Bachelor Thesis (BSc)

A supersymmetric approach to the Gribov problem within an instanton field configuration

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Abstract

In non-Abelian gauge theories it is necessary to fix a gauge in order to calculate gauge dependant quantities. Beyond perturbation theory this is however no longer possible since gauge conditions do not possess unique solutions. From this stems the Gribov-Singer ambiguity. A possible way to resolve this, is to restrict to field configuration lying within the first Gribov region where the Faddeev-Popov operator is positive semi-definite. In principle it should also be possible to average over all independent solutions, however this leads to significant cancellations. One therefore hopes to shed more light on the behaviour of the Faddeev-Popov operator beyond the first Gribov region. This thesis attempts to solve this problem by using Supersymmetry. After introducing the necessary mathematical tools of Supersymmetry, the supersymmetric partner potentials to the Faddeev-Popov operator in an instanton field configuration is calculated. From there certain approximations in conjunction with a power series Ansatz are used to find a possible solution. A closed form to each of the potential terms is found and some considerations are given as to how a further approach would look like.

Contents

1 Introduction

The aim of this thesis will be to use Supersymmetry on the Zero mode solutions of the Faddeev-Popov operator in an instanton field configuration. This is done to find a way for resolving the Gribov-Singer ambiguity. If the behaviour of the supersymmetric operator is simple enough, one could determine its spectrum and expand this knowledge to a method for handling negative Eigenvalues of the initial Faddeev-Popov operator. First an introduction to the necessary tools of Supersymmetry will be given. Then we find a general approach to gauge theories which will connect us to the Gribov problem and the aforementioned Faddeev-Popov operator. Sections 2 and 3 then deal with the concrete calculations. [1]

1.1 Supersymmetry

Supersymmetry is in some sense a generalization of the physical notion of symmetry. Since symmetry is omnipresent not only in our physical theories but also in the phenomenological description of the world around us, it is no wonder a more complex mathematical construct such as Supersymmetry has huge potential in shedding light on current and also future questions physics and mathematics might have to offer. The following introduction is aimed at understanding the core concepts of Supersymmetry in context to this thesis.

As mentioned before, symmetric principles usually allow us a more clear mathematical description of a physical system. In general this is achieved through the correlation between

Symmetry - Conservation law - Degeneracy.

Supersymmetry works similar and thus predicts new kind of particles called *SUSY particles.* [5] The first approach to Supersymmetry came through Quantum field theory in order to better understand the relation between fermions and bosons as will be shown below. Later this was expanded to Quantum and also Classical mechanics. [5]

Since Supersymmetry as a model must contain fermions and bosons, the most simple application is the product space spanned by

$$|n_B n_F\rangle = |n_B\rangle\,|n_F\rangle\,;\ \text{where } n_B = 0, 1, ..., \infty \text{ and } n_F = 0, 1$$

where n_B is the number of bosons and n_F the number of fermions. It is now necessary to introduce Operators which change between these states. [1]

$$\begin{aligned} Q_+\,|n_B n_F\rangle &\propto |n_B - 1, n_F + 1\rangle \\ Q_-\,|n_B n_F\rangle &\propto |n_B + 1, n_F - 1\rangle \end{aligned} \tag{1}$$

Where the normalisation factor is left out. For a theory to be supersymmetric its Hamilton Operator H must be invariant under transformation with these

[1]In this thesis we work with natural units $\hbar = c = 1$

Operators

$$[H, Q_\pm] = 0 \tag{2}$$

Thus in accordance with (1) $Q_\pm$ can be defined as

$$\begin{aligned} Q_+ &= ab^\dagger \\ Q_- &= a^\dagger b \end{aligned} \tag{3}$$

Where a is the bosonic and b the fermionic creation operator. The fermionic operators possess nilpotency

$$b^\dagger b^\dagger = bb = 0$$

From (3) it follows directly that $Q_\pm$ inherit this nilpotency which is a very important aspect of Supersymmetry and the key to calculate the supersymmetric Hamilton Operator as will be shown below.[5]

To obtain a non trivial theory we have to generalize (3)

$$\begin{aligned} Q_+ &= Ab^\dagger \\ Q_- &= A^\dagger b \end{aligned} \tag{4}$$

Where the operators A and $A^\dagger$ are arbitrary bosonic functions of the original operators a and $a^\dagger$[1]
It can be shown that the Ansatz $H = \{Q_+, Q_-\}$ fulfills our necessary condition in (2). Now with our more general definition in (4) it no longer holds, that $[H, N_B] = 0$, while it is still true that $[H, N_F] = 0$. Thus we must characterize our state solely by the quantum number n_F and it follows that

$$\begin{aligned} H\,|E_{nF}\rangle &= E\,|E_{nF}\rangle \\ N_F\,|E_{nF}\rangle &= n_F\,|E_{nF}\rangle \end{aligned}$$

Since there are only two states for n_F (0 and 1) we can rewrite these states as two dimensional vectors [5]

$$|E_{nF}\rangle = \begin{bmatrix} |E_0\rangle \\ |E_1\rangle \end{bmatrix} \Leftrightarrow \begin{bmatrix} \text{Boson} \\ \text{Fermion} \end{bmatrix}$$

As a consequence any supersymmetric Hamilton operator has to be diagonal in this basis and takes the form [1]

$$H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} := \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{5}$$

Which can also be written as

$$H = \frac{1}{2}\{A, A^\dagger\}\sigma_0 - \frac{1}{2}[A, A^\dagger]\sigma_3 \tag{6}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We now define A and $A^\dagger$ more specifically in accordance to a and $a^\dagger$ which read as

$$a = \sqrt{\frac{m}{2}}(q + i\frac{p}{m})$$
$$a^\dagger = \sqrt{\frac{m}{2}}(q - i\frac{p}{m})$$

where we define $W(q)$ as the *superpotential*

$$A = \sqrt{\frac{m}{2}}(W(q) + i\frac{p}{m})$$
$$A^\dagger = \sqrt{\frac{m}{2}}(W(q) - i\frac{p}{m})$$

Note that $W(q)$ is not strictly speaking a potential in the sense that its dimension is not that of energy. [1]
Using (6) it can be shown that H now reads as

$$H = \frac{1}{2}(\frac{p^2}{m} + W(q)^2)\sigma_0 - \frac{1}{2\sqrt{m}}\frac{dW(q)}{dq}\sigma_3 \tag{7}$$

We are interested in how H acts on the ground state wave function ψ_0 where the Energy $E = 0$ for H_1. Through some calculation a formula for the superpotential $W(q)$ can be derived with the before defined framework. It is given by

$$W(q) = -\frac{1}{\sqrt{m}}\frac{d}{dx}\ln(\psi_0) \tag{8}$$

Using equations (7) and (8) will be the main focus of this thesis.

1.2 Gauge Theories

To describe elementary particle physics the most important quantum field theories are gauge theories. The classical example being the theory of electromagnetism. So to start we remind the reader of the Maxwell equations of electrodynamics in four-vector notation. [6]

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}; \qquad \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{9}$$

Where F is the *field strength tensor* defined by

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$
$$G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

Here $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor in four dimensions and A the four vector potential. A gauge transformation is then given by

$$A^{\mu} \Rightarrow A^{'\mu} = A^{\mu} - \partial^{\mu}\chi$$

where χ is an arbitrary function. It now follows directly that under this transformation the field strength tensor remains unchanged.

$$F^{'\mu\nu} = \partial^{\mu} A^{'\nu} - \partial^{\nu} A^{'\mu} = F^{\mu\nu}$$

$F^{\mu\nu}$ is called *gauge invariant.*[2]

In this sense one can define the term **gauge invariance** as the property of a class of vector potentials $\{A^{(k)}, k \in \mathbb{N}\}$, related by these gauge transformations which describe the same electric and magnetic fields.[7] Applying this principle in a general sense, it turns out that, when a given global invariance is generalized to a local one (here, the electromagnetic gauge invariance) a resulting field is a necessity. This insight gave rise to Gauge Theories, with one of the most prevalent ones being Yang Mills theory which will be briefly discussed below. [2]

Electrodynamics is an example of an abelian gauge theory. Yang-Mills theory, also known as non-abelian gauge theory is primarily used to describe the electroweak as well as the strong nuclear force. [6] In fact the entire standard model of particle physics is described using Yang-Mills theory. The Lagrangian of Yang-Mills theory reads as

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} \tag{10}$$

$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu} \tag{11}$$

Where A^{a}_{μ} are the gauge fields, and the parameters of the Lagrangian are the coupling constant g and the structure constants f^{abc} of the associated gauge algebra. [4] Since we will be working in SU(2) Yang Mills theory the structure

constants will take the form of the Levi-Civita tensor ϵ^{abc}.
The Lagrangian (10) is invariant under local gauge transformations, which take the infinitesimal form

$$A^a_\mu \to A^a_\mu + D^{ab}_\mu \phi_b$$
$$D^{ab}_\mu = \delta^{ab}\partial_\mu + gf^{abc}A^c_\mu$$

where ϕ_b are arbitrary functions. [4]

