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Comment

A comment on "Egalitarianism and efficiency in repeated symmetric games" by V. Bhaskar [Games Econ. Behav. 32 (2000) 247–262] ☆

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ABSTRACT

We identify an error in Bhaskar's (2000) Proposition 4. We provide counterexamples to this result and demonstrate that it is not correctable.

1. The setting

Consider a finitely repeated game with discounting and perfect monitoring, in which the stage game is a symmetric 2×2 coordination game with a conflict of interest, i.e. a version of the well-known Battle-of-the-Sexes. For the purpose of this comment, it suffices to focus attention on the stage game with payoffs described by

	α	β
α	$0, 0$	$0, 1$
β	$1, 0$	$-1, -1$

In a thought-provoking paper, Bhaskar (2000) studies the properties of symmetric equilibria of such a game. The main finding (Proposition 1 for the twice repeated version; Proposition 2 for any number of repetitions) is that in any optimal symmetric equilibrium it must be that payoffs are as egalitarian as possible. To understand this, note that in any symmetric strategy profile, in the first stage, both players must use the same distribution over actions α and β. If both players' realized actions are the same, then the symmetry of their repeated game strategies implies that in the next stage they again both use the same distribution over actions,[1] causing delay in successful coordination. If, however, the realized actions are different (Bhaskar, 2000 calls this event the "breaking" of symmetries), future play is essentially unrestricted by the symmetry requirement. That is, any continuation of the subgame is feasible. Bhaskar's (2000) Propositions 1 and 2 state that any optimal symmetric subgame perfect equilibrium must have the feature that continuation payoffs after the event

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[1] Note that this distribution, however, may differ from the one used by both in the first stage.

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that symmetries are broken are as close as possible to being equal across players. Suppose the continuation payoffs are exactly equal. Then both players have an incentive to randomize optimally in the first stage, i.e. attach probability $\frac{1}{2}$ to both actions. This results in the fastest possible breaking of symmetries. Suppose, on the other hand, that the continuation payoffs are asymmetric. For instance, the person whose realized action is α continues to play α, while her opponent, with realized action β continues to play β for the remainder of the game. This is part of a subgame perfect equilibrium, but it is not the optimal one. In this case, both players prefer to be the one whose realized action is β when symmetries are broken. Thus, in equilibrium, both players attach too much probability (more than the optimal $\frac{1}{2}$) on β in rounds before symmetries are broken, resulting in costly delay of the breaking of symmetries.

This result of Bhaskar (2000) (Propositions 1 and 2) is perfectly sound. Bhaskar (2000) then turns to the question of what might constitute an optimal continuation play. That is, how should players play after symmetries are broken in order to achieve the highest degree of egalitarianism in the finitely (or infinitely) repeated game with discounting[2]?

2. The error

We first introduce a bit of notation. With *continuation play*, in this note, we always refer to the part of a strategy profile that describes play (on the equilibrium path) after a given history of some finite length t with the property that symmetries are broken at this time t. We shall focus on *conditionally efficient* continuation play. A continuation play is conditionally efficient if, in any stage after symmetries are broken, with probability one play is either (α,β) or (β,α). It is easy to see that a strategy profile needs to prescribe conditionally efficient continuation play after every history in which symmetries are broken in order to be efficient.[3] However, as Bhaskar (2000) notes (in his Propositions 1 and 2) and as discussed above, not every equilibrium with conditionally efficient continuation play is also efficient.

Now fix a particular finite history at the end of which symmetries are first broken. Let T denote the number of periods remaining in the repeated game at the stage when symmetries are broken. We renormalize time such that the stage at which symmetries are broken is time 1. A conditionally efficient continuation play can then be fully described by a sequence $z = \langle z_s\rangle_{s=1}^{T}$, where $z_s = 1$ if the continuation play at stage s is (β,α) and $z_s = -1$ if it is (α,β). Without loss of generality we take $z_1 = 1$.

We now define the difference in payoffs between the two players as

$$\Delta^\delta(z,T) = \sum_{s=1}^{T} \delta^{s-1} z_s.$$

The appropriate measure[4] of asymmetry in payoffs (i.e. of anti-egalitarianism) associated with the continuation play described by z is defined by $|\Delta^\delta(z,T)|$. For all continuations of length $T \in \{1,2,\ldots,\infty\}$ and all discount rates $\delta \in (0,1]$ (except the combination $T=\infty$ and $\delta=1$), define $Z^*(\delta,T)$ to be the set of sequences that minimizes $|\Delta^\delta(z,T)|$. This is the set of Bhaskar's (2000) *egalitarian conventions*.

