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Symmetric play in repeated allocation games ☆

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Abstract

We study symmetric play in a class of repeated games when players are patient. We show that, while the use of symmetric strategy profiles essentially does not restrict the set of *feasible* payoffs, the set of *equilibrium* payoffs is an interesting proper subset of the feasible and individually rational set. We also provide a theory of how rational individuals play these games, identifying particular strategies as focal through the considerations of Pareto optimality and simplicity. We report experiments that support many aspects of this theory.

JEL classification: C72; C73; C92; D01

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1. Introduction

We are interested in the economic implications of how people behave in symmetric situations with repeated interaction. We explore this issue by studying what we call allocation games. An allocation game involves two issues: a coordination issue and a competition issue. Much of our analysis concerns the two player case, which is essentially the battle-of-the-sexes, and is described as follows. There are two prizes (x_1, x_2) with $x_1 > x_2$. Each player simultaneously demands an element of x. If the demands are distinct, payoffs are awarded according to the chosen demands; otherwise payoffs are zero to both.

The efficient Nash equilibria of an allocation game deliver asymmetric payoffs, thus providing motivation for players to coordinate on asymmetric play.[1] We study the process by which ex-ante symmetric players eventually come to occupy different roles and receive different payoffs. But we require the process to operate within the confines of a symmetric equilibrium of the repeated game. One reason to focus on symmetric play is that players may have no commonly understood labels or characteristics that could serve, via a norm or convention, to coordinate their actions on an asymmetric outcome. More generally, our goal is to respect the symmetric structure of the underlying game, and understand how strategic behavior dictates the development of asymmetric outcomes along the path of play.

The first to study symmetric equilibria of the repeated battle-of-the-sexes game was Bhaskar [7].[2] A key event in both Bhaskar's [7] and our analyses is the breaking of symmetries. As players are ex-ante symmetric, and do not communicate directly, the only means they have to distinguish their roles is the realized history of the game itself. As long as players play the same action in the stage game, they remain in symmetric positions. Once one player plays one action and the other a different action, symmetries are said to be broken.

Bhaskar [7] showed that, for finite repetitions of the game, as well as for the infinitely repeated game with discounting, among all symmetric equilibria of the game, those and only those that promise equal continuation payoffs at the history at which symmetries are broken are (Pareto-) efficient. All other equilibria yield lower ex-ante expected payoffs. The intuition for this result is simple. A promise of equal payoffs allows players to randomize over the two actions with equal probability prior to the breaking of symmetries. Such uniform randomization then leads to the fastest possible breaking of symmetries. Notice, thus, that in an efficient symmetric equilibrium of this repeated game, it may well be that an asymmetric outcome is realized in a particular stage. At the first such realization, it must be that the asymmetry in stage game payoffs is exactly compensated by payoffs in later stages, at least in expectation.

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1 We do observe such asymmetric outcomes. For instance, the norm in some societies that men hold the door open to allow women to pass through first can be seen as an instance of an efficient Nash equilibrium with asymmetric payoffs in the battle-of-the-sexes game.

2 There are few papers on similar topics before Bhaskar [7]. Crawford and Haller [19] study repeated pure coordination games, in which not only players but also strategies are symmetric. Crawford and Haller [19] show that, even if players cannot a priori condition their choice of actions on their names, the repeated game has symmetric subgame perfect equilibria where players expect to coordinate after very few rounds. Farrell [20] studies a situation in which one instance of the battle-of-the-sexes game is preceded by possibly infinite rounds of cheap-talk communication. Farrell [20] shows that, under symmetric strategic behavior, even infinite rounds of cheap talk do not allow players to always coordinate on an efficient equilibrium in the battle-of-the-sexes game.

1.1. Four questions we address

Bhaskar's [7] result raises several questions that we address in this paper. First, while it is true that for any discount factor the only efficient equilibria are those that promise equal continuation payoffs when symmetries are broken, we ask how inefficient are other equilibria with asymmetric continuation payoffs, especially as δ tends to 1. Asymmetric continuation payoffs lead to delay in the breaking of symmetries. But, as players are very patient, such delays should perhaps not decrease payoffs too much. Are there equilibria that after the breaking of symmetries lead to very asymmetric continuations that are, nevertheless, close to efficient for patient players? That is, is Bhaskar's [7] result of the inefficiency of non-egalitarian continuation payoffs economically significant?

Second, as our answer to the first question is affirmative, suppose players did indeed want to play an efficient equilibrium. How would they do it, in light of the fact that there are infinitely many efficient symmetric equilibria in the repeated game? If we want to develop an informative positive theory of play, it is not enough to say that players will somehow manage to play one of these equilibria. We want to understand how players might select among these equilibria and what this implies for the actual play. In other words, is there a focal strategy, in the sense of Schelling [35] as a "strategy that is suggested by the structure of the game itself"?

Third, as we provide an answer to the second question and identify a focal strategy, we ask whether the theory we propose is descriptive of actual play. We report results from a laboratory experiment designed to assess whether observed behavior is similar to the focal strategy that we derive from first principles.

Fourth, and finally, we ask to what extent the arguments from our analysis have force in more general games. We extend some of our results in two directions: (i) to n-player allocation games, and (ii) to other 2-player games, including those of the prisoners' dilemma variety.

1.2. Our approach and results

An n player allocation game is characterized by a set of n prizes with distinct non-negative payoffs. Each player simultaneously demands one of these prizes. If the demands are all distinct, then prizes are distributed according to the demands. Upon any other realization, in which two or more players demand the same prize, payoffs are zero to all.[3]

We consider all symmetric strategy profiles of all repeated allocation games with discounting and the following monitoring structure.[4] After each stage of play, players observe only whether

[3] There are numerous applications for which allocation games are stylized models. Many of these have been discussed in the literature for the 2-person case, i.e., the Battle of the Sexes. For larger n, the allocation game captures important elements of role assignments in a team or organization, where efficient role assignments do not depend on idiosyncratic characteristics of the members of the team. The team needs exactly one member to occupy each role in order to function. Applications come from a broad range of organizations, including committee assignments in legislatures, allocation of routine chores in a marriage or household, assignment of territories in a sales force, appointment of department chairs, and duties on an assembly line. Other examples of joint production abound. The vector x captures the values or costs to the players assigned to different roles, with the simplifying assumption that joint production breaks down if any of the assignments are unfilled.

[4] The term "symmetric equilibrium" usually implies that the game under consideration is one that is completely symmetric. That is, all players have the same set of strategies and the same payoff function. This is probably the reason that lead Crawford and Haller [19] to refer to a somewhat generalized concept by the different name "attainable equilibrium". Indeed, there are games that are not completely symmetric, and yet exhibit symmetries. Restrictions of symmetries in the

or not the task has been executed, i.e., whether symmetries have been broken. For two player allocation games, i.e., for battle-of-the-sexes games, this is equivalent to perfect monitoring. For general n player games this monitoring structure simplifies the analysis, without, we conjecture, fundamentally influencing the results.

In order to discuss our first question, regarding inefficiency of asymmetric continuation payoffs, we introduce one subtle piece of terminology, conditional payoff profiles. At the history at which symmetries are broken, players can be uniquely identified by the action they played at this stage. We are then interested in the possible payoffs to these players, so identified. For each symmetric strategy profile we can compute the ex-ante expected payoff of a player *conditional* on this player playing a given action at the breaking of symmetries. We are then interested in understanding how different these conditional expected payoffs can be for the n positions each of the players might find themselves in.

We establish three facts. First, the restriction to symmetric strategy profiles, in itself, does not essentially restrict the space of feasible conditional payoff profiles. In particular, as the discount factor tends to one, the space of feasible conditional payoff profiles converges to the convex hull of all stage game payoff profiles (Proposition 1).

Second, we study what we call symmetric stationary semi-public equilibria of the repeated game. One way to express a stationary semi-public strategy is that a player forms an incomplete history: it is empty prior to the breaking of symmetries (and, thus, the player must play the same strategy after every history prior to the breaking of symmetries) and the history starts only when symmetries are broken. After that point no restrictions are imposed. We show (Proposition 2) that for every asymmetric point on the Pareto-frontier of the set of feasible conditional payoff profiles, there is a neighborhood of this point with the property that no symmetric stationary semi-public equilibrium can generate a payoff profile within this neighborhood, even as the discount factor tends to one. In other words, even if players are arbitrarily patient, any symmetric stationary semi-public equilibrium with an asymmetric conditional payoff profile has an ex-ante expected payoff (the same to all players) that is bounded away from the most efficient payoff.

Third, for two-player allocation games, we are able to dispense with the stationary semi-public qualification and are, moreover, able to show a stronger result. For a particular two-player allocation game we provide an upper bound (Proposition 3) and a lower bound (Proposition 4) for the set of conditional payoff profiles for *all* symmetric equilibria of the repeated game. The lower bound demonstrates the existence of asymmetric, but positive, payoffs sustained in a symmetric equilibrium. The upper bound shows that a player cannot expect, conditional on taking either role, more than three quarters of the highest available payoff, thereby demonstrating the *substantial* inefficiency of very asymmetric payoff-promises.

While this result is for one particular game, it would not be difficult, but tedious, to extend it to all two-player allocation games. For the paper, we, therefore, adopted a short-cut. In Appendix E we show how this result can be directly applied to a wide class of two-player allocation games. The appendix also shows that this result has similar implications for many games that are not allocation games, including, e.g., games of the prisoners' dilemma variety.

game, following Crawford and Haller [19], have been further studied by Alos-Ferrer and Kuzmics [3]. See also Harsanyi and Selten [23] and Casajus [14,15] for related notions. In this form, even if perhaps not with this name, attainable equilibria have been studied in various contexts by e.g. Farrell [20] (see also Farrell and Rabin [21]) in symmetric repeated cheap talk games, Crawford and Haller [19] in repeated pure matching (coordination) games, Bhaskar [7] in repeated battle-of-the-sexes games, Blume [8] in a study of language, and Blume and Gneezy [10,11], and Blume, Duffy, and Franco [9] in lab experiments with symmetric (non-repeated) games.

This answers the first question: asymmetric conditional payoff profiles are meaningfully inefficient.

Turning to the issue of how players will play an allocation game, we find the Pareto criterion a compelling element of a theory of focal points.[5] Given a choice between two (equilibrium) outcomes, Pareto optimality is precisely the condition that aligns the incentives of all agents. We find it all the more compelling in a fully symmetric environment. In Corollary 1 to Proposition 2, we show that there is a *unique most efficient payoff profile* sustainable by a symmetric equilibrium, even when the discount factor tends to one. That is, Pareto efficiency selects a unique payoff profile, even with perfectly patient players.[6]

However, while Pareto optimality pins down a particular payoff profile, there remains an infinite number of symmetric equilibria with efficient payoffs. We thus turn to the question as to which of the efficient symmetric equilibria is focal. We look for a symmetric equilibrium that is not only ex-post payoff-symmetric in the limit, but also particularly ex-post payoff-symmetric among all such equilibria for all high discount factors. Perhaps surprisingly, there does exist a unique strategy profile with this feature. In Proposition 6 we show that it is such that continuation play at every history at which symmetry is broken is given by the Thue–Morse sequence.[7] Proposition 5 characterizes the Thue–Morse sequence in economic terms, as the limit sequence that is obtained for patient players, when play is such that at every stage, the player with the currently lowest accumulated discounted payoff receives the highest stage game payoff.

As it happens, though, that strategy might be considered rather complex. For example, the strategy profile that gives rise to the Thue–Morse sequence cannot be represented by a finite automaton. In order for a particular strategy profile to be focal, it must be sufficiently plausible that, not only can agents identify it, but (at a minimum) they must have faith that their opponents can do so as well.[8] We seek a descriptive theory and, as such, simplicity forms the final element of our approach. If, in the minds of players, simplicity of a strategy is appropriately captured by the state-complexity of the smallest automaton that represents this strategy, one must abandon the Thue–Morse continuation play in favor of a simpler strategy. As the discount factor tends to one, there is a simplest strategy in the automaton sense that induces efficient payoffs in equilibrium. This strategy adopts a rotation scheme once symmetry is broken.[9,10] These results are formally stated as Propositions 7 and 8.

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5 There may generally be competing concerns, such as risk-dominance, but this has no bite in our setting.

6 This payoff profile is ex-post payoff-symmetric. Bhaskar [7] establishes a similar result for 2-player repeated allocation games that are either finitely repeated or have a fixed discount factor strictly less than one.

7 See Thue [38,39], and Morse [31]. Allouche and Shallit [2] characterize some interesting mathematical properties of this sequence.

8 See Blume and Gneezy [11] for a nice experiment highlighting this point.

9 The notion of rotation schemes in repeated interactions dates back to Luce and Raiffa [29]. Prisbey [33] first reported the observation of rotation strategies in laboratory experiments, in the context of 2-player games. Blume and Heidhues [12] study rotation schemes arising from a symmetric equilibrium of a repeated auction environment. Lau and Mui [27,28] study a particular equilibrium in 2-player repeated games involving “turn taking” as a way to implement nearly symmetric payoffs, which is a specific instance of our rotation schemes. Cason, Lau, and Mui [16] observe rotation schemes in 2-player assignment games, and demonstrate that subjects are able to teach such behavior to future opponents.

10 A familiar example for rotation schemes is the assignment of positions in a pickup soccer or hockey game. Players must be assigned roles. Nearly everyone would rather be a forward than the other roles and nobody wants to be the goalkeeper. A common convention is to rotate players through the positions so that nobody is required to be the goalkeeper all the time and everyone spends some time at the forward position. One can also think about the assignment of unpleasant chores on a camping trip among scouts, etc.

This answers our second question. We analyze a repeated game with an infinitely many equilibria, and yet are able to deduce a unique focal-point from simple normative principles.

The theory suggests that symmetric payoffs can be justified purely through an efficiency criterion, without any inherent preference for "fairness".[11] Delivering symmetric payoffs, though, requires a greater level of complexity than delivering asymmetric payoffs, in that players have to manage to equally share the payoffs following the breaking of symmetry, by use of a scheme that delivers different stage game payoffs over time.

To answer our third question we ask: do players manage to do this? Our experiments, reported in Section 7, demonstrate that the answer is, overwhelmingly, "yes". More specifically, nearly symmetric payoffs are almost always delivered, and they are delivered via rotation schemes. This provides a positive answer to our fourth question: even in the much more difficult 3-person allocation games, where multiple efficient rotation schemes are possible, the simple rotation schemes emerge. We interpret this as strong evidence that Pareto-efficiency and, secondarily, simplicity, are elements of focal strategies observed in our experiment. Further, this result has implications for how players should mix before symmetry is broken. In particular, anticipating ex post symmetric payoffs, players should mix close to uniformly, thereby breaking symmetry as fast as possible. This implication about mixed strategies prior to coordination is also supported by the data.

2. Model

2.1. The stage game

While many of our results can be generalized to other settings, we focus attention on a particular class of games.

Definition 1. A symmetric normal form stage game $\Gamma = (I, A, \hat{u})$ is an *n-player allocation game* if $A = \{1, ..., n\}$, i.e., $|A| = |I| = n$, $\hat{u}(a) = (0, ..., 0)$ for all action profiles $a \in A^n$ with the property that there are two players $i \neq j$ such that $a_i = a_j$, and there exists an $x \in \mathbb{R}^n$ such that (i) $x_1 > \ldots > x_n \geq 0$, and (ii) $\hat{u}_i(a) = x_{a_i}$ for all $i \in I$ if a is a permutation of $(1, 2, ..., n)$.