1.3 Gribov Problem

This section follows closely [4]

To calculate gauge dependant quantities it is necessary to fix a gauge. However this is not possible beyond perturbation theory since gauge conditions like the Landau gauge

$$\partial_\mu A^a_\mu = 0 \tag{12}$$

do no longer possess a unique solution for a given configuration. The resulting independent solutions are called *Gribov copies* while the associated ambiguity of the gauge condition is named *Gribov-Singer ambiguity.* To obtain a well-defined gauge non-perturbatively, it is now necessary to apply further constraints in addition to (12). Thus there will be a remaining set of Gribov copies called the *residual gauge orbit.* A way to resolve the Gribov ambiguity is to restrict the residual gauge orbit to the *first Gribov region*, which is defined by the region the *Faddeev-Popov operator*

$$M^{ab} = -\partial_\mu D^{ab}_\mu \tag{13}$$

is strictly positive semi-definite. This region is convex-bounded and has zero eigenvalues only on its boundary the so called *Gribov horizon.*

1.4 Faddeev-Popov Operator

Stated here again in Euclidean space the Faddeev-Popov operator reads as

$$M^{ab} = -\partial_\mu(\partial_\mu\delta^{ab} + gf^{abc}A^c_\mu) \tag{14}$$

The Faddeev-Popov operator has been studied mostly in its behaviour around the first Gribov region where its eigenvalues remain positive up to its boundary. Beyond the first Gribov region the remainder of the residual gauge orbit is a set of additional Gribov regions, seperated by concentric Gribov horizons, having successively more negative eigenvalues. The number of these negative eigenvalues increases by one for each horizon. To resolve the Gribov-Singer ambiguity it should in principle also be possible to average over all Gribov copies, however because of the aforementioned behavior of (14) it comes to significant cancellation during this process. Therefore one hopes to shed light on the behaviour of (14) beyond the first Gribov horizon and its negative eigenvalue solutions.[4]

2 The Supersymmetric Approach

The topological field configuration of an instanton is given by the algebra elements

$$\begin{aligned} A_\mu &= \frac{2}{r^2+\lambda^2}\tau_{\mu\nu}r_\nu \\ \tau_{\mu\nu} &= \frac{1}{4i}(\tau_\mu\bar{\tau}_\nu - \tau_\nu\bar{\tau}_\mu) \\ \tau_\mu &= (i\boldsymbol{\tau}, 1) \\ \bar{\tau}_\mu &= (-i\boldsymbol{\tau}, 1) \end{aligned} \tag{15}$$

With τ^i the Pauli matrices and where λ characterizes the size of the instanton. The corresponding gauge fields are given by

$$A^a_\mu = \frac{1}{g}\frac{2}{r^2+\lambda^2}r_\nu\zeta^a_{\nu\mu} \tag{16}$$

With ζ^a being the t'Hooft tensors. [3]
With that the eigenvalue equation can be written as

$$\partial^2\phi^a + f^{abc}\frac{2}{r^2+\lambda^2}r_\mu\zeta^b_{\mu\nu}\partial_\nu\phi^c = -\omega^2\phi^a \tag{17}$$

The zero modes to (17) apart from the trivial ones are taken from [3] and will be stated below.

$$\phi_{\frac{1}{2}4}(r) = \frac{-\frac{r^2}{\lambda^2} + (1+\frac{r^2}{\lambda^2})\ln(1+\frac{r^2}{\lambda^2})}{\frac{r^3}{\lambda^3}} \tag{18}$$

$$\phi_{18}(r) = \frac{\frac{r^4}{\lambda^4} + 2\frac{r^2}{\lambda^2} - 2(1+\frac{r^2}{\lambda^2})\ln(1+\frac{r^2}{\lambda^2})}{\frac{r^4}{\lambda^4}} \tag{19}$$

Note that $\phi_{\frac{1}{2}4}(r)$ has multiplicity two while $\phi_{18}(r)$ has multiplicity one, leaving three trivial solutions of constant nature which will not be discussed further.

With these solutions given we now find the corresponding superpotential which is given by equation (8). To achieve this we first make the following substitution.

$$\frac{r}{\lambda} = \tilde{r}$$

$$\Rightarrow \frac{\partial f(\tilde{r})}{\partial r} = \frac{\partial f(\tilde{r})}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial r} = \frac{1}{\lambda}\frac{\partial f(\tilde{r})}{\partial \tilde{r}}$$

Thus W is now given by

$$W = -\frac{1}{\sqrt{m}\lambda}\frac{d}{d\tilde{r}}\ln(\phi(\tilde{r}))$$

The supersymmetric Operator H is then given by equation (7).
We seperate our calculations for each of the zero modes.

2.1 Results for $\phi_{\frac{1}{2}4}$

For $W_{\frac{1}{2}4}(\tilde{r})$ we find:

$$W_{\frac{1}{2}4}(\tilde{r}) = \frac{1}{\sqrt{m}\lambda} \frac{(3+\tilde{r}^2)\ln(1+\tilde{r}^2) - 3\tilde{r}^2}{(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2) - \tilde{r}^3} \tag{20}$$

For simplification we define $\hat{W}_{\frac{1}{2}4}(\tilde{r})$ as

$$\hat{W}_{\frac{1}{2}4}(\tilde{r}) = \frac{(3+\tilde{r}^2)\ln(1+\tilde{r}^2) - 3\tilde{r}^2}{(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2) - \tilde{r}^3}$$

$$\Rightarrow W_{\frac{1}{2}4}(\tilde{r}) = \frac{1}{\sqrt{m}\lambda} \hat{W}_{\frac{1}{2}4}(\tilde{r})$$

Thus the equations for H_1 and H_2 become

$$H_1 = \frac{1}{2m\lambda^2}(p^2\lambda^2 + \hat{W}(\tilde{r})^2 - \frac{d\hat{W}(\tilde{r})}{d\tilde{r}}) \tag{21}$$

$$H_2 = \frac{1}{2m\lambda^2}(p^2\lambda^2 + \hat{W}(\tilde{r})^2 + \frac{d\hat{W}(\tilde{r})}{d\tilde{r}}) \tag{22}$$

With (21) and (22) we find

$$H_1 = \frac{1}{2m\lambda^2}\left(p^2\lambda^2 + \frac{2(1 + \frac{6}{\tilde{r}^2} - \frac{3\tilde{r}^2}{-\tilde{r}^2+(1+\tilde{r}^2)\ln(1+\tilde{r}^2)})}{1+\tilde{r}^2}\right) \tag{23}$$

$$H_2 = \frac{1}{2m\lambda^2}\left(p^2\lambda^2 + \frac{2\tilde{r}^4(3+5\tilde{r}^2)}{(1+\tilde{r}^2)(\tilde{r}^3 - (\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2))^2} - \frac{2(1+\tilde{r}^2)\ln(1+\tilde{r}^2)(\tilde{r}^2(6+\tilde{r}^2) + (-3+\tilde{r}^2)\ln(1+\tilde{r}^2))}{(1+\tilde{r}^2)(\tilde{r}^3 - (\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2))^2}\right) \tag{24}$$

For better illustration the potential terms in (23) and (24) are visualized below. Note that the graph around zero is not accurate due to the occurring singularity.

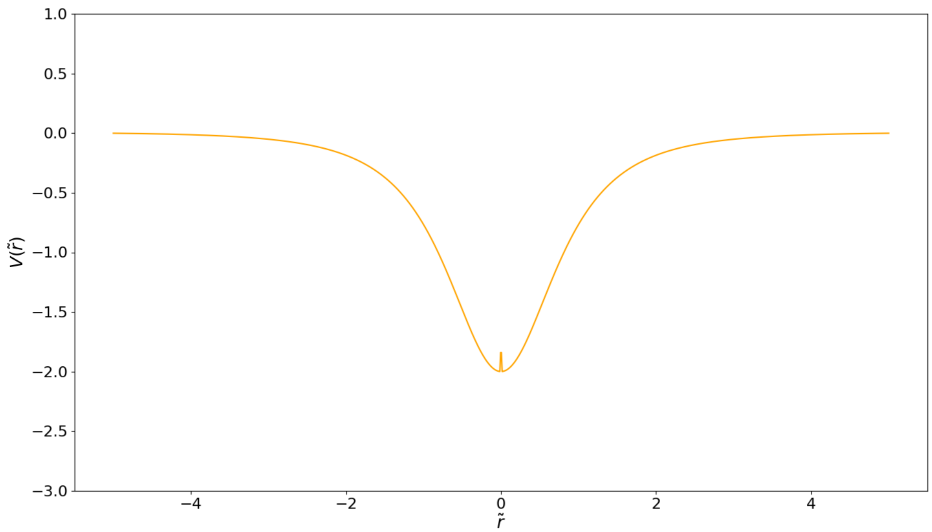

Figure 1: Potential of (23)

