathbf{V}(t) = (\mathbf{Q}(t), \mathbf{P}(t))$ and their time derivatives $\mathbf{d}_t\mathbf{V}(t) = (\dot{\mathbf{Q}}(t), \dot{\mathbf{P}}(t))$. The morphisms $f$ and identity morphisms are constructed from $\mathbf{V}$ and $\mathbf{d}_t\mathbf{V}$ based on eq. (29). Therefore, an explicit construction of the categories is not necessary, and $\mathbf{V}$, $\mathbf{d}_t\mathbf{V}$ are summarized for $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ in the following table table 2.

**Table 2:** Objects $\mathbf{Q}$ and $\mathbf{P}$ in the Newtonian $\mathcal{N}$, Lagrangian $\mathcal{L}$, and Hamiltonian $\mathcal{H}$ category. The time derivatives $\dot{\mathbf{Q}}$, $\dot{\mathbf{P}}$ are required for the morphisms based on eq. (29). Explicit expression of the objects and time derivatives for the mathematical pendulum in fig. 1.

| **Category** | **Q** | **P** | $\dot{\mathbf{Q}}$ | $\dot{\mathbf{P}}$ | **Applied relations** |
|---|---|---|---|---|---|
| $\mathcal{N}$ | $\mathbf{r} = (x, y)$ | $\dot{\mathbf{r}} = (\dot{x}, \dot{y})$ | $\dot{\mathbf{r}} = (\dot{x}, \dot{y})$ | $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}) = \mathbf{g}$ | $\mathbf{F} = m\ddot{\mathbf{r}} = m\mathbf{g}$ |
| | $\mathbf{r} = \theta$ | $\dot{\mathbf{r}} = \dot{\theta}$ | $\dot{\mathbf{r}} = \dot{\theta}$ | $\ddot{\mathbf{r}} = -\frac{g}{l}\sin(\theta)$ | $\frac{dE}{dt} = 0$ |
| $\mathcal{L}$ | $\mathbf{q} = \theta$ | $\dot{\mathbf{q}} = \dot{\theta}$ | $\dot{\mathbf{q}} = mgl\sin(\theta)$ | $-\frac{g}{l}\sin(\theta)$ | $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$ |
| $\mathcal{H}$ | $\mathbf{q} = \theta$ | $\mathbf{p} = p_\theta$ | $\dot{\mathbf{q}} = \dot{\theta} = \frac{p_\theta}{ml^2}$ | $\dot{\mathbf{p}} = \dot{p}_\theta = mgl\sin(\theta)$ | $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$, $\dot{p}_\theta = \frac{\partial H}{\partial \theta}$ |

Table 2 makes it seem as if the states of the pendulum are just a renaming of the objects in the three categories. However, an important distinction between the objects of the categories $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$, and the states of a physical system in the respective formulations needs to be made. The objects of $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ are defined in section 4 in the full Euclidean, configuration, or phase space. In contrast, specific physical systems will in general not share the same overall spaces, as their states differ from the objects in $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ due to constraints imposed by the system. The categories themselves however do not know of any constraints, since they are defined for the full set of variables.

Therefore, the above definition for the categories in table 2 fails to accurately describe the pendulum, as it includes objects that represent impossible states for the physical system.

For example, consider the Newtonian category $\mathcal{N}$. The objects $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N}$ represent the full set of coordinates in $\mathbb{R}^3$. However, the pendulum does not need to be described in $\mathbb{R}^3$. As a first simplification, the objects can be reduced to the position and velocity in $\mathbb{R}^2$, where the same categorical structure of section 4.1 applies with $\mathbb{R}^2 \subset \mathcal{N}$. Additionally, and even more significantly, not all objects $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{R}^2$ are valid for describing the pendulum. Due to the constraint eq. (55), only those objects that satisfy the constraint can be used for the time evolution. Other objects that do not fulfill these conditions have no physical interpretation, since the system cannot be in those states. For example, a position of the pendulum's mass $m$ that has an absolute distance smaller or greater than $l$ from the origin $O$ is not possible in this description because $l$ is incompressible. These invalid states should not contribute to

the time evolution, as the time evolution of an impossible state has no meaning in this description.
Furthermore, not all objects $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{N}$ can be achieved by the pendulum. The mapping from a valid state to an invalid one should not be possible, or give the identity mapping, since physically the pendulum cannot have a position or velocity that does not lie on its trajectory. This contradicts the definition of morphisms from definition 3.1, since for every object, a mapping to any other object shall exist. By the given pendulum, this condition cannot be fulfilled with the constraint $x^2 + y^2 = l^2$ with $l = constant$.

Thus, to accurately describe the physical system, the category must be restricted to objects that satisfy the given constraints. The valid objects form a subset of all objects in the category. This suggests defining a subcategory with these valid objects, with the time evolution according to the equations of motion as morphisms. In this sense, a subcategory is a collection of some objects and their morphisms from the original category [13].

For objects that do not adhere to the constraints, and therefore have no sensible physical interpretation, several approaches can be taken. They can remain in the larger category, with no meaningful time evolution or composition for a concrete system, or they can be placed in another subcategory of invalid objects. In the latter case, morphisms acting on invalid objects would produce meaningless results that contradict the equations of motion of a special system. Invalid objects could also be mapped into an empty set, which then must be included in the original category, as the objects of a subcategory must be included in the original category [13]. Another approach would be to define mappings and compositions with invalid objects to give the identity morphisms. This could be interpreted as "when one tries to transform a state from or into an invalid state, nothing happens, as the state does not exist for the system".

Here, as the specific example of the pendulum is studied, only valid solutions of the equations of motion eq. (60) will be further considered. A proper description of a physical system within a categorical framework should include only valid states of the system as objects, while the morphisms represent time evolutions satisfying the system's equation of motion. To formalize this, a subcategory is defined, which selects those objects from the original category that satisfy the constraints on the system. In the following section 6.5.1, this approach is explicitly performed for the Newtonian category $\mathcal{N}$ in consideration of the constraint eq. (55) for the pendulum.