That is, every one of n players requests any one of n amounts of money. If demands are consistent, such that every player requests a different amount, then players are paid their respective demands. Otherwise players are paid zero. Note that all players in an allocation game are symmetric, while there are no symmetric strategies.[12] Note that for $n = 2$ this reduces to the battle-of-the-sexes.

[11] It has been suggested to us that an alternative explanation of observing rotation schemes in repeated allocation games is that players have an intrinsic preference for fairness. This is, however, not true. Even with preferences for fairness, the repeated game, especially when players are very patient, still typically has an infinite number of symmetric equilibria with symmetric ex ante payoffs, but where players will receive asymmetric continuation payoffs upon the breaking of symmetries. In order to identify a unique focal point one would still have to appeal to the same three considerations as we do in this paper: symmetric equilibrium, Pareto-efficiency, and simplicity.

[12] In these games we restrict attention to payoff vectors that have distinct elements. This is not a crucial assumption for our results. However, it implies that, while all players are symmetric, there are no symmetric stage game actions. This avoids expository complications. See Crawford and Haller [19], where symmetry among actions is an essential element of the model and analysis.

2.2. The repeated game

The game is played repeatedly at discrete points in time $t = 0, 1, 2, \ldots$. Players discount future payoffs with a common discount factor $\delta < 1$. In each period players observe an element of $Y = \{c, n\}$ (as well as their own realized action). If $a^t \in A^n$ is played at stage t then players observe c if a^t is a permutation of $(1, 2, \ldots, n)$. Otherwise they observe n. Thus c is the "event" that players achieved coordination (a non-zero payoff vector) and n is the event that they did not achieve coordination and thus obtained 0 payoffs all. The payoff matrix is assumed to be common knowledge. Players, thus, know what payoff they received at each stage. Note that for 2-player allocation games, this information structure is equivalent to perfect monitoring. Otherwise, monitoring is less than perfect.

We can thus describe public and private histories for the repeated game. The *set of public histories* is given by $\mathcal{H} = \bigcup_{t=0}^{\infty} Y^t$ with $Y^0 = \emptyset$. Player i's *set of private histories* is given by $\mathcal{H}_i = \bigcup_{t=0}^{\infty} (A \times Y)^t$. Given the symmetry we have $\mathcal{H}_i = \mathcal{H}_j = \mathcal{H}^*$ for all $i, j \in I$. A pure strategy is a mapping $\sigma_i : \mathcal{H}^* \to A$. A behavioral strategy is a mapping $\sigma_i : \mathcal{H}^* \to \Delta(A)$.[13] For convenience we shall also define the set of all play paths by $\mathcal{P} = (A^n)^{\infty}$.

For a pure strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ payoffs are given by $u_i(\sigma) = (1 - \delta) \times \sum_{t=0}^{\infty} \delta^t \hat{u}_i(a^t(\sigma))$, where $a^t(\sigma)$ is the action profile induced by strategy profile σ in period t. For mixed strategy profiles we extend u_i by taking expectations.

2.3. The solution concept

We study symmetric Nash equilibria. Formally, given the notation above, a behavioral strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is symmetric if, simply, $\sigma_i = \sigma_j$ for all $i, j \in I$. Note that this does not imply that two players necessarily behave in the same way after a particular given public history, even if symmetries have not yet been broken, as they may well have different private histories. If two players, up to a particular history, ended up playing the same action in every stage, say both played action a in stage 1, both action b in stage 2, and so on, then they must indeed play the same mixed action in the stage following that history. If, on the other hand, they have not played in identical fashion up to a given history, then their play after that can differ arbitrarily.

2.4. Notions of payoff symmetry

Consider an n-player allocation game $\Gamma = (I, A, \hat{u})$. Consider a behavioral strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$. This strategy profile σ induces a probability distribution over the set of all play paths $\mathcal{P}$. Recall that $u_i(\sigma)$ denotes player i's expected (ex ante discounted) payoff from the repeated game strategy profile σ.

Definition 2. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is ex-ante payoff-symmetric if $u_i(\sigma) = u_j(\sigma)$ for all $i, j \in I$.

[13] The monitoring structure is not crucial to the results. Its main purpose is to keep the analysis of the experiment simple. The theory could easily be developed (and would not be very different) for other structures such as perfect monitoring at all times. Under perfect monitoring, e.g., there are faster, but more complicated, ways to break symmetry, leveraging the partial breaking of symmetries that tends to occur on a path of play.

If σ is a symmetric strategy profile then it is ex-ante payoff-symmetric, i.e., if $\sigma_i = \sigma_j$ for all i, j, then $u_i(\sigma) = u_j(\sigma)$ for all i, j. But not every ex-ante payoff-symmetric strategy profile is also symmetric.

For a given play-path $p \in \mathcal{P}$ denote $T(p) = \min_{t \geq 0}\{t \mid a_i^t \neq a_j^t \text{ for all } i, j \in I\}$, the time period in which symmetry is broken along this path. Consider player $i \in I$. Consider all play-paths in which player i plays action $a \in A$ when symmetries are broken. This set is given by $\mathcal{P}_i(a) = \{p \in \mathcal{P} \mid a_i^{T(p)} = a\}$, and henceforth we refer to player i as the a-player in this event.

Define player i's ex ante (discounted repeated game) payoff conditional on player i being the a-player, that is, conditional on the event $p \in \mathcal{P}_i(a)$, by $w_i(a, \sigma) = \mathbb{E}_0[u_i(\sigma) \mid p \in \mathcal{P}_i(a)]$. That is, $w_i(a, \sigma)$ is an expectation evaluated ex ante, under the distribution over play paths induced by σ, conditioning on player i taking action a when symmetries are broken.

Note that if $\sigma = (\sigma_1, ..., \sigma_n)$ is symmetric, then, necessarily, $w_i(a, \sigma) = w_j(a, \sigma)$ for all $i, j \in I$ and for all $a \in A$. In this case, we can drop the subscript and denote by $w(a, \sigma)$ the expected payoff of the a-player.

Definition 3. A symmetric strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$ is ex-post payoff-symmetric if $w(a, \sigma) = w(b, \sigma)$ for all $a, b \in A$.

2.5. Example: symmetries in the battle-of-sexes

To further clarify the definitions of the previous section, and how we will use them, consider a 2-player repeated allocation game, i.e., a symmetrized version of the Battle-of-the-Sexes, with stage game given by

	H	L
H	0, 0	2, 1
L	1, 2	0, 0

Both players have a high action H that pays $x_1 = 2$ and a low action L that pays $x_2 = 1$. This stage game is played repeatedly with perfect monitoring and a common discount factor $\delta < 1$. The game is symmetric. Informally, a strategy profile is symmetric if the strategies of the two players are the same mapping from histories to actions. A first implication of symmetry is that at the empty history both players need to use the same mixed action. It cannot be, for instance, that in the first stage one player chooses H (with probability one) and the other player chooses L (with probability one), i.e., the efficient equilibria are precluded by symmetry.[14] In the second stage there are three possible histories: both players played H, both players played L, and the players took distinct actions in the first stage. In the first two cases, the players must use the same mixed action as each other in the second stage, as symmetries have not been broken. Notice though, that this mixed action, used by both players, can be different from the mixed action used in the first stage, and can depend as well on whether the history is (H, H) or (L, L). In the remaining case, symmetries are broken: it is common knowledge between the players that they have taken different actions, and this fact distinguishes them. At this history, and forever after,

[14] This is a natural implication, as even if players have observable differences (gender, age, etc.) there often is no convention mapping the list of observables into asymmetric recommendations for play, and preplay communication cannot generally be relied upon to accomplish this, as demonstrated by Farrell [20].

the two players can use different distributions over actions. Symmetry implies, however, that whatever mixed action is taken in the second stage by the player who in the first stage played H, would have also been taken by his opponent, had the first stage realizations been exchanged. Of course, it could happen that symmetry is broken only later, say in stage T. Then it must be the case that in every stage preceding T both players employ the same distribution over actions. Finally, at stage T it must have been that, for the first time in the game, one player's action realized as L, while the other's realized as H. Because symmetry means only that the players map histories to future actions in the same way, once symmetry is broken, joint continuation play is unrestricted at subsequent stages.

Naturally, if two players employ the same strategy in a symmetric game they must expect the same payoff, which is to say that a symmetric strategy profile is ex ante payoff symmetric. However, on the path of play they may very well obtain different payoffs. For instance, they could randomize $\frac{1}{2}H + \frac{1}{2}L$ until symmetry is broken and then, once an (H, L) has realized, continue to play those same actions (H, L) forever after. Thus, while they use symmetric strategies and expect the same payoff ex-ante, they also expect a difference in their payoffs. In other words, conditional on being the H-player, i.e. conditional on being the player who plays H at the stage when symmetry is broken, a player expects a continuation payoff of 2, which is different than the payoff of 1 that the L-player, the player who plays L when symmetry is broken, expects. In this regard we refer to the *expected payoff of the H-player*, $w(H)$, i.e., the ex ante expected discounted payoff to a player conditional on this player being the H-player as just described. Similarly we have $w(L)$, the *expected payoff of the L-player*. Symmetry ensures these are well-defined, as payoffs do not depend on the identity of the player who so happens to become the H-player along a path of play. Under the strategy $\frac{1}{2}H + \frac{1}{2}L$, e.g., we have that $w(H) = 2\frac{1}{2}\sum_{t=0}^{\infty}(\frac{\delta}{2})^t = \frac{2}{2-\delta}$ and, similarly, $w(L) = 1\frac{1}{2}\sum_{t=0}^{\infty}(\frac{\delta}{2})^t = \frac{1}{2-\delta}$.

For a given strategy σ, ex post payoff symmetry means that the expected payoff (discounted back to time 0) conditional on being the H-player is the same as the payoff conditional on being the L-player, i.e., $w(H,\sigma) = w(L,\sigma)$. More generally, ex-post payoff-symmetry requires that every player expects the same payoff independent of which action he takes when symmetries are broken, which is to say that each player expects no difference between his payoff and the payoff of any other player. Observe that symmetric strategy profiles are not generally ex-post payoff-symmetric.

3. Feasible payoffs

Consider a repeated n-player allocation game. Consider the following strategy, represented in automaton form, $(\mathcal{W}, w^0, f, \tau)$.[15] Let $w^0 = R \in \mathcal{W}$ be the initial state. Let $f(R)$ be the uniform distribution over all actions in A. Let $\tau(R, a, y) = R$ if $y = n$ (symmetry was not broken), and $\tau(R, a, y) = S(a)$ for some $S(a) \in \mathcal{W}$ with $S(a) \neq S(b)$ for all $a, b \in A$ if $y = c$ for the first time (symmetry was broken). This automaton represents a strategy in which all players initially randomize uniformly over all actions in every period until coordination is achieved (i.e., all players use a different action) and, hence, symmetry is broken. After that, if all players use this automaton, all players' automata will now be in different states. Thus, from that point on they can, in principle, play any (possibly asymmetric) strategy profile of the repeated game. Let us assume

[15] The set $\mathcal{W}$ is the set of states for the automaton, w^0 is the initial state, $f : \mathcal{W} \to A$ is the (possibly random) action chosen as a function of states (which could also depend on observables from the game, but this is not needed for our purposes), and finally $\tau : \mathcal{W} \times A \times Y \to \mathcal{W}$ is the state transition function.

that the continuation played does not depend on the particular history at which symmetry is broken. Thus, let $v(a)$ denote the continuation payoff for the player who played action $a \in A$ when symmetry was broken (exclusive of the payoff at the date symmetries are broken). We will show that this class of strategies is sufficient to generate essentially all feasible payoffs of the repeated game without symmetries.

Let $\nu \in \mathbb{R}^n$ denote the vector of these continuation payoffs. Given the uniform distribution over actions the probability of coordination in any given stage is $q = \frac{n!}{n^n} > 0$. Thus, players will eventually coordinate, symmetry will be broken, and players obtain the payoff-vector ν from the continuation play in the now unrestricted repeated game. Let $w(a)$ denote the ex-ante expected payoff to the player who eventually plays action a when symmetry is broken. Then

$$w(a) = \big[(1-\delta)x_a + \delta v(a)\big] q \sum_{t=0}^{\infty} \delta^t (1-q)^t = \big[(1-\delta)x_a + \delta v(a)\big] \frac{q}{1-\delta(1-q)}.$$

Let the set of payoff-vectors that are feasible through the use of all (including asymmetric) strategy profiles of the repeated game be denoted by $\mathcal{F}_\Gamma \subset \mathbb{R}^n$. Note that $\mathcal{F}_\Gamma$, while in principle dependent on the discount factor δ, is actually constant for all $\delta \geq \bar{\delta}$ for some $\bar{\delta} < 1$. This follows from a result in Sorin [36], also stated as Lemma 1 in Fudenberg and Maskin [22] and as Lemma 3.7.1 in Mailath and Samuelson [30]. In the 2-player allocation game, in fact, we have that $\mathcal{F}_\Gamma(\delta) = \mathcal{F}_\Gamma$ for all $\delta \geq \frac{1}{2}$. Let $\mathcal{F}_\Gamma^s(\delta) \in \mathbb{R}^n$ denote the set of feasible payoff-vectors under symmetric strategy profiles, which contains the set of payoffs delivered by the automaton construction above. A typical element $w \in \mathcal{F}_\Gamma^s(\delta)$ is, thus, a vector of payoffs, in which each coordinate corresponds to an action $a \in A$ and represents the expected discounted payoff, $w(a)$, for some underlying symmetric strategy profile, to a player conditional on this player being the a-player (the player who plays action a when symmetry is broken). Then

$$\mathcal{F}_\Gamma^s(\delta) \supseteq \left\{ w \in \mathbb{R}^n \;\middle|\; w = \big[(1-\delta)x + \delta \nu\big] \frac{q}{1-\delta(1-q)} \text{ for some } \nu \in \mathcal{F}_\Gamma \right\}.$$

It is thus obvious that, as δ tends to 1, i.e., as players become increasingly patient, the set of feasible payoffs under symmetric strategies coincides with the set of feasible payoffs under all strategies.

Proposition 1. *Let Γ be an n-player allocation game. As the discount factor tends to one in the associated repeated game, the set of payoff profiles that are feasible under symmetric strategies tend to the set of feasible (unrestricted) payoff profiles. That is*

$$\lim_{\delta \to 1} \mathcal{F}_\Gamma^s(\delta) = \mathcal{F}_\Gamma.$$

Note that Proposition 1 immediately generalizes to all symmetric n-player games as long as it is possible to break all symmetries in finite time with some symmetric strategy profile with probability 1.

4. Equilibrium payoffs

While symmetry, thus, hardly poses a restriction on the set of feasible payoffs, this section demonstrates that symmetry does impose interesting restrictions on the set of equilibrium payoffs for all discount factors, even as δ tends to 1.

Consider a repeated n-player allocation game with coordinated payoff vector x. Recall that x_a is the stage payoff a player receives when playing a and all other players play in such a way that no two players choose the same action. We can translate this payoff vector of the stage game to payoff profiles in the repeated game as follows. Let Π be the space of action permutations with typical element $\pi : A \to A$. Let w^π be a payoff profile such that $w^\pi(a) = x_{\pi(a)}$, i.e., the repeated game payoff to the a-player is given by the stage game payoff a player gets from playing action $\pi(a)$ (in the action profile in which no two players play the same two actions).