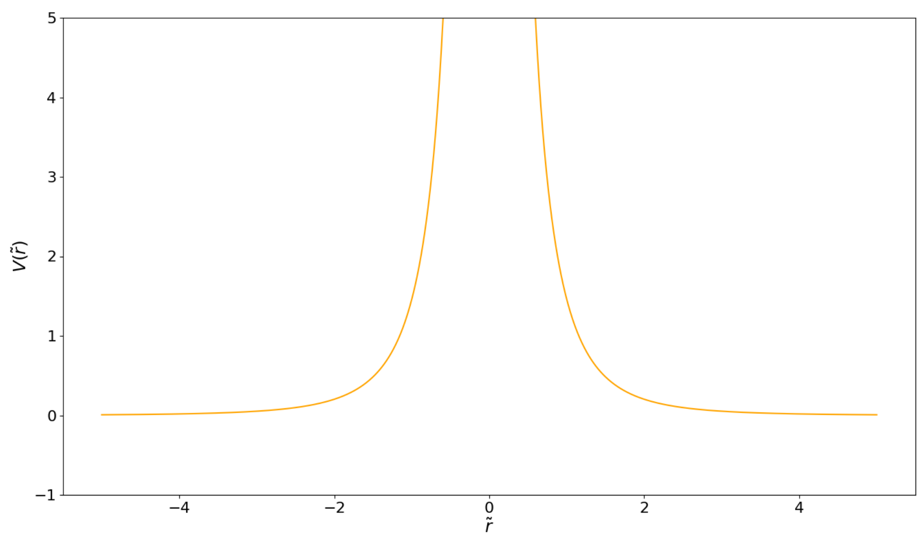

Figure 2: Potential of (24)

2.2 Results for ϕ_{18}

For $W_{18}(\tilde{r})$ we find:

$$W_{18}(\tilde{r}) = \frac{1}{\sqrt{m}\lambda} \frac{8\tilde{r}^2 - 4(2+\tilde{r}^2)\ln(1+\tilde{r}^2)}{\tilde{r}^3(2+\tilde{r}^2) - 2(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2)} \tag{25}$$

Similar to before we define $\hat{W}_{18}$ as

$$\hat{W}_{18} = \frac{8\tilde{r}^2 - 4(2+\tilde{r}^2)\ln(1+\tilde{r}^2)}{\tilde{r}^3(2+\tilde{r}^2) - 2(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2)}$$

and thus

$$W_{18} = \frac{1}{\sqrt{m}\lambda}\hat{W}_{18}$$

Now we can again use equations (21) and (22) to find

$$H_1 = \frac{1}{2m\lambda^2}\Big(p^2\lambda^2 + \frac{8\tilde{r}^2(5+4\tilde{r}^2) - 4(1+\tilde{r}^2)(10+3\tilde{r}^2)\ln(1+\tilde{r}^2)}{\tilde{r}^4(1+\tilde{r}^2)(2+\tilde{r}^2) - 2(\tilde{r}+\tilde{r}^3)^2\ln(1+\tilde{r}^2)}\Big) \tag{26}$$

$$\begin{aligned} H_2 =& \frac{1}{2m\lambda^2}\Big(p^2\lambda^2 + \frac{8\tilde{r}^4(6+3\tilde{r}^2-4\tilde{r}^4)}{(1+\tilde{r}^2)(\tilde{r}^3(2+\tilde{r}^2) - 2(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2))^2} + \\ & \frac{4(1+\tilde{r}^2)\ln(1+\tilde{r}^2)\big(3\tilde{r}^2(\tilde{r}^4-8) + 2(6+3\tilde{r}^2+\tilde{r}^4)\ln(1+\tilde{r}^2)\big)}{(1+\tilde{r}^2)(\tilde{r}^3(2+\tilde{r}^2) - 2(\tilde{r}+\tilde{r}^3)\ln(1+\tilde{r}^2))^2}\Big) \end{aligned} \tag{27}$$

Below the potential terms of (26) and (27) are again shown.

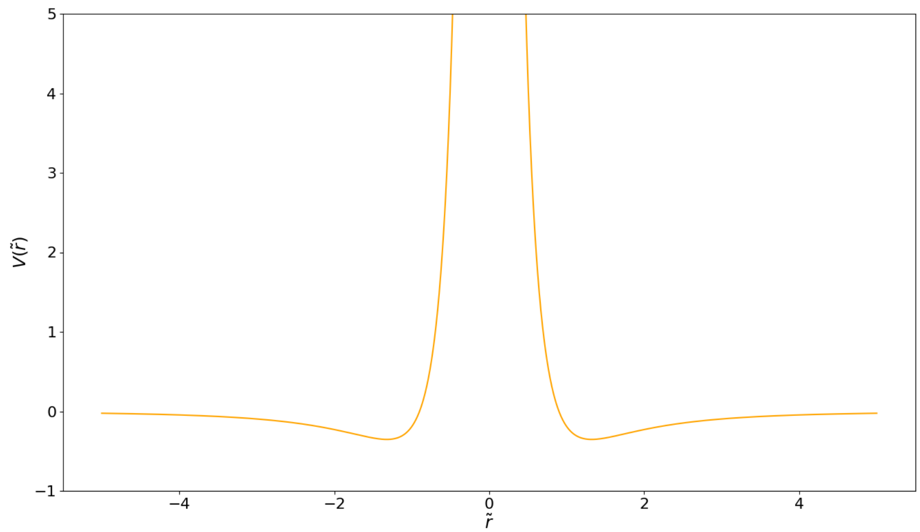

Figure 3: Potential of (26)

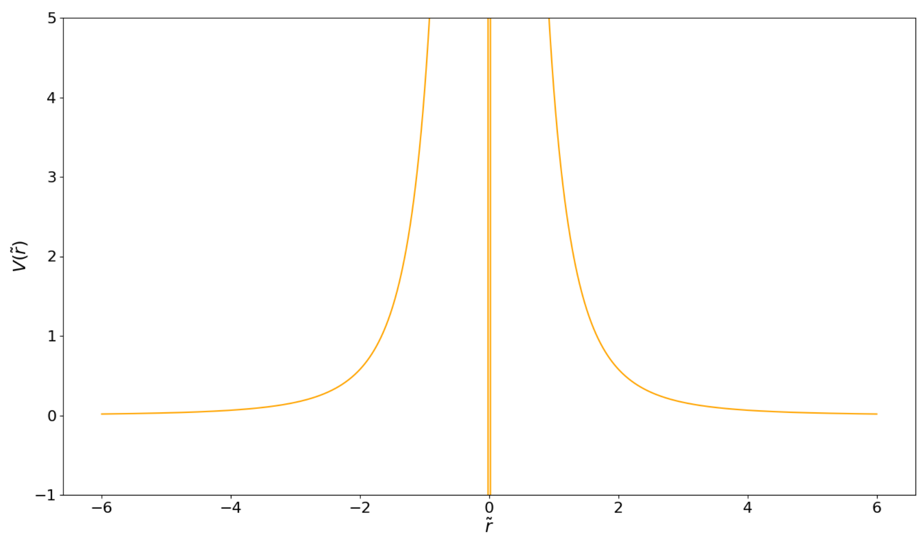

Figure 4: Potential of (27)

2.3 Finding approximate solutions

We can express the solutions we found in the following form.

$$H = \frac{1}{2m\lambda^2}(p^2\lambda^2 + V) \tag{28}$$

Where V is the potential of the Hamilton Operator. To understand these results better we now develop this Potential V in a Maclaurin series. The first terms are given below up to order $\tilde{r}^6$.

$$\phi_{\frac{1}{2}4}(\tilde{r})$$

$$V_{H1} = -2 + \frac{8\tilde{r}^2}{3} - \frac{134}{45}\tilde{r}^4 + \frac{427}{135}\tilde{r}^6 ...$$

$$V_{H2} = \frac{2}{\tilde{r}^2} - \frac{2}{3} + \frac{58}{135}\tilde{r}^4 - \frac{301}{405}\tilde{r}^6 ...$$

$$\phi_{18}(\tilde{r})$$

$$V_{H1} = \frac{2}{\tilde{r}^2} - 5 + \frac{59}{10}\tilde{r}^2 - \frac{127}{20}\tilde{r}^4 + \frac{9277}{1400}\tilde{r}^6 ...$$

$$V_{H2} = \frac{6}{\tilde{r}^2} - 3 + \frac{17}{10}\tilde{r}^2 - \frac{17}{20}\tilde{r}^4 + \frac{303}{1400}\tilde{r}^6 ...$$

The leading term in these solutions is either $\tilde{r}^2$ or $\frac{1}{\tilde{r}^2}$. If we make an approximation for small $\tilde{r}$ and discard terms of higher order the following observations can be made. The Hamilton Operator is now one of two forms.

$$H = \frac{1}{2m\lambda^2}(p^2\lambda^2 + \alpha\tilde{r}^2) \tag{29}$$

$$H = \frac{1}{2m\lambda^2}(p^2\lambda^2 + \frac{\alpha}{\tilde{r}^2}) \tag{30}$$

Where α is an arbitrary constant.
Equation (29) just describes the system of an harmonic oscillator. This will be discussed afterwards. The case of (30) will be discussed below.