Bhaskar (2000) makes use of the following construction. Define an infinite sequence $\hat{z} = \langle \hat{z}_s\rangle_{s=1}^{\infty}$ by the following recursive formula. Set $\hat{z}_1 = 1$. Set $\hat{z}_s = 1$ if $\Delta^\delta(\hat{z}, s-1) \leqslant 0$, and $\hat{z}_s = -1$ otherwise. Obviously, $\hat{z}$ depends on δ. This construction can be thought of as the continuation play that, at every period, awards the high payoff to the player whose current cumulative discounted payoff in the repeated game is lower. Denote by $\hat{z}^T$ the truncation of $\hat{z}$ to length T.

Proposition 4 of Bhaskar (2000) makes, in part, the following claim: For any $T < \infty$ and for any $\delta < 1$, $\hat{z}^T \in Z^*(\delta,T)$.[5] That is, the convention of awarding the high payoff to the currently disadvantaged player results in a maximally egalitarian outcome. This statement is not correct. In fact it is not easy to state a less general correction as the following Proposition proves.

Proposition 1. *Let T be either odd and at least 5, or two times such a number. Then there is a $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ the sequence $\hat{z}^T$ is not an egalitarian convention, i.e. $\hat{z}^T \notin Z^*(\delta,T)$.*

Proof. First, consider the case $T = 5$ and $\delta > 0.62$. Then $\hat{z}^5 = (1,-1,-1,1,-1)$. This can readily be verified to be less egalitarian than the sequence $\tilde{z}^5 = (1,1,-1,-1,-1)$. That is $|\Delta^\delta(\tilde{z}^5,5)| < |\Delta^\delta(\hat{z}^5,5)|$ for all $\delta > 0.62$. The reason is simple: one player must be awarded the prize fewer times than the other; when δ is sufficiently large, to best equalize payoffs this player should be compensated by having his wins as early as possible. The same logic applies, for δ close enough to 1, to all odd $T \geqslant 5$. The sequence $\hat{z}^T$ must coincide with $\hat{z}^5$ for the first 5 periods. Yet, it is most egalitarian (provided

[2] In the case of no discounting, this is a less interesting question.

[3] Note that different histories can, of course, lead to different conditionally efficient continuation play. Suppose symmetries are broken in stage 2. There are two histories with that property. One in which in the first stage both players played α, and one in which both played β. The continuation could, in principle, depend on which of the two histories realized. For example, it could prescribe always (α,β) after the first and (β,α) after the second history.

[4] This is the quantity that governs mixing incentives in the stages prior to symmetries being broken.

[5] The remainder of Bhaskar's (2000) Proposition 4 is incorrect if the first claim is incorrect, as it states that, generically, $Z^*(\delta,T) = \{\hat{z}^T\}$, i.e. that $\hat{z}^T$ is the *unique* egalitarian convention.

δ is large enough) that the player who gets the fewer prizes should receive them in the first $\frac{T-1}{2}$ periods. This logic is not restricted to odd T. Let $T = 10$ and $\delta > 0.79$. Then $\hat{z}^{10}$ is now extended to produce $(1, -1, -1, 1, -1, 1, 1, -1, -1, 1)$. Imagine pairing adjacent periods together, so that $\hat{z}^{10} = ((1, -1), (-1, 1), (-1, 1), (1, -1), (-1, 1))$. Each of these two-period pairs can be thought of as awarding a single prize (of value $1 - \delta$) to the first (second) player if the first (second) element in the pair is positive. The effective discount rate across these two-period prizes is then δ^2. We can then view this example as exactly equivalent to the previous five-period example with a discount rate of δ^2. This argument demonstrates that a more egalitarian[6] sequence is $((1, -1), (1, -1), (-1, 1), (-1, 1), (-1, 1))$. This argument readily generalizes to all T that are two times an odd number that is at least 5. □

There are more instances of pairs of T and δ, not covered by Proposition 1, in which it is also true that $\hat{z}^T$ is not an egalitarian convention, i.e. not in $Z^*(\delta, T)$. We shall not discuss this here. Proposition 1 suffices to make the argument that not only is Bhaskar's (2000) Proposition 4 incorrect, it is also not really possible to prove a similar statement under more stringent conditions.

The error in logic from the proof in Bhaskar (2000) is that the optimization of the egalitarian objective is done, at any period, conditional on the partial sequence already generated before that period. In other words, it is indeed true that, conditional on the sequence $\hat{z}^{T-1}$, the play in the subsequent period that results in the most egalitarian payoffs after one more period of play is described by $\hat{z}_T$. But the optimization problem of finding a sequence in $Z^*(\delta, T)$ is a global one, and may well involve a sequence that bears no resemblance to $\hat{z}^T$.[7]

References

Bhaskar, V., 2000. Egalitarianism and efficiency in repeated symmetric games. Games Econ. Behav. 32, 247–262.

[6] However, not necessarily most egalitarian!

[7] It is true, however, that for almost all values of $\delta > \frac{1}{2}$, there is a unique most egalitarian convention. We are grateful to an anonymous referee for pointing this out.