The evolution of taking roles
Online Supplement

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Abstract

This note contains supplementary material for our paper entitled "The evolution of taking roles".

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1 The meta game with 4 types

Next consider the case $|\Theta| = 4$. With four types there is again the hierarchical structure. Since the number of types is odd, there is no egalitarian structure, but there is a circular structure that is approximate egalitarian.

	L_1	L_2	L_3	L_4
L_1	u^*	a	a	a
L_2	b	u^*	a	a
L_3	b	b	u^*	a
L_4	b	b	b	u^*

hierarchical

	L_1	L_2	L_3	L_4
L_1	u^*	a	a	b
L_2	b	u^*	a	a
L_3	b	b	u^*	a
L_4	a	b	b	u^*

approximate egalitarian

It will turn out to be useful to consider a further reduces form type game in which some labels are summarized in sub-groups which are treated equally by all other labels. For instance:

	G_{12}	L_3	L_4
G_{12}	h_2	a	a
L_3	b	u^*	a
L_4	b	b	u^*

hierarchical

	L_1	G_{23}	L_4
L_1	u^*	a	b
G_{23}	b	h_2	a
L_4	a	b	u^*

approximate egalitarian

For both type structure, full label support equilibria exist only for anti-coordination games, but not for conflict games.

Furthermore, for $|\Theta| \geq 4$ there are also structures with a partial hierarchy among some intra-egalitarian groups. Consider the case $|\Theta| = 4$:

	L_1	L_2	L_3	L_4
L_1	u^*	a	a	a
L_2	b	u^*	a	b
L_3	b	b	u^*	a
L_4	b	a	b	u^*

Top label and egalitarian-group

	L_1	L_2	L_3	L_4
L_1	u^*	a	b	a
L_2	b	u^*	a	a
L_3	a	b	u^*	a
L_4	b	b	b	u^*

Egalitarian-group - bottom label

Considering in the first type-game the types L_1-L_3 as one egalitarian group G_T, and in the second type-game the types L_2-L_4 a further reduction of the type-structures is given by

	T	G_B
T	u^*	a
G_B	b	v_3

Top label and egalitarian-group

	G_T	B
G_T	v_3	a
B	b	u^*

Egalitarian-group - bottom label

Analogous pre-stable type structures with a hierarchy between a single type and an egalitarian group of $k \equiv |\Theta| - 1$ labels exist, of course, for any even number of labels $|\Theta|$ and lead to the correspondingly further reduced type-structure:

	T	G_B
T	u^*	a
G_B	b	v_k

	G_T	B
G_T	v_k	a
B	b	u^*

Top label and egalitarian-group   Egalitarian-group - bottom label

Remember that $v_k \in [u^*, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, u^*]$. In conflict games with $u^* \geq b$ it follows for all $k \geq 3$ that $v_k > b$. Hence, the top-label (or the labels of the top egalitarian group in the second reduced type-game) dominates the labels of the bottom egalitarian group (or the bottom label, respectively). In equilibrium all probability weight must therefore be on the top-label, or, respectively, on the labels of the top egalitarian group. In anti-coordination games with $u^* < b$ in the first game there is a full label-support equilibrium with a top label and an egalitarian group at the bottom of the hierarchy. In the second game it depends: For sufficiently small k the expected payoff within the egalitarian group v_k is still smaller than b and there is a full -label-support equilibrium, yet there must be a $\bar{k}$ such that for all $k \geq \bar{k}$ the payoff $v_k \geq b$, the payoff of the top-group dominates the payoff of the bottom label payoff and the bottom label cannot be played in equilibrium.

2 Group sub-structures

These arguments can be generalized for more hierarchies among groups with different sub-structures.