Note that for all π we have $w^\pi \in \mathcal{F}_\Gamma$, and, thus, by Proposition 1, also in $\lim_{\delta\to 1} \mathcal{F}^s_\Gamma(\delta)$. This is to say that payoff profile x (and all its permutations) are feasible under a symmetric strategy profile, at least in the limit in which the discount factor tends to 1. Furthermore, there are many payoff profiles y that are close to some w^π that are also feasible under symmetric strategy profiles for δ sufficiently close to 1.

Denote by $\mathcal{Z}_\Gamma = \text{conv}\{w^\pi\}_{\pi\in\Pi}$ the Pareto frontier of $\mathcal{F}_\Gamma$, that is, the convex hull of the permutations of x.

For a given payoff z and a small positive number $\epsilon > 0$ let $\mathcal{U}^z_\epsilon = \{y \in \mathcal{F}_\Gamma \mid \|y - z\|_\infty \le \epsilon\}$, where $\|\cdot\|_\infty$ is the infinity norm (the difference between y and z is the maximal difference between their coordinates). Note that for any $z \in \mathcal{Z}_\Gamma$ and any $\epsilon > 0$ there is a $\bar{\delta} < 1$ such that for all $\delta \ge \bar{\delta}$ we have that $\mathcal{U}^z_\epsilon \cap \mathcal{F}^s_\Gamma(\delta) \ne \emptyset$.

We say that a strategy is *stationary semi-public* if it has two properties: (i) for every history at which symmetry is not yet broken, the mixed action (used by all players, as required by symmetry) is identical, and (ii) the continuation play upon the breaking of symmetry depends only on the realized action at the stage in which symmetry is broken. That is, the continuation plan is dictated by the action played when symmetry breaks, and the plan can be carried out without any additional information, such as reference to the period in which symmetry is broken.

Let $\mathcal{E}^s_\Gamma(\delta)$ denote the set of payoff profiles (as a vector to the respective a-players) that are sustainable under symmetric stationary semi-public equilibria of the repeated game.[16]

Proposition 2. *Consider the repeated game associated with any n-player allocation game Γ and any Pareto efficient payoff profile $z \in \mathcal{Z}_\Gamma$ that is not completely symmetric, i.e., such that there exist a and b for which $z_a \ne z_b$. There is an $\epsilon > 0$ such that for all $\delta \in (0, 1)$ we have $\mathcal{U}^z_\epsilon \cap \mathcal{E}^s_\Gamma(\delta) = \emptyset$.*

In words Proposition 2 states that, for any asymmetric payoff profile that is close to the Pareto frontier, there is no symmetric stationary semi-public equilibrium that supports those payoffs for any discount factor, even as δ tends to one. Its proof is given in Appendix A. We provide here an intuition for the result. Consider a symmetric stationary semi-public strategy profile that is not ex-post payoff-symmetric, but for which the payoffs are close to the Pareto frontier. The proposition focuses on the following unilateral deviation. Consider playing, at every round before symmetry is broken, an action which promises the highest continuation payoff (and then playing as prescribed in the continuation game). How much will this cost in terms of slowing down the process of breaking symmetry? Since payoffs are, by assumption,nearly efficient, it must be that players are breaking symmetry fast, relative to the discount factor. Using this fact, the proof

[16] The result applies only to stationary semi-public equilibria. We conjecture that the same conclusion holds for all symmetric equilibria. Proving so may rely, in part, on characterizing the optimal way for n players to break symmetries in the repeated game. The proof for stationary semi-public equilibria is short and elegant as it uses only a few key aspects of the structure of these equilibria.

shows that the delay cost of the deviation can be made as small as needed through the choice of ϵ. Now, since the strategy profile is not ex-post payoff-symmetric, the deviation also promises the player a strictly higher continuation payoff once symmetry is broken, thus making it profitable.

While the above argument uses the stationary semi-public structure of strategies, we conjecture that the conclusion of Proposition 2 holds for all symmetric strategies. Proposition 3 will show that in a particular 2-player allocation game, no symmetric equilibrium (whether or not it is stationary semi-public) can deliver an individual payoff exceeding three fourths the total available surplus.[17] This shows, in particular, that ϵ can be taken rather large for highly asymmetric payoffs, and that the conclusion applies to all symmetric equilibria.

We remark that ϵ cannot be chosen uniformly with respect to payoff profiles $z \in \mathcal{Z}_\Gamma$ along the Pareto frontier. Let us define $z^* \in \mathcal{Z}_\Gamma$ to be the (unique) payoff-symmetric point on the Pareto frontier, i.e., $z^*_a = 1/n \sum_{a \in A} x_a$ for all $a \in A$. Roughly speaking, the more symmetric is z, the smaller ϵ must be chosen, with the particular property that as z approaches z^*, the associated choices of ϵ must converge to zero.

This observation has two important implications. First, there is a unique efficient payoff profile that can be sustained by a symmetric equilibrium as δ tends to one, and that profile is the one that is exactly ex-post symmetric. We have the following.

Corollary 1. $\mathcal{Z}_\Gamma \cap \lim_{\delta \to 1} \mathcal{E}^s_\Gamma(\delta) = z^*$.

In words, as δ tends to one, the only efficient payoff profile sustained by a symmetric equilibrium is that which is ex-post payoff-symmetric.

In light of Proposition 2, which implies that the intersection cannot be bigger than z^*, this can be proven by construction. Consider a strategy profile that promises continuation payoffs z^* from every history at which symmetry is broken. It is easily verified that uniform mixing over A at all preceding histories constitutes an equilibrium. So as $\delta \to 1$, this equilibrium delivers payoffs z^*.

The second implication is that payoff profiles that are more asymmetric necessitate more efficiency loss to implement. This reduction in equilibrium efficiency will tend to be severe as z approaches the least symmetric points on the Pareto frontier, given by $\{w^\pi\}_{\pi \in \Pi}$. While Proposition 2 does not precisely characterize this tradeoff for arbitrary allocation games, the next section demonstrates the magnitude of this effect by means of the simplest possible example.

4.1. A 2-player example

Consider the 2-player allocation game with $x_1 = 1$ and $x_2 = 0$. I.e., the stage game is given by

	H	L
H	0, 0	1, 0
L	0, 1	0, 0

where we label the actions $A = \{H, L\}$. Note that H (for "high") weakly dominates L (for "low") within the stage game.[18] As with all 2-player allocation games, observing c or n, i.e., whether

[17] Further, we show in Appendix E that this argument extends to a large class of symmetric 2-player games.

[18] This dominance relation between the two actions, however, is not important in the analysis and does not drive the results. The analysis is similar if x_2 is a small positive number.

or not coordination has been achieved, along with one's own action, is sufficient for players to know exactly what has been played. Thus, the repeated game is one of perfect monitoring.

Denote, as before, by $\mathcal{F}$ the set of payoff vectors that are feasible when considering all strategy profiles of the repeated game, and denote by $\mathcal{F}^s(\delta)$ those payoff vectors that are feasible under the restriction that strategy profiles be symmetric.The set $\mathcal{F}$ does not depend on δ, as long as $\delta \geq \frac{1}{2}$, and is equal to the triangle given by the convex hull of payoff pairs $(0, 0)$, $(0, 1)$, and $(1, 0)$. The set $\mathcal{F}^s(\delta)$ is a sub-triangle, which, by Proposition 1, tends to $\mathcal{F}$ as δ tends to 1.

It is well-known and, in fact, immediate for the game at hand, that the set of equilibrium payoff pairs, without the symmetry qualification, is equal to $\mathcal{F}$ for high discount factors. Denote by $\mathcal{E}^s(\delta)$ the set of symmetric equilibrium payoff pairs (here without the stationary semi-public requirement). An element of $\mathcal{E}^s(\delta)$ is a pair of payoffs, the first is the expected discounted payoff to the H-player, the second the expected discounted payoff to the L-player. We shall show that $\mathcal{E}^s(\delta)$ includes many payoff-pairs that are not ex-post payoff-symmetric, but yet is not nearly as large as $\mathcal{F}^s(\delta)$.

To obtain an interesting upper bound on the set of symmetric equilibrium payoff pairs in this repeated game, we appeal to a fixed point argument of an appropriate function in the set of potential symmetric equilibrium payoff pairs.[19] Consider time 0 or any time period in which symmetry has not yet been broken. The potential outcomes of play in this stage are the pure strategy combinations HH, HL, LH, and LL.

Suppose HL occurs. Then the H-player obtains a payoff of 1, the L-player one of 0, and symmetry is broken. Thus, the continuation play can be any (possibly asymmetric) equilibrium of the repeated game and so the continuation payoffs can be any element in $\mathcal{F}$.

We can, in fact, combine the (relatively negligible) one-period payoff and the later discounted per-period continuation-payoff in $\mathcal{F}$, by assigning the two players an appropriate payoff-pair in $\mathcal{F}$ right at this time 0. That is, upon observing outcome HL, continuation payoffs can be (almost) any pair $(w^H, w^L) \in \mathcal{F}$.

After outcomes HH and LL symmetry is not broken. Thus, the continuation payoffs can be any symmetric *equilibrium* payoff-pair. Thus, after HH a continuation might be any $(w^H_{HH}, w^L_{HH}) \in \mathcal{E}^s(\delta)$, which, as we shall see, is a more severe restriction than just $w^H_{HH}, w^L_{HH} \geq 0$ and $w^H_{HH} + w^L_{HH} \leq 1$. After LL continuation payoffs might be any $(w^H_{LL}, w^L_{LL}) \in \mathcal{E}^s(\delta)$, possibly different from (w^H_{HH}, w^L_{HH}).

These expectations induce particular incentives for the two players governing their mixed action at time 0. Together with the continuation profile, this determines their expected repeated game payoffs at stage 0. With this in mind, we define a function $f : \mathcal{G} \to \mathcal{G}$, with $\mathcal{G}$ denoting the set of all subsets of $\mathcal{F}$, which assigns to a candidate symmetric equilibrium set $\mathcal{E}^s \in \mathcal{G}$, used for continuation payoffs after HH and LL, the set of all possible equilibrium payoff-pairs at time 0, obtained by working through the appropriate incentives, as outlined above.

The set of symmetric equilibrium payoff-pairs $\mathcal{E}^s(\delta)$ must be a subset of the largest fixed point of f. The proof of the following proposition uses this fact in order to derive an upper bound $\bar{\mathcal{E}}^s(\delta)$ such that $\mathcal{E}^s(\delta) \subset \bar{\mathcal{E}}^s(\delta) \subset \mathcal{F}$.

[19] As far as we are aware, this proof technique for equilibrium payoffs of repeated games appears first in Abreu, Pearce, and Stacchetti [1] as the idea of self-generating sets.

Proposition 3. *In any symmetric equilibrium of the repeated allocation game characterized by* $x = (1, 0)$*, as the discount factor tends to one, no player can expect, conditional on being either the* H*- or* L*-player, a payoff that exceeds* $\frac{3}{4}$*. That is*

$$\left(w^H, w^L\right) \in \lim_{\delta \to 1} \mathcal{E}^s(\delta) \quad \Rightarrow \quad w^H, w^L \leq \frac{3}{4}$$

The proof is given in Appendix B and generalized to a broader class of 2-player games in Appendix E, which includes games outside the class of allocation games. We provide here a brief sketch. We assume that the set of symmetric equilibrium payoffs satisfies the restriction that any individual payoff (to the H-player, or the L-player) cannot exceed a certain threshold $\bar{w}$. This assumption is certainly true for $\bar{w} = 1$. Now suppose we fix $\bar{w} \in (\frac{3}{4}, 1]$ and accordingly take continuations after HH and LL to satisfy that each payoff is less than or equal to $\bar{w}$, while, of course, the continuation after HL and LH is any pair in $\mathcal{F}$. We show that for every such profile of continuations the expected payoff pair to the H- and L-player at time 0 is strictly less than $\bar{w}$. This implies, in particular, that in order for a candidate set of payoff-pairs to be a fixed point of the mapping f, it must satisfy that no individual payoff exceeds $\frac{3}{4}$.

Note that Proposition 3 does not state that eventually, along the path of play, the payoffs to the H- and L-player cannot be highly asymmetric. What Proposition 3 does imply, though, is that, in this case, the expected time at which symmetry is broken, and those continuation payoffs are realized, is so far in the future (relative to δ) that the players' discounted payoffs viewed from time 0 are nowhere near efficient. This is true even if the discount factor is arbitrarily close to 1. The more patient players become, the longer it takes for symmetry to break, which, it turns out, is not compensated by the increased patience with which players view their payoffs.

Having shown that certain payoff-pairs are not possible to sustain with a symmetric equilibrium, we turn now to demonstrating that there are actually many payoff-pairs that *can* be supported, including asymmetric ones. In fact the next proposition completely characterizes the set of payoff-pairs sustainable in any symmetric stationary semi-public equilibrium.

Proposition 4. *As* δ *tends to one, the set of payoff-pairs sustainable by symmetric stationary semi-public equilibria has Lebesgue-measure* $\frac{1}{6}$*, which is* $\frac{1}{3}$ *the measure of the feasible set.*

The proof is given in Appendix C. Proposition 3 and 4 are summarized in Fig. 1. One can show, by example, that the set of payoff-pairs sustainable by all symmetric equilibria lies strictly between the bounds we provide. That is, one can sustain payoff-pairs (slightly) outside the lens depicted in Fig. 1 via the use of equilibria that are not stationary semi-public.

5. Implementing symmetric continuations

Corollary 1 says that, in repeated allocation games, the Pareto criterion identifies a particular unique payoff profile. Our view is that, at least in this context, strict Pareto optimality is a powerful basis on which to view such payoffs as focal and, therefore, to expect coordination on an equilibrium that delivers those payoffs. Nonetheless, if one is interested in a theory of play, which is to say a description of how to play the game, then there remains a multiplicity of strategies, in fact of symmetric stationary semi-public equilibrium strategies, that deliver those payoffs. This section addresses the question as to how players might implement the ex-post symmetric payoffs for repeated 2-player allocation games.

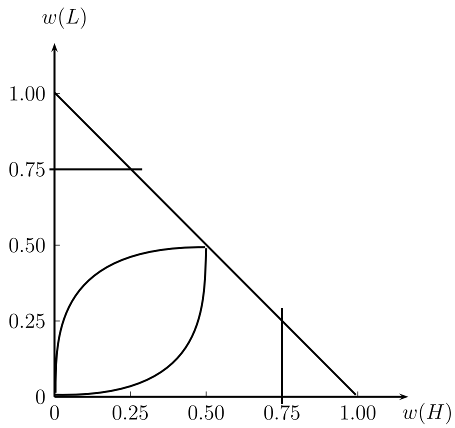

Fig. 1. The lens represents the set of payoff-pairs sustained by symmetric stationary semi-public equilibria (Proposition 4). The corners represent payoff-pairs not achievable in any symmetric equilibrium (Proposition 3).

Consider the stage at which symmetry is broken and players commence with an ex-post payoff-symmetric continuation strategy. Take any discount factor $\delta < 1$. It follows from Lemma 3.7.1 in Mailath and Samuelson [30] originally due to Sorin [36] that an exactly payoff-symmetric continuation can be constructed from an appropriate sequence of pure stage game action profiles, provided δ is large enough. Indeed it is easy to see that there are many such possible constructions (see, e.g., the proof of Proposition 3 in Bhaskar [7]).