We can rewrite (30) as

$$H = \frac{p^2}{2m} + \frac{c}{\tilde{r}^2}\ , \qquad c = \frac{\alpha}{2m\lambda^2}$$

Thus writing out the time independent Schrödinger equation we get

$$H\psi = E\psi$$

$$\Leftrightarrow \frac{p^2}{2m}\psi(\tilde{r}) + \frac{c}{\tilde{r}^2}\psi(\tilde{r}) = E\psi(\tilde{r})$$

$$\Leftrightarrow \frac{1}{2m}\psi^{''}(\tilde{r}) + (\frac{c}{\tilde{r}^2} - E)\psi(\tilde{r}) = 0$$

This equation has the following general solution.

$$\psi(\tilde{r}) = C_1\sqrt{\tilde{r}}J(\frac{1}{2}\sqrt{1-8mc}, -i\sqrt{2Em}\tilde{r}) + C_2\sqrt{\tilde{r}}Y(\frac{1}{2}\sqrt{1-8mc}, -i\sqrt{2Em}\tilde{r}) \tag{31}$$

Where J and Y are the Bessel functions of first and second kind and C_1, C_2 are arbitrary constants. Note that the argument inside of either Bessel function becomes real if we assume $2E < 0$

We now look at the Schrödinger Equation for H in (29). Similarly to above this becomes

$$\frac{1}{2m}\psi^{''}(\tilde{r}) + (c\tilde{r}^2 - E)\psi(\tilde{r}) = 0$$

Which, as mentioned is the Schrödinger Equation for the harmonic oscillator. The general solution at ground state energy is of the form

$$\psi(\tilde{r}) = Ae^{k\tilde{r}^2}$$

with

$$A, k \in \mathbb{R}$$

To determine a possible Ansatz for $\psi(\tilde{r})$ we now also want to understand our solutions for large $\tilde{r}$. We therefore first make the substitution $\tilde{r} = \frac{1}{x}$ and again develop the solution in a Maclaurin Series. The results for $\phi_{\frac{1}{2}4}(\tilde{r})$ are shown below.

$$V_{H1}(x) = \frac{2(1 + 6x^2 - \frac{3}{-1+(1+x^2)\ln(1+\frac{1}{x^2})})}{1 + \frac{1}{x^2}} \tag{32}$$

$$\begin{aligned} V_{H2}(x) = \frac{2x^2}{(1+x^2)(-1+(1+x^2)\ln(1+\frac{1}{x^2})}\Big(5 + 3x^2 + \\ \ln(1+\frac{1}{x^2})\big(-1 - 7x^2 - 6x^4 + x^2(-1 + 2x^2 + 3x^4)\ln(1+\frac{1}{x^2})\big)\Big) \end{aligned} \tag{33}$$

Developing this in a Maclaurin series we get

$$V_{H1} = \frac{2(-4 + \ln(\frac{1}{x^2}))}{-1 + \ln(\frac{1}{x^2})}x^2 + \frac{2(5 - 4\ln(\frac{1}{x^2}) + 5\ln(\frac{1}{x^2})^2)}{(-1 + \ln(\frac{1}{x^2}))^2}x^4 + ...$$

$$V_{H2} = -\frac{2(-5 + \ln(\frac{1}{x^2})}{(-1 + \ln(\frac{1}{x^2})^2}x^2 - \frac{2(7 + 5\ln(\frac{1}{x^2}) + 3\ln(\frac{1}{x^2})^2 + \ln(\frac{1}{x^2})^3)}{(-1 + \ln(\frac{1}{x^2}))^3}x^4 - ...$$

We now resubstitute $\tilde{r} = \frac{1}{x}$ and cancel after the first term in our series. Thus we arrive at

$$V_{H1} \cong \frac{4(-2 + \ln(\tilde{r}))}{\tilde{r}^2(-1 + 2\ln(\tilde{r}))}$$

$$V_{H2} \cong -\frac{2(-5 + 2\ln(\tilde{r}))}{\tilde{r}^2(-1 + 2\ln(\tilde{r}))^2}$$

If we repeat the same calculations for ϕ_{18} we find

$$V_{H1} \cong -\frac{8(-4 + 3\ln(\tilde{r}))}{\tilde{r}^4}$$

$$V_{H2} \cong \frac{8(-4 + 3\ln(\tilde{r}))}{\tilde{r}^4}$$

Note that for large $\tilde{r}$ these results all tend towards 0.

3 Further Calculations

In the following calculations the before made substitution $\tilde{r} = \frac{r}{\lambda}$ still holds but for readability r now means $\tilde{r}$.

3.1 Determining the correct Ansatz

With our results from section 2 we now try to find a possible approximate solution to (28) which is split into four different equations depending on V. It is now convenient to look at the before found solutions for very large r as well as very small r.
For $V_{H1}(\phi_{\frac{1}{2}4})$ we in both cases have a solution of the form e^{-kr^2}. We combine this observation with the general power series Ansatz

$$u(r) = \sum_j b_j r^j \tag{34}$$

And obtain the following Ansatz for $\psi(r)$

$$\psi(r) = e^{-kr^2} u(r) \tag{35}$$

Combining this Ansatz with the Schrödinger-Equation yields.

$$(\frac{1}{2m}p^2 + \frac{1}{2m\lambda^2}V(r))\psi(r) = E\psi(r) \tag{36}$$

Since in our case $V(r)$ is always a fraction it is convenient to write it in the form

$$V(r) = \frac{f(r)}{g(r)}$$

Thus (36) becomes

$$\psi^{''}(r) + (\frac{1}{\lambda^2}\frac{f(r)}{g(r)} - 2mE)\psi(r) = 0 \tag{37}$$

Now all that remains is following the Ansatz (35) in the above equation (37) which yields

$$\begin{aligned} g(r)e^{-kr^2}u^{"}(r) - g(r)e^{-kr^2}4kru^{'}(r) \\ + e^{-kr^2}\Big(g(r)\big(4k^2r^2 - 2k - 2mE\big) + \frac{1}{\lambda^2}f(r)\Big)u(r) = 0 \end{aligned}$$

Let $\Omega := 2k + 2mE$. Thus after some simplification the above equation becomes

$$g(r)\Big(u(r)^{"} - 4kru^{'}(r) + \big(4k^2r^2 - \Omega\big)u(r)\Big) + \frac{1}{\lambda^2}f(r)u(r) = 0 \tag{38}$$

3.2 Results for $H_1(\phi_{\frac{1}{2}4})$

We find for V_{H1}

$$f(r) = (2r^2+12)\Big((1+r^2)\ln(1+r^2)-r^2\Big)-6r^4$$
$$g(r) = r^2\Big((1+r^2)\ln(1+r^2)-r^2\Big)(1+r^2) \tag{39}$$

For $u(r)$ it is true that

$$u^{"}(r) = \sum_j (j+2)(j+1)b_{j+2}r^j \ ; \ u^{'}(r) = \sum_j (j+1)b_{j+1}r^j \tag{40}$$

Thus using (39) and (40) with (38) yields

$$r^2\big((1+r^2)\ln(1+r^2)-r^2\big)(1+r^2)\Big(\sum_j (j+2)(j+1)b_{j+2}r^j$$
$$-4kr\sum_j (j+1)b_{j+1}r^j + (4k^2r^2-\Omega)\sum_j b_jr^j\Big)$$
$$+\frac{1}{\lambda^2}\Big((2r^2+12)\big((1+r^2)\ln(1+r^2)-r^2\big)-6r^4\Big)\sum_j b_jr^j = 0$$

If we assume $r < 1$ we can replace the natural logarithm with its power series

$$\ln(1+r^2) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}r^{2n} := \sum_{n=0}^{\infty} a_nr^{2n} \tag{41}$$