## 6.5 Subcategories $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, $\mathcal{H}^{pendulum}$

### 6.5.1 Subcategory $\mathcal{N}^{pendulum}$

To recover the notion that a category in analytical mechanics should be able to describe a physical system, where the objects represent the possible states of the system and the morphisms perform time evolutions consistent with the equations of motion, a subcategory $\mathcal{N}^{pendulum}$ of $\mathcal{N}$ is defined.

Informally, a subcategory is a collection of some objects and some arrows of the original category, such that the subcategory itself fulfills the requirements of a category [13]. Formally, a subcategory is defined in the following way: [16]

**Definition 6.1 (Subcategory)** *A subcategory $\mathcal{C}^{sub}$ of a category $\mathcal{C}$ consists of:*

1. ***Objects:*** *a collection of objects, such that* $\mathbf{Obj}(\mathcal{C}^{sub}) \subseteq \mathbf{Obj}(\mathcal{C})$
2. ***Morphisms:*** *For every pair of objects $A$, $B \in \mathcal{C}^{sub}$ the morphism $f \in \mathcal{C}^{sub}(\mathrm{A}, \mathrm{B})$ with $f : \mathrm{A} \to \mathrm{B}$ represents the mapping from $A$ to $B$. Additionally, the collection of morphisms $f$ is a subcollection of the morphisms in $\mathcal{C}$:*

$$\hom_{\mathcal{C}^{sub}}(\mathrm{A}, \mathrm{B}) \subseteq \hom_{\mathcal{C}}(\mathrm{A}, \mathrm{B})$$

3. ***Composition:*** *The composition in $\mathcal{C}^{sub}$ is the composition in $\mathcal{C}$. For two morphisms $f \in \mathcal{C}^{sub}(\mathrm{A}, \mathrm{B})$, $g \in \mathcal{C}^{sub}(\mathrm{B}, \mathrm{C})$, the composition $f \circ g$ in $\mathcal{C}^{sub}$ is the composition in $\mathcal{C}$. If the compositions of $f \circ g$ in $\mathcal{C}^{sub}$ and $\mathcal{C}$ are equal to each other, the subcategory $\mathcal{C}^{sub}$ is full.*

    - ***Associativity:*** *Associativity is defined in $\mathcal{C}^{sub}$ in the same way as in $\mathcal{C}$.*

4. ***Identity morphism*** *For an object $A \in \mathcal{C}^{sub}$, the identity morphism $1_A \in \mathcal{C}^{sub}(\mathrm{A}, \mathrm{A})$ is equivalent to the identity morphism in $\mathcal{C}$:*

$$1_A \in \mathcal{C}^{sub}(\mathrm{A}, \mathrm{A}) = 1_A \in \mathcal{C}(\mathrm{A}, \mathrm{A})$$

5. ***Inclusion functor:*** *[13] [14] The inclusion functor $I : \mathcal{C}^{sub} \to \mathcal{C}$ is a functor according to definition 3.2, that maps every object in $\mathcal{C}^{sub}$ to itself in $\mathcal{C}$:*

$$I(A) = A \in \mathcal{C}^{sub} \qquad\qquad I(f) = f \in \mathcal{C}^{sub}$$

In the specific case of the Newtonian category, $\mathcal{N}$ includes all coordinates and associated velocities $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{R}^3$. In the aim to describe a pendulum, the subcategory $\mathcal{N}^{pendulum}$ selects only those $(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{R}^2$ that fulfill the constraint posed by eq. (55). The objects in $\mathcal{N}^{pendulum}$ are then given by:

$$\mathbf{Obj}(\mathcal{N}^{pendulum}) = \{(\mathbf{r}, \dot{\mathbf{r}}) \in \mathcal{N} \mid \mathbf{r} = (x, y),\, \dot{\mathbf{r}} = (\dot{x}, \dot{y}),\, x^2 + y^2 = l^2\, x\dot{x} + y\dot{y} = 0\}$$

where $l$ is the length of the pendulum, and the condition $x\dot{x} + y\dot{y} = 0$ ensures the velocities $\dot{x}, \dot{y}$ correspond to their respective coordinates $x, y$, such that $(\dot{x}, \dot{y}) \perp (x, y)$. This ensures that the subcategory adheres to the physical constraints described in section 6.1.2.

The conditions for a subcategory, definition 6.1, are automatically fulfilled, as all objects fulfilling the constraints are also coordinates and velocities in $\mathcal{N}$. All morphisms $f_{\mathcal{N},n} \in \mathcal{N}$ from eq. (20), where $n$ scales the stepsize $ndt$ and is sufficiently small to accurately give the time evolution, are also morphism in $\mathcal{N}^{pendulum}$. Composition and identity are equivalent to those in $\mathcal{N}$.

Similarly, the Lagrangian category $\mathcal{L}$ and Hamiltonian category $\mathcal{H}$ need to be effectively reduced to consider only solutions of the specific pendulum at hand. As for the Newtonian case, the valid objects for the pendulum will be included in $\mathcal{L}^{pendulum}$ and $\mathcal{N}^{pendulum}$.

### 6.5.2 Subcategory $\mathcal{L}^{pendulum}$

The generalized coordinate and velocity $(\mathbf{q} = \theta, \dot{\mathbf{q}} = \dot{\theta})$ in the Lagrangian category $\mathcal{L}$, obtained from eq. (57), automatically fulfill the constraint in eq. (55) and the requirement that the velocity is perpendicular to the position:

$$\begin{aligned} &x^2 + y^2 = l^2 cos^2(\theta) + l^2 sin^2(\theta) = l^2 \\ &x\dot{x} + y\dot{y} = l^2\ cos(\theta)\ \dot{\theta} sin(\theta) + l^2\ sin(\theta)\ \dot{\theta}(-cos(\theta)) = 0 \end{aligned}$$

This is true for any $\theta, \dot{\theta}$ where $\dot{\theta} = \frac{d\theta}{dt}$. This would suggest that the Lagrangian category $\mathcal{L}$ describes all possible states of a pendulum with length $l$ at once.