Normalize the time at which symmetry is broken to zero. Taking, for simplicity, and without loss of generality, the stage game of Section 4.1, one can describe any Pareto efficient continuation play by a sequence $y = (y_t)_{t=0}^{\infty} \in \{-1, 1\}^{\infty}$, where, without loss of generality, $y_0 = 1$.[20] The interpretation is that $y_t = 1$ corresponds to players using the same actions as in stage 0 (i.e., in which the H-player receives the high payoff), while $y_t = -1$ corresponds to players using the opposite actions (in which the L-player receives the high payoff). For any continuation y, define the normalized difference in payoffs between the two players by

$$\Delta^y(\delta) = (1-\delta)\sum_{t=0}^{\infty} \delta^t y_t.$$

We will also make use of the difference in payoffs corresponding to partial sequences: $\Delta^y(\delta|_T) = (1-\delta)\sum_{t=0}^{T-1} \delta^t y_t$.

It is convenient to reproduce a version of Lemma 3.7.1 in Mailath and Samuelson [30] here.

Lemma 1. *Let $\frac{1}{2} \le \delta < 1$. There is a continuation play y such that $\Delta^y(\delta) = 0$.*

That is, for any $\delta \ge \frac{1}{2}$, one can find an exactly ex-post payoff-symmetric continuation play. However, the set of symmetric continuations is generally very sensitive to the discount factor δ. Thus under even a small degree of uncertainty over the precise value of δ, selecting an ex-post payoff-symmetric strategy is impossible. We are thus interested in finding a particular fixed sequence that is nearly ex-post payoff-symmetric for *all* large values of δ. Such a sequence would

[20] We abuse notation by using y and z to denote continuation plays, rather than payoff profiles, in this section.

have the desirable property of guaranteeing approximate ex-post payoff-symmetry if all that is known is that the discount factor will be close to one, without any further information about a specific distribution for δ or common knowledge thereof. In other words, for patient players, such a sequence represents the obvious way to play if one desires that the meta-norm of Pareto optimality be robust to small perturbations or uncertainty regarding the discount factor.

More precisely, this section accomplishes two goals. First, we prove that there exists a continuation play with a particularly strong payoff symmetry property for high discount factors. As it happens, this continuation is the well-known Thue–Morse sequence, which we denote throughout this section by z.[21] Specifically, we show that z satisfies the following property. For any k, and any sequence y with periodicity k, there exists a $\bar{\delta}$ such that for all $\bar{\delta} < \delta < 1$, $|\Delta^z(\delta)| < |\Delta^y(\delta)|$. Second, we provide a new characterization of z. For any δ, construct the continuation $\hat{z}$ that, at every t, awards the payoff to the player who currently has the smaller total accumulated payoff.[22] That is, $\hat{z}_t = 1$ if $\Delta^{\hat{z}}(\delta|_t) \leq 0$ and otherwise $\hat{z}_t = -1$.

The Thue–Morse sequence, z, is defined as follows. Set $z_0 = 1$, and define the sequence recursively by $z_{2s} = z_s$ and $z_{2s+1} = -z_s$ for all s.[23] We prove that the limit of sequences $\hat{z}$ as δ tends to one is the Thue–Morse sequence, z. That is, the Thue–Morse sequence z has the property that, for sufficiently large δ, it awards the high payoff to the player who has the lower present discounted payoff.

Proposition 5. *For every t there exists a $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ the following is true. If $z_t = 1$ then $\Delta^z(\delta|_t) < 0$, and if $z_t = -1$ then $\Delta^z(\delta|_t) > 0$.*

The proofs of results in this section are relegated to Appendix D.

This result provides an economic interpretation of the Thue–Morse sequence. In particular, z is described by the limit of the sequence that, in every period, assigns the payoff to the currently disadvantaged player, as δ tends to one.

Recall that for any δ there exist sequences that are exactly payoff-symmetric. However, those sequences may become highly asymmetric for nearby values of δ. We now show that z has the desirable property of being very nearly payoff-symmetric for all δ close to one.

A sequence y has periodicity k if $y_{t+k} = y_t$ for all $t \geq 0$. We now provide the main result of this section.

Proposition 6. *For every k there exists a $\bar{\delta} < 1$ such that, for every sequence y with periodicity k, $|\Delta^z(\delta)| < |\Delta^y(\delta)|$ whenever $\delta > \bar{\delta}$.*

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[21] See Thue [38,39] and Morse [31]. Allouche and Shallit [2] provide a useful discussion.

[22] Bhaskar [7] utilizes the same construction for a given, fixed, discount factor, showing that it is efficient in the infinitely repeated game. He does not, however, consider the limiting properties of this sequence as the discount factor tends to one. All of our results in this section are about this limiting case, where we identify the limit as the Thue–Morse sequence, and show that it has a robust optimality property for all high discount rates.

[23] It is well known that this sequence can alternatively be constructed as follows. Set $z_0 = 1$. Proceed iteratively, at each step replacing every instance of 1 in z_k with $(1, -1)$ in z_{k+1}, and replacing every instance of -1 in z_k with $(-1, 1)$ in z_{k+1}. Another alternative construction is the following. Let us generally call $\{z_t\}_{t=0}^{2^k-1}$ the block of size 2^k. Let $\{-z_t\}_{t=0}^{2^k-1}$ denote the inverse of this block (i.e., 1's are replaced by -1's and vice versa). We have the block of size 1. The block of size 2 is then given by the block of size 1 followed by the inverse of the block of size 1. In general, the block of size 2^k is given by the block of size 2^{k-1} (which accounts for its first half) followed by the inverse of this very same block (which accounts for the second half). Yet another well-known equivalent characterization of the Thue–Morse sequence is the following. Let $z_t = 1$ whenever the binary expansion of t has an even number of 1's and let $z_t = -1$ otherwise.

The proposition applies immediately to all 2-player allocation games. One can extend the characterization of z to n-player allocation games, with $n \geq 3$. The most obvious way to do so is to consider, at any stage t in the continuation play, the ranking of players, π^t, defined by increasing order of their present discounted payoffs in the repeated game. The play at stage t, then, is defined by $a^t_{\pi(i)} = i$, i.e., the action of the player with the i-th lowest payoff at stage t is the one that pays the i-th largest amount according to x.

Consider high values of δ. For any allocation game defined by a given vector x, this sequence is, generically, well-defined. For such δ, this continuation play is ex-post payoff-symmetric. It also has the property, as in the 2-player case, that in every successive block of n periods, each player receives each of the payoffs x_a exactly once. The exact sequence to which this play converges, as δ tends to one, will generally depend on x. We conjecture that for any repeated allocation game, that limit sequence retains the property ascribed to the Thue–Morse sequence by Proposition 6, i.e., that it is more payoff-symmetric than any periodic sequence for sufficiently high discount factors.

6. Simplicity

We now consider the matter of the complexity of the strategy that is to serve as the focal point of the repeated allocation game.

According to Kalai's [24] survey on bounded rationality and complexity in repeated games, Aumann [4] was the first to point to viewing repeated game strategies as automata as an aid to measuring complexity of those strategies. The most common measure of complexity of a given repeated game strategy was then formally provided (simultaneously) by Ben Porath [6], Neyman [32], and Rubinstein [34] as *state-complexity*, which is given as the minimal number of states of any (finite) automaton that encodes the given repeated game strategy. The higher this number the more complex is a strategy. Kalai and Stanford [25] then showed that this notion of complexity of a repeated game strategy is equivalent to the number of continuation strategies this repeated game strategy can generate. Thus, state-complexity can be identified without reference to automata.

We shall use (low) complexity very differently from the work reviewed by Kalai [24]. In Ben Porath [6] and Neyman [32], among others, players are allowed to use only strategies of bounded complexity. In Rubinstein [34] and the literature following it, as reviewed in Chatterjee and Sabourian [17], players have, in addition to preferences over outcomes, also a preference in favor of less complex strategies. We here appeal to low complexity simply as a selection device (choosing from all complexity-unrestricted symmetric equilibria) to complete our description of a focal point in repeated allocation games and, thus, follow a third strand of the complexity literature which was initiated by Baron and Kalai [5]. This choice matters since, in our setting, all best responses to a complex strategy are complex, and so a lexicographically small preference for simplicity does not serve as a useful notion to derive a focal strategy.

We thus define the complexity of a strategy σ as the smallest number of states in any automaton that implements this strategy.

Proposition 7. *Consider any repeated n-player allocation game Γ. Let σ denote a strategy with the property that the symmetric strategy profile $\{\sigma\}^n$ is efficient in the limit as δ tends to one. Let $(\mathcal{W}, w^0, f, \tau)$ be an automaton representation of σ. Then $|\mathcal{W}| \geq n + 1$.*

Proof. Note first that there must be a state $w \in W$ such that $f(w)$ is totally mixed over A. Note also that in order for this automaton to eventually lead to an efficient continuation, it must be that all actions must be played purely after some history. I.e., for every action $a \in A$ we must have a state w_a such that $f(w_a)$ attaches probability 1 to a single action. $\square$

There are, of course, less complex automata. For instance consider the automaton with just a single state describing to play action 1. The symmetric profile of such a strategy is, in fact, an equilibrium (unless $n = 2$ and $x \gg 0$) and it yields a payoff of zero. There are also less complex automata that implement strategies with strictly positive payoff profiles. For example, the single-state automaton that prescribes the totally mixed stage game symmetric Nash mixture at every history constitutes a symmetric equilibrium. What Proposition 7 says is that all symmetric strategy profiles (whether or not they are equilibria) with fewer than $n + 1$ states are necessarily inefficient for patient players.

As the discount factor tends to one, how do players select a focal strategy that implements the efficient payoff profile? The answer that we provide in Section 5 results in the Thue–Morse sequence. The Thue–Morse sequence is simple according to some notions, such as Kolmogorov complexity and in the sense that it can be represented by a very simple Turing Machine.[24]

But the Thue–Morse sequence cannot be represented by a finite automaton. Supposing that complexity, in players' minds, is well-captured by state complexity, we now ask the following question. Among symmetric equilibria that deliver efficient payoffs as δ tends to one, which strategy is simplest and, thus, a candidate focal point of the game?

Fix an allocation game Γ. Fix an order over, or labeling of, actions $g : A \to \{1, \dots, n\}$ that is surjective.

Definition 4. A *rotation scheme* for Γ with respect to order g is a strategy with an automaton representation $(\mathcal{W}, w^0, f, \tau)$ with state space $\mathcal{W} = R \cup A$, initial state $w^0 = R$, output function f with $f(R)$ specifying a totally mixed distribution over A, and $f(a) = a$ for each $a \in A$, and transition function τ that produces $\tau(R, a, n) = R$ (for the case when symmetry is not broken), and $\tau(R, a, c) = \tau(a, \cdot, \cdot) = g^{-1}(g(a) + 1 \bmod n)$ (otherwise), where a is the realized action of the player.

In words, a rotation scheme randomizes with a fixed distribution until symmetry is broken, and then rotates among the pure actions according to g in the continuation. Note that a rotation scheme achieves the lower bound on state complexity identified in Proposition 7. Generically, the simplest efficient symmetric equilibria for patient players are rotation schemes of Γ. We focus here on allocation games with the following property.

Definition 5. A vector of payoffs x for an n-player allocation game is *strongly distinct* if, given two non-identical sets of non-negative integers $\{\alpha_a\}_{a \in A}$ and $\{\beta_a\}_{a \in A}$, we have $\sum_{a \in A} \alpha_a x_a \neq \sum_{a \in A} \beta_a x_a$.

[24] To do so one needs one memory cell with an initial element, one function of the memory to $\{0, 1\}$, and one function of the memory to itself that tells us how to update the memory. For instance, we can use an initial memory $m_0 = 0$; output function $f : \mathbb{N}_0 \to \{0, 1\}$ given by $f(m) = 1$ if m's binary expansion has an even number of zero's (and $f(m) = 0$ otherwise); and memory update function $g : \mathbb{N}_0 \to \mathbb{N}_0$ given by $g(m) = m + 1$. Alternatively, and close to the motivation of Section 5, we can use initial memory $m_0 = 1$; output function $f : \mathbb{R} \to \{0, 1\}$ given by $f(m) = 1$ if $m < 0$ (and $f(m) = 0$ otherwise); and memory update function $g : \mathbb{R} \to \mathbb{R}$ such that $g(m) = g(m) + D$, where D is the current payoff difference.

The following result formalizes the characterization.

Proposition 8. *Consider an n-player allocation game with strongly distinct payoff-vector x. Consider the set of strategies Σ^* such that their associated symmetric strategy profiles constitute efficient equilibria in the limit as δ tends to one. If $\sigma \in \Sigma^*$ has an automaton representation with state-complexity $n+1$, then σ is a rotation scheme.*

Proof. Efficiency of the strategy implies that there must be at least one state in which the play is totally mixed. By Corollary 1 an efficient equilibrium must be ex-post payoff-symmetric. Since x is strongly distinct, as δ tends to one, the requirement that every player achieve the same discounted payoff implies that every player must receive each of the $\{x_a\}_{a\in A}$ payoffs infinitely often. The unique simplest way to do this is by rotating through actions according to a surjective rotation g. □

Proposition 8 identifies as rotation schemes those strategies that are the simplest efficient symmetric equilibria for patient players. We remark that rotation schemes remain simpler than the Thue–Morse sequence even under alternative notions of complexity, such as the Turing notion. Uniqueness of rotation schemes, though, is determined only up to the order, g. For a rotation scheme to qualify as a focal point of the repeated allocation game, it must be that in players' minds there is a conventional order over the actions.

That order must be defined with reference only to the game itself, which is to say that g should depend only on x, the vector of payoffs of the allocation game. Among the ways to define such a function, there is an attractive construction, which might be called "increasing." That is, upon successful breaking of symmetry, players coordinate by rotating through the payoffs of x in increasing order (mod n).[25] Once "increasing" is determined to be the conventional ordering, Proposition 8 identifies a unique way to play the game. While it is not exactly efficient for $\delta < 1$, it has the desirable property of being most simple among all nearly efficient continuation protocols.

7. Experimental evidence

The theory of focal points that we have developed makes a number of testable predictions for play in repeated allocation games. In order to assess the empirical content of the theory, we report data from a battery of laboratory experiments implementing these games.

7.1. Description of experiments

The experimental sessions were conducted at the Social Science Experimental Laboratory (SSEL) at Caltech between February and November, 2010. We recruited undergraduate subjects who had no previous experience with related experiments. For each session, subjects entered the laboratory and were randomly assigned to private computer terminals separated by dividers. All interactions among subjects were computerized, using an extension of the open source software package Multistage Games.[26] Instructions were read aloud for everyone to hear. The exchange

[25] While "increasing" is, in our view, the most natural candidate for a simple rotation scheme, it may be argued that "decreasing" is equally as natural a choice. As we demonstrate below, in our laboratory experiments, whenever a rotation scheme is adopted, which is very often, it is always "increasing".

[26] Documentation and instructions for downloading the software can be found at http://multistage.ssel.caltech.edu.

rate from experimental points to dollars varied across sessions so that the predicted earnings of each subject was \$15–25, inclusive of a \$10 show-up payment. Each experimental session lasted approximately one hour, including instructions (which are available as supplementary material) and payment.

Each experiment consisted of ten matches. In each match subjects were (uniformly and independently) randomly allocated into groups to play the repeated allocation game. The stage games were implemented as follows. In the 2-player games there were two actions, chosen with on-screen buttons labeled L and H. In the 3-player games there were three buttons, labeled E, W, and H. In either case, the experimental screen displayed these buttons in order from left to right. Unless the action profile was a permutation of the set of pure actions, all payoffs were zero. Otherwise, positive payoffs were awarded.

We varied several aspects of the repeated allocation game across sessions. The main design variables are the number of players, the stage game payoffs, the stopping rule, discounting, and feedback. These parameters are summarized in Table 1. We turn now to a detailed description of these treatments.

