Bachelor-thesis

Metric reconstruction with causal dynamical triangulation

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Contents

1 Introduction

Quantum gravity is an approach to unite the two big theories of the 20th century, namely general relativity and quantum-mechanics. General relativity describes how space and time are behaving in the presence of mass, an affect which plays a role if one takes a look on heavy objects, such as stars. Different in quantum-mechanics were the one concentrates on small objects, such as atoms or molecules.
The other three fundamental forces, namely: strong interaction, weak interaction and electromagnetism can be well described on a quantum-mechanical point of view. Uniting gravitation with quantum-mechanics is a major goal for today's science, so many theories have already been set up to make this possible. A candidate for "the theory of everything" is quantum gravity.
The goal in this bachelor thesis is to reconstruct the metric of a space-time grid with causal dynamical triangulation and analyzing the problems which can occur. It is structured in such a way that first the theoretical background is referred to, after that the metric of three space-time grids are calculated as well as discussed and at the end there will be a discussion on the evaluated results and a brief outlook.

2 Theoretical background

The aim of the following subchapters is to give a brief insight into the topic and to address the theoretical background of the bachelor thesis.

2.1 Observables in quantum gravity

Observables in physics need to be invariant under the choice of the coordinate system as well as under gauge transformations. In quantum gravity this condition leads to invariance under diffeomorphism.[1] Therefore the metric is not physical. The simplest quantities are operators which are scalar and therefore invariant. Two possible and interesting examples are given below:[1]

$$O_1(x) = \phi^\dagger(x)\phi(x) \tag{1}$$

$$O_2(x) = R(x) \tag{2}$$

Where $O_1(x)$ describes the Higgs particle and $O_2(x)$ describes the local curvature and x stands for an event. Another fundamental quantity to describe the particle is the propagator, which can be defined as:

$$D(x, y) = \langle O(y)O(x)\rangle \tag{3}$$

[1] Invertible function that maps one differentiable manifold to another differntiable manifold that such the function and its inverse are differentiable

Since the propagator is depending on events and not on coordinates, it is diffeomorphism-invariant and therefore one can give this a physical meaning. In quantum gravity space-time is expected to be isotropic on average, but it is more difficult to find one quantity which fulfills the diffeomorphism - invariance condition, since the distances depend on the metric and are therefore dynamical as well and become itself an expectation value. For every event there should exist a unique geodesic length connecting two events:

$$r(X,Y) = \langle min_{z(t)} \int_X^Y dt g^{\mu\nu} \frac{\mathrm{dz(t)}_\mu}{\mathrm{dt}} \frac{\mathrm{dz(t)}_\nu}{\mathrm{dt}} \rangle \tag{4}$$

Where g stands for the metric, $z(t)$ describes a specific path and X and Y are events. The minimum of equation 4 leads to the geodesic length. The propagator could be seen as an function of expectation values of the geodesic distance.[2] Scalar curvature invariants, which are from a quantumfield theory point of view composite operators, which are difficult to evaluate and therefore the Fröhlich-Morchio-Strocchi-formalism (FMS) is used. For more details see [1] and [2].

2.2 FMS formalism and dynamical triangulation

The FMS mechanism is based on constructing observables by splitting the metric in a classical term g^c, which is dominating the system and a fluctuation term γ. The classical part does not have a preferred event, different to the Friedman-Robertson-Walker space-time[2], which has such an event, namely the big-bang and has therefore limitations. Since γ is small in comparison to g^c, a linear split can be made, which leads to the following:

$$g = g^c + \gamma \tag{5}$$

Taking the existence of a cosmological constant into account leads to the fact, that g^c can only be (anti) de-Sitter or Minkowski space-time.[3] Since the γ^{-1} fulfills a Dyson-equation [4] one is able to state a polynomial in γ and γ^{-1} up to a certain order using the formalism of perturbation theory.
Dynamical triangulation supports the assumption that the geometry is de-Sitter space-time.[1] In this work the causal dynamical triangulation (CDT) is used. This designates a nonperturbative path-integral approach to quantum gravity. The main idea is based on piecewise flat space which allows to introduce causal features into gravitational paths. CDT tries to build a theory of quantum gravity with a suitable scaling limits of lattice theory, where as their dynamics are given in nonperturbative pathintegrals which approximately represent the curved space time. [3]

[2]describes a homogeneous, isotropic,expanding universe

[3]de-Sitter $\rightarrow$ positive cosmological constant, anti de-Sitter $\rightarrow$ negative cosmological constant, flat spacetime $\rightarrow$ cosmological constant is zero

[4]$\gamma^{-1} = -(g^c)^{-1}\gamma(g^c+\gamma)^{-1}$[1]

3 Determination of metric in a two-dimensional grid

In order to clearly solve the metric on every event one has to introduce a suitable gauge, the path integral, invertibility of the metric as well as appropriate boundary conditions. In determination of the metric one has to introduce a gauge in order to have the same number of equations and variables. There are more possible gauges to use but in this thesis the Haywood gauge is used, which is listed below.

$$\sum_{\mu,\nu} g^{\mu\nu} \partial_\nu g_{\mu\rho} = 0 \tag{6}$$

Where $g^{\mu\nu}$ stands for the elements of the metric and $g_{\mu\rho}$ represents the elements of the inverse metric. The derivative since one deals with linear vectors, can be expressed:

$$\partial_1 g(\vec{A}) = g\begin{pmatrix} A_1 + 1 \\ A_2 \end{pmatrix} - g(\vec{A}) \tag{7}$$

Where $\vec{A}$ stands for the current position in the grid, where one wants to calculate the derivative. The same procedure has to be done if one wants to look at the derivative of the second coordinate, only with the difference that one now has a slope of one along the second direction.
The mentioned path integral above can be described by the following equation:

$$s(x,y)^2 = \int_x^y g^{\mu\nu} \frac{\mathrm{dz(t)}_\mu}{\mathrm{dt}} \frac{\mathrm{dz(t)}_\nu}{\mathrm{dt}} dt \tag{8}$$

Where $z(t)$ stands for the time dependent path, x and y are the placeholders for the time span and $g^{\mu\nu}$ are the elements of the current metric. Last condition is the invertibility of the metric, which can be mathematically expressed:

$$g^{\mu\nu} g_{\mu\rho} = \mathbb{I} \tag{9}$$

Since invertibility is required, the matrix must not have a determinant which is equal to zero. The boundary conditions in this work are periodic. Taking equations 6 - 9 into account and also making the exception that one is looking for a symmetric metric one is clearly able to solve the metric on every event in the grid. In the following chapters the formalism is going to be demonstrated with three examples.