Inserting this into the above equation and doing some simple calculation yields

$$r^2(1+r^2)^2\sum_n a_nr^{2n}\sum_j\Big((j+2)(j+1)b_{j+2}r^j - 4kjb_jr^j + 4k^2b_{j-2}r^j - \Omega b_jr^j\Big)$$
$$-r^4(1+r^2)\sum_j\Big((j+2)(j+1)b_{j+2}r^j - 4kjb_jr^j + 4k^2b_{j-2}r^j - \Omega b_jr^j\Big)$$
$$+\frac{1}{\lambda^2}\Big((2r^2+12)\big((1+r^2)\sum_n a_nr^{2n}-r^2\big)-6r^4\Big)\sum_j b_jr^j = 0$$

To make further calculations simpler we introduce some abbreviations. We get

$$c\sum_n a_nr^{2n}\sum_j \Pi_jr^j - d\sum_j \Pi_jr^j +$$
$$\frac{1}{\lambda^2}\Big(e\big(f\sum_n a_nr^{2n}-r^2\big)-6r^4\Big)\sum_j b_jr^j = 0$$

where

$$c = r^2(1+r^2)^2; \qquad d = r^4(1+r^2); \qquad e = (2r^2+12); \qquad f = (1+r^2)$$
$$\Pi_j = (j+2)(j+1)b_{j+2} - 4kjb_j + 4k^2 b_{j-2} - \Omega b_j$$

We expand the second part of this equation.

$$c\sum_n a_n r^{2n} \sum_j \Pi_j r^j - d\sum_j \Pi_j r^j +$$
$$\frac{1}{\lambda^2} ef \sum_j \sum_n b_j a_n r^{j+2n} - \frac{1}{\lambda^2}(er^2 + 6r^4)\sum_j b_j r^j = 0$$

To make further calculations simpler we split this into three separate sums such that

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

where

$$\sigma_1 = c\sum_j \sum_n \Pi_j a_n r^{j+2n}$$
$$\sigma_2 = \frac{1}{\lambda^2} ef \sum_j \sum_n b_j a_n r^{j+2n}$$
$$\sigma_3 = -d\sum_j \Pi_j r^j - \frac{1}{\lambda^2}(er^2+6r^4)\sum_j b_j r^j$$

To continue we first show that

$$\sum_j \sum_n a_j b_n r^{j+2n} = \sum_k \sum_n a_{k-2n} b_n r^k \tag{42}$$

Proof: For a given k we look for all coefficients $a_j b_n$ such that:

$$j + 2n = k \qquad \Rightarrow \qquad j = k - 2n$$

Assuming $n \in \mathbb{N}, a_x = 0 \qquad \forall x < 0$

$$\begin{aligned} &\Rightarrow a_{k-2n} b_n \text{ is coefficient for } r^k \\ &\Leftrightarrow \forall n \in \mathbb{N} : a_{k-2n} b_n \text{is coefficient for } r^k \end{aligned} \tag{43}$$

It follows that for a given k:

$$\sum_n a_{k-2n} b_n$$

Is the sum of all coefficients for r^k And since

$$\forall j \ \exists k, n : j = k - 2n$$

It follows that:

$$\sum_j \sum_n a_j b_n r^{j+2n} = \sum_k \sum_n a_{k-2n} b_n r^k$$

□

Using (42) σ_1 and σ_2 become

$$\sigma_1 = c \sum_l \sum_n \Pi_{l-2n} a_n r^l = c \sum_l \alpha_l r^l$$
$$\sigma_2 = \frac{1}{\lambda^2} ef \sum_l \sum_n b_{l-2n} a_n r^l = \frac{1}{\lambda^2} ef \sum_l \beta_l r^l$$

where

$$\alpha_l = \sum_n \Pi_{l-2n} a_n$$
$$= \sum_n \Big((l-2n+2)(l-2n+1) b_{l-2n+2} - 4k(l-2n) b_l - 2n + 4k^2 b_{l-2n-2} - \Omega b_l - 2n \Big) a_n$$
$$\beta_l = \sum_n \beta_{l-2n} a_n$$

If we resubstitute the before made abbreviations and use index shifting we find for these sums respectively

$$\sigma_1 = \sum_l (\alpha_{l-2} + 2\alpha_{l-4} + \alpha_{l-6}) r^l := \sum_j \mu_j r^j$$
$$\sigma_2 = \sum_l (2\beta_{l-4} + 14\beta_{l-2} + 12\beta_l) r^l := \sum_j \xi_j r^j$$

where we defined μ_j and ξ_j as the prefactors.
Similarly σ_3 becomes

$$\sigma_3 = \sum_j \Big(-\Pi_{j-6} - \Pi_{j-4} - \frac{1}{\lambda^2}(8b_{j-4} + 12b_{j-2}) \Big) r^j := \sum_j \nu_j r^j$$

where we again defined ν_j as the prefactor.

Thus our equation finally can be written down as

$$\sum_j (\xi_j + \mu_j + \nu_j) r^j = 0 \tag{44}$$

3.3 Modifying the Ansatz

For $V_{H2}(\phi_{\frac{1}{2}4})$, $V_{H1}(\phi_{18})$ and $V_{H2}(\phi_{18})$ we follow the same logic as before but must modify our Ansatz in the following way.
Since for small r we found (31) we notice that this approaches a solution of the form $\sqrt{r}c_1$ where c_1 is an arbitrary constant. For large r the solution is of the form c_2e^{-kr}. Thus our Ansatz becomes

$$\psi(r) = \sqrt{r}e^{-kr}u(r). \tag{45}$$

Where $u(r) = \sum_j b_j r^j$ swallows the constants c_1 and c_2. Inserting this into (37) we get

$$g(r)\psi'' + (\frac{1}{\lambda^2}f(r) - 2mg(r)E)\psi(r) = 0$$

$$\psi''(r) = \frac{1}{4r^{\frac{3}{2}}}(e^{-kr}(4r^2u''(r) + (4r - 8kr^2)u' + (4k^2r^2 - 4kr - 1)u(r))$$

$$\Rightarrow \frac{g(r)}{4r^{\frac{3}{2}}}(e^{-kr}(4r^2u''(r) + (4r - 8kr^2)u' + (4k^2r^2 - 4kr - 1)u(r)) + (\frac{1}{\lambda^2}f(r) - 2mg(r)E)\sqrt{r}e^{-kr}u(r) = 0$$

After short calculation this becomes.

$$g(r)\Big(\sqrt{r}u''(r) + (\frac{1}{\sqrt{r}} - 2k\sqrt{r})u'(r) + ((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r})u(r)\Big) + f(r)\Big(\frac{1}{\lambda^2}\sqrt{r}u(r)\Big) = 0 \tag{46}$$

Now all that is left to do is insert the specific Potential $V(r) = \frac{f(r)}{g(r)}$ into eq. (46) as it is done in the following calculations.

3.4 Results for $H_2(\phi_{\frac{1}{2}4})$

$$f(r) = 2r^4(3 + 5r^2) - 2(1 + r^2)\ln(1 + r^2)\big(r^2(6 + r^2) + (r^2 - 3)\ln(1 + r^2)\big)$$

$$g(r) = (1 + r^2)\big(r^3 - (r + r^3)\ln(1 + r^2)\big)^2$$

Using (41) and (40) and inserting $f(r)$, $g(r)$ into (46) we get

$$(1 + r^2)\big(r^3 - (r + r^3)\sum_n a_n r^{2n}\big)^2\Big(\sqrt{r}\sum_j (j + 2)(j + 1)b_{j+2}r^j + (\frac{1}{\sqrt{r}} - 2k\sqrt{r})\sum_j (j + 1)b_{j+1}r^j + \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)\sum_j b_j r^j\Big) + \big(2r^4(3 + 5r^2) - 2(1 + r^2)\sum_n a_n r^{2n}\big(r^2(6 + r^2) + (r^2 - 3)\sum_n a_n r^{2n}\big)\big)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big) = 0$$

We split this sum into two parts such that

$$\sigma_1 + \sigma_2 = 0$$

where

$$\sigma_1 = (1+r^2)\big(r^3 - (r+r^3)\sum_n a_n r^{2n}\big)^2\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j$$
$$+ (\frac{1}{\sqrt{r}} - 2k\sqrt{r})\sum_j (j+1)b_{j+1}r^j + \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)\sum_j b_j r^j\Big)$$
$$\sigma_2 = \big(2r^4(3+5r^2) - 2(1+r^2)\sum_n a_n r^{2n}\big(r^2(6+r^2) + (r^2-3)\sum_n a_n r^{2n}\big)\big)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big)$$

To continue we first determine $(\sum_n a_n r^{2n})^2$

$$\begin{aligned} &= \sum_n a_n r^{2n} \sum_k a_k r^{2k} = \sum_{n,k} a_n a_k r^{2(n+k)} \\ &\text{let } n+k=\eta \\ &= \sum_n \sum_k a_{\eta-k} a_k r^{2\eta} = \sum_\eta \tilde{a}_\eta r^{2\eta} \\ &\text{where } \tilde{a}_\eta = \sum_k a_{\eta-k} a_k \end{aligned} \qquad (47)$$