However, it can be argued that based on section 4.2, either $\theta$ or $\dot{\theta}$ could be chosen from the set of all possible angles and derivatives such that $\frac{d\theta}{dt} \neq \dot{\theta}$ (a position or velocity is selected for a different velocity or position), and the constraints would still be fulfilled. This contradicts physical interpretations and valid states. For example, a pendulum with a maximum deflection $\theta > 0$ cannot be in the state $(\theta = 0, \dot{\theta} = 0)$. A mapping $f_{\mathcal{L}}(\theta = 0, \dot{\theta} = 0) = (\theta \neq 0, \dot{\theta} \neq 0)$ is not compatible with the equations of motion, and cannot exist for the pendulum, even though without any restrictions such objects could be chosen from $\mathcal{L}$. The equations of motion in section 2 require $\frac{d\mathbf{q}}{dt} \equiv \dot{\mathbf{q}}$.
If this were not given, and any $\theta, \dot{\theta}$ could be put together in a state $(\theta, \dot{\theta})$, the valid states $(\theta, \dot{\theta} = \frac{d\theta}{dt})$ can be selected in a subcategory $\mathcal{L}^{pendulum}$ to ensure a correct description of the dynamical system.
Considering this, the objects of the subcategory $\mathcal{L}^{pendulum}$ can thus be defined as:

$$\mathbf{Obj}(\mathcal{L}^{pendulum}) = \left\{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{L} \mid \mathbf{q} = \theta,\, \dot{\mathbf{q}} = \dot{\theta},\, \dot{\theta} = \frac{d\theta}{dt}\right\}$$

Or, when the total energy $E$ is introduced, the selection can also be performed by the requirement, that for all valid states $(\theta, \dot{\theta})$, the total energy $E$ is conserved:

$$\mathbf{Obj}(\mathcal{L}^{pendulum}) = \left\{ (\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{L} \mid \mathbf{q} = \theta,\, \dot{\mathbf{q}} = \dot{\theta},\, E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - cos(\theta)) = constant \right\}$$

The subcategory $\mathcal{L}^{pendulum}$ fulfills the requirements in definition 6.1 with the morphisms from eq. (19), and the composition, associativity, and identity from section 4.2. The inclusion functor $I$ could be interpreted as a reversion of the selection of the states $(\theta, \dot{\theta})$

### 6.5.3 Subcategory $\mathcal{H}^{pendulum}$

$\mathcal{H}$ includes all phase space points, and thus describes all phase portraits and with that the set of all trajectories of the pendulum.
As before, the length $l$ needs to be determined to focus on the study of a single pendulum based on the constraints. To ensure that the mapping of any state onto every other state can be performed by $f_{\mathcal{H}}$, only those states satisfying the equations of motion eq. (65) are selected. To ensure this, $\theta$ and $p_\theta$ can be related to each other via $p_\theta = ml^2\dot{\theta} = ml^2 \frac{d\theta}{dt}$, or by requiring the conservation of energy through $H = E = const.$

Thus, the objects in $\mathcal{H}^{pendulum}$ can be defined once via the canonical momentum $\mathbf{p}$ or the total energy $E$:

$$\mathbf{Obj}(\mathcal{H}^{pendulum}) = \left\{ (\mathbf{q}, \mathbf{p}) \in \mathcal{H} \mid \mathbf{q} = \theta,\, \mathbf{p} = p_\theta,\, p_\theta = ml^2 \frac{d\theta}{dt} \right\}$$

$$\mathbf{Obj}(\mathcal{H}^{pendulum}) = \left\{ (\mathbf{q}, \mathbf{p}) \in \mathcal{H} \mid \mathbf{q} = \theta,\, \mathbf{p} = p_\theta,\, H = E = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta) = constant \right\}$$

Again, since only certain $(\mathbf{q}, \mathbf{p}) \in \mathcal{H}$ are selected to be objects in $\mathcal{H}^{pendulum}$, the mapping through morphisms, composition, associativity, and the identity morphism are given by those in section 4.3.

### 6.5.4 Initial Conditions

When the dynamics of a specific pendulum is studied, the initial conditions are traditionally given by the pendulums position and velocity at the maximum deflection. In terms of the angle $\theta$, the initial conditions would read as

$$\theta_0 = max(|\theta|) \qquad \dot{\theta}_0 = 0 \tag{66}$$

So far, the categories $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$, from section 4 do not consider one specific object $(\mathbf{Q}, \mathbf{P})$ as an exceptional initial condition. Instead, the morphisms from eq. (26) interpret the state $(\mathbf{Q}, \mathbf{P})$ on which the mapping is performed as an initial state, and thus the full set of objects is the full set of possible initial conditions for the pendulum. Based on any one state $(\mathbf{Q}, \mathbf{P})$, all other states corresponding to those initial conditions can be reached by varying n in $f_{\mathcal{C},n}$. To consider only the motion of a single pendulum with length $l$ and maximum deflection $\theta_0$ in a categorical framework, additional subcategories need to be defined to satisfy this condition without violating the equations of motion of the system. Those subcategories could select only the objects for which the equations of motion are satisfied considering a set of initial conditions $(\theta_0, \dot{\theta}_0)$.

With the definition of the subcategories $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, and $\mathcal{H}^{pendulum}$, which include only the states and morphisms that satisfy the equations of motion for a pendulum with constant length $l$, functors can be defined that perform the mappings between the descriptions of the pendulum in $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, $\mathcal{H}^{pendulum}$. By selecting the objects into these subcategories, it is ensured that the functors described in section 5 preserve the relationships between Newtonian, Lagrangian, and Hamiltonian descriptions while adhering to the system's constraints.