The vector of coordinated payoffs, x, takes values $(30, 10)$, $(30, 1)$, $(30, 20, 10)$, $(30, 20, 1)$, and $(30, 5, 1)$, denominated in experimental points, as shown in the third column of Table 1. The main purpose of varying x is to vary the magnitude of possible asymmetry in payoffs. In sessions 4 and 5, payoffs (marked by an asterisk in Table 1) were discounted with a factor $\delta = 11/12$ across rounds. In all other sessions payoffs were not explicitly discounted.

The stopping rule was either Fixed or Random in each session. In the Fixed Stopping condition, each match consisted of twenty rounds. In the Random Stopping condition, a pair of dice were rolled at the end of each round. If the sum of the dice was less than or equal to three (probability $3/36$), the match was terminated; otherwise it continued to a next round. Notice that the implied discount factor under Random Stopping is analogous to the rate at which payoffs are discounted in sessions 4 and 5. As we discuss below, for a given continuation play, the predicted mixing probabilities before symmetry is broken depends on the stopping rule.

Finally, we varied the feedback (monitoring) across sessions. The main feedback treatment is called Mixed. Under Mixed Feedback, from the onset of a match, subjects observe only their own payoff after each stage game, until symmetry is broken. In every stage of the continuation in which symmetry is broken, subjects observe both their own payoff and the actions taken by the other players. The second feedback treatment is called Payoff. Under Payoff Feedback, subjects observe only their own payoff at every stage of the game.

In the 2-player games, the treatments are equivalent, since one's own payoff implies a certain action for the opponent. But Payoff Feedback provides strictly less information than Mixed Feedback in the 3-player games. Notice though that the distinction is irrelevant theoretically, since the breaking of symmetry is, on its own, enough information to allow unrestricted joint play. Nevertheless, we shall see that the distinction is empirically relevant. In particular, under Mixed Feedback, to the extent subjects do not have a shared meta-norm, seeing the actions of others helps coordinate expectations of future play.

There is a third feedback treatment called Difference, used in a single 2-player session. Under Difference Feedback, subjects were implicitly given the same information as in Mixed Feedback, but it was framed differently. The summary statistic that subjects were given on-screen was the difference between their current cumulative payoff in the match and that of their opponent. The idea behind Difference Feedback was to encourage continuation play that resembled the Thue–Morse sequence.

Table 1
Summary of experiment sessions.

Session	Players	Payoffs	Stopping rule	Feedback	Subjects
1	2	(30, 10)	Random	Mixed	10
2	2	(30, 1)	Random	Mixed	10
3	2	(30, 10)	Fixed	Mixed	12
4	2	(30, 10)*	Fixed	Mixed	16
5	2	(30, 10)*	Fixed	Difference	14
6	3	(30, 20, 10)	Random	Payoff	12
7	3	(30, 20, 1)	Random	Payoff	15
8	3	(30, 5, 1)	Random	Payoff	15
9	3	(30, 20, 10)	Random	Mixed	15
10	3	(30, 5, 1)	Random	Mixed	15
11	3	(30, 20, 10)	Fixed	Mixed	15
12	3	(30, 20, 10)	Fixed	Mixed	15
13	3	(30, 5, 1)	Fixed	Mixed	15

Under all feedback treatments, the computer screen contained a history panel giving feedback from previous stages in the match, according to the specific treatment for that session. At the end of every session, points were summed from the subjects' earnings across the ten matches and converted to dollars. Subjects were paid privately in cash at the end of the session.

7.2. Symmetry in the laboratory

Before proceeding with the analysis of the data, it is important to comment on the relationship between the theory of focal points and the conditions in the laboratory.

The informational features of the laboratory setting we implement ensure that the restrictions of symmetric play are appropriate for a strategic analysis of observed behavior. By design, all subjects in a given group are in an exactly symmetric position with each other at the start of each match. The matching process is random and anonymous, so subjects have no information about who their opponents are. They, therefore, cannot use any norms that operate on such distinctions between players. Players continue to be in a symmetric position until a realization of positive payoffs in the group. At this point, under all of our feedback treatments, it is common knowledge among the group that there is a particular subject who got each available distinct positive payoff. For the remainder of the match, no two players are symmetric, and symmetry thus places no restriction on the continuation play.

Once symmetry is broken, there is of course the matter of what, if any, continuation play is focal. If there is any lack of common knowledge of the focal play, then further information could be useful in coordinating play. As we will see, the Mixed Feedback treatment has a powerful effect in this direction.

Of course, it is possible that subjects in exactly symmetric positions will choose different strategies. This could arise for a variety of behavioral reasons, potentially leading to non-equilibrium play. Most important from our point of view, however, is that all rational calculations are indeed subject to the restrictions of symmetry. In particular, there is no public or private information available to subjects that would allow them to coordinate on asymmetric play when symmetries are present. Even if such a signal were available, it would be extremely difficult to use as a coordination device unless there was a norm associated with the signal that served as a way to commonly interpret it.

Even so, the theory that we develop above is not directly applicable in all respects to the setting of the experiments. The analysis above proves results for infinitely repeated allocation games with discounting, focusing on the limiting results as players become perfectly patient. These conditions are not feasible to implement in the laboratory. Most of the strategic effects we are interested in, however, do not change dramatically when moving from the theoretical setting to the laboratory.

In particular, the theoretical argument for focusing on Pareto efficient continuations once symmetries are broken is equally compelling in the finite horizon case or the case of impatient players as in the infinitely repeated version with a high discount factor. Similarly, the observation that the closer an equilibrium is to delivering ex-post symmetric payoffs, the more efficient will be the mixing probabilities before symmetries are broken, remains true in finite horizon games.

In general, the equilibrium set of a finitely repeated game may be very different from its infinitely repeated counterpart. But the effects in our setting are relatively minor. This is because, for the allocation games we study, in every symmetric equilibrium, once symmetries are broken, the continuation play in every stage of the infinitely repeated game is a stage game Nash equilibrium. There is no unraveling of candidate equilibria due to backwards induction. In addition, there are some new theoretical predictions that arise in the case of finite horizon games, and these are readily testable with our data.

7.3. *Efficiency of continuation play*

We assess first our contention that continuation play following the breaking of symmetry will be Pareto efficient. Our hypothesis is that in allocation games Pareto efficiency is likely to be a salient meta-norm. This entails that once positive payoffs are achieved in a match, the play in every subsequent round will be coordinated and also result in positive payoffs.

Pooling all of the 2-player sessions, we find that in all of the rounds that occur after symmetry has been broken, 93% (4082/4398) successfully achieve coordination. Given that there are many feasible efficient continuations, and achieving one of them requires coordination between the players, we view this as very strong evidence in favor of our hypothesis.

Turning to the 3-player sessions, we find that the feedback treatment is very important. The reason is that coordination is a fundamentally harder objective to achieve in 3-player games. Notice that only 6 of the 27 (22%) possible action profiles result in positive payoffs. Pooling the sessions from the baseline treatment, in which subjects receive Payoff Feedback, 34% (300/876) of rounds in which symmetry was broken achieve coordination.

While this is significantly better than random, part of the reason that play was not more efficient is due to miscoordination between players on how to proceed in the continuation play. The Mixed Feedback treatment is meant to address this issue, whereby players can see what actions their opponents are choosing when coordination fails, and use this information to help coordinate in future rounds. Pooling across the Mixed Feedback sessions, 68% (2417/3554) of rounds in which symmetry was broken achieve coordination.

7.4. *Ex-post payoff symmetry*

The meta-norm of Pareto efficiency suggests not only that Pareto efficient continuations will be used. Beyond that, nearly payoff-symmetric continuations should be used.

Table 2
Efficiency and ex-post payoff-symmetry of continuation play.

Session	Efficiency of continuation play				Ex-post payoff symmetry			
	# Rounds	# Coord.	Ratio	Totals	S	Avg.	S'	Avg.
1	223	215	96%		0.084		0.101	
2	446	437	98%		0.009		0.044	
3	1088	995	91%	93%	0.040	0.050	0.043	0.050
4	1412	1369	97%		0.029		0.024	
5	1229	1066	87%		0.087		0.040	
6	356	134	38%		0.662		0.539	
7	289	92	32%	34%	0.604	0.591	0.447	0.435
8	231	74	32%		0.508		0.318	
9	844	610	72%		0.276		0.218	
10	421	274	65%		0.050		0.088	
11	753	456	61%	68%	0.101	0.134	0.072	0.121
12	778	563	72%		0.223		0.173	
13	758	514	68%		0.021		0.052	

To assess the extent to which this is true in our data, we examine how close payoffs are to being ex-post symmetric. We denote by $\pi = (\pi_1, \dots, \pi_n)$ the vector of realized payoffs for a repeated allocation game, with $\pi_1 \geq \dots \geq \pi_n$. Define for a given repeated allocation game

$$S = \frac{(\pi_1 - \pi_n)/\pi_n}{(x_1 - x_n)/x_n},$$

which measures a normalized percentage difference in ex-post payoffs between the highest and lowest paid player. $S = 0$ when payoffs are ex-post symmetric, and S is normalized to have maximal value at unity, which corresponds to the maximally asymmetric continuation play in which every subject takes the same action in every round following the breaking of symmetry. We aggregate S across groups and matches within a session by taking a weighted average across observations, with weights given by the total payoff of the group in the repeated game.

The mean value of S across the five 2-player sessions is 0.05. In the 3-player games we find again that the feedback treatment has a large effect. Under Payoff Feedback, in which it is harder to play complex continuations due to lack of information about the action choices of others, subjects more frequently played the simplest continuation in which players use the same actions in subsequent rounds, delivering very asymmetric payoffs. This is captured by the large average figure of $S = 0.51$ for the three Payoff Feedback sessions. In the remaining 3-players sessions, which used Mixed Feedback, the average value of S is 0.13.

As a robustness check we consider an alternative measure of payoff asymmetry given by

$$S' = \frac{(\pi_1 - \pi_n)/R}{x_1 - x_n},$$

where R is the number of rounds in the continuation game from the point at which symmetry is broken. One difference between S' and S is that they treat differently the normalization between games with a very low value of x_n and those with more equal payoffs. The results that we obtain, though, are qualitatively similar using both measures. These results, along with those on the efficiency of continuation play, are depicted at the session level in Table 2.

All of these results show that subjects clearly manage to use sufficiently complex continuations in order to reduce ex-post asymmetry in payoffs. This is true even in the 3-player Payoff

Feedback sessions, and much more so in the other sessions. Our theory explains this play as arising from a norm of ex-ante Pareto efficiency.

7.5. Continuation strategies: 2-player games

Thus far we have provided evidence that, as predicted, subjects in our experiments play repeated allocation games in such a way as to achieve nearly ex-post payoff-symmetric outcomes. We turn now to analyzing which strategies are used to deliver such payoffs. A useful starting point for analyzing the continuation play after symmetry is first broken is to determine which continuations are most payoff-symmetric in the experimental treatments. The answer to this question depends on the design variables of the experiment.

We consider first the 2-player games under Random Stopping (sessions 1 and 2). Payoff symmetry requires that the expected continuation payoffs to playing H and L, conditional on symmetry being broken, are equal. This is equivalent to the requirement that, in the infinitely repeated game with discount factor equal to the probability of continuation under Random Stopping, the (non-random) continuation payoffs to playing H and L, conditional on symmetry being broken, are equal.Recall that there are many such continuations that deliver exactly symmetric payoffs in the infinitely repeated game, provided δ is large enough, which holds in our experiments. One example is the construction of awarding the high payoff to the player with the smaller current cumulative payoff. A different construction that is also exactly symmetric works by awarding the high payoff to the player who won the high payoff at the previous stage, provided that his new normalized discounted payoff would not exceed $(x_1 + x_2)/2$, in which case the payoff is awarded to the opponent.

However both of these continuations, and every other exactly egalitarian continuation at $\delta = 11/12$, are very complex in the sense of automaton state complexity.[27] The multiplicity of ex-post symmetric continuations, together with the complexity of each one of them, makes such continuations unlikely candidates for observed behavior. There are much simpler strategies that are very nearly payoff-symmetric. The simplest among these are rotation schemes or, in the case of two players, turn-taking.[28]

We consider next the 2-player games under Fixed Stopping without discounting (session 3). All two player games have a High payoff (H) and a Low payoff (L). Any continuation play that splits the H and L payoffs equally between the two players, and only these continuations, are most payoff-symmetric. It does not matter in what order the H and L payoffs are awarded. When there are an even number of rounds left at the moment symmetries are broken, each player must receive the H payoff in exactly half of the remaining rounds. When there are an odd number of rounds remaining, one player must receive H exactly one more time than the other player.

Thus, as under Random Stopping, many continuation profiles deliver payoffs that are maximally ex-post symmetric. Efficiently implementing a payoff-symmetric outcome, though, requires the players to coordinate on a particular continuation play from the many acceptable ones. Our analysis above suggests that, among all such continuation plays, simplicity is the natural focal criterion for selecting a continuation play in this setting. As proven above, rotation is the

[27] To illustrate, the first 20 elements of the sequences are, respectively, $(1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, \ldots)$ and $(1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)$.

[28] For $\delta = 11/12$, in all of the games we study, the ex-ante efficiency loss in equilibrium associated with rotation, relative to the exactly payoff-symmetric strategies, is never more than 1%; see Table 5.

Table 3
The columns represent the maximally payoff-symmetric continuation for 2-player games under Fixed Stopping with payoffs discounted at rate $\delta = 11/12$. "1" represents the H-player playing H and "0" represents the H-player playing L.

Round	Rounds remaining in continuation play upon the breaking of symmetry																			
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	
3	0	0	0	0	1	0	0	1	0	0	1	1	0	1	0	0	0	0		
4	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1			
5	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0				
6	1	1	0	0	0	0	0	0	1	0	1	1	1	0	1					
7	1	1	1	0	0	1	0	0	1	1	0	0	1	0						
8	1	0	1	0	1	1	1	0	0	0	1	0	0							
9	0	1	0	1	0	1	1	1	0	0	0	0								
10	0	0	1	0	0	0	0	1	1	0	1									
11	1	1	0	1	0	0	0	0	1	1										
12	1	0	0	0	1	1	1	0	0											
13	1	1	1	0	1	1	0	1												
14	1	1	1	1	1	0	1													
15	0	0	1	0	1	1														
16	0	0	0	0	0															
17	1	1	1	0																
18	1	0	1																	
19	1	0																		
20	0																			

simplest such continuation, in the sense of minimal state complexity of an automaton that implements the strategy.[29]

The remaining case for 2-players games is Fixed Stopping with discounting (sessions 4 and 5). Here, the maximally payoff-symmetric continuation play is unique, and it depends on the round in which symmetry is broken. These continuations are presented in Table 3.[30] It is obvious that this continuation protocol is highly complex and changes unpredictably with the round at which symmetry is broken. We thus predict that a much simpler but nearly payoff-symmetric continuation play will be used instead. Rotation is, again, one obvious candidate. Session 5 was designed to encourage the continuation play in which the player with the lower current cumulative payoff plays H, in order to assess how robust the convention of rotating is in these experiments.

Summarizing, our hypothesis is that rotation schemes will be observed frequently in sessions 1–4. In session 5 we expect to observe either rotation or the awarding of the high payoff to the currently disadvantaged player. We examine two summary statistics regarding rotation schemes. The first is very demanding. We look at every observation in which a group broke symmetry with at least one round of continuation play remaining. Most of these observations,

[29] The proof of Proposition 8 is easily adapted to the finite horizon case. We remark that, in the case of non-discounted payoffs, rotation also coincides with a version of awarding the high payoff to the player with the lower cumulative payoff, in which ties are always awarded to the same player.