Definition S1. *Group sub-structures:*

(a) A pre-stable type structure has a **group sub-structure** *if the set of labels Θ can be partitioned into non-empty sets $\Theta_1, ..., \Theta_M$ with $M < |\Theta|$ such that for all $i, j \in \{1, \ldots, M\}$ holds $x_{\theta_i}(\theta_j) = x_{\theta'_i}(\theta'_j)$ for all $\theta_i, \theta'_i \in \Theta_i$ and $\theta_j, \theta'_j \in \Theta_j$.*

(b) A pre-stable structure has a **hierarchy among groups** *if Θ can be partitioned into two nonempty sets Θ_T and Θ_B such that $x_\theta(\theta') = 1$ for all $\theta \in \Theta_T$ and $\theta' \in \Theta_B$.*

(c) In a pre-stable type structure a label θ_T is called a **top label** *if $x_{\theta_T}(\theta) = 1$ for all $\theta \in \Theta \setminus \{\theta_T\}$ and a* **top label within subgroup** Θ_g *if $x_{\theta_T}(\theta) = 1$ for all $\theta \in \Theta_g \setminus \{\theta_T\}$.*

(c) *In a pre-stable type structure a label θ_B is called a* **bottom label** *if $x_{\theta_B}(\theta) = 0$ for all $\theta \in \Theta \setminus \{\theta_B\}$ and a* **bottom label within subgroup** Θ_g *if $x_{\theta_B}(\theta) = 0$ for all $\theta \in \Theta_g \setminus \{\theta_B\}$.*

Consider the type game of a pre-stable structure with a group sub-structure. A full support strategy in this type game can only be on equilibrium if within each group the payoffs are equilibrated. More precisely, for any subset of labels $\Theta_j \subset \Theta$ let for all $\theta \in \Theta_j$

$$\sigma|_{\Theta_j}(\theta) = \frac{\sigma(\theta)}{\sum_{\theta' \in \Theta_j} \sigma(\theta')}. \tag{1}$$

Definition S2. *Consider the type game of a pre-stable structure with set of labels Θ and with a group sub-structure $(\Theta_1, \ldots, \Theta_M)$.*

(a) *Let for each Θ_j the* **sub-group type game** $\mathfrak{G}_{\Theta_j}$ *denote the $|\Theta_j| \times |\Theta_j|$ game derived from the full type game by eliminating all rows and all columns for labels $\theta_k \notin \Theta_j$.*

(b) *A full support strategy σ of the type game with full set of labels Θ is called* **equilibrated within group** Θ_j *if under $\sigma|_{\Theta_j}$ every label $\theta \in \Theta_j$ obtains exactly the same expected payoff w_j in the sub-group type game $\mathfrak{G}_{\Theta_j}$.*

(c) *A full support strategy of the type game with full set of labels $\Theta = \biguplus_{j=1}^{M} \Theta_j$ is called* **within sub-group equilibrated** *if for every Θ_j with $j \in \{1, \ldots, M\}$ it is equilibrated within group Θ_j.*

For a strategy of the meta game σ that induces a pre-stable type structure that is within sub-group equilibrated we can now introduce a further reduced **inter-group type game** that has one pure strategy ϑ_j, $j \in \{1, \ldots M\}$, for every sub-group and a $M \times M$ payoff matrix with payoff w_j on the diagonal and payoffs a and b as induced by the original type game. Note further that a strategy σ of the original meta game induces a strategy $\hat{\sigma}$ in the inter-group type game via

$$\hat{\sigma}(\vartheta_j) = \sum_{\theta' \in \Theta_j} \sigma(\theta'). \tag{2}$$

Definition S3. *Consider a full label-support strategy of the meta game with set of labels Θ inducing a type game of a pre-stable structure with a group sub-structure $(\Theta_1, \ldots, \Theta_M)$, within sub-group equilibrated. The induced full support strategy σ of the type game with full set of labels Θ is called* **inter-group equilibrated** *if every induced strategy in the inter-group type game earns the same expected payoff.*

Definition S4. *A full support strategy σ of the type game induced by a pre-stable structure is called* **equilibrated** *if every pure strategy in the type game earns the same expected payoff under σ.*

Lemma S1. *Consider a pre-stable type structure with a group sub-structure $\Theta_1, ..., \Theta_M$. A full label support strategy σ of the type game with set of types $\Theta = \biguplus_{j=1}^{M} \Theta_j$ is equilibrated if and only if*

- *it is within sub-group equilibrated, and*
- *it is inter-group equilibrated.*

Note that a full support strategy σ of the type game is equilibrated if and only if it is a Nash equilibrium of the type game. Lemma S1, in conjunction with Lemma 4, gives us therefore a clear picture when when a full label support NSS or ESS exists for a pre-stable structure with sub-group structure.