## 6.6 Functors between $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, $\mathcal{H}^{pendulum}$

The functors between the subcategories $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, $\mathcal{H}^{pendulum}$ are defined in the same way as the functors $F$ from section 5. This means $F_{\mathcal{A},\mathcal{B}} = F_{\mathcal{A}^{pendulum},\mathcal{B}^{pendulum}}$, the only difference is that $F_{\mathcal{A}^{pendulum},\mathcal{B}^{pendulum}}$ is only applied to objects satisfying the constraints of the respective subcategory. Therefore, in the following the discussion *pendulum* will be dropped, and all following $F_{\mathcal{A},\mathcal{B}}$ are understood to perform the transformations of the respective subcategories $F_{\mathcal{A},\mathcal{B}} : \mathcal{A}^{pendulum} \to \mathcal{B}^{pendulum}$.

In section 6.6.1 to section 6.6.3, the functors between $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, $\mathcal{H}^{pendulum}$ are written out explicitly. This serves as a framework for transforming the different formulations of the pendulum's dynamics into each other, but also identifies inconsistencies and inaccuracies in the functors defined in section 5.

### 6.6.1 Lagrangian $\mathcal{L}^{pendulum}$ $\leftrightarrow$ Newtonian $\mathcal{N}^{pendulum}$

**Object Part**

The transformation from the Cartesian coordinate and velocity to the generalized variables is given by the coordinate transformation from Cartesian coordinates to polar coordinates eq. (57), as motivated in section 6.1.2.

From the polar coordinates in eq. (57) follows the transformation $\mathbf{R}$ of the position and velocity: [17]

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l\,cos(\theta) \\ l\,sin(\theta) \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} -l\,sin(\theta)\dot{\theta} \\ l\,cos(\theta)\dot{\theta} \end{pmatrix} \tag{67}$$

The reduction from the four variables $(x, y, \dot{x}, \dot{y})$ to the angle and angular velocity $(\theta, \dot{\theta})$ is performed through the following relations:

$$\theta = arctan\left(\frac{y}{x}\right) \qquad l^2 = x^2 + y^2 \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{l^2}$$

Where $0 \leq \theta \leq 2\pi$ and the coordinates $(x, y)$ are chosen such that the arctangent is defined.

The mapping between Cartesian and polar coordinates can be inverted to $\mathbf{R}'$ via eq. (67).

The action of the functors $F^{obj}_{\mathcal{N},\mathcal{L}}$, $F^{obj}_{\mathcal{L},\mathcal{N}}$ on the objects in $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$ are thus given by:

$$F^{obj}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = l \begin{pmatrix} cos(\theta) \\ sin(\theta) \\ -sin(\theta)\dot{\theta} \\ cos(\theta)\dot{\theta} \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \text{ via } \begin{array}{c} l = \sqrt{x^2 + y^2} \\ \theta = arctan(\frac{y}{x}) \\ \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{l^2} \end{array} \quad 0 \leq l < \text{inf},\ 0 \leq \theta \leq 2\pi$$

$$F^{obj}_{\mathcal{L},\mathcal{N}} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = F^{obj}_{\mathcal{L},\mathcal{N}}\ l \begin{pmatrix} cos(\theta) \\ sin(\theta) \\ -sin(\theta)\dot{\theta} \\ cos(\theta)\dot{\theta} \end{pmatrix} = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \text{ from eq. (67)}$$

**Arrow Part**

The arrow parts transform the morphisms $f$ from $\mathcal{N}^{pendulum}$ to those in $\mathcal{L}^{pendulum}$ and vice versa. With the transformation of the objects from above, and the functors from section 5.3 and section 5.3.2, the arrow parts of the functors $F^{\text{arr}}_{\mathcal{L},\mathcal{N}}$ and $F^{\text{arr}}_{\mathcal{N},\mathcal{L}}$ for the categories $\mathcal{N}^{pendulum}$ and $\mathcal{L}^{pendulum}$ transform as:

$$F^{arr}_{\mathcal{N},\mathcal{L}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \ddot{\mathbf{r}}\, dt \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \mathbf{g}\, dt \end{pmatrix} =$$
$$= \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \frac{m}{l^2}\frac{\partial L}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} - \frac{g}{l} sin(\theta)\, dt \end{pmatrix} = \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \ddot{\theta}\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix} \tag{68}$$

$$F^{arr}_{\mathcal{L},\mathcal{N}}\ f_{\mathcal{L}} \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \frac{m}{l^2}\frac{\partial L}{\partial \theta}\, dt \end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{N}} \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} - \frac{g}{l} sin(\theta)\, dt \end{pmatrix} =$$
$$= \begin{pmatrix} \mathbf{r} + \frac{1}{m}\frac{\partial T}{\partial \dot{\mathbf{r}}}\, dt \\ \dot{\mathbf{r}} + \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \mathbf{g}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \ddot{\mathbf{r}}\, dt \end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} \tag{69}$$

Where $\mathbf{r} = (x, y)$ and $\dot{\mathbf{r}} = (\dot{x}, \dot{y})$, and eq. (60) and eq. (63) have been used. Furthermore, additional prefactors have been introduced into eq. (68) and eq. (69) to obtain the correct accelerations $\ddot{\theta}$ and $\ddot{\mathbf{r}}$. Strictly applying eq. (44) would give:

$$F^{arr}_{\mathcal{N},\mathcal{L}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \ddot{\mathbf{r}}\, dt \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{L}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} + \mathbf{g}\, dt \end{pmatrix} =$$
$$= \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \frac{\partial L}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} - mglsin(\theta)\, dt \end{pmatrix} = \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + ml^2\ddot{\theta}\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix}$$

While based on the equations of motion eq. (60), the equivalence $-mlgsin(\theta) = ml^2\ddot{\theta}$ in the lower row is true, it does not comply with the originally intended time evolution from eq. (11) since $ml^2\ddot{\theta} \neq \ddot{\theta}$ for all $m$, $l$.