[30] We remark that this finding contradicts Proposition 4 in Bhaskar [7], which claims incorrectly that for any finite horizon and any discount factor, the convention of awarding the higher payoff to the currently disadvantaged player is most egalitarian. See Kuzmics and Rogers [26] for details.

Table 4
Frequency of rotation schemes.

Session	# Obs.	Perfect rotation or flipping			Ending rotation or flipping		
		# Rotate/flip	Ratio	Totals	# Rotate/flip	Ratio	Totals
1	44	39	89%		40	91%	
2	45	41	91%		43	96%	
3	60	41	68%	81%	51	85%	89%
4	80	70	88%		74	93%	
5	69	51	74%		56	81%	
6	22	0	0%		2	9%	
7	28	1	4%	3%	1	4%	5%
8	25	1	4%		1	4%	
9	43	10	23%		19	44%	
10	38	10	26%		16	42%	
11	49	12	24%	30%	26	53%	52%
12	50	13	26%		29	58%	
13	50	23	46%		29	58%	

of course, have many rounds remaining. Among all such groups, we count the number of groups that immediately enter a rotation scheme and maintain it perfectly until the end of the match.

The findings are summarized in Table 4. Pooling all of the 2-player sessions, we find that 81% of these groups play a perfect rotation scheme in the continuation play. While already high, this statistic omits many groups who have nearly perfect rotation, or who have perfect rotation from some point on that is strictly after the round when symmetry was broken. To account for such observations, we look also at the frequency of groups who end the match with $n = 2$ rounds of rotation. In this case the observed frequency of success raises to 89%. We thus take this as strong evidence that rotation schemes are the dominant choice of continuation play. Even in session 5 81% of groups end with rotation, and 74% of groups rotate perfectly upon the first breaking of symmetry, despite the framing of the feedback. Simplicity, paired with Pareto efficiency, is indeed a powerful criterion for focality.[31]

7.6. Continuation strategies: 3-player games

The main thrust of our argument and the conclusions from the 2-player games hold also in 3-player games. We discuss them separately only because there is an additional subtlety that arises, namely, that the rotation scheme is not unique when $n > 2$. As discussed in Section 6, there is one rotation scheme for every ordering over actions. One particular such rotation scheme must be coordinated on in order to achieve positive payoffs.

[31] These criteria are not special to the laboratory, as suggested by the example of rotation schemes in pickup sports games mentioned in footnote 10.

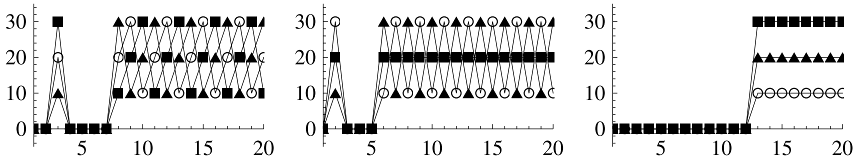

Fig. 2. Examples of play from three matches, showing a rotation scheme (left), flipping (center) and the simplest, but payoff-asymmetric, play of remaining with the same actions (right). The data is from Session 11, match 3 and group 4 (left), match 4 and group 3 (center), and match 4 and group 5 (right).

In every session and every group within each session that plays a rotation scheme, the observed order over actions is that of increasing payoffs. The unanimity of this convention is striking.[32]

Four of the sessions (6, 9, 11, and 12) have payoffs given by $x = (30, 20, 10)$. This is a knife-edge case that violates the generic condition of strongly distinct payoffs. As a result, in these sessions it is not true that rotation schemes are the uniquely simplest strategies that are ex-post payoff-symmetric as δ tends to one. Indeed, there is also the "flipping" convention in which $30 \to 10 \to 30$ and $20 \to 20$. In fact, there is a sense in which this scheme may be considered simpler than rotation. There is an automaton representation of flipping that can be reduced to two states after a finite amount of time, which is not true of a rotation scheme.

To illustrate the typical patterns of play in our data we refer to Fig. 2. Each panel depicts the play of one group in one match from one session (Session 11). The horizontal axis codes the rounds within the given match, while the vertical axis codes the payoff of subjects in each round. The left panel shows a rotation scheme, the center panel shows an example of flipping, and the right panel shows a group that, upon the breaking of symmetry, played the same actions in all remaining stages of the game, resulting in asymmetric payoffs. Notice that in this final case, it happened to take longer for subjects to break symmetry.

As noted above, the 3-player sessions with Payoff Feedback (sessions 6–8) suffer from an inability of subjects to effectively coordinate their continuation play. As a result, the proportion of groups that end the match with a round of rotation or flipping is very small – 5% in aggregate. The groups in the remaining sessions (9–12) average 52%, with 30% rotating or flipping perfectly starting when symmetry is broken. These proportions are lower than the corresponding figures for 2-player games, and in this sense coordination is less successful in the 3-player games. Of course, this pattern is to be expected, as the coordination problem is significantly harder in the 3-player games. Given the difficulty of the problem, and the observation that perfect rotation is a demanding requirement, we view these figures as a very positive result, demonstrating that coordination is achieved frequently even in the more difficult 3-player games.

The groups that coordinate efficiently in sessions with $x = (30, 20, 10)$ use exclusively the flipping scheme and never a rotation scheme. Moreover, these sessions have weakly higher rates of perfect flipping than the rotation rates in sessions with strongly distinct payoffs. Both of these findings are consistent with our observation that flipping may be considered more simple than rotation.

32 We should point out that the order of the action buttons on subjects' screens from left to right coincided with increasing payoffs. So it is possible that the convention was "left to right." We cannot identify which of these conventions drives our finding.

Table 5
Mixing probabilities before symmetry is broken under Random Stopping. Mixing probabilities are then used to compute the value of the game, which is also listed as a proportion of the value obtained under optimal uniform mixing.

	Payoff vector				
	(30, 10)	(30, 1)	(30, 20, 10)	(30, 20, 1)	(30, 5, 1)
Strategy	(.569, .413)	(.625, .375)	(.349, .333, .317)	(.347, .356, .297)	(.378, .339, .283)
Value	18.41	14.16	11.99	10.16	7.13
Efficiency	99.7%	99.0%	99.9%	99.6%	99.0%

7.7. Breaking symmetry and equilibrium mixing: theoretical predictions

Having presented the findings on continuation play from the stage in which symmetry is broken, we turn now to analyzing play in stages where symmetry is present. Our analysis provides an explicit link between the continuation play and mixing probabilities in the presence of symmetry. Indeed, one of the main points of the analysis is that continuation strategies that deliver more egalitarian payoffs have mixing probabilities that are closer to uniform, and hence are more efficient. Before presenting the empirical findings, we first extend the theory to the games that are implemented in the lab.

Given the predominance of rotation and, in the case of $x = (30, 20, 10)$, flipping, observed in the data, we conduct our main analysis of mixing probabilities under the assumption that all players anticipate this continuation play upon the initial breaking of symmetry.

We first discuss Random Stopping. Mixing probabilities in the symmetric equilibrium depend on the continuation payoffs and the discount factor. We describe the computation of equilibrium mixing for the case of the 2-player games; the three player games work similarly. The utility of action H is computed as

$$u(H) = \alpha\delta v + (1-\alpha)\frac{x_1 + \delta x_2}{1-\delta^2},$$

where α is the probability the opponent plays H and v is the value of the game. The utility of L, $u(L)$, is computed similarly. The symmetric equilibrium values of α and v are those which solve the system $u(H) = u(L) = v$. Under a rotation or flipping scheme, equilibrium mixing probabilities are stationary. Finally, the (normalized discounted) value of the game can be compared to the value that would obtain under optimal uniform mixing. Table 5 summarizes these computations applied to our sessions.

In the Fixed Stopping treatment, equilibrium mixing is not stationary. The mixing probabilities are easily computed, though, via backward induction. We again describe the procedure for the two player games; the three player games work similarly. In the final round, round twenty, the mixing probabilities are simply those of the (Pareto efficient) symmetric Nash equilibrium. This equilibrium has associated with it a value, $v(20)$. In the penultimate round, the utility of action H is computed as

$$u(H|19) = \alpha v(20) + (1-\alpha)(x_1 + x_2),$$

where α is the probability the opponent chooses H. This is because when playing H, with probability α symmetry is not broken and the total value of the game reflects the value at the last stage, $v(20)$, while with probability $(1-\alpha)$ symmetry is broken and a payoff of x_1 is obtained in stage 19 and, given the rotation scheme, a payoff of x_2 is obtained in the last stage. The payoff to action L is computed similarly, and the equilibrium mixing probability is that which equates the

Table 6
Mixing probabilities before symmetries are broken under Fixed Stopping. Bold numbers indicate probabilities that exceed uniform.

Round	Payoff vector		
	(30, 10)	(30, 20, 10)	(30, 5, 1)
1	uniform	uniform	(**.357**, **.370**, .273)
2	(**.622**, .378)	(**.354**, .333, .313)	(**.394**, .310, .296)
3	uniform	uniform	uniform
4	(**.622**, .378)	(**.355**, .333, .312)	(**.358**, **.372**, .270)
5	uniform	uniform	(**.397**, .308, .295)
6	(**.623**, .377)	(**.356**, .333, .311)	uniform
7	uniform	uniform	(**.359**, **.374**, .267)
8	(**.623**, .377)	(**.357**, .333, .310)	(**.402**, .307, .291)
9	uniform	uniform	uniform
10	(**.623**, .377)	(**.359**, .333, .308)	(**.362**, **.379**, .259)
11	uniform	uniform	(**.411**, .303, .286)
12	(**.623**, .377)	(**.362**, .333, .305)	uniform
13	uniform	uniform	(**.367**, **.387**, .246)
14	(**.623**, .377)	(**.366**, .333, .301)	(**.428**, .296, .276)
15	uniform	uniform	uniform
16	(**.626**, .374)	(**.375**, .333, .292)	(**.379**, **.406**, .215)
17	uniform	uniform	(**.474**, .279, .247)
18	(**.638**, .362)	(**.395**, .333, .272)	uniform
19	uniform	uniform	(**.431**, **.487**, .082)
20	(**.750**, .250)	(**.500**, .333, .167)	(**.833**, .139, .028)

expected utilities to the two actions. The value of the game at this stage, $v(19)$ is then computed as the expected utility of either action at the equilibrium value of α. The mixing probability in the previous stage can now be computed using this value and the maintained assumption of the rotation scheme. The equilibrium is determined recursively in this manner. Table 6 lists the mixing probabilities for each of the three games we study.

Notice first that there is an effect across rounds modulo the periodicity of the anticipated rotation or flipping scheme. In each game, equilibrium mixing is uniform when the number of rounds remaining is divisible by the periodicity of the continuation play. In other periods, mixing is non-uniform, and reflects the relative positions of the actions over the remaining finite horizon, conditional on symmetries being broken in that period. These positions are not necessarily ordinally the same as the stage game payoffs to the actions. Notice, e.g., the last column of Table 6 in which in the first round (and every three rounds thereafter) the middle action is played with higher probability than the high action.

7.8. Breaking symmetry and equilibrium mixing: evidence

The first issue we address is whether subjects indeed play (totally) mixed strategies in the presence of symmetries. Though we do not observe mixed strategies directly, there is evidence that subjects do indeed mix. Symmetry is broken in almost all of the groups that play repeated allocation games. For instance, focusing on Fixed Stopping, there are but 2 groups out of 360 that never coordinate before the end of the match, receiving a total payoff of zero for the match. Under Random stopping, there are more instances of groups never coordinating (24 out of 200 under Mixed Feedback and 51 out of 140 under Payoff Feedback), but most of these observations are simply due to the match having very few rounds.

For each treatment, under the presumption that subjects have in mind the rotation or flipping scheme that is empirically observed to be the dominant play in that session, our theory makes an explicit prediction for how subjects should mix before symmetry is broken. These are contained in Tables 5 and 6. We estimate a mixing distribution in the following way. For every session, we look at all rounds in which symmetry has not been broken, and we take the empirical frequencies over pure actions from these observations as an estimate of the mixing distribution. Aggregating the data in this way assumes stationarity across rounds, which is theoretically justified only under Random Stopping.

The data is summarized in Table 7. The panel labeled "All Rounds" lists the frequencies of action choices from all rounds in which symmetry had not yet been broken in the group. These distributions can be compared to the equilibrium mixed strategies computed above. For the Random Stopping sessions, this comparison is straightforward.

For the Fixed Stopping sessions, we observe first that there is no evidence in the data of the predicted periodicity effect. This can be seen by looking at the last panel of Table 7, labeled "Rounds with Uniform Prediction." There is no systematic or significant difference between play in all rounds and play in those rounds in which the prediction is uniform.[33]

For all sessions except session 1, Pearson chi-squared tests reject the hypothesis that the empirical frequencies come from the theoretical distribution of the model. But the differences are mostly qualitatively small. In most sessions, the main discrepancy is that H is overplayed. This could be due to uncertainty in the continuation play that will be used. There is one notable exception: in session 2 L is overplayed. This session has payoffs $x = (30, 1)$. One possible explanation, of the form studied first by Stahl and Wilson [37] and taken up by Costa-Gomez, Crawford, and Broseta [18] and Camerer, Ho, and Chong [13], is that, anticipating a rotation scheme, and anticipating that an opponent is likely to play H (as is the case in other sessions), a level-k reasoner of the next level would optimally play L.

The estimated mixing distributions in Table 7 directly imply a probability of breaking symmetry, q, in any given round. These figures are listed under "Empirical q" in Table 8. These probabilities are maximized at the uniform distribution, and so they are bounded above by $\frac{1}{2}$ for 2-player games and by $\frac{6}{27}$ for 3-player games. Assuming independence across rounds, which empirically is not a bad approximation, any value of q induces a (geometric) distribution over the round in which symmetry is broken.

More directly, in the data we explicitly observe the breaking of symmetry. Fig. 3 depicts the distribution of times at which symmetry breaks for the Fixed Stopping sessions in both two-player (left) and three-player (right) games.[34] The empirical distributions are in blue, whereas the distributions implied by independent uniform mixing are in orange. As is clear in the figure, the distribution of times at which symmetry is broken is nearly as efficient as is feasible.

33 The rounds included in the last panel of Table 7 depend on the session. In all 2-player games they are the rounds in which an even number of periods remain. In the 3-player games with $x = (30, 20, 10)$ they are again the rounds in which an even number of periods remain, since the flipping scheme has periodicity two. In all other 3-player games they are the rounds in which the number of remaining periods is divisible by three.

34 The last bin represents groups that never broke symmetry, of which there is one observation in each panel. The Random Stopping sessions are similar, but are not as easily represented graphically because of the need to account for groups that do not break symmetry in games of different lengths.

Table 7
Aggregate action frequencies in rounds before symmetry is broken. "All Rounds" lists data from all such rounds. For sessions with Fixed Stopping, the right panel lists frequencies from the subset of rounds in which the theoretical prediction is uniform mixing.