Proposition S1. *Consider a conflict base-game. For the corresponding meta game with $|\Theta| \geq 2$ no ESS and no full label support NSS can exists with ...*

(a) ... a top label, or

(b) ... a bottom label, or

(c) ... a hierarchy among groups, or

(d) ... a group sub-structure in which one group with more than one label has has a top player (within that group), or

(e) ... a group sub-structure in which one group with more than one label has has a bottom player(within that group).

3 There is no ESS when $|\Theta| = 4$ in the conflict case

Suppose the base game is one of conflict with $\frac{a+b}{2} > d > b$ and $|\Theta| = 4$. We now show that this meta game has no ESS. By Lemma 2 any ESS must have full support on all four types. We then show that any candidate ESS that satisfies properties a and b of Lemma 2, necessarily has a dominated type, and thus cannot have full support, violating property c of Lemma 2.

We have to go through a series of cases. First, suppose that there is one type who plays H against all other types. Then this type dominates all other types, and we arrive at a contradiction. Second, suppose that there is one type who plays D against all other types. Then this type is dominated by all other types, a contradiction. The only case remaining is such that all types play H against at least one other type and at most two other types. This pins down a unique type structure (subject to relabeling), the unique approximate egalitarian structure (subject to relabeling) given by the following matrix.

	L_1	L_2	L_3	L_4
L_1	u^*	a	a	b
L_2	b	u^*	a	a
L_3	b	b	u^*	a
L_4	a	b	b	u^*

Given $d > b$ and thus $u^* > b$, type L_3 is dominated by type L_2.

4 The only two ESS when $|\Theta| = 5$ in the conflict case

Suppose the base game is one of conflict with $\frac{a+b}{2} > d > b$ and $|\Theta| = 5$. This game has exactly two ESS. One is the egalitarian one. The other is as follows.

	L_1	L_2	L_3	L_4	L_5
L_1	u^*	b	a	a	a
L_2	a	u^*	b	b	b
L_3	b	a	u^*	a	b
L_4	b	a	b	u^*	a
L_5	b	a	a	b	u^*

with the types L_3, L_4, and L_5 receiving equal probability weight and the other two also positive probability weight.

5 There is no ESS when $|\Theta| = 6$ in the conflict case

Suppose the base game is one of conflict with $\frac{a+b}{2} > d > b$ and $|\Theta| = 6$. We now show that this meta game has no ESS. By Lemma 2 any ESS must have full support on all six types. We then show that any candidate ESS that satisfies properties a and b of Lemma 2 either has a dominated type or has type game equilibrium that does not have full support. In either case it then follows that the type game cannot have a full support ESS. This is immediate in the dominated type case and true in the other case by the fact that a full support ESS is necessarily the unique Nash equilibrium of a game (see e.g. Weibull (1995, Proposition 2.2)).

There are a series of cases to go through. First, consider the case that there is one type who plays H against all other types. Then this type dominates all other types, and we arrive at a contradiction. Second, suppose that there is one type who plays D against all other types. Then this type is dominated by all other types, a contradiction.

Third, consider the case that one type plays H against all but one other type. Then, if we want to avoid having dominated types, the type game must have the following substructure.

	L_1	L_2	L_3	L_4	L_5	L_6
L_1	u^*	b	a	a	a	a
L_2	a	u^*	b	b	b	b
L_3	b	a	u^*			
L_4	b	a		u^*		
L_5	b	a			u^*	
L_6	b	a				u^*

Note that types the four types 3 to 6 are all treated equally by types 1 and 2. They can only differ in how they play against each other. The problem, thus, reduces to considering these four types only and by the argument above there is no type structure with four types in which there is no dominated type.