3.1 Formalism demonstrated in Minkowski-space

In order to demonstrate the procedure of the algorithm, the Minkowski metric is calculated in this chapter. The following picture illustrates the grid:

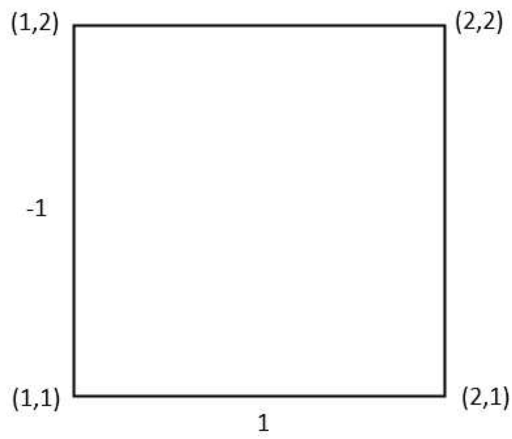


Figure 1: Space-time-grid for Minkowski-space
every horizontal connection equals 1 and every vertical connection equals -1

Using equation 8 one is able to calculate g^{11} and g^{22} with the help of vectors $z_1(t)$ and $z_2(t)$ as well as geodesic length, which can be taken from Figure 1.[5] This procedure respectively the vectors could be done on every event in the grid, which means the stated conditions above are the same for every point $\vec{A}$. So far the following elements have been found: $g^{11} = 1$ and $g^{22} = -1$.
One is left with the fact, that the elements g^{12} and g^{21} (which are the same, since we allow only the case of a symmetrical metric) can not be calculated with the definition of the geodesic length, since the grid has not any diagonal connection.

To solve this one can use the invertibilty condition together with the gauge. For the sake of simplicity in calculation and because this calculation only serves to demonstrate the algorithm, the gauge is simplified to:

$$g^{22}(\partial_1 g^{11} - \partial_1 g^{12}) = g^{11}(\partial_2 g^{22} - \partial_2 g^{21}) \tag{10}$$

Since the aim is demonstrate that $g^{21}(1,1) = g^{12}(1,1) \stackrel{!}{=} 0$ a condition for g^{12} can be obtained from equation 10 and together with the invertibilty (eq. 9), the following equation can be set up:

[5] $z_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ and $z_2(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$

$$-g^{12}(1,1)\cdot(\tfrac{g^{12}(1,2)+g^{12}(2,1)}{2}) = -g^{12}(1,1)^2 \rightarrow g_1^{12}(1,1) = 0, g_2^{12}(1,1) = \tfrac{g^{12}(1,2)+g^{12}(2,1)}{2}$$

As one can see the equation above is quadratic which means one obtains two solutions. But one can show through the same formalism on the other points in the grid that those two solutions are equally. Which means that the Minkowski metric is obtainable with this formalism. The calculation on the other points, for instance (2,1), can be determined with the same algorithm and should also lead to the same metric, because one has on every point in the grid the same conditions. As boundary conditions one can use periodic.

3.2 Determination of metric with one cross connection and four events

In this chapter one wants to concentrate on the following grid which is shown below.

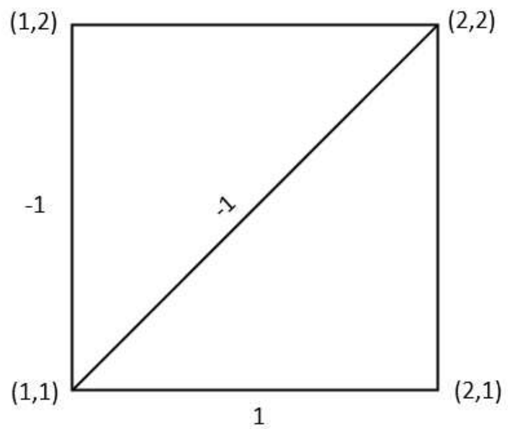


Figure 2: Space-time-grid with one diagonal connection and 4 events
every horizontal connection equals 1 and every vertical and diagonal connection equals -1

As discussed in the last chapter, one is able to solve the non diagonal matrix elements through the equation 8, as long as one has at least one diagonal connection from one event to another. If one has more than one there can be some contradiction in the results if one takes a look on different events, but this is discussed in the next chapter, where one has to deal with such a problem.
The metric elements g^{11} and g^{22} can be calculated through the same procedure and vectors which is discussed in chapter 3.1. Furthermore there are also having the same value as in Minkowski-space.[6] One is now able to calculate the metric elements $g^{12}(1,1)$ and $g^{12}(2,2)$ with equation 8, since there is a connection between those two events. The vectors which one has to use are listed below:

[6] $g^{11} = 1, g^{22} = -1$

$$z(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} t \\ t \end{pmatrix} \text{ and } z(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -t \\ -t \end{pmatrix}$$

By evaluating one is left with the condition that $g^{12}(1,1) = g^{12}(2,2) = -1$. In order to able to evaluate the non diagonal metric elements for the events (1,2) and (2,1) one has to use the Haywood gauge (eq. 6). The boundary conditions are periodic. After the calculation, the values for the elements are $g^{12}(2,1) = g^{12}(1,2) = -1$.
One has determined on every point in the grid the metric which are listed below:

$$g(1,1) = g(1,2) = g(2,1) = g(2,2) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Once one has found a metric that satisfies the gauge it is necessary to counter-check if the metric also fulfills: *The geodetic length calculated using the metric must be same for all paths together as if one directly took the geodetic distance of the individual points together.*
A picture of the possible paths one is going to look at is given below.