### 6.6.2 Newtonian $\mathcal{N}^{pendulum}$ $\leftrightarrow$ Hamiltonian $\mathcal{H}^{pendulum}$

**Object Part**

The mapping $F^{obj}_{\mathcal{N},\mathcal{H}} : (x, y) \to \theta$ is given by the same transformation into polar coordinates eq. (57) as in section 6.6.1. Furthermore, the transformation $F^{obj}_{\mathcal{N},\mathcal{H}} : (\dot{x}, \dot{y}) \to p_\theta$ can be achieved by applying eq. (48):

$$F^{obj}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} \overset{\mathbf{R}}{\Rightarrow} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \to \begin{pmatrix} \theta \\ \frac{\partial H}{\partial \dot{\theta}} \end{pmatrix} = \begin{pmatrix} \theta \\ p_\theta \end{pmatrix}$$

Where $\mathbf{R}$ is the coordinate transformation from eq. (67) and $p_\theta = mgl^2\dot{\theta}$ has been applied.

The reverse transformation is performed by $F^{obj}_{\mathcal{H},\mathcal{N}}$:

$$F^{obj}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} = \begin{pmatrix} \theta \\ \frac{\partial H}{\partial p_\theta} \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \overset{\mathbf{R}'}{\Rightarrow} \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix}$$

Additionally, the mappings by $F^{obj}_{\mathcal{H},\mathcal{N}}$ and $F^{obj}_{\mathcal{N},\mathcal{H}}$ for the pendulum can also be described via a canonical transformation and the coordinate transformation from section 6.6.1. Then, the transformation from Cartesian coordinates and velocities $(x, y, \dot{x}, \dot{y})$ to the angular coordinate $\theta$ and the canonical momentum $p_\theta$, as well as the reverse transformation, is given as:

$$F^{obj}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} \quad \text{via} \quad \begin{matrix} \theta = \arctan\left(\frac{y}{x}\right) \text{ and eq. (67)} \\ p_\theta = m(x\dot{y} - y\dot{x}) \end{matrix}$$

$$F^{obj}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} = F^{obj}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} l\cos(\theta) \\ l\sin(\theta) \\ \frac{p_\theta}{ml}\sin(\theta) \\ -\frac{p_\theta}{ml}\cos(\theta) \end{pmatrix} = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \quad \text{via eq. (67) and } p_\theta = ml^2\dot{\theta}$$

**Arrow Part**

According to eq. (51) and eq. (52), the arrow parts of the functors $F^{arr}_{\mathcal{N},\mathcal{H}}$ and $F^{arr}_{\mathcal{H},\mathcal{N}}$ act as:

$$F^{arr}_{\mathcal{N},\mathcal{H}}\ f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix} = F^{arr}_{\mathcal{N},\mathcal{H}} \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \frac{\partial T}{\partial p_\theta}\, dt \\ p_\theta - \frac{\partial U}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \frac{p_\theta}{ml^2}\, dt \\ p_\theta - mglsin(\theta)\, dt \end{pmatrix} = \begin{pmatrix} \theta + \frac{\partial H}{\partial p_\theta}\, dt \\ p_\theta - \frac{\partial H}{\partial \theta}\, dt \end{pmatrix} = f_{\mathcal{H}} \begin{pmatrix} \theta(t) \\ p_\theta(t) \end{pmatrix}$$

$$F^{arr}_{\mathcal{H},\mathcal{N}}\ f_{\mathcal{H}} \begin{pmatrix} \theta(t) \\ p_\theta(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{N}} \begin{pmatrix} \theta + \frac{\partial H}{\partial p_\theta}\, dt \\ p_\theta - \frac{\partial H}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \frac{\partial T}{\partial p_{\mathcal{N}}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \dot{\mathbf{r}}}\, dt \end{pmatrix} = \begin{pmatrix} \mathbf{r} + \dot{\mathbf{r}}\, dt \\ \dot{\mathbf{r}} - \frac{1}{m}\frac{\partial U}{\partial \mathbf{r}}\, dt \end{pmatrix} = f_{\mathcal{N}} \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}$$

### 6.6.3 Lagrangian $\mathcal{L}^{pendulum}$ $\leftrightarrow$ Hamiltonian $\mathcal{H}^{pendulum}$

The transformations between Lagrangian mechanics and Hamiltonian mechanics are given by the Legendre transformation eq. (36).

**Object Part**

The generalized coordinate $\mathbf{q} = \theta$ and canonical coordinate $\mathbf{q} = \theta$ are the same in the Lagrangian and the Hamiltonian framework. The mapping between the generalized velocity $\dot{\mathbf{q}} = \dot{\theta}$ and the canonical momentum $\mathbf{p} = p_\theta = ml^2\dot{\theta}$ is given through the definition of the canonical momentum $\frac{\partial L}{\partial \dot{\theta}} = p_\theta$. Thus, the transformation of the objects in $\mathcal{L}^{pendulum}$ and Hamiltonian $\mathcal{H}^{pendulum}$ by $F^{obj}_{\mathcal{L},\mathcal{H}}$ and $F^{obj}_{\mathcal{H},\mathcal{L}}$ reads as:

$$F^{obj}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \theta \\ \frac{\partial L}{\partial \dot{\theta}} \end{pmatrix} = \begin{pmatrix} \theta \\ ml^2\dot{\theta} \end{pmatrix} = \begin{pmatrix} \theta \\ p_\theta \end{pmatrix}$$

$$F^{obj}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} = \begin{pmatrix} \theta \\ \frac{p_\theta}{ml^2} \end{pmatrix} = \begin{pmatrix} \theta \\ \frac{\partial L}{\partial \dot{\theta}} \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