After some calculation and using (47) σ_1 becomes:

$$\sigma_1 = (1+r^2)\big(r^6 - 2(r^4+r^6)\sum_n a_n r^{2n} + (r^2+2r^4+r^6)\sum_\eta \tilde{a}_\eta r^{2\eta}\big)$$
$$\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + (\frac{1}{\sqrt{r}} - 2k\sqrt{r})\sum_j (j+1)b_{j+1}r^j$$
$$+ \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)\sum_j b_j r^j\Big)$$

Defining some abbreviations we can simplify this in the following way

$$\sigma_1 = (1+r^2)\big(a - b\sum_n a_n r^{2n} + c\sum_\eta \tilde{a}_\eta r^{2\eta}\big)$$
$$\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + A\sum_j (j+1)b_{j+1}r^j + B\sum_j b_j r^j\Big)$$

where

$$a = r^6; \qquad b = 2(r^4+r^6); \qquad c = (r^2+2r^4+r^6)$$
$$A = (\frac{1}{\sqrt{r}} - 2k\sqrt{r}); \qquad B = \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)$$

Multiplying this out we get

$$\sigma_1 = (1+r^2)\big(a\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + aA\sum_j (j+1)b_{j+1}r^j$$
$$+ aB\sum_j b_j r^j - b\sqrt{r}\sum_n\sum_j a_n(j+2)(j+1)b_{j+2}r^{j+2n} - bA\sum_n\sum_j a_n(j+1)b_{j+1}r^{j+2n}$$
$$- bB\sum_n\sum_j a_n b_j r^{j+2n} + c\sqrt{r}\sum_\eta\sum_j \tilde{a}_\eta(j+2)(j+1)b_{j+2}r^{j+2\eta}$$
$$+ cA\sum_\eta\sum_j \tilde{a}_\eta(j+1)b_{j+1}r^{j+2\eta} + cB\sum_\eta\sum_j \tilde{a}_\eta b_j r^{j+2\eta}\big)$$

If we introduce $l = j + 2n$ and $m = j + 2\eta$ and use (42) we can write this as:

$$\sigma_1 = (1+r^2)\Big(a\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + aA\sum_j (j+1)b_{j+1}r^j + aB\sum_j b_j r^j$$
$$- b\sqrt{r}\sum_l \alpha_l r^l - bA\sum_l \beta_l r^l - bB\sum_l \gamma_l r^l$$
$$+ c\sqrt{r}\sum_m \tilde{\alpha}_m r^m + cA\sum_m \tilde{\beta}_m r^m + cB\sum_m \tilde{\gamma}_m r^m\Big)$$

where

$$\alpha_l = \sum_n a_n(l-2n+2)(l-2n+1)b_{l-2n+2}$$
$$\beta_l = \sum_n a_n(l-2n+1)b_{l-2n+1}$$
$$\gamma_l = \sum_n a_n b_{l-2n}$$
$$\tilde{\alpha}_m = \sum_\eta a_\eta(m-2\eta+2)(m-2\eta+1)b_{m-2\eta+2}$$
$$\tilde{\beta}_m = \sum_\eta a_\eta(m-2\eta+1)b_{m-2\eta+1}$$
$$\tilde{\gamma}_m = \sum_\eta a_\eta b_{m-2\eta}$$

Thus if we use the same summation index and take the sum out we can also write this as

$$\sigma_1 = (1+r^2)\sum_j \Big(a\sqrt{r}(j+2)(j+1)b_{j+2} + aAj(j+1)b_{j+1} + aBb_j$$
$$- b\sqrt{r}\alpha_j - bA\beta_j - bB\gamma_j + c\sqrt{r}\tilde{\alpha}_j + cA\tilde{\beta}_j + cB\tilde{\gamma}_j\Big)r^j$$

If we resubstitute our before defined abbreviations and use index-shifting this sum becomes after some calculation

$$\sigma_1 = \sum_j \Big(\big((j-4)(j-5) + (j-4) - \frac{1}{4}\big) b_{j-4} - \big(2k(j-5) + k\big) b_{j-5} + k^2 b_{j-6}$$

$$- 2\big(\alpha_{j-4} + \alpha_{j-6} + \beta_{j-3} - 2k\beta_{j-4} + \beta_{j-5} - 2k\beta_{j-6} - \frac{1}{4}\gamma_{j-2} - k\gamma_{j-3}$$

$$+ (k^2 - \frac{1}{4})\gamma_{j-4} - k\gamma_{j-5} + k^2\gamma_{j-6}\big) + \tilde{\alpha}_{j-2} + 2\tilde{\alpha}_{j-4} + \tilde{\alpha}_{j-6} + \tilde{\beta}_{j-1}$$

$$- 2k\tilde{\beta}_{j-2} + 2\tilde{\beta}_{j-3} - 4k\tilde{\beta}_{j-4} + \tilde{\beta}_{j-5} - 2k\tilde{\beta}_{j-6} - \frac{1}{4}\tilde{\gamma}_j - k\tilde{\gamma}_{j-1} + (k^2 - \frac{1}{4})\tilde{\gamma}_{j-2}$$

$$- k\tilde{\gamma}_{j-3} + (2k^2 - \frac{1}{4})\tilde{\gamma}_{j-4} - k\tilde{\gamma}_{j-5} + k^2\tilde{\gamma}_{j-6}\Big) \sqrt{r}(1+r^2) r^j$$

If we define the prefactor as ξ_j this sum can be expressed as:

$$\sigma_1 = \sum_j \xi_j \sqrt{r}(1+r^2) r^j = \sum_j (\xi_j + \xi_{j-2}) \sqrt{r} r^j$$

We now try to bring σ_2 into a similar form. Written out σ_2 becomes

$$\sigma_2 = 2r^4(3+5r^2)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big) - 2(1+r^2)\big(\sum_n a_n (6r^{2n+2} + r^{2n+4})$$

$$+ (r^2-3)(\sum_n a_n r^{2n})^2\big)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big)$$

$$= 2r^4(3+5r^2)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big) - 2(1+r^2)\frac{1}{\lambda^2}\sqrt{r}\big(\sum_n\sum_j a_n b_j (6r^{j+2n+2} + r^{j+2n+4})$$

$$+ (r^2-3)\sum_\eta\sum_j \tilde{a}_\eta b_j r^{j+2\eta}\big); \qquad \text{(using (47))}$$

$$= \frac{1}{\lambda^2}\sqrt{r}\Big(2r^4(3+5r^2)\sum_j b_j r^j - 2(1+r^2)\big(\sum_k \alpha_k 6r^k + \sum_l \beta_l r^l + (r^2-3)\sum_m \gamma_m r^m\big)\Big)$$

Where we used the index-shift $k = j + 2n + 2; \quad l = j + 2n + 4; \quad m = j + 2\eta$ in conjunction with (42) as well as

$$\alpha_k = \sum_n a_n b_{k-2n-2}$$

$$\beta_l = \sum_n a_n b_{l-2n-4}$$

$$\gamma_m = \sum_\eta \tilde{a}_\eta b_m$$

Summing over the same index this becomes, after some calculation

$$= \frac{1}{\lambda^2}\sqrt{r}\sum_j \Big(6b_{j-4} + 10b_{j-6} - 18\alpha_j - 12\alpha_{j-2} - 3\beta_j - 2\beta_{j-2} + 9\gamma_j + 3\gamma_{j-2} - 2\gamma_{j-4}\Big) r^j$$

If we define the prefactor as μ_j this sum can be expressed as

$$\sigma_2 = \sum_j \mu_j \sqrt{r} r^j$$

Combining σ_1 and σ_2 we thus arrive at the equation

$$\sum_j (\xi_j + \xi_{j-2} + \mu_j) \sqrt{r} r^j = 0 \tag{48}$$

3.5 Results for $H_1(\phi_{18})$

$$f(r) = 8r^2(5+4r^2) - 4(1+r^2)(10+3r^2)\ln(1+r^2)$$
$$g(r) = r^4(1+r^2)(2+r^2) - 2(r+r^3)^2\ln(1+r^2)$$

Using (41) and (40) and inserting $f(r)$, $g(r)$ into (46) we get

$$\big(r^4(1+r^2)(2+r^2) - 2(r+r^3)^2\sum_n a_n r^{2n}\big)\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j$$
$$+ (\frac{1}{\sqrt{r}} - 2k\sqrt{r})\sum_j (j+1)b_{j+1}r^j + \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)\sum_j b_j r^j\Big)$$
$$+ \big(8r^2(5+4r^2) - 4(1+r^2)(10+3r^2)\sum_n a_n r^{2n}\Big)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big) = 0$$

We again define certain abbreviations and simplify this expression.