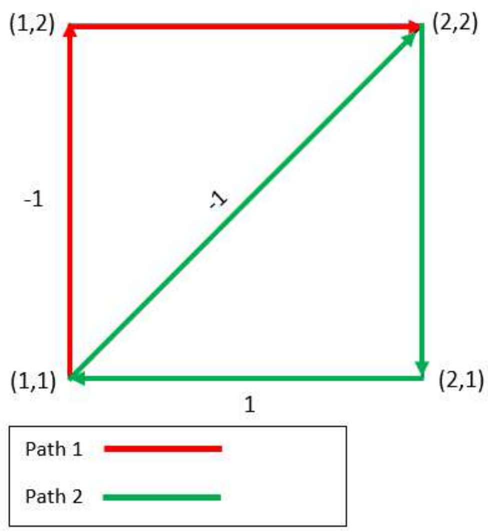


Figure 3: Possible paths
every horizontal connection equals 1 and every vertical and diagonal connection equals -1

Firstly one can concentrate on the green path, where following steps are need to be taken. The path can be divided into three 'subpaths', namely the one from the event (1,1) to (2,2), to the event (2,1) and back to (1,1).
The three paths can be described by the following vectors:

$$\vec{z(t)}_{green1} = \begin{pmatrix} t+1 \\ t+1 \end{pmatrix}, where \text{ t} \in (0,1)$$

$$\vec{z(t)}_{green2} = \begin{pmatrix} 2 \\ -t+3 \end{pmatrix}, where \ \mathrm{t} \in (1,2)$$
$$\vec{z(t)}_{green3} = \begin{pmatrix} -t+4 \\ 1 \end{pmatrix}, where \ \mathrm{t} \in (2,3)$$

As one can see from Figure 3 if one sums up the three geodesic lengths it leads to the value -1.
It is important to work with normalized vectors in order to get the correct solution. The normalized differential of the vectors lead to:

$$\frac{\mathrm{d}\tilde{\mathrm{z(t)}}_{\mathrm{green1}}}{\mathrm{dt}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$
$$\frac{\mathrm{d}\tilde{\mathrm{z(t)}}_{\mathrm{green2}}}{\mathrm{dt}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\frac{\mathrm{d}\tilde{\mathrm{z(t)}}_{\mathrm{green3}}}{\mathrm{dt}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Below an example is given of how the calculation was done with each subpath. Important is that the addition of all the three subpaths above leads to the value -1, because that is the value which one take from Figure 2.

$$\int_0^1 g^{11}(1,1) \cdot \frac{\mathrm{dz_{green1_1}(t)}}{\mathrm{dt}} \cdot \frac{\mathrm{dz_{green1_1}(t)}}{\mathrm{dt}} + 2 \cdot g^{12}(1,1) \cdot \frac{\mathrm{dz_{green1_1}(t)}}{\mathrm{dt}} \cdot \frac{\mathrm{dz_{green1_2}(t)}}{\mathrm{dt}} \\ + \mathrm{g}^{22}(1,1) \cdot \frac{\mathrm{dz_{green1_2}(t)}}{\mathrm{dt}} \cdot \frac{\mathrm{dz_{green1_2}(t)}}{\mathrm{dt}} dt$$

Once calculated one is left with the fact that the solution of the calculated length is the same as one sums up all the geodesic lengths. Same procedure can be done if one looks at the red path, which is shown in Figure 3. If we are going from event (1,1) via a detour to event (2,2) the lengths should be add up to zero. The following paths were used in the calculation and are listed below:

$$\vec{z(t)}_{red1} = \begin{pmatrix} 1 \\ t+1 \end{pmatrix}, where \ \mathrm{t} \in (0,1)$$
$$\vec{z(t)}_{red2} = \begin{pmatrix} t \\ 2 \end{pmatrix}, where \ \mathrm{t} \in (1,2)$$

The same calculation was done, as mentioned above and the solution for this sum is zero, as expected. It should also be mentioned that path 1 is light-like and can therefore only influence the event at (2,2) causally if it moves at the speed of light, while path 2 is a time-like path, so the results can be affected even if the carrier is not moving at the speed of light.

3.3 Determination of metric with more cross connections and nine events

The following situation is now considered, whereby the bachelor's student came up with this grid because the author was interested in a non symmetric case. A picture of the situation is given below.

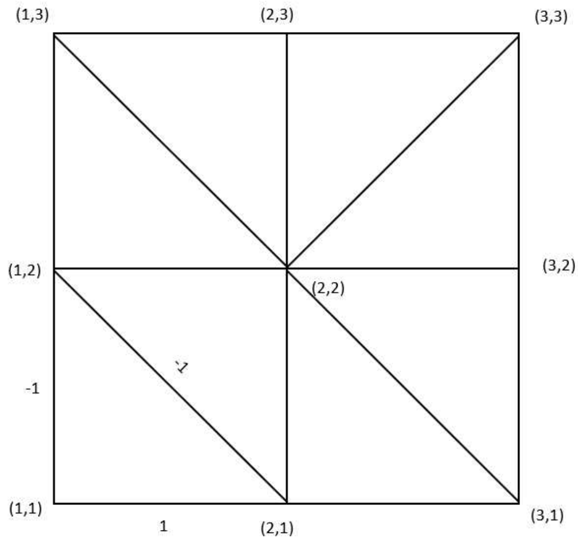


Figure 4: Space-time-grid with more cross connections and nine events
every horizontal connection equals 1 and every vertical and diagonal connection equals -1

To evaluate the metric on every event, the author wrote a program with "Mathematica". The program is included in the last chapter of the thesis. The boundary conditions are in this case also periodic. With the help of the program the author came up with the following metric:

$$g(1,1) = g(1,2) = g(1,3) = g(2,2) = g(2,1) = g(3,1) = g(2,3) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$g(3,3) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$g(3,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The main thought by creating the program was to write it based on differences between every event in the grid. Therefore one gets also values for g^{12} where there is no connection between the events and therefore these values are simply not existing. This could be

improved that the program is able filter only relevant solutions but this would go beyond the scope of this work.
As one sees in Figure 4 the event (2,2) is overdetermined, which leads to problems and are therefore discussed in the next chapter. On event (2,3) there were also some difficulties, because the gauge (eq. 6) leads to the condition that every number fulfills it. But this is not the main issue because one can use (since the gauge has to be true for every event in the grid) the gauge on other events, such as (2,2) and (1,3) and evaluate $g^{12}(2,3)$ over these applicable terms.

3.3.1 Problems with the calculation

As mentioned in the chapter before, the event (2,2) is causing a few problems because there are three diagonal connections which contradict by considering equation 8. The connections from (2,2) to (3,1) and (1,3) have the same direction and therefore don't contradict. Whereas the connection to (3,3) has another direction and leads to another value for $g^{12}(2,2)$. The gauge is also collapsing, because one tries to differentiate although they are different sets and therefore a transfer function would have to be developed to convey correctly. But this procedure is very complicated and would therefore go beyond the scope of this work. One can utter that a problem of contradiction will appear if one has at least five connections from one event to the others. A picture of a situation that would also cause problems according to the author is given below:

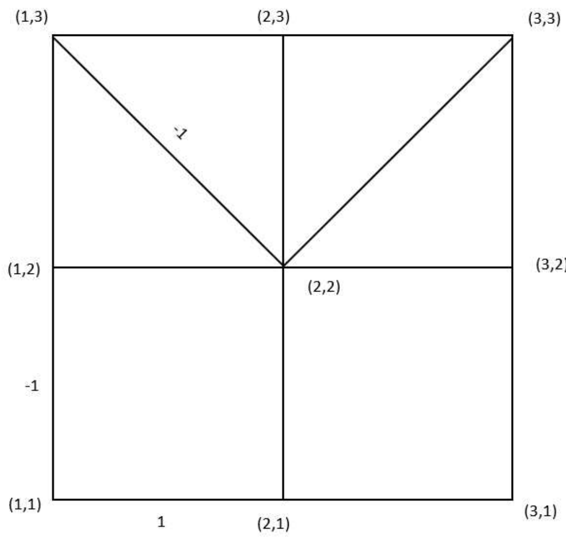


Figure 5: Space-time-grid that would also cause problems in determination of the metric on event (2,2)
every horizontal connection equals 1 and every vertical and diagonal connection equals -1

It is well illustrated by Figure 5 that the main problem are the two connections facing in opposite directions, namely the one from event (1,3) to (2,2) and from (3,3) to (2,2).

If one thinks of the definition of the path integral it also makes sense that it causes contradiction, since one gets an extra minus-sign by comparing these two connections. To be able to determine the 'correct' solution for $g^{12}(2,2)$ one has to calculate it through the proper time, which is discussed in the next chapter.

3.3.2 Paths

In order to determine the $g^{12}(2,2)$ one has to determine it by using the definition of the proper time, which is defined:

$$d\tau = -\sqrt{ds^2} \tag{11}$$

The proper time is always positive, since there is no negative time, which leads to the fact that the sign of the inputs does not influence as it does in the definition of equation 8. To determine $g^{12}(2,2)$ it is necessary to find a path which does not cross event (2,2) but has the same proper time if one choose a path through event (2,2). The metric element $g^{12}(2,2)$ is also necessary to evaluate $g^{12}(2,3)$, because one has to use the gauge on (2,2) in order to get a clearly result for it. A picture of the situation is given below:

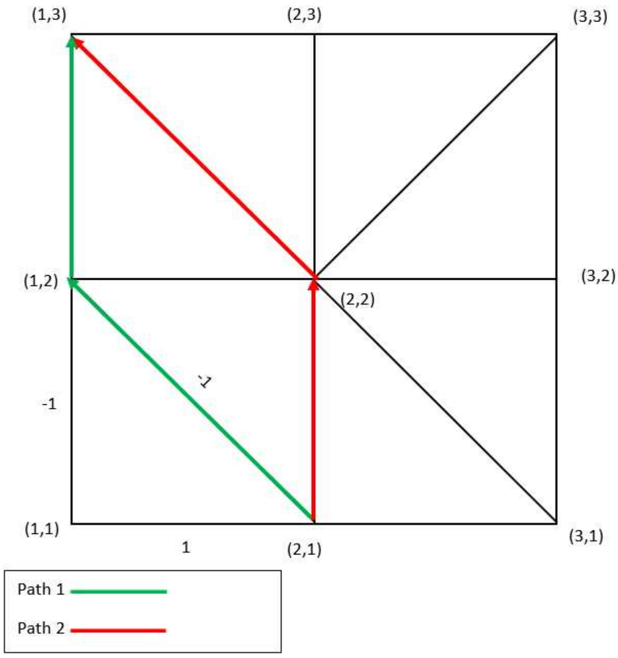


Figure 6: Paths with the same proper time
every horizontal connection equals 1 and every vertical and diagonal connection equals -1

For path 1 the metric on every event is known, so one can calculate the proper time. Since the two paths should have the same proper time, one can determine which value $g^{12}(2,2)$ should have in order to be correct according to this definition. Executing the described leads to the result that $g^{12}(2,2)$ is equal to 1. The evaluated proper time can be found in the Mathematica program which can be found in chapter 6.
Together with the gauge (eq.6) and the evaluated $g^{12}(2,2)$ one is able to clearly determine $g^{12}(2,3)$, which leads to the value -1. Trying to calculate $g^{12}(2,3)$ thorough the gauge on event (1,3) was also tried but it also led to the fact that every number can be a solution for the element $g^{12}(2,3)$.
Since it has not yet been proven that there is only one solution for a metric element, it would not be dramatic if the gauge via the event (2,2) also resulted in a not clearly defined solution for the metric element $g^{12}(2,3)$.

4 Results and Outlook

After the calculation one left with the following metric for each space-time grid:[7]

$$g(1,1) = g(1,2) = g(2,1) = g(2,2) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$g(1,1) = g(1,2) = g(1,3) = g(2,2) = g(2,1) = g(3,1) = g(2,3) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$g(3,3) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$g(3,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The determination of the metric for the grid containing four events, it was relatively straight forward, because there was no contradiction between the connections to the events. Different to the calculation of the metric of the grid, which is containing nine events, which results in contradiction in the value of $g^{12}(2,2)$. In the course of the thesis one comes to the conclusion that there is always a contradiction between the values when one has more than five connections, however this can also occur when such a situation as in Figure 5 is given. It is very interesting what is happening at the event (2,2) from a physical point of view. If one has given a space-time grid with more connections, there is a curvature in space, which is singular and can therefore be seen as a black hole. As already mentioned in this thesis, one can use a transfer function to describe what happens there, or one can replace the points with a pentagon, which on the one hand causes fewer points but on the other hand results in a new length scale. It is also possible to define the metric as a limit value process. The next step would be to test these three mechanisms and thus trying to solve this problem.

[7] see chapter 3.2 and 3.3

5 Summary

In course of processing on the thesis, interesting details emerged with regard to the calculation of the metric at the events in the space-time grid.
First, the formalism was demonstrated using the Minkowski-metric, then a grid containing a cross-connection was considered, and finally a space-time grid with nine events was considered using a self-written Mathematica program. With the latter, problems arose with the determination of $g^{12}(2,2)$ with regard to a contradiction resulting from equation 8. This problem results from overdetermination of the event (2,2). Since it would go beyond the scope of the bachelor thesis, no transfer function or limit value process was developed that would solve the problem of the undefined derivatives (which occur in equation 6) over the set. Instead, the element $g^{12}(2,2)$ was calculated using the proper time of another path, since the paths used in Figure 6 require the same proper time.
In the course of processing it was found out that it leads to a contradiction if one has more than five connections to events. The contradiction can also occur if one has less than five connections but the they are in the opposite diagonal direction, like in Figure 5.