**Arrow Part**

The arrow parts $F^{arr}_{\mathcal{L},\mathcal{H}}$, $F^{arr}_{\mathcal{H},\mathcal{L}}$ map the time evolutions in the configuration space of the pendulum to those in phase space, and vice versa. According to eq. (39) and eq. (38) they transform as:

$$F^{arr}_{\mathcal{L},\mathcal{H}}\ f_{\mathcal{L}} \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \frac{\partial L}{\partial \theta}\, dt \end{pmatrix} = F^{arr}_{\mathcal{L},\mathcal{H}} \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} - mglsin(\theta)\, dt \end{pmatrix} =$$
$$= \begin{pmatrix} \theta + \frac{\partial H}{\partial p_\theta}\, dt \\ p_\theta - \frac{\partial H}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \frac{p_\theta}{ml^2}\, dt \\ p_\theta + mglsin(\theta)\, dt \end{pmatrix} = f_{\mathcal{H}} \begin{pmatrix} \theta(t) \\ p_\theta(t) \end{pmatrix}$$

$$F^{arr}_{\mathcal{H},\mathcal{L}}\ f_{\mathcal{H}} \begin{pmatrix} \theta(t) \\ p_\theta(t) \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \theta + \dot{\theta}\, dt \\ p_\theta + \dot{p}_\theta\, dt \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \theta + \frac{\partial H}{\partial p_\theta}\, dt \\ p_\theta - \frac{\partial H}{\partial \theta}\, dt \end{pmatrix} = F^{arr}_{\mathcal{H},\mathcal{L}} \begin{pmatrix} \theta + \frac{p_\theta}{ml^2}\, dt \\ p_\theta + mglsin(\theta)\, dt \end{pmatrix} =$$
$$= \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} + \frac{\partial L}{\partial \theta}\, dt \end{pmatrix} = \begin{pmatrix} \theta + \dot{\theta}\, dt \\ \dot{\theta} - mglsin(\theta)\, dt \end{pmatrix} = f_{\mathcal{L}} \begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \end{pmatrix}$$

# 7 Discussion and Outlook

The following shall address potential inconsistencies between the physics formulations and the application of category theory on Newtonian, Lagrangian, and Hamiltonian mechanics. Some choices in the formulation of the categories and the functors are further motivated. Additionally, the Outlook section 7.4 mentions further concepts in category theory and physics that could be investigated in an extended study of the here developed concepts.

## 7.1 Morphism Requirements and Valid States

When a specific physical system is studied, the requirement "based on any one initial state $(\mathbf{Q}, \mathbf{P})$, all other states in the category $\mathcal{C}$ can be reached by varying n in $f_{\mathcal{C},n}$" from definition 3.1 and section 4.4 is fulfilled only under certain conditions.

In a general case, the category $\mathcal{C}$ has all states $(\mathbf{Q}, \mathbf{P})$ as objects, and does not consider constraints on the system, energy conservation, or whether the states are solutions of the equations of motion for a specific system or not. For example, the category $\mathcal{N}$ itself does not require its objects to satisfy the constraint $x^2 + y^2 = l^2$. Without that constraint, the requirement that $f_{\mathcal{C},n}$ maps all states in $\mathcal{N}$ to any state in $\mathcal{N}$ can be fulfilled, however for some mappings a physical meaning is lost, as explained in section 6.4.

To fulfill the requirement when a specific system is studied under consideration of its constraints, subcategories are defined. Those provide a way to ensure that only valid physical states are included, and the mapping between states is limited to those that adhere to the constraints, conservation laws and equations of motion.They also clarify that the full original category cannot realistically represent the behaviour of constrained systems without modification or relaxation of its physical assumptions.

## 7.2 Coordinate Transformations and Redundant Information

The transformation from generalized coordinates to Newtonian ones essentially introduces redundant information. By construction, generalized coordinates provide a reduced description of the system, using the minimal set of independent variables in consideration of the system's constraints. Whenever applicable, they reduce the degrees of freedom and redundancies present in the full set of Cartesian coordinates. This oftentimes simplifies the equations of motion.

Transforming back to Cartesian coordinates reintroduces all original degrees of freedom, including those that are not independent. This reintroduction of redundancies often leads to a more complex description of the system, and can make the understanding of the dynamics of the system more complicated by relating constrained and unconstrained

motions. For example, the derivation of the equations of motion for the pendulum in Cartesian coordinates in section 6.1.1 is more lengthy and less straightforwards than when generalized coordinates are used in section 6.1.2 or section 6.2.

Determining how many variables are needed for the back transformation to Cartesian coordinates, as well as the exact form of the transformation, depend on the system's constraints and the dimensionality of its configuration space. Those constraints need to be included in the transformation to obtain the correct variables. For example, the description of the pendulum in Newtonian mechanics, section 6.1, as well as the transformations $\mathcal{N}^{pendulum} \leftrightarrow \mathcal{L}^{pendulum}$ and $\mathcal{N}^{pendulum} \leftrightarrow \mathcal{H}^{pendulum}$ require the formulation of the constraint equation eq. (55) for consistency.

## 7.3 Time Evolution in $\mathcal{N}$ and Transformations by Functors $F$

The functors $F$ presented in section 5 are not the only ways to relate the different objects and morphisms to each other. In section 2.1, the time evolution in Newtonian mechanics has been stated both via the Newtonian momentum $\mathbf{p}$ and via the total energy $E = T + U$. In the expression of the functors in section 5 however, only the time evolution in terms of the energy eq. (8) is used, even though the formulation via $\mathbf{p}$ and $\frac{d\mathbf{p}}{dt}$ could be considered more fundamental and central to Newtonian mechanics, as it is derived from Newton's laws.

This decision has been made for multiple reasons. Once, for specific cases where the Lagrangian is the difference between the system's kinetic and potential energy $L = T - U$ eq. (9) and the Hamiltonian as the sum $H = T + U$ eq. (15), the Newtonian formulation via the energies closer resembles the structure of the Lagrangian and Hamiltonian descriptions. This perspective highlights structural similarities and provides simplified expressions for the functors between the mechanics frameworks. However, eq. (9) and eq. (15) do not hold for any system. For relativistic systems, in the presence of an electromagnetic field, or when nonholonomic constraints are present, the Lagrangian $L$ cannot be written in the form eq. (9). For such systems, alternative approaches are required to perform the transformations $\mathcal{L} \leftrightarrow \mathcal{N}$ and $\mathcal{L} \leftrightarrow \mathcal{H}$.