$$\Leftrightarrow \quad \big(a - b\sum_n a_n r^{2n}\big)\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + A\sum_j (j+1)b_{j+1}r^j + B\sum_j b_j r^j\Big)$$
$$+\big(c - d\sum_n a_n r^{2n}\big)\big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\big) = 0$$

here

$$\begin{aligned} &a = r^4(1+r^2)(2+r^2); \qquad b = 2(r+r^3)^2; \\ &c = 8r^2(5+4r^2); \qquad d = 4(1+r^2)(10+3r^2) \\ &A = (\frac{1}{\sqrt{r}} - 2k\sqrt{r}); \qquad B = \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big) \end{aligned} \tag{49}$$

We can split this into two sums σ_1, σ_2 such that

$$\sigma_1 + \sigma_2 = 0$$

where

$$\sigma_1 = \sum_j \big(a\sqrt{r}(j+2)(j+1)b_{j+2} + aA(j+1)b_{j+1} + aBb_j + c\frac{1}{\lambda^2}\sqrt{r}b_j\big)r^j$$
$$\sigma_2 = \sum_j\sum_n \big((-ba_n\sqrt{r}(j+2)(j+1)b_{j+2}) - ba_nA(j+1)b_{j+1} - ba_nBb_j - da_n\frac{1}{\lambda^2}\sqrt{r}b_j\big)r^{j+2n}$$

Using $l = j - 2n$ with (42) σ_2 becomes

$$\sigma_2 = \sum_l\sum_n \Big((-ba_n\sqrt{r}(l-2n+2)(l-2n+1)b_{l-2n+2} - ba_nA(l-2n+1)b_{l-2n+1}$$
$$- ba_nBb_{l-2n} - da_n\frac{1}{\lambda^2}\sqrt{r}b_{l-2n}\Big)r^l$$

If we introduce one more simplification this can be written as:

$$\sigma_2 = \sum_l \left(-\sqrt{r} b\alpha_l - bA\beta_l - bB\gamma_l - d\sqrt{r}\delta_l\right) r^l$$

where

$$\alpha_l = \sum_n a_n(l-2n+2)(l-2n+1)b_{l-2n+2}$$
$$\beta_l = \sum_n a_n(l-2n+1)b_{l-2n+1}$$
$$\gamma_l = \sum_n a_n b_{l-2n}$$
$$\delta_l = \sum_n a_n \frac{1}{\lambda^2} b_{l-2n}$$

Thus if we resubstitute our parameters and use some index shifting this becomes

$$\sum_l \Big(-2\alpha_{l-2} - 4\alpha_{l-4} - 2\alpha_{l-6} + 4k\beta_{l-2} - 4\beta_{l-3} + 8k\beta_{l-4} - 2\beta_{l-5} + 4k\beta_{l-6}$$
$$+ \frac{1}{2}\gamma_l + 2k\gamma_{l-1} - (2(k^2 - 2mE) - 1)\gamma_{l-2} + 4k\gamma_{l-3} - (4(k^2 - 2mE) - \frac{1}{2})\gamma_{l-4}$$
$$+ 2k\gamma_{l-5} - 2(k^2 - 2mE)\gamma_{l-6} - 40\delta_l - 52\delta_{l-2} - 12\delta_{l-4}\Big)\sqrt{r} r^l$$

We define this sum as

$$\sigma_2 := \sum_l \xi_l \sqrt{r} r^l$$

Similarly σ_1 takes after some calculation the following form:

$$\sigma_1 = \sum_j \Big(\big(2(j-2)(j-3) + 2(j-2) - \frac{1}{2} + \frac{40}{\lambda^2}\big)b_{j-2} - \big(4k(j-3) + 2k\big)b_{j-3}$$
$$+ \big(3(j-4)(j-5) + 3(j-4) + (2k^2 - \frac{3}{4} - 4mE)\big)b_{j-4}$$
$$+ \big((j-6)(j-7) - 6k(j-5) - 3k\big)b_{j-5} + \big((j-6) + (3k^2 - 6mE)\big)b_{j-6}$$
$$- \big(2k(j-7) + k\big)b_{j-7} + \big(k^2 - \frac{1}{4} - 2mE\big)b_{j-8}\Big)\sqrt{r} r^j$$

Which we define as

$$\sigma_1 := \sum_j \mu_j \sqrt{r} r^j$$

Combining these two sums we now arrive at the equation

$$\sum_j (\mu_j + \xi_j)\sqrt{r} r^j = 0 \tag{50}$$

3.6 Results for $H_2(\phi_{18})$

$$f(r) = 8r^4(6{+}3r^2{-}4r^4){+}4(1{+}r^2)\ln(1{+}r^2)\Big(3r^2(r^4{-}8){+}2(6{+}3r^2{+}r^4)\ln(1{+}r^2)\Big)$$

$$g(r) = (1+r^2)\Big(r^3(2+r^2) - 2(r+r^3)\ln(1+r^2)\Big)^2$$

Using (41) and (40) and inserting $f(r),\ g(r)$ into (46) we arrive at

$$\Big((1+r^2)\big(r^3(2+r^2) - 2(r+r^3)\sum_n a_n r^{2n}\big)^2\Big)\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j$$
$$+ (\frac{1}{\sqrt{r}} - 2k\sqrt{r})\sum_j (j+1)b_{j+1}r^j + \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big)\sum_j b_j r^j\Big)$$
$$+ \Big(8r^4(6+3r^2-4r^4) + 4(1+r^2)(\sum_{n=0}^{\infty} a_n r^{2n})(3r^2(r^4-8) + 2(6+3r^2+r^4)\sum_{n=0}^{\infty} a_n r^{2n})\Big)$$
$$\Big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\Big) = 0$$

We simplify this by introducing some abbreviations in the following way

$$\Leftrightarrow \Big(a\big(b - c\sum_n a_n r^{2n}\big)^2\Big)\Big(\sqrt{r}\sum_j (j+2)(j+1)b_{j+2}r^j + A\sum_j (j+1)b_{j+1}r^j + B\sum_j b_j r^j\Big)$$
$$+ \Big(d + e\sum_{n=0}^{\infty} a_n r^{2n}\big(f + g\sum_{n=0}^{\infty} a_n r^{2n}\big)\Big)\Big(\frac{1}{\lambda^2}\sqrt{r}\sum_j b_j r^j\Big) = 0$$

where

$$\begin{aligned} &a = (1+r^2); \qquad b = r^3(2+r^2); \qquad c = 2(r+r^3) \\ &d = 8r^4(63r^2 - 4r^4); \qquad e = 4(1+r^2) \\ &f = 3r^2(r^4-8); \qquad g = 2(6+3r^2+r^4) \\ &A = (\frac{1}{\sqrt{r}} - 2k\sqrt{r}); \qquad B = \big((k^2\sqrt{r} - \frac{k}{\sqrt{r}} - \frac{1}{4r^{\frac{3}{2}}}) - 2mE\sqrt{r}\big) \end{aligned} \tag{51}$$

Using (47) and after some calculation the above equation becomes

$$\begin{aligned} &\sum_j \Big(ab^2\big(\sqrt{r}(j+2)(j+1)b_{j+2} + A(j+1)b_{j+1} + Bb_j\big) + df\frac{1}{\lambda^2}\sqrt{r}b_j\Big)r^j \\ &+ \sum_j \Big((-2abc\sum_n a_n)\big(\sqrt{r}(j+2)(j+1)b_{j+2} + A(j+1)b_{j+1} + Bb_j\big) \\ &+ (dg+ef)\sum_n a_n\big(\frac{1}{\lambda^2}\sqrt{r}b_j\big)\Big)r^{j+2n} \\ &+ \sum_j \Big(ac^2\sum_\eta \tilde{a}_\eta\big(\sqrt{r}(j+2)(j+1)b_{j+2} + A(j+1)b_{j+1} + Bb_j\big) \\ &+ eg\sum_\eta \tilde{a_\eta}\big(\frac{1}{\lambda^2}\sqrt{r}b_j\big)\Big)r^{j+2\eta} = 0 \end{aligned} \tag{52}$$

For further calculations we distinguish between the three sums such that

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

To further simplify σ_2 we first define $l = j + 2n \;\Rightarrow\; j = l - 2n$ With this we can use (42). Thus σ_2 becomes

$$\sigma_2 = \sum_l \sum_n \Big(a_n(-2abc)\big(\sqrt{r}(l-2n+2)(l-2n+1)b_{l-2n+2} + A(l-2n+1)b_{l-2n+1} + Bb_{l-2n}\big) + (dg+ef)a_n\big(\frac{1}{\lambda^2}\sqrt{r}b_{l-2n}\big)\Big) r^l$$

We can further simplify this by introducing certain parameters.