References

[1] A. Maas. *Towards testing the Fröhlich-Morchio-Strocchi mechanism in quantum gravity.* 41st International Conference on High Energy physics - ICHEP2022 **6** (2022).
doi:https://doi.org/10.1016/j.jpowsour.2016.04.073

[2] A. Maas. *The Fröhlich-Morchio-Strocchi mechanism and quantum gravity.* SciPost Physics **8** (2020).
doi:10.21468/scipostphys.8.4.051

[3] R. Loll. *Quantum gravity from causal dynamical triangulations: a review.* Classical and Quantum Gravity **37** (2019) 013002.
doi:10.1088/1361-6382/ab57c7

6 Mathematica program

$\mathrm{NRows} = 3$

$\mathrm{NCols} = 3$

$\mathrm{NPoints} = \mathrm{NRows} * \mathrm{NCols}$

$\mathrm{Coords} = \mathrm{Table}[\{i, j\}, \{i, \mathrm{NRows}\}, \{j, \mathrm{NCols}\}]$

3

3

9

$\{\{\{1,1\},\{1,2\},\{1,3\}\},\{\{2,1\},\{2,2\},\{2,3\}\},\{\{3,1\},\{3,2\},\{3,3\}\}\}$

$\mathrm{SpaceLike} = \mathrm{Flatten}[\mathrm{Table}[i + (j-1) * \mathrm{NRows}{<}{-}{>}i + j * \mathrm{NRows}, \{i, 1, \mathrm{NRows}\}, \{j, 1, \mathrm{NCols} - 1\}]]$

$\{14, 47, 25, 58, 36, 69\}$

$\mathrm{TimeLike} = \ \mathrm{Flatten}[\mathrm{Table}[i + (j-1) * \mathrm{NRows}{<}{-}{>}i + 1 + (j-1) * \mathrm{NRows}, \{i, 1, \mathrm{NRows}$
$-1\}, \{j, 1, \mathrm{NCols}\}]]$

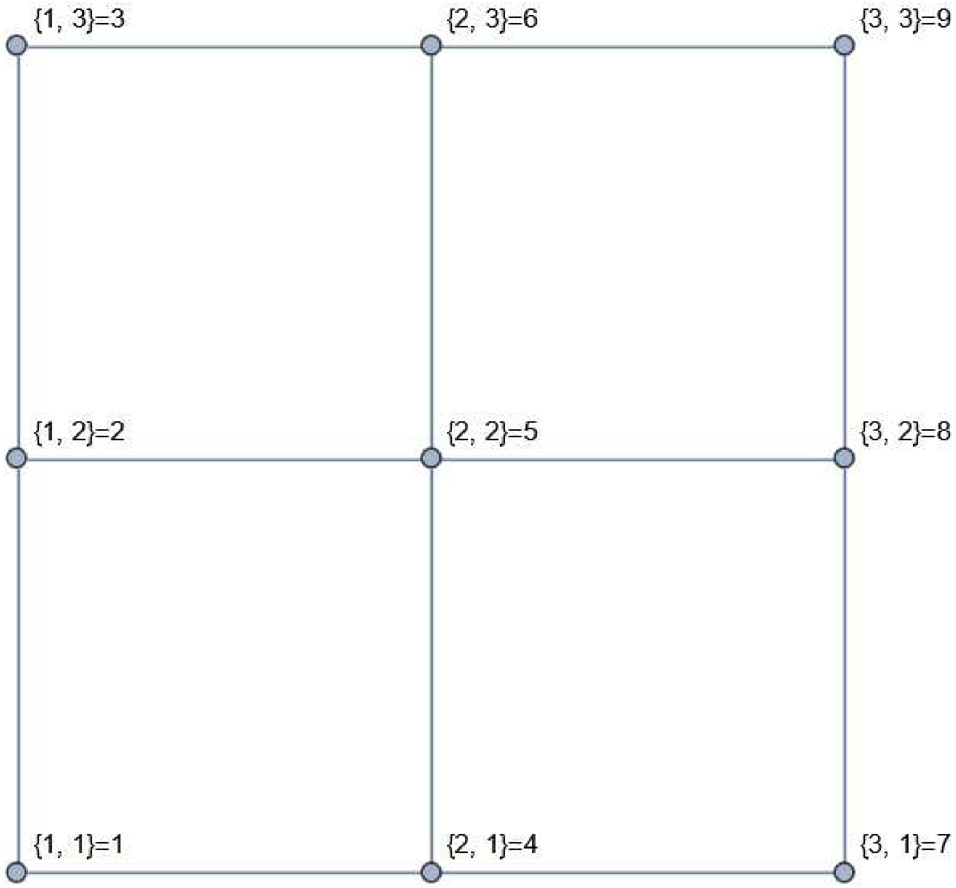
$\{12, 45, 78, 23, 56, 89\}$

$\mathrm{Graph}[$

$\mathrm{Table}[\mathrm{Labeled}[i, \mathrm{ToString}[\mathrm{Flatten}[\mathrm{Coords}, 1][[i]]] <> \text{“=”} <> \mathrm{ToString}[i]], \{i, \mathrm{NPoints}\}],$

$\mathrm{Flatten}[\{\mathrm{SpaceLike}, \mathrm{TimeLike}\}],$

$\mathrm{GraphLayout} \to \text{“GridEmbedding”}]$



DiagLike = {2<->4, 3<->5, 5<->7, 5<->9}

$\{24, 35, 57, 59\}$

Connections = Flatten[{SpaceLike, TimeLike, DiagLike}]

$\{14, 47, 25, 58, 36, 69, 12, 45, 78, 23, 56, 89, 24, 35, 57, 59\}$

sSpaceLike = 1

sTimeLike = −1

sDiagLike = −1

InvariantLengths = Flatten[{

Table[sSpaceLike, {i, Length[SpaceLike]}],

Table[sTimeLike, {i, Length[TimeLike]}],

```
Table[sDiagLike, {i, Length[DiagLike]}]
}]
```