Fourth, a similar argument can be made when we consider the case that one type plays D against all but one other type. This also leads to the existence of a dominated type in much the same way as in the previous case.

The remaining cases must then all have that every type plays H against at least two and at most three opponents. Given that the total number of H plays in the matrix must be 15 we must have exactly three types who play H against two opponent types and exactly three types who play H against three opponent types. Let us call the first group the $2H$-group and the latter the $3H$-group. There are now, without loss of generality, four cases. Each group (of three types each) amongst themselves can only be either egalitarian or hierarchical. Each case leads to a different type structure, all are approximate egalitarian.

We omit reproducing the four possible type games here. We only describe the results. If both groups are hierarchical amongst themselves, this leads to the approximate egalitarian type structure in which types are allocated on a circle in a specific way (as used in the proof of Proposition 3). This type game has a dominated type. If the $3H$-group is hierarchical and the $2H$-group is egalitarian then any induced type structure has the non-egalitarian Nash equilibrium with five types in its support as described above. If the $3H$-group is egalitarian and the $2H$-group is hierarchical then any induced type structure has the egalitarian Nash equilibrium with five types in its support. If both groups are egalitarian then any induced type structure has the egalitarian Nash equilibrium with three types in its support.

A Proof of Lemma S1

Proof of Lemma S1: Note that the expected payoff of any label θ_i, $i \in \{1, \ldots, |\Theta|\}$, in some group of labels Θ_j, $j \in \{1, \ldots, M\}$, can be decomposed in the probability of playing against a label in its own group Θ_j times the conditional expected payoff w_j in that case, and the probability of playing against any label not in the group and the conditional expectation in that case.

Consider a full support strategy σ of the type game induced by a pre-stable structure.

Proof of "only if" statement: Suppose there is a group Θ_j which is not within equilibrated. Then there are at least two labels which earn a different expected payoff conditional on playing in that group. But since all labels outside the group play identically against both labels, this implies that they also earn different expected payoffs overall and thus the full support strategy of the full type game cannot be equilibrated. Thus being within sub-group equilibrated is a necessary condition for σ to be equilibrated. Next we show that σ can only be equilibrated if it is inter group equilibrated. Suppose not. Then pick two labels from different groups Θ_i and Θ_j. Then both labels earn different payoffs, contradicting that σ is equilibrated.

Proof of the "if" statement: Suppose the full label support strategy σ is within sub-group equilibrated and inter-group equilibrated. Then, because of within-subgroup equilibration every label $\theta_i in \Theta_j$ earns the same expected payoff as Θ_j in the inter group type game. Furthermore all Θ_j, $j \in \{1, \ldots, M\}$ earn the same expected payoff (since σ is inter-group equilibrated), all labels earn the same expected payoff and σ is equilibrated, q.e.d.

A.1 Proof of Proposition S1

(a) Since $u^* \geq b$ for conflict games the top label (who plays hawk and earns the largest possible payoff a against all other labels) is a (at least weakly) dominant strategy in the type game and would under full label support earn strictly more than any other strategy of the type game.

(b) Since $u^* \geq b$ for conflict games a bottom label (who plays dove against all other labels) is weakly dominated by all other strategies and it thus cannot be part of any full support equilibrium of the type game.

(c) The same argument as (a) now applies to the top group in the inter-group type game.

(d) The same argument as in (a) now applies to the sub-group type game $\mathfrak{G}_{\Theta_j}$ of such a sub-group Θ_j with a top label.

(e) The same argument as in (b) now applies to the sub-group type game $\mathfrak{G}_{\Theta_j}$ of such a sub-group Θ_j with a bottom label.

References

WEIBULL, J. W. (1995): *Evolutionary Game Theory*. MIT Press, Cambridge, Mass.