Instead of transformations performed on the coordinates used in $T$ and $U$, canonical transformations could be used to perform the mappings with $\mathbf{p}$. This however is less explicit for general cases, and could include a formulation via the metric tensor for example. Due to the structural difference to the Legendre transformation, this was not considered in greater detail in section 5. However, for the transformation $F^{obj}_{\mathcal{N},\mathcal{H}} : \mathcal{N}^{pendulum} \to \mathcal{H}^{pendulum}$ and $F^{obj}_{\mathcal{H},\mathcal{N}} : \mathcal{H}^{pendulum} \to \mathcal{N}^{pendulum}$ this has been performed to give an example of a transformation of objects other than via the energy consideration.

Additionally, the time evolution in eq. (3) includes the time derivative $\frac{d\mathbf{p}}{dt}$, while the Lagrangian and Hamiltonian formalisms primarily involve derivatives with respect to the variables of the system, which also leads to a structurally different framework. Using the derivatives of the total energy $E$ of the system to describe the time evolution in eq. (8) most closely resembles the derivatives of the Lagrangian $L$ and the Hamiltonian $H$, and even coincides with $L$ or $H$ or both in special cases.

## 7.4 Outlook

### 7.4.1 Further Considerations in Category Theory

In this work, category theory was introduced by defining categories and functors. The categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ were constructed to represent Newtonian, Lagrangian, and Hamiltonian mechanics according to section 3. However, this only provides one example of the potential applications of category theory. Many concepts related to category theory have neither been mentioned nor studied here. Categories and functors represent only two fundamental concepts of category theory as a much broader mathematical framework.

To further investigate the structure of analytical mechanics from a categorical framework, *natural transformations* could be studied. Natural transformations describe the mappings between functors [13]. For a more formal definition, consider two functors $F_1, F_2 : \mathcal{A} \to \mathcal{B}$. A natural transformation $\tau : F_1 \to F_2$ is a mapping which assigns arrows to each object $a \in \mathbf{Obj}(\mathcal{A})$ such that $\tau a : F_1 a \to F_2 a \in \mathcal{B}$, and for every morphism $f : a \in C \to a' \in \mathcal{C}$ the following

diagram commutes: [13]

$$\begin{array}{ccc} F_1(a) & \xrightarrow{\tau_a} & F_2(a) \\ {\scriptstyle F_1 f}\downarrow & & \downarrow{\scriptstyle F_2 f} \\ F_1(a') & \xrightarrow[\tau_{a'}]{} & F_2(a') \end{array}$$

Natural transformations between for $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ and their functors could exist, but need not to. If it possible to construct such $\tau$, they could reveal further structural similarities between Newtonian, Lagrangian, and Hamiltonian mechanics. They could represent a systematic mapping between the functors connecting the respective categories.

Building on the foundation of categories, functors, and natural transformations, further investigations of additional concepts in category theory could be considered. Examples include the principle of duality, which studies dual categories that represent a 'reversion of all arrows' with respect to the original category [14]. Additionally, adjoint functors, limits, monads, and much more (see [14]) could be explored in relation to $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ and their functors.

### 7.4.2 Further Physics Considerations

A closer study of the categorical description of analytical mechanics could reveal new insights into the structure of mechanics, and highlight fundamental structures.
On the other hand, a more rigorous examination of the spaces of $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ could also lead to further inconsistencies and discrepancies between category theory and physics, requiring additional refinements of the categorical description of mechanics, or could lead to a disqualification of the proposed categories as accurate physics representations.

Assuming that the categorical framework holds, additional dynamics formulations, for example the Hamilton-Jacobi formalism, could be studied for their similarities to Newtonian, Lagrangian, and Hamiltonian mechanics via their respective categories.

The application of more advanced concepts in category theory could highlight new aspects of analytical mechanics. These investigations might not only further unify the frameworks of Newtonian, Lagrangian, and Hamiltonian mechanics but could also provide a foundation for exploring broader theoretical questions in physics.

# 8 Conclusion

The motivation for applying category theory to analytical mechanics arises from the observation that Newtonian, Lagrangian, and Hamiltonian mechanics could be represented as categories. In mathematics, categories are abstract structures which are used to describe relations between mathematical objects [13]. Newtonian, Lagrangian, and Hamiltonian mechanics show similarities in their descriptions of the states and time evolutions of a physical system, which suggests that those shared structures could be further investigated within the framework of category theory.

All three mechanics formulations use a kind of position and a velocity-dependent quantity (a velocity or momentum) to describe the states of a system. The transition from one state into another is described as a trajectory by a time evolution, which is governed by the respective equations of motion of the dynamical object. In a categorical framework, the states can be interpreted as objects, and the time variations as morphisms. The composition of morphisms describes a succession of timesteps, and the identity morphism leaves the state unchanged, thus representing no change in time. Based on this structure, the categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ are defined, which are supposed to represent the respective mechanics formulations.

In category theory, mappings between categories are described via functors. To further develop a categorical view on analytical mechanics, functors mapping between $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ are defined. Between Lagrangian and Hamiltonian mechanics, this mapping is well-known and typically expressed as the Legendre transformation. The transformations between Newtonian and Lagrangian, and Newtonian and Hamiltonian mechanics are less familiar, and require a closer consideration of the structural properties of the relations between their mathematical spaces. However, simplifications can be made to be able to express those transformations more explicitly for certain system.

Nevertheless, when aiming to describe a concrete physical system with those categories and functors, the exact transformations between the frameworks will also depend on the constraints of the system. This is where the previously developed categorical representation of $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$ fails. As an example, a simple mathematical pendulum is considered. In the case of the pendulum, the fixed rod length $l$ poses a constraint on the system and its possible states. However, constraints are not inherently included in the definition of the categories $\mathcal{N}$, $\mathcal{L}$, and $\mathcal{H}$, which leads to the inclusion of non-physical states and morphisms that do not correspond to valid trajectories for the pendulum.