Session	Action frequencies before symmetry is broken							
	All rounds				Rounds with uniform prediction			
	H	*M*	*L*	# Obs.	*H*	*M*	*L*	# Obs.
1	0.549		0.451	144				
2	0.486		0.514	208				
3	0.589		0.411	224	0.587		0.413	150
4	0.617		0.383	376	0.575		0.425	240
5	0.728		0.272	342	0.745		0.255	208
6	0.456	0.313	0.231	528				
7	0.402	0.363	0.235	498				
8	0.569	0.253	0.178	687				
9	0.375	0.406	0.219	753				
10	0.463	0.270	0.268	627				
11	0.329	0.362	0.309	741	0.292	0.380	0.328	411
12	0.407	0.422	0.171	666	0.408	0.440	0.152	375
13	0.483	0.262	0.255	726	0.497	0.246	0.256	195

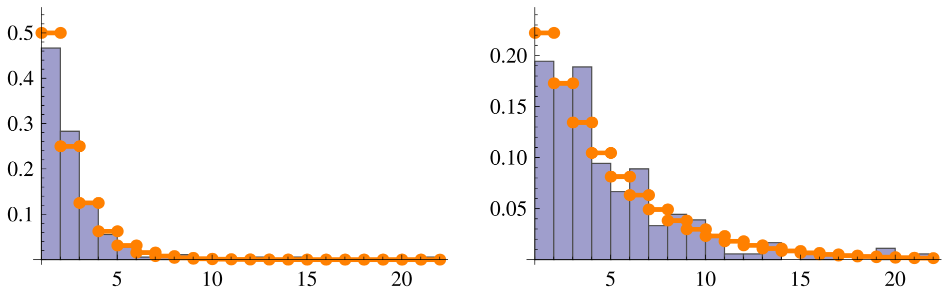

Fig. 3. Distribution of times at which symmetry is broken under Fixed Stopping for two-player (left) and three-player Mixed Feedback (right) sessions. The empirical distributions are in blue, and the theoretical distributions corresponding to uniform mixing are in orange.

More rigorously, we use this data to obtain an estimated value of q through a maximum likelihood approach. Some groups never break symmetry, and those observations are thus treated as truncated data. The likelihood function is easily derived as

$$\mathcal{L}(q) = -n \log(q) - \left(\sum_{m=1}^{M} n_m^* T_m + \sum_{g=1}^{n} (t_g - 1) \right) \log(1 - q),$$

where $t = (t_1, \ldots, t_n)$ is a vector denoting the round at which symmetry was broken for every group that indeed broke symmetry at some point, T_m is the number of rounds that were played in match m, and n_m^* is the number of groups in match m that never broke symmetry. The results of this estimation are presented in the last panel of Table 8.

For the most part, the values of q estimated from the empirical mixing frequencies are very close to those estimated from the distribution over the round in which symmetry was broken. Nor

Table 8
Coordination probabilities. "Empirical q" lists probabilities induced by the distributions in Table 7. "Maximum Likelihood" lists estimates obtained from the observations of when symmetry is broken, with associated negative log likelihoods.

Session	Empirical q	Maximum likelihood	
		q	$\mathcal{L}(q)$
1	0.495	0.653	46.49
2	0.500	0.452	71.61
3	0.484	0.536	77.35
4	0.473	0.426	128.22
5	0.396	0.404	115.32
6	0.198	0.165	78.76
7	0.206	0.199	82.79
8	0.154	0.118	83.06
9	0.200	0.171	114.95
10	0.200	0.187	100.58
11	0.221	0.198	123.04
12	0.176	0.225	118.42
13	0.193	0.206	123.28

Table 9
Correlations between payoff symmetry and mixing efficiency.

		Mixing efficiency	
		Empirical q	MLE q
Payoff	S	−0.237	−0.490
Symmetry	S'	−0.141	−0.413

is there a systematic pattern of differences between the two estimation procedures. We take this as evidence that our estimation procedures are informative.

A striking feature of Table 8 is that, except in the Payoff Feedback sessions, the estimated values of q are very efficient. That is, they are close to $\frac{n!}{n^n}$, which is the value obtained under uniform mixing. This is precisely what is required for equilibrium mixing, given that we observe a preponderance of very nearly ex-post payoff-symmetric continuation play.

There is some variation across sessions with respect to how quickly subjects break symmetry. Our theory suggests that the sessions in which symmetry is broken most quickly will be those in which payoffs are most nearly symmetric. This is in fact precisely what we observe. For example, notice that the 3-player games with Payoff Feedback (sessions 6–8) have both low estimated values of q and high measures of payoff asymmetry, as subjects more frequently simply repeat the actions that led to positive payoffs. To express this relationship, we compare the measures of payoff symmetry S and S' from Table 2 with the estimated values of q from Table 8. Table 9 shows the correlations between payoff symmetry and efficiency of mixing probabilities for all 3-player games (sessions 6–13). Our hypothesis is that these correlations should be negative. In all four cases, this prediction obtains in our data.[35] In summary, the evidence for how subjects

[35] When looking at only the 3-player games with Mixed Feedback, the conclusion strengthens, with all correlations below −0.5.

play before symmetry is broken is very much in line with our theoretical predictions, based on Pareto efficiency and simplicity in the context of repeated allocation game.

8. Conclusion

This article provides a theory of behavior in an economically relevant class of symmetric repeated games, and reports results from a set of laboratory experiments that study behavior in these games. The theory elicits a focal point as a way to play the game suggested by the structure of the game itself. For such a theory to be meaningful, it should be based on principles that are broadly applicable, which we think of as conventions or meta-norms. The criteria that we leverage in this regard are Pareto optimality and simplicity. For the allocation games that we study, under the constraints of symmetric play, these criteria are capable of making an explicit prediction for equilibrium play in repeated allocation games.

This play consists of two phases. In the first phase, subjects mix symmetrically over pure actions until coordination is achieved. In the second phase, a continuation game ensues in which play can be asymmetric, delivering continuation payoffs to the players conditional on the action they played when symmetry was broken.

As first suggested by Bhaskar [7], our results highlight a general tradeoff between Pareto efficiency at the ex ante stage, and the extent of payoff asymmetry ex post. In particular, our finding that not all feasible payoffs can be supported by symmetric equilibria (Proposition 1) derives from the fact that ex-post payoff asymmetry induces inefficient mixing initially as players compete to obtain the higher continuation payoffs, resulting in costly delay. Thus, Pareto optimality dictates ex post symmetric payoffs (Corollary 1). In this sense, egalitarianism can be justified on the basis of efficiency alone, without reference to fairness per se.

The experiments that we run in the laboratory resoundingly support simplicity as a criterion to dictate how to deliver the efficient payoff profile.[36] Rotation schemes are, by far, the predominant type of continuation play. We explain this observation by virtue of the fact that rotation schemes deliver nearly symmetric payoffs, thereby incentivizing players to mix efficiently and minimize the delay cost imposed by symmetry.

Our analysis leaves open a number of questions. First, it would be interesting to explore the extent to which the link between payoff symmetry and Pareto efficiency extends to other symmetric games. The main intuition driving our results suggests that, in many contexts, there will be a tradeoff between payoff symmetry and ex ante efficiency. More generally, it would be useful to have a better understanding of the restrictions implicit in symmetric play in repeated games.

Second, it should be possible to extend our characterization of the Thue–Morse sequence, as the limit of sequences that award high payoffs to disadvantaged players, to games with more than two players. In particular, it would be interesting to know if its property of being most nearly ex-post payoff-symmetric for high discount factors is true more generally. Also, Proposition 6 could potentially be strengthened so as to dispense with the requirement of periodicity. Finally, it is potentially interesting to understand if there are other repeated games for which Pareto efficiency at high discount factors implies similar play to that described by the Thue–Morse sequence.

[36] The experiments in fact support simplicity as a criterion that comes after efficiency. Players do not use the simplest strategies. They instead use the simplest strategies among those that are close to efficient.

Appendix A. Proof of Proposition 2

Fix an allocation game Γ. Let $X = \sum_i x_{a_i}$. Fix any payoff profile $z \in \mathcal{P}_\Gamma$ on the Pareto frontier of Γ that is not completely symmetric, i.e., for which there exist i and j with $z_i \neq z_j$. In fact, let, without loss of generality, $z_1 \geq \cdots \geq z_n$. Fix an $\epsilon > 0$ and, finally, consider any payoff profile $w \in \mathcal{U}^z_\epsilon$.

We will demonstrate that ϵ can be chosen small enough such that for every δ large enough to allow $w \in \mathcal{F}^z(\delta)$, and every symmetric stationary semi-public strategy profile σ that delivers payoffs w, it must be that σ is not an equilibrium.

Since σ is stationary semi-public it induces a particular probability of coordination, q, at every stage in which symmetries are unbroken. It prescribes, as well, a given $\bar{w}$ as the continuation payoff profile upon breaking of symmetries. The most stringent case to consider is when σ prescribes an efficient (always coordinated) continuation play upon the breaking of symmetries, so that $\sum_i \bar{w}_i = X$. Defining

$$V(q) = q \sum_t (1-q)^t \delta^t = \frac{q}{1 - \delta(1-q)},$$

we have, by definition, that $w = \bar{w} V(q)$.

Consider a unilateral deviation from σ to the strategy σ' that plays a_1 with probability one at every history before symmetries are broken, and then plays according to σ in the continuation. For a contradiction, we want to show ϵ can be chosen small enough such that this deviation is profitable for every δ at which w is feasible.

Observe that $q = n!(1 - \sum_{i>1} \sigma_{a_i}) \prod_{i>1} \sigma_{a_i}$. Similarly, the coordination probability induced when one player, say i, uses σ' and the other players all use σ, is $q' = (n-1)! \prod_{i>1} \sigma_{a_i}$. Thus, $q = n(1 - \sum_{i>1} \sigma_{a_i}) q'$ implying, since σ is totally mixed, that $q' > q/n$.

For $w \in \mathcal{U}^z_\epsilon$ it must be that $V(q) \geq 1 - \epsilon$, i.e.,

$$q \geq \frac{(1-\delta)(1-\epsilon)}{1 - \delta(1-\epsilon)} \equiv T.$$

The intuition is that the deviation to σ' is profitable because feasibility of w requires δ to be large enough that the reduction in coordination probability from q to q' is not too costly, whereas the gain, upon the breaking of symmetries, is significant.

We have

$$u(\sigma'_i, \sigma_{-i}) = \bar{w}_{a_1} V(q') > \bar{w}_{a_1} V(q/n) \geq \bar{w}_{a_1} V(T/n) \tag{1}$$

$$> \bar{w}_{a_1} \frac{1}{n(1-\delta)+\delta} \tag{2}$$

$$> \left(\frac{1}{n} \sum_i \bar{w}_{a_i} \right) V(q) \tag{3}$$

$$= u(\sigma). \tag{4}$$

Eq. (1) holds because V is increasing. Eq. (2) is true for sufficiently small ϵ because $\lim_{\epsilon \to 0} V(T/n) = \frac{1}{n(1-\delta)+\delta}$. There exists a $\bar{\delta}$ so that (3) holds for all $\delta > \bar{\delta}$, since as $\lim_{\delta \to 1} \frac{1}{n(1-\delta)+\delta} = 1 \geq V(q)$. Since $\bar{w}_{a_1} > \frac{1}{n} \sum_i \bar{w}_{a_i}$ by a given constant amount, ϵ and $\bar{\delta}$ can be chosen without reference to q, achieving the desired contradiction. Finally, notice that as $\epsilon \to 0$, feasibility requires that δ converge to one, while $\bar{\delta}$ can remain bounded away from one, completing the proof.

Appendix B. Proof of Proposition 3

Let $\bar{\mathcal{E}}(\bar{w}) \subset \mathcal{F}$ be such that any $(w^H, w^L) \in \bar{\mathcal{E}}(\bar{w})$ satisfies $w^H, w^L \leq \bar{w}$. We have $\bar{\mathcal{E}}(1) = \mathcal{F}$, but for $\bar{w} < 1$ $\bar{\mathcal{E}}(\bar{w})$ is a proper subset of $\mathcal{F}$.

The proof proceeds by showing that whenever $\bar{w} > \frac{3}{4}$, there is a $\bar{w}' < \bar{w}$ such that $f(\bar{\mathcal{E}}(\bar{w})) \subset \bar{\mathcal{E}}(\bar{w}')$. This directly implies that any $G \in \mathcal{G}$ with a payoff greater than $\frac{3}{4}$ cannot be a fixed point of f.

Let $(w^H_{HH}, w^L_{HH}) \in \bar{\mathcal{E}}(\bar{w})$ be the continuation after HH and $(w^H_{LL}, w^L_{LL}) \in \bar{\mathcal{E}}(\bar{w})$ be the continuation after LL. Furthermore let $(w^H, w^L) \in \mathcal{F}$ be the (immediately paid out) continuation after HL and LH. Let $\alpha \in [0, 1]$ denote the probability players attach to pure action H in stage 0, which must be the same for both players by symmetry.

At stage 0, when players contemplate their choice of α, expected payoffs from choosing pure action H and L are given by

$$u(H, \alpha) = \frac{w^H_{HH} + w^L_{HH}}{2}\delta\alpha + (1-\alpha)w^H,$$

and

$$u(L, \alpha) = w^L\alpha + (1-\alpha)\frac{w^H_{LL} + w^L_{LL}}{2}\delta.$$

This follows from the continuation payoffs and the fact that after HH and LL both players are equally likely, given attainability, to end up being the H-player or L-player at the moment when symmetries are broken.

Players now choose α such that neither of them has an incentive to deviate to another (mixed) action. Thus, we are looking for a symmetric equilibrium of the following symmetric 2×2 game.

	H	L
H	$\delta\frac{w^H_{HH}+w^L_{LL}}{2}, \delta\frac{w^H_{HH}+w^L_{LL}}{2}$	w^H, w^L
L	w^L, w^H	$\delta\frac{w^H_{LL}+w^L_{LL}}{2}, \delta\frac{w^H_{LL}+w^L_{LL}}{2}$

.

Note that if this game has only pure equilibria (such as H being a dominant strategy) then the ex-ante, at time 0, expected payoff must be less than or equal to $\delta\bar{w} < \bar{w}$. Thus the desired conclusion would hold in fact for any $\bar{w} > 0$.

So, the interesting case involves continuations such that this game has a strictly mixed equilibrium.

The unique completely mixed symmetric equilibrium is given by

$$\alpha^* = \frac{2w^H - (w^H_{LL} + w^L_{LL})\delta}{2(w^H + w^L) - (w^H_{HH} + w^L_{HH} + w^H_{LL} + w^L_{LL})\delta}.$$

Given the continuation profile and the induced α^* we then have that the ex-ante, at stage 0, expected payoff to the (eventual) H-player at the (possibly later) moment symmetries are broken is given by

$$w^* = (\alpha^*)^2 w^H_{HH}\delta + 2\alpha^*(1-\alpha^*)w^H + (1-\alpha^*)^2 w^H_{LL}\delta. \tag{5}$$

We are now trying to show that $w^* < \bar{w}$ whenever $\bar{w} > \frac{3}{4}$, given $w^H_{HH}, w^H_{LL} \leq \bar{w}$, as well as the aforementioned restrictions on the continuation profile, and subject to the incentive constraints.

In order to do so we distinguish two cases. Suppose first that $\alpha^* \leq \frac{1}{2}$. That is,

$$\frac{2w^H - (w^H_{LL} + w^L_{LL})\delta}{2(w^H + w^L) - (w^H_{HH} + w^L_{HH} + w^H_{LL} + w^L_{LL})\delta} \leq \frac{1}{2},$$

or, equivalently,

$$2w^H - (w^H_{LL} + w^L_{LL})\delta \leq 2w^L - (w^H_{HH} + w^L_{HH})\delta.$$

Thus,

$$w^H - w^L \leq \frac{1}{2}\big[(w^H_{LL} + w^L_{LL})\delta - (w^H_{HH} + w^L_{HH})\delta\big].$$

Given $w^H + w^L \leq 1$, we finally have,

$$w^H \leq \frac{1}{2} + \frac{1}{4}\delta \leq \frac{3}{4}.$$

From Eq. (5), as long as $w^H \leq \bar{w}\delta$ we must have $w^* \leq \bar{w}\delta$ as well. This is definitely true if $\frac{1}{2} + \frac{1}{4}\delta \leq \bar{w}\delta$, i.e., if $\bar{w} \geq \frac{\frac{1}{2}+\frac{1}{4}\delta}{\delta}$. Thus, if $\bar{w} > \frac{3}{4}$ (in the limit when $\delta \to 1$), we have $w^* < \bar{w}$, as desired.