$$= \sum_l \Big((-2abc)\big(\sqrt{r}\alpha_l^{\sigma_2} + A\beta_l^{\sigma_2} + B\gamma_l^{\sigma_2}\big) + (dg+ef)\sqrt{r}\delta_l^{\sigma_2}\Big) r^l$$

where

$$\alpha_l^{\sigma_2} = \sum_n a_n(l-2n+2)(l-2n+1)b_{l-2n+2}$$
$$\beta_l^{\sigma_2} = \sum_n a_n(l-2n+1)b_{l-2n+1}$$
$$\gamma_l^{\sigma_2} = \sum_n a_n b_{l-2n}$$
$$\delta_l^{\sigma_2} = \sum_n a_n \frac{1}{\lambda^2} b_{l-2n}$$

Similarly σ_3 becomes

$$\sum_l \Big(ac^2\big(\sqrt{r}\alpha_l^{\sigma_3} + A\beta_l^{\sigma_3} + B\gamma_l^{\sigma_3}\big) + eg\sqrt{r}\delta_l^{\sigma_3}\Big) r^l$$

where

$$\alpha_l^{\sigma_3} = \sum_\eta \tilde{a}_\eta(l-2\eta+2)(l-2\eta+1)b_{l-2\eta+2}$$
$$\beta_l^{\sigma_3} = \sum_\eta \tilde{a}_\eta(l-2\eta+1)b_{l-2\eta+1}$$
$$\gamma_l^{\sigma_3} = \sum_\eta \tilde{a}_\eta b_{l-2\eta}$$
$$\delta_l^{\sigma_3} = \sum_\eta \tilde{a}_\eta \frac{1}{\lambda^2} b_{l-2\eta}$$

Now we can resubstitute our before defined abbreviations.

With that σ_1 becomes

$$\sigma_1 = \sum_j \Big((4r^6 + 8r^8 + 5r^{10} + r^{12})\sqrt{r}(j+2)(j+1)b_{j+2}$$
$$+ \sqrt{r}(4r^5 - 8kr^6 + 8r^7 - 16kr^8 + 5r^9 - 10kr^{10} + r^{11})(j+1)b_{j+1}$$
$$+ \sqrt{r}(-r^4 - 4kr^5 + (4k^2 - 2 - 8mE)r^6 - 8kr^7 + (8k^2 - \frac{5}{4} - 16mE)r^8 - 5kr^9$$
$$+ (5k^2 - \frac{1}{4} - 10mE)r^{10} - kr^{11} + (k^2 - 2mE)r^{12})b_j$$
$$+ \sqrt{r}(-1152r^6 - 576r^8 + 912r^{10} + 72r^{12} - 96r^{14})\frac{1}{\lambda^2}b_j \Big) r^j$$

Using index shifting and collecting the same prefactors this becomes

$$\sigma_1 = \sum_j \Big((4(j-4)(j-5) + 4(j-4) - 1)b_{j-4} + (-8k(j-5) - 4k)b_{j-5}$$
$$+ (8(j-6)(j-7) + 8(j-6) + (4k^2 - 2 - 8mE) - 1152\frac{1}{\lambda^2})b_{j-6}$$
$$+ (-16k(j-7) - 8k)b_{j-7} + (5(j-8)(j-9) + 5(j-8) + (8k^2 - \frac{5}{4} - 16mE) - 576\frac{1}{\lambda^2})b_{j-8}$$
$$+ (-10k(j-9) - 5k)b_{j-9} + ((j-10)(j-11) + (j-10) + (5k^2 - \frac{1}{4} - 10mE) + 912\frac{1}{\lambda^2})b_{j-10}$$
$$- kb_{j-11} + ((k^2 - 2mE) + \frac{72}{\lambda^2})b_{j-12} - \frac{96}{\lambda^2}b_{j-14} \Big)\sqrt{r}r^j$$

Similarly σ_2 becomes:

$$\sigma_2 = \sum_l \Big(-8\alpha^{\sigma_2}_{l-4} - 20\alpha^{\sigma_2}_{l-6} - 16\alpha^{\sigma_2}_{l-8} - 4\alpha^{\sigma_2}_{l-10} - 8\beta^{\sigma_2}_{l-3} + 16k\beta^{\sigma_2}_{l-4}$$
$$- 20\beta^{\sigma_2}_{l-5} + 40k\beta^{\sigma_2}_{l-6} - 16\beta^{\sigma_2}_{l-7} + 32k\beta^{\sigma_2}_{l-8} - 4\beta^{\sigma_2}_{l-9} + 8k\beta^{\sigma_2}_{l-10}$$
$$+ 2\gamma^{\sigma_2}_{l-2} + 8k\gamma^{\sigma_2}_{l-3} - (8k^2 - 5 - 16mE)\gamma^{\sigma_2}_{l-4} + 20k\gamma^{\sigma_2}_{l-5} + 4\gamma^{\sigma_2}_{l-6}$$
$$- (20k^2 - 40mE)\gamma^{\sigma_2}_{l-6} + 16k\gamma^{\sigma_2}_{l-7} - (16k^2 - 1 - 32mE)\gamma^{\sigma_2}_{l-8} + 4k\gamma^{\sigma_2}_{l-9}$$
$$- (4k^2 - 8mE)\gamma^{\sigma_2}_{l-10} - 96\delta^{\sigma_2}_{l-2} + 480\delta^{\sigma_2}_{l-4} + 588\delta^{\sigma_2}_{l-6} - 132\delta^{\sigma_2}_{l-8} - 144\delta^{\sigma_2}_{l-10} - 64\delta^{\sigma_2}_{l-12} \Big)\sqrt{r}r^l$$

And σ_3 becomes:

$$\sigma_3 = \sum_l \Big(4\alpha^{\sigma_3}_{l-2} + 12\alpha^{\sigma_3}_{l-4} + 12\alpha^{\sigma_3}_{l-6} + 4\alpha^{\sigma_3}_{l-8}$$
$$+ 4\beta^{\sigma_3}_{l-1} - 8k\beta^{\sigma_3}_{l-2} + 12\beta^{\sigma_3}_{l-3} - 24k\beta^{\sigma_3}_{l-4} + 12\beta^{\sigma_3}_{l-5}$$
$$- 24k\beta^{\sigma_3}_{l-6} - 4\beta^{\sigma_3}_{l-7} - 8k\beta^{\sigma_3}_{l-8} - \gamma^{\sigma_3}_{l} - 4k\gamma^{\sigma_3}_{l-1} + (4k^2 - 3 - 8mE)\gamma^{\sigma_3}_{l-2}$$
$$- 12k\gamma^{\sigma_3}_{l-3} + (12k^2 - 3 - 24mE)\gamma^{\sigma_3}_{l-4} - 12k\gamma^{\sigma_3}_{l-5} + (12k^2 - 1 - 24mE)\gamma^{\sigma_3}_{l-6}$$
$$- 4k\gamma^{\sigma_3}_{l-7} + (4k^2 - 8mE)\gamma^{\sigma_3}_{l-8} + 48\delta^{\sigma_3}_{l} + 72\delta^{\sigma_3}_{l-2} + 32\delta^{\sigma_3}_{l-4} + 8\delta^{\sigma_3}_{l-6} \Big)\sqrt{r}r^l$$

If we define the above sums as follows:

$$\begin{aligned}
\sigma_1 &:= \sum_j \xi_j \sqrt{r} r^j \\
\sigma_2 &:= \sum_j \mu_j \sqrt{r} r^j \\
\sigma_3 &:= \sum_j \nu_j \sqrt{r} r^j
\end{aligned} \tag{53}$$

We arrive at the equation

$$\sum_j (\xi_j + \mu_j + \nu_j)\sqrt{r} r^j = 0 \tag{54}$$

4 Conclusion

We presented a supersymmetric approach to the zero modes of the Faddeev-Popov operator in order to understand whether a more simple solution can be found, which in turn could lead to a better understanding of the Gribov-Singer ambiguity. This was done within an instanton field configuration through calculating the supersymmetric partner potentials. Our results brought no direct improvement in solvability, however to each potential term in (28) an approximate solution for the wavefunction either of the form

$$\sum_j \chi_j \sqrt{r} r^j = 0 \tag{55}$$

or of the form

$$\sum_j \chi_j r^j = 0 \tag{56}$$

has been found. To now further check for solvability it must be noted, that in each case the prefactor χ_j has to be zero for all j. As shown in section 3, χ_j is of the general form

$$\chi_j = \sum_i \alpha_{ij} b_i = 0 \tag{57}$$

where b_i are the initial coefficients we are interested in. Equation (57) is a matrix equation which can also be expressed as

$$\chi = A\mathbf{b} \tag{58}$$

One now would have to try to calculate the determinant of A in order to check for solvability and possibly find the solutions depending on the structure of A. However this would be beyond the scope of this thesis so we will end here.
As is evident, further research is necessary to build upon the findings presented in this thesis.

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