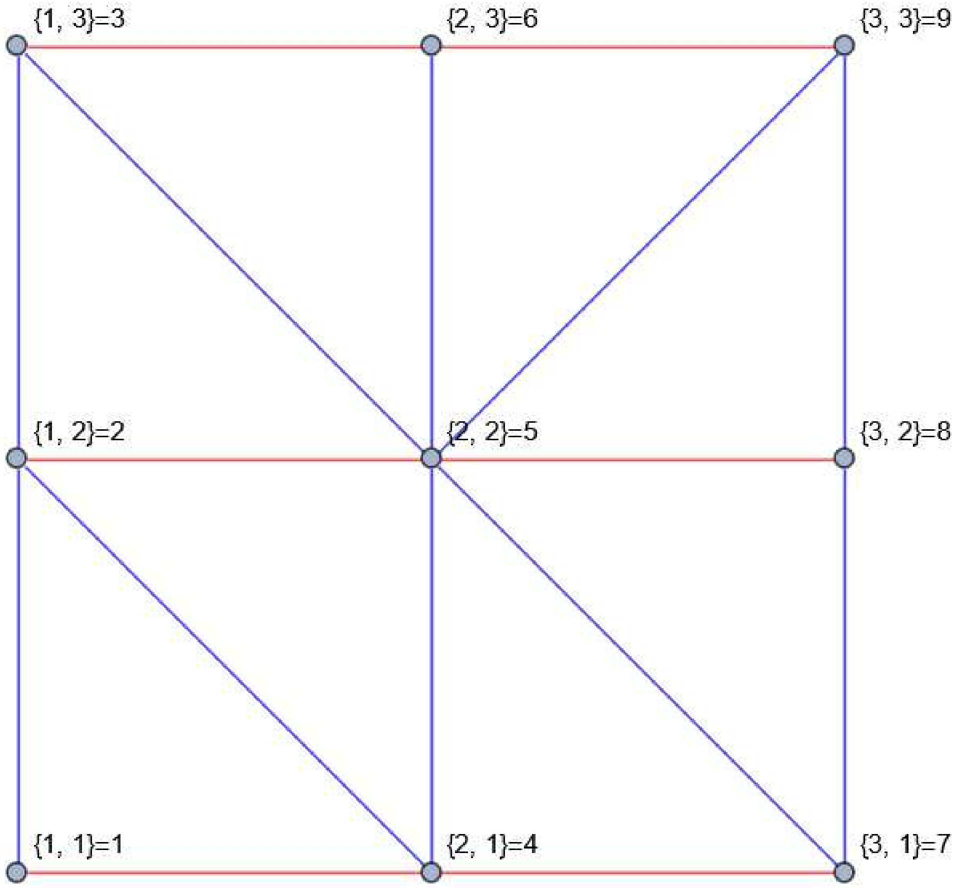
1

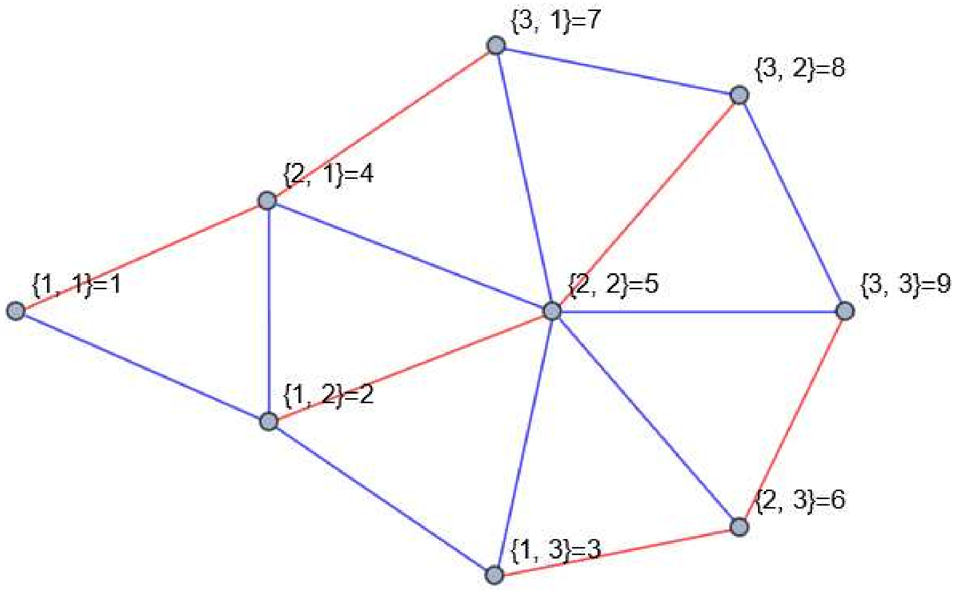
−1

−1

{1, 1, 1, 1, 1, 1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1}

```
Coloring[l_]:=Switch[l, sSpaceLike, Red, sTimeLike, Blue, sDiagLike, Green];
Graph[
Table[Labeled[i, ToString[Flatten[Coords, 1][[i]]] <> "=" <> ToString[i]], {i, NPoints}],
Table[Style[Connections[[i]], Coloring[InvariantLengths[[i]]]], {i, Length[InvariantLengths]}],
EdgeWeight → InvariantLengths,
GraphLayout → "GridEmbedding"]
(* Plot Graph with all Edges and Links in automatic shape *)
Graph[
Table[Labeled[i, ToString[Flatten[Coords, 1][[i]]] <> "=" <> ToString[i]], {i, NPoints}],
Table[Style[Connections[[i]], Coloring[InvariantLengths[[i]]]], {i, Length[InvariantLengths]}],
EdgeWeight → InvariantLengths,
GraphLayout → Automatic]
(*Obtainds^2MatrixfromPlot*)
ds2 = WeightedAdjacencyMatrix[%];
ds2//MatrixForm
```


{1, 3}=3
{2, 3}=6
{3, 3}=9
{1, 2}=2
{2, 2}=5
{3, 2}=8
{1, 1}=1
{2, 1}=4
{3, 1}=7


{3, 1}=7
{3, 2}=8
{2, 1}=4
{1, 1}=1
{2, 2}=5
{3, 3}=9
{1, 2}=2
{2, 3}=6
{1, 3}=3

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \end{pmatrix}$$

(*listing all possible differences*)

$A = \text{Coords}$

$B = \text{Coords}$

subfora11 $= \text{Table}[\{1,1\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora12 $= \text{Table}[\{1,2\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora13 $= \text{Table}[\{1,3\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora21 $= \text{Table}[\{2,1\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora22 $= \text{Table}[\{2,2\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora23 $= \text{Table}[\{2,3\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora31 $= \text{Table}[\{3,1\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora32 $= \text{Table}[\{3,2\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

subfora33 $= \text{Table}[\{3,3\} - B[[i,j]], \{i,1,3\}, \{j,1,3\}]$

$\{\{\{1,1\},\{1,2\},\{1,3\}\},\{\{2,1\},\{2,2\},\{2,3\}\},\{\{3,1\},\{3,2\},\{3,3\}\}\}$

$\{\{\{1,1\},\{1,2\},\{1,3\}\},\{\{2,1\},\{2,2\},\{2,3\}\},\{\{3,1\},\{3,2\},\{3,3\}\}\}$

