Note: A necessary condition for symmetric completely mixed Nash-equilibria*

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Abstract

We establish a necessary condition for an arbitrary finite symmetric two player game to have a symmetric completely mixed Nash equilibrium.

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1 Definitions and Results

Consider an arbitrary finite symmetric two player game with $n \times n$ payoff matrix A. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_+$ with $\sum_{i=1}^n x_i = 1$ denote a mixed strategy. In a symmetric completely mixed Nash equilibrium of such a game all pure strategies must earn the same expected payoff. More precisely, a mixed

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strategy profile $x = (x_1, \dots, x_n)^T \in \mathrm{I\!R}^n_{\geq 0}$ with is a symmetric completely mixed Nash equilibrium if and only if $x_i > 0$ for $1 \leq i \leq n$, $\sum_{i=1}^n x_i = 1$, and

$$Ax = (c, c, \dots, c)^T, \tag{1}$$

for some constant $c \in \mathrm{I\!R}$.

In order to rephrase this we need a few definitions. For any $m \times n$ matrix A, let $D = D(A)$ denote A-induced payoff difference matrix given by the $(m-1) \times n$ matrix obtained from A as follows.[1] The k-th row of D is the difference between rows k and $k+1$, for $k = 1, 2, ..., n-1$. Further, denote by $\bar{D} = \bar{D}(A)$ the $m \times n$ matrix coincides with D for the first $n-1$ rows and has the unit vector (vector of all ones) in row m and define $b = (0, 0, ..., 0, 1)^T \in \mathrm{I\!R}^m$. Further, we write $x \geq 0$ iff $x_i \geq 0 \; \forall \; 0 \leq i \leq n$ and $\exists$ i such that $x_i > 0$. We write $x > 0$ iff $x_i > 0 \; \forall \; 0 \leq i \leq n$.

A vector $x \in \mathrm{I\!R}^n$ represents a completely mixed Nash equilibrium of the finite symmetric two player game with $n \times n$ payoff matrix A game if $x > 0$ (that is each coordinate satisfies $x_i > 0$) and

$$\bar{D}x = b. \tag{2}$$

We can break down this characterization into two conditions:

Lemma 1. *A vector $x \in \mathrm{I\!R}^n$ represents a completely mixed Nash equilibrium of the finite symmetric two player game with $n \times n$ payoff matrix A if and only if the following two conditions hold:*

(I) **Equal Payoff Condition**
$\bar{D}x = b$ *and* $x \geq 0$.

(II): **Full Support Condition**
$x_i > 0$ *for* $1 \leq i \leq n$.

The proof is obvious. The next lemma characterizes when there exists an solution x satisfying the Equal Payoff Condition (I) - generalized to an arbitrary $m \times n$ matrix A:

[1]For the payoff matrix of a symmetric game $m = n$.

Lemma 2.

(3) $\exists\, x \in \mathbb{R}^n$ *such that* $\bar{D}x = b, x \geq 0 \Leftrightarrow \nexists\, w \in \mathbb{R}^{m-1}$ *such that* $w^T D < 0.$

Lemma 2 can equivalently be stated in the following form:[2]

Lemma 3.

(4) $\exists\, x \in \mathbb{R}^n$ *such that* $\bar{D}x = b, x \geq 0 \Leftrightarrow \nexists\, w \in \mathbb{R}^{m-1}$ *such that* $w^T D > 0.$

Proof of Lemma 2:
Farkas' lemma states that either $\bar{D}x = b, x \geq 0$ has a solution $x \in \mathbb{R}^n$ or $v^T\bar{D} \leq 0, v^T b > 0$ has a solution $v = (v_1, v_2, ..., v_m)^T \in \mathbb{R}^m$, but not both.[3]

"Only if": By Farkas' lemma the existence of a solution to the Equal Payoff Condition (I) implies that there is no $v = (v_1, v_2, ..., v_m)^T \in \mathbb{R}^n$ with $v^T b > 0$ such that $v^T\bar{D} \leq 0$. Note that the condition $v^T b > 0$ is satisfied if and only if $v_m > 0$. Let $w = w(v)$ be the vector in $\mathbb{R}^{(m-1)}$ that consists of the first $m-1$ coordinates of v. Note that the condition $v^T\bar{D} \leq 0$ is satisfied if and only if $w^T D \leq -v_m$.

Thus, there exists a solution to the Equal Payoff Condition (I) only if there is no $(w_1, \ldots, w_{m-1}, v_m)$ such that $w^T D \leq -v_m$ with $v_m > 0$.
This implies that there is no $w \in \mathbb{R}^{m-1}$ with $w^T D < 0$.

"If": Suppose $\nexists w \in \mathbb{R}^{m-1}$ such that $w^T D < 0$.
Then for all $v_m > 0$ there is no $w \in \mathbb{R}^{m-1}$ such that $w^T D \leq -v_m$.
This implies that $\nexists v = (v_1, \ldots, v_{m-1}, v_m) \in \mathbb{R}^m$ such that $v^T\bar{D} \leq 0, v^T b > 0$.
This implies by Farkas' lemma the Equal Payoff Condition (I). QED

A few more definitions are necessary. For an $m \times n$ matrix A let $\mathrm{col}(A)$ denote the set of column vectors of A. For any vector $a \in \mathbb{R}^m$ let $\mathrm{HS}(a)$ denote the *half space* induced by a, given by the set of all vectors $v \in \mathbb{R}^m$ such that $v^T a \leq 0$. For an $m \times n$ matrix A let $\mathrm{HS}(A) = \bigcup_{a \in \mathrm{col}(A)} \mathrm{HS}(a)$ denote the union of all half spaces of columns of A. Note that for every

[2] Just set $w' = -w$.

[3] See Farkas (1901) or read e.g. Vohra (2005) for a modern textbook treatment of Farkas' lemma.

element $v \in \text{HS}(A)$ there is a $a \in \text{col}(A)$ such that $v^T a \leq 0$. In fact:

$$v \in HS(A) \Leftrightarrow \exists_{a \in col(A)} \text{ such that } v^T a \leq 0. \tag{5}$$

Now, we provide a necessary condition for the Equal Payoff Condition (I).

Lemma 4. *Consider a finite symmetric two player normal form game with $n \times n$ payoff matrix A. The Equal Payoff Condition (I) has a solution $x \in \mathbb{R}^n$ only if $HS(D(A)) = \mathbb{R}^{n-1}$, i.e. the union of half-spaces induced by the set of columns of the payoff difference matrix D covers the whole set $\mathbb{R}^{n-1}$.*

Proof: The existence of a solution x to the Equal Payoff Condition (I) implies, by Lemma 3 that for every $w \in \mathbb{R}^{n-1}$ there must also exist a vector $d' \in \text{col}(D)$ such that $w^T d' \leq 0$. Thus, $w \in \text{HS}(D)$. This implies $\text{HS}(D) = \mathbb{R}^{(n-1)}$. QED

Our main results follow now immediately from Lemmata 1 and 4:

Proposition 1. *Consider a finite symmetric two player normal form game with $n \times n$ payoff matrix A. If this game has a symmetric completely mixed Nash equilibrium then $HS(D(A)) = \mathbb{R}^{n-1}$, i.e. the union of half-spaces induced by the set of columns of the payoff difference matrix covers the whole set $\mathbb{R}^{n-1}$.*

Proposition 1 and Lemma 2 turned out to be helpful to prove some results in a recent paper of ours (Herold and Kuzmics (2017)) and we hope they turn out to be useful for other game theorists as well.

References

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