Suppose now that $\alpha^* > \frac{1}{2}$. Obviously,

$$w^* \leq w^{**} = \max_{\alpha \in [\frac{1}{2},1], w^H, w^H_{HH}, w^H_{LL}} \alpha^2 w^H_{HH}\delta + 2\alpha(1-\alpha)w^H + (1-\alpha)^2 w^H_{LL}\delta,$$

subject to the given restrictions on w^H, w^H_{HH}, and w^H_{LL}. Further, the indifference condition[37] required to induce mixing implies that $w^H \leq 1 - \frac{w^H_{HH}+w^L_{HH}}{2}\delta$. w^{**} is increasing in each of w^H, w^H_{HH}, w^H_{LL}. Thus, $w^{**} \leq \max_{\alpha \in [\frac{1}{2},1], w^H_{HH} \in [0,\bar{w}]} \alpha^2 w^H_{HH}\delta + 2\alpha(1-\alpha)(1 - \frac{w^H_{HH}}{2}\delta) + (1-\alpha)^2\bar{w}\delta$. But for $\alpha > \frac{1}{2}$, this expression is maximized at $w^H_{HH} = \bar{w}$. Thus we have

$$w^{**} \leq \max_{\alpha \in [\frac{1}{2},1]} \alpha^2\bar{w}\delta + 2\alpha(1-\alpha)\left(1 - \frac{\bar{w}}{2}\delta\right) + (1-\alpha)^2\bar{w}\delta.$$

Therefore, $w^{**} \leq \bar{w}\delta$ if $(1 - \frac{\bar{w}}{2}\delta) < \bar{w}\delta$. This, in turn, is true if $\bar{w}\delta > \frac{2}{3}$.

To summarize, for δ close enough to 1, $\bar{w} > \frac{3}{4}$ ensures that in order for a set $\mathcal{E}^a(\delta)$ to be a fix point of the mapping f it has to satisfy $\mathcal{E}^a(\delta) \subset \bar{\mathcal{E}}(\frac{3}{4})$. □

Appendix C. Proof of Proposition 4

Note that each stationary semi-public symmetric strategy profile has associated with it a (normalized discounted) payoff $\bar{w}^H$, the continuation payoff to the player who plays H when

[37] There are two possibilities for a mixed equilibrium: either the game is a coordination game or of the Hawk–Dove variety. For a coordination game we must have $w^H \leq \frac{w^H_{LL}+w^L_{LL}}{2}\delta < \frac{1}{2}$ and $w^L \leq \frac{w^H_{HH}+w^L_{HH}}{2}\delta < \frac{1}{2}$. For a game to be of the Hawk–Dove variety we must have $w^H \geq \frac{w^H_{LL}+w^L_{LL}}{2}\delta$ and $w^L \geq \frac{w^H_{HH}+w^L_{HH}}{2}\delta$, which, by $w^H + w^L \leq 1$, implies that $w^H \leq 1 - \frac{w^H_{HH}+w^L_{HH}}{2}\delta > \frac{1}{2}$ (as long as δ sufficiently close to 1). Thus, for w^H the Hawk–Dove case is less restrictive than the coordination case.

symmetries are broken. $\bar{w}^H$ must be consistent with the initial mixing probability α. Notice that, provided $\delta \geq \frac{1}{2}$, any continuation $\bar{w}^H \in [0, 1]$ is feasible.[38] Given α, at any period in which players are symmetric, the probability of the symmetry breaking is $q(\alpha) = 2\alpha(1-\alpha)$. Ex ante, when mixing initially with α and playing an efficient continuation once symmetries are broken, each player has expected payoff

$$u(\alpha, \delta) = \frac{1}{2}\sum_{t=0}^{\infty} q(1-q)^t \delta^t = \frac{q}{2(1-\delta(1-q))}.$$

In order to incentivize the players to mix initially, we must have equal expected payoffs from either action, which requires

$$\alpha\delta u + (1-\alpha)\bar{w}^H = (1-\alpha)\delta u + \alpha\big(1-\bar{w}^H\big).$$

On the left hand side, when choosing H, there are two possibilities. With probability α the other player chooses H, symmetries are not broken, and the game continues at the next date with continuation payoff u. On the other hand, with probability $(1-\alpha)$ the other player chooses L, in which case symmetry is broken and the continuation $\bar{w}^H$ is realized. Similarly, when choosing L, with probability $(1-\alpha)$ symmetry is not broken, and when it is, a continuation of $1-\bar{w}^H$ is realized. Solving, we obtain

$$\bar{w}^H(\alpha, \delta) = \frac{\alpha(1-\delta\alpha)}{1-\delta(1-q(\alpha))}.$$

Even though players both expect u at the beginning of the game, ex post they may obtain different payoffs. We will show which payoff profiles (w^H, w^L) can be supported by symmetric stationary semi-public equilibria.

We have that

$$w^H(\alpha, \delta) = \bar{w}^H \sum_{t=0}^{\infty} q(1-q)^t \delta^t = \bar{w}^H \frac{q}{1-\delta(1-q)}$$

and

$$w^L(\alpha, \delta) = \big(1-\bar{w}^H\big)\sum_{t=0}^{\infty} q(1-q)^t \delta^t = \big(1-\bar{w}^H\big)\frac{q}{1-\delta(1-q)}.$$

For a given δ, these expressions trace out a parametric curve of equilibrium payoff profiles as α varies from zero to one, starting and ending at the origin. Notice that, given a pair (w^H, w^L), any payoff along the ray pointing to the origin is also supportable, by using an inefficient continuation that gives total payoff $F < 1$. We would then have $u = \frac{F}{2}\frac{q}{1-\delta(1-q)}$ and continuation payoffs once symmetries are broken of $(\bar{w}^H, F - \bar{w}^H)$. Thus, the region defined by the parametric curve represents the set of symmetric stationary semi-public equilibrium payoffs.

We now show that as δ tends to one, one third of feasible payoffs are supported by a symmetric stationary semi-public equilibria. First notice that $\lim_{\delta\to 1}(w^H(\frac{1}{2}, \delta), w^L(\frac{1}{2}, \delta)) = (\frac{1}{2}, \frac{1}{2})$. Define next the area

[38] This follows from Sorin [36], and appears as Mailath and Samuelson [30, Lemma 7.3.1].

$$A' = \lim_{\delta \to 1} \int_{x=0}^{\frac{1}{2}} w^H(\alpha, \delta) \frac{\partial w^L(\alpha, \delta)}{\partial \alpha} d\alpha,$$

which gives the limiting area under the upper lobe of equilibrium payoffs. Thus, the limiting area sustained by symmetric stationary semi-public equilibrium payoffs is

$$A = 2\left(A' - \frac{1}{8}\right) = \frac{1}{6},$$

where the area A' is computed with some straightforward but tedious calculus and algebra. □

Appendix D. Proofs of results for the Thue–Morse sequence

A simple lemma will prove useful.

Lemma 2. *Let z be the Thue–Morse sequence. Let $\boldsymbol{H}(t)$ be the number of 1's in the sequence z up to and including stage t. Define $\boldsymbol{L}(t)$, analogously, as the number of -1's in z through stage t. If t is odd then*

1. $z_t = -z_{t-1}$, *and*
2. $\boldsymbol{H}(t) = \boldsymbol{L}(t)$.

Proof. The first statement follows from directly from the definition, $-z_{2s+1} = z_{2s} = z_s$. The second statement follows from the first (by induction). □

Proposition 5. *For every t there exists a $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ the following is true. If $z_t = 1$ then $\Delta^z(\delta|_t) < 0$, and if $z_t = -1$ then $\Delta^z(\delta|_t) > 0$.*

Proof. The statement is obviously true for $t = 1$ and $t = 2$. Suppose now that $t \geq 3$ is odd. Then $\boldsymbol{L}(t-2) = \boldsymbol{H}(t-2)$ by the second part of Lemma 2. By the first part we then have $z_t = -z_{t-1}$. Suppose $z_{t-1} = 1$. Then $\boldsymbol{H}(t-1) = \boldsymbol{L}(t-1) + 1$ and, thus, there is a $\bar{\delta}$ such that for all $\delta > \bar{\delta}$ we have $\Delta^z(\delta|_t) > 0$. Given $z_t = 0$ the statement is true in this case. Now suppose that $z_{t-1} = -1$. Then $\boldsymbol{H}(t-1) = \boldsymbol{L}(t-1) - 1$ and, thus, there is a $\bar{\delta}$ such that for all $\delta > \bar{\delta}$ we have $\Delta^z(\delta|_t) < 0$. Given $z_t = 1$ the statement is true also in this case. This proves the statement for t odd.

We now turn to t even. Again, we know the statement is true for $t = 1$ and $t = 2$. Suppose the statement is true for all $\tau \leq t - 1$. As t is even set $s = t/2$. We have

$$\begin{aligned}
\sum_{i=0}^{2s-1} \delta^i z_i &= \sum_{i=0}^{s-1} \delta^{2i} z_{2i} + \sum_{i=0}^{s-1} \delta^{2i+1} z_{2i+1} \\
&= \sum_{i=0}^{s-1} (\delta^2)^i z_i - \delta \sum_{i=0}^{s-1} (\delta^2)^i z_i \\
&= (1-\delta) \sum_{i=0}^{s-1} (\delta^2)^i z_i
\end{aligned}$$

Since $s \leq t-1$, we know that if $z_s = 1$, then there is a $\bar{\delta}$ such that $\Delta^z(\delta^2|_s) < 0$ for all $\delta^2 > \bar{\delta}$ and so under the same condition, by the preceding development, $\Delta^z(\delta|_{2s}) < 0$. By definition, $z_t = z_{2s} = z_s = 1$, and the case is satisfied for all $\delta > \sqrt{\bar{\delta}}$. The argument is parallel for $z_s = -1$. □

Proposition 6. *For every k there exists a* $\bar{\delta} < 1$ *such that, for every sequence y with periodicity k,* $|\Delta^z(\delta)| < |\Delta^y(\delta)|$ *whenever* $\delta > \bar{\delta}$.

Proof. A sequence of periodicity k is characterized by its first k entries. Given its structure it is straightforward to express $\Delta^y(\delta)$ as a function of these first k entries. In fact,

$$\Delta^y(\delta) = \frac{1-\delta}{1-\delta^k} \sum_{t=0}^{k-1} \delta^t y_t.$$

Note that $\sum_{t=0}^{k-1} \delta^t y_t = \Delta^y(\delta|_k)$ is some polynomial in δ of degree $k-1$. We are interested in whether and, if so, at what rate, it converges to 0 as δ tends to 1. Let $A^0(\delta) = \Delta^y(\delta|_k)$. Suppose first that $A^0(1) \neq 0$. Then this sum, obviously, does not tend to 0. So suppose that $A^0(1) = 0$. Then $A^0(\delta)$ can be factored by $(1-\delta)$. Thus, let $A^0(\delta) = (1-\delta)A^1(\delta)$, where $A^1(\delta)$ is another polynomial in δ, but of degree $k-2$. We can, again, either have $A^1(1) = 0$ or not. In the latter case, the sum of interest then tends to zero at the same rate as $1-\delta$. In the former case, we can factor out another $1-\delta$. Repeating this argument $k-1$ times we obtain that there is a $\bar{\delta} < 1$ such that for all $\delta \geq \bar{\delta}$

$$\left|\Delta^y(\delta)\right| \geq \frac{(1-\delta)^k}{1-\delta^k}.$$

It remains to be shown that $\Delta^z(\delta)$ tends to zero faster than any such $\Delta^y(\delta)$. We have that $|\Delta^z(\delta)| = (1-\delta)\prod_{j=0}^{\infty}(1-\delta^{2^j})$.[39] Set $\Delta^z_s(\delta) = \prod_{j=0}^{s-1}(1-\delta^{2^j})$. Note that for any $j \geq 1$ the expression $(1-\delta^{2^j})$ can be written as the product of $(1-\delta^{2^{j-1}})$ and $(1+\delta^{2^{j-1}})$. The former term can then, by the same argument, be factorized into another two such terms. Repeating, and applying this factorization to all terms in the above product we obtain an alternative representation of $\Delta^z_s(\delta)$, given by

$$\Delta^z_s(\delta) = (1-\delta)^s \prod_{j=0}^{s-1} \left(1+\delta^{2^j}\right)^{s-j-1}.$$

Since, clearly, $\Delta^z_s(\delta) > \Delta^z_{s+1}(\delta)$ for all s, and $\Delta^z(\delta) = (1-\delta)\lim_{s\to\infty} \Delta^z_s(\delta)$, it is the case that $\Delta^z(\delta) < (1-\delta)\Delta^z_k(\delta)$.

Finally, we thus have

$$\begin{aligned}\Delta^z(\delta) &< (1-\delta)\Delta^z_k(\delta) \\ &= (1-\delta)^{k+1} \prod_{j=0}^{k-1} \left(1+\delta^{2^j}\right)^{k-j-1}\end{aligned}$$

[39] Calling $\{z_t\}_{t=0}^{2^j-1}$ the j-th block, this can be verified by observing that the difference in payoffs after the first block is $\Delta^z_0(\delta) = 1$. After the second block it is $\Delta^z_1(\delta) = (1-\delta)\Delta^z_0(\delta)$, after the third $\Delta^z_2(\delta) = (1-\delta^2)\Delta^z_1(\delta)$, and generally after the $j+1$-st it is $\Delta^z_j(\delta) = (1-\delta^{2^{z-1}})\Delta^z_{j-1}(\delta)$. Finally the desired difference is $\Delta^z(\delta) = (1-\delta)\lim_{j\to\infty} \Delta^z_j(\delta)$, which can be expressed as the infinite product given above.

$$< (1-\delta)^{k+1} \prod_{j=0}^{k-1} 2^{k-j-1}$$
$$= (1-\delta)^{k+1} 2^{\sum_{j=0}^{k-1} k-j-1}$$
$$= (1-\delta)^{k+1} 2^{\frac{k(k-1)}{2}}.$$

Thus, as δ tends to 1, $\Delta^z(\delta)$ tends to zero at least an order faster than $\Delta^y(\delta)$ when y has a given, but arbitrary, periodicity k. This completes the proof. □

Appendix E. On more general repeated symmetric 2-player games

In this section we demonstrate that symmetric equilibrium (without the stationary semi-public qualification) poses a restriction on the set of possible payoff profiles, of the kind demonstrated in Proposition 3, for a large set of repeated symmetric 2-player games with perfect monitoring. In particular, this class contains a large set of repeated 2-player allocation games, as well as a large set of repeated symmetric games of the prisoners' dilemma variety.

The proof is simple and uses only Proposition 3, the observation that for all games the set of individually rational and feasible payoff profiles can be made to be a subset of the set of feasible payoff profiles for the 2-player allocation game with payoff vector $x = (0, b)$, where b is some appropriately chosen positive real number, as well as the following lemma, which uses notation spelled out in Sections 3 and 4.

Lemma 3. *Let $I = \{1, 2\}$, let $A = \{H, L\}$ and let $\