$\{\{\{0,0\},\{0,-1\},\{0,-2\}\},\{\{-1,0\},\{-1,-1\},\{-1,-2\}\},\{\{-2,0\},\{-2,-1\},\{-2,-2\}\}\}$

$\{\{\{0,1\},\{0,0\},\{0,-1\}\},\{\{-1,1\},\{-1,0\},\{-1,-1\}\},\{\{-2,1\},\{-2,0\},\{-2,-1\}\}\}$

$\{\{\{0,2\},\{0,1\},\{0,0\}\},\{\{-1,2\},\{-1,1\},\{-1,0\}\},\{\{-2,2\},\{-2,1\},\{-2,0\}\}\}$

$\{\{\{1,0\},\{1,-1\},\{1,-2\}\},\{\{0,0\},\{0,-1\},\{0,-2\}\},\{\{-1,0\},\{-1,-1\},\{-1,-2\}\}\}$

$\{\{\{1,1\},\{1,0\},\{1,-1\}\},\{\{0,1\},\{0,0\},\{0,-1\}\},\{\{-1,1\},\{-1,0\},\{-1,-1\}\}\}$

$\{\{\{1,2\},\{1,1\},\{1,0\}\},\{\{0,2\},\{0,1\},\{0,0\}\},\{\{-1,2\},\{-1,1\},\{-1,0\}\}\}$

$\{\{\{2,0\},\{2,-1\},\{2,-2\}\},\{\{1,0\},\{1,-1\},\{1,-2\}\},\{\{0,0\},\{0,-1\},\{0,-2\}\}\}$

$\{\{\{2,1\},\{2,0\},\{2,-1\}\},\{\{1,1\},\{1,0\},\{1,-1\}\},\{\{0,1\},\{0,0\},\{0,-1\}\}\}$

$\{\{\{2,2\},\{2,1\},\{2,0\}\},\{\{1,2\},\{1,1\},\{1,0\}\},\{\{0,2\},\{0,1\},\{0,0\}\}\}$

```
(*listing the differences (except (2,2),(3,2) and(2,3))*)
(*(1,2)*)

a12diffa1b1 = Table[subfora12[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]
a12diffa1b1 = ArrayReshape[a12diffa1b1, {1, 9}]
a12diffa2b2 =  Table[subfora12[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]
a12diffa2b2 = ArrayReshape[a12diffa2b2, {1, 9}]
```

$\{\{0,0,0\},\{-1,-1,-1\},\{-2,-2,-2\}\}$

$\{\{0,0,0,-1,-1,-1,-2,-2,-2\}\}$

$\{\{1,0,-1\},\{1,0,-1\},\{1,0,-1\}\}$

$\{\{1,0,-1,1,0,-1,1,0,-1\}\}$

(*(1, 3)*)

a13diffa1b1 = Table[subfora13[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]

a13diffa1b1 = ArrayReshape[a13diffa1b1, {1, 9}]

a13diffa2b2 = Table[subfora13[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]

a13diffa2b2 = ArrayReshape[a13diffa2b2, {1, 9}]

$\{\{0, 0, 0\}, \{-1, -1, -1\}, \{-2, -2, -2\}\}$

$\{\{0, 0, 0, -1, -1, -1, -2, -2, -2\}\}$

$\{\{2, 1, 0\}, \{2, 1, 0\}, \{2, 1, 0\}\}$

$\{\{2, 1, 0, 2, 1, 0, 2, 1, 0\}\}$

(*(3, 3)*)

a33diffa1b1 = Table[subfora33[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]

a33diffa1b1 = ArrayReshape[a33diffa1b1, {1, 9}]

a33diffa2b2 = Table[subfora33[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]

a33diffa2b2 = ArrayReshape[a33diffa2b2, {1, 9}]

$\{\{2, 2, 2\}, \{1, 1, 1\}, \{0, 0, 0\}\}$

$\{\{2, 2, 2, 1, 1, 1, 0, 0, 0\}\}$

$\{\{2, 1, 0\}, \{2, 1, 0\}, \{2, 1, 0\}\}$

$\{\{2, 1, 0, 2, 1, 0, 2, 1, 0\}\}$

(*(3,1)*)

a31diffa1b1 = Table[subfora31[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]

a31diffa1b1 = ArrayReshape[a31diffa1b1, {1, 9}]

a31diffa2b2 = Table[subfora31[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]

a31diffa2b2 = ArrayReshape[a31diffa2b2, {1, 9}]

$\{\{2, 2, 2\}, \{1, 1, 1\}, \{0, 0, 0\}\}$

$\{\{2, 2, 2, 1, 1, 1, 0, 0, 0\}\}$

$\{\{0, -1, -2\}, \{0, -1, -2\}, \{0, -1, -2\}\}$

$\{\{0, -1, -2, 0, -1, -2, 0, -1, -2\}\}$

(*(2,1)*)

a21diffa1b1 = Table[subfora21[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]

a21diffa1b1 = ArrayReshape[a21diffa1b1, {1, 9}]

a21diffa2b2 = Table[subfora21[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]

a21diffa2b2 = ArrayReshape[a21diffa2b2, {1, 9}]

$\{\{1, 1, 1\}, \{0, 0, 0\}, \{-1, -1, -1\}\}$

$\{\{1, 1, 1, 0, 0, 0, -1, -1, -1\}\}$

$\{\{0, -1, -2\}, \{0, -1, -2\}, \{0, -1, -2\}\}$

$\{\{0, -1, -2, 0, -1, -2, 0, -1, -2\}\}$

(*(1,1)*)

a11diffa1b1 = Table[subfora11[[i, j, 1]], {i, 1, 3}, {j, 1, 3}]

```
a11diffa1b1 = ArrayReshape[a11diffa1b1, {1,9}]
a11diffa2b2 =  Table[subfora11[[i, j, 2]], {i, 1, 3}, {j, 1, 3}]
a11diffa2b2 = ArrayReshape[a11diffa2b2, {1, 9}]
```

$\{\{0,0,0\},\{-1,-1,-1\},\{-2,-2,-2\}\}$

$\{\{0,0,0,-1,-1,-1,-2,-2,-2\}\}$

$\{\{0,-1,-2\},\{0,-1,-2\},\{0,-1,-2\}\}$

$\{\{0,-1,-2,0,-1,-2,0,-1,-2\}\}$

```
(*function to solve for g12, g11 and g22 is not included because they are always 1 and -1*)
(*the divided by two and the missing 2*g12 is due to the norm*)

g12solver[a1_, a2_, g12_]:=a1^2/2 + g12 * a1 * a2 - a2^2/2

(*(1, 2)*)

g12fora12 = Table[Solve[g12solver[a12diffa1b1, a12diffa2b2, g12][[1, i]]==ds2[[2, i]], g12], {i, 1, 9}]
```