To address this discrepancy between the mathematical formulation of the categories and the observable valid states, and to recover an accurate physical description subcategories are introduced. The subcategories $\mathcal{N}^{pendulum} \subseteq \mathcal{N}$, $\mathcal{L}^{pendulum} \subseteq \mathcal{L}$, and $\mathcal{H}^{pendulum} \subseteq \mathcal{H}$ select those objects from the respective categories that satisfy the system's constraints and conservation laws, and comply with the equations of motion. They ensure that only valid states and transformations are considered, preserving the physical meaning of the time evolutions. Additional subcategories can be introduced to incorporate a chosen set of initial conditions to only study one trajectory and a specific set of solutions to the equations of motion.

While these subcategories provide a way to resolve the issue of impossible states of the system, they rely on prior knowledge of the physical description of the system. This does not necessarily simplify the description of a physical system, yet still offers a structured way of representing the dynamics of the system and the transformations between Newtonian, Lagrangian, and Hamiltonian mechanics.

Based on those subcategories, the same functors as for $\mathcal{N}$, $\mathcal{L}$, $\mathcal{H}$ can be applied to map between $\mathcal{N}^{pendulum}$, $\mathcal{L}^{pendulum}$, and $\mathcal{H}^{pendulum}$

# References


[1] Samson Abramsky and Bob Coecke. *A categorical semantics of quantum protocols*. 2007. arXiv: quant-ph/0402130 [quant-ph]. URL: https://arxiv.org/abs/quant-ph/0402130.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. 2nd ed. Vol. 60. Graduate Texts in Mathematics. Originally published by Nauka, Moscow, 1974. New York, NY: Springer-Verlag New York, 1989, pp. XVI, 520. ISBN: 978-1-4757-2063-1. DOI: 10.1007/978-1-4757-2063-1.

[3] J. Baez and M. Stay. "Physics, Topology, Logic and Computation: A Rosetta Stone". In: *New Structures for Physics*. Springer Berlin Heidelberg, 2010, pp. 95–172. ISBN: 9783642128219. DOI: 10.1007/978-3-642-12821-9_2. URL: http://dx.doi.org/10.1007/978-3-642-12821-9_2.

[4] Bruce H. Bartlett. *Categorical Aspects of Topological Quantum Field Theories*. 2005. arXiv: math/0512103 [math.QA]. URL: https://arxiv.org/abs/math/0512103.

[5] Andrei Bernevig and Titus Neupert. *Topological Superconductors and Category Theory*. 2015. arXiv: 1506.05805 [cond-mat.str-el]. URL: https://arxiv.org/abs/1506.05805.

[6] Bob Coecke and Aleks Kissinger. *Categorical Quantum Mechanics I: Causal Quantum Processes*. 2016. arXiv: 1510.05468 [quant-ph]. URL: https://arxiv.org/abs/1510.05468.

[7] Bob Coecke and Eric Oliver Paquette. *Categories for the practising physicist*. 2009. arXiv: 0905.3010 [quant-ph]. URL: https://arxiv.org/abs/0905.3010.

[8] Shawn X. Cui. "Four dimensional topological quantum field theories from $G$-crossed braided categories". In: *Quantum Topology* 10.4 (Sept. 2019), pp. 593–676. ISSN: 1664-073X. DOI: 10.4171/qt/128. URL: http://dx.doi.org/10.4171/QT/128.

[9] Mark Feshbach and Alexander A. Voronov. *A higher category of cobordisms and topological quantum field theory*. 2011. arXiv: 1108.3349 [math.QA]. URL: https://arxiv.org/abs/1108.3349.

[10] Herbert Goldstein, Charles P. Poole, and John L. Safko. *Classical Mechanics*. 3rd ed. © 2002, Published by Pearson (June 15, 2001). Pearson, 2001. ISBN: 9780201657029.

[11] Anton Kapustin. *Topological Field Theory, Higher Categories, and Their Applications*. 2010. arXiv: 1004.2307 [math.QA]. URL: https://arxiv.org/abs/1004.2307.

[12] Liang Kong and Zhi-Hao Zhang. *An invitation to topological orders and category theory*. 2022. arXiv: 2205.05565 [cond-mat.str-el]. URL: https://arxiv.org/abs/2205.05565.

[13] Saunders Mac Lane. *Categories for the Working Mathematician*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. Published 25 September 1998 (Hardcover), 19 November 2010 (Softcover). Springer New York, NY, 1978, pp. XII, 318. ISBN: 978-0-387-98403-2. DOI: 10.1007/978-1-4757-4721-8. URL: https://doi.org/10.1007/978-1-4757-4721-8.

[14] Tom Leinster. *Basic Category Theory*. 2016. arXiv: 1612.09375 [math.CT]. URL: https://arxiv.org/abs/1612.09375.

[15] David Morin. *Introduction to Classical Mechanics: With Problems and Solutions*. Cambridge, UK: Cambridge University Press, 2008. ISBN: 9780521876223. DOI: 10.1017/CBO9780511808951.

[16] Steven Roman. *An Introduction to the Language of Category Theory*. 1st ed. Vol. 1. Compact Textbooks in Mathematics. 171 b/w illustrations, 5 illustrations in colour. Cham: Birkhäuser Cham, 2017, pp. XII, 169. ISBN: 978-3-319-41917-6. DOI: 10.1007/978-3-319-41917-6.

[17] Ernst Schmutzer. *Grundlagen der Theoretischen Physik*. Wiley-VCH Verlag GmbH & Co. KGaA, 2005. ISBN: 9783527405558. DOI: 10.1002/9783527623662.

# Declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly indicated all material which has been quoted either literally or by content from the sources used.

*Graz, 16.01.2025*

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