$\{\{\},\{\{\}\},\{\},\{\{g12\to 1\}\},\{\},\{\{g12\to 0\}\},\left\{\left\{g12\to \frac{3}{4}\right\}\right\},\{\},\left\{\left\{g12\to -\frac{3}{4}\right\}\right\}\}$

```
(*(1, 3)*)

g12fora13 = Table[Solve[g12solver[a13diffa1b1, a13diffa2b2, g12][[1, i]]==ds2[[3, i]], g12], {i, 1, 9}]
```

$\{\{\},\{\},\{\{\}\},\left\{\left\{g12\to -\frac{3}{4}\right\}\right\},\{\{g12\to 1\}\},\{\},\{\{g12\to 0\}\},\left\{\left\{g12\to \frac{3}{4}\right\}\right\},\{\}\}$

(*$(3,3)$*)

g12fora33 = Table[Solve[g12solver[a33diffa1b1, a33diffa2b2, g12][[1, i]]==ds2[[9, i]], g12], {i, 1, 9}]

$\{\{\{\text{g12}\to 0\}\},\{\{\text{g12}\to -\frac{3}{4}\}\},\{\},\{\{\text{g12}\to \frac{3}{4}\}\},\{\{\text{g12}\to -1\}\},\{\},\{\},\{\},\{\{\}\}\}$

(*$(3,1)$*)

g12fora31 = Table[Solve[g12solver[a31diffa1b1, a31diffa2b2, g12][[1, i]]==ds2[[7, i]], g12], {i, 1, 9}]

$\{\{\},\{\{\text{g12}\to \frac{3}{4}\}\},\{\{\text{g12}\to 0\}\},\{\},\{\{\text{g12}\to 1\}\},\{\{\text{g12}\to -\frac{3}{4}\}\},\{\{\}\},\{\},\{\}\}$

(*$(2,1)$*)

g12fora21 = Table[Solve[g12solver[a21diffa1b1, a21diffa2b2, g12][[1, i]]==ds2[[4, i]], g12], {i, 1, 9}]

$\{\{\},\{\{\text{g12}\to 1\}\},\{\{\text{g12}\to -\frac{3}{4}\}\},\{\{\}\},\{\},\{\},\{\},\{\{\text{g12}\to 0\}\},\{\{\text{g12}\to \frac{3}{4}\}\}\}$

(*now we need to include the gauge in order to get (1,1)*)

eich1[g12known_, g12sol_]:=(g12known − g12sol) + g12sol ∗ (−g12known + g12sol)(*gauge x-direc.*)

eich2[g12known_, g12sol_]:=(−(g12known) + g12sol) + g12sol ∗ (−g12known + g12sol)(*gauge y-direc.*

(*$(1,1)$*)

g12fora11 = Solve[eich1[1, g12sol]==eich2[1, g12sol], g12sol]

$\{\{\text{g12sol}\to 1\}\}$

(*points where boundary condition is needed (2,3) and (3,2)*)

(*(2, 3)*)

g12fora23 = Solve[eich1[−1, g12sol]==eich2[1, g12sol], g12sol]

$\{\{\}\}$

(*Placeholder Matrix to solve the following*)

save = $\{\{1, x\}, \{x, -1\}\}$

save//MatrixForm

$\{\{1, x\}, \{x, -1\}\}$

$$\begin{pmatrix} 1 & x \\ x & -1 \end{pmatrix}$$

(*there the gauge does not help because every number fulfills it, trying it with invertion condition*)

g12fora23Invertion = Solve[Dot[save, Inverse[save]]==$\{\{1, 0\}, \{0, 1\}\}$, x]

$\{\{\}\}$

(*Inverse condition not working - calculation will be done through gauge from (2,2)*)

(*(3, 2)*)

g12fora32 = Solve[eich1[1, g12sol]==eich2[−1, g12sol], g12sol]

$\{\{\text{g12sol} \to 0\}\}$

(*Creating two paths to determine g12(2,2) together with the proper time*)

$$\textbf{propertime[g11\_, g12\_, g22\_, z\_]:=}$$
$$(-\text{g11} * D[z[[1]], \text{lambda}] * D[z[[1]], \text{lambda}] - 2 * \text{g12} * D[z[[1]], \text{lambda}] * D[z[[2]], \text{lambda}]$$
$$-\text{g22} * D[z[[2]], \text{lambda}] * D[z[[2]], \text{lambda}])\text{^}(0.5)$$

(*Path1 : (2, 1)->(1, 2)->(1, 3)*)

$$\text{firstPath1} = \{2 - \text{lambda}, 1 + \text{lambda}\}$$
$$\text{firstPath2} = \{1, 1 + \text{lambda}\}$$

$\{2 - \text{lambda}, 1 + \text{lambda}\}$

$\{1, 1 + \text{lambda}\}$

$$\text{timePath1} = 0.5 * (\text{Integrate}[\text{propertime}[1, 1, -1, \text{firstPath1}], \{\text{lambda}, 0, 1\}]$$
$$+\text{Integrate}[\text{propertime}[1, 1, -1, \text{firstPath2}], \{\text{lambda}, 1, 2\}])$$

1.20711

(*Path2 : (2, 1)->(2, 2)->(1, 3)*)

(*Calculating the metricelement g12 of (2,2)*)

$$\text{secondPath1} = \{2, 1 + \text{lambda}\}$$
$$\text{secondPath2} = \{3 - \text{lambda}, 1 + \text{lambda}\}$$

$\{2, 1 + \text{lambda}\}$

{3 − lambda, 1 + lambda}

```
g12fora22 =
Solve[
timePath1==0.5 * (Integrate[propertime[1, 1, −1, secondPath1], {lambda, 0, 1}]
+Integrate[propertime[1, g12, −1, secondPath2], {lambda, 1, 2}]),
g12]
```

{{g12 → 1.}}

(*Listing every metric in the grid*)

(*$g(1,1) = g(2,1) = g(1,2) = g(1,3) = g(2,2) = g(3,1) = \{\{1,1\},\{1,-1\}\}$,
$g(3,2) = g(2,3) = \{\{1,0\},\{0,-1\}\}$,
$g(3,3) = \{\{1,-1\},\{-1,-1\}\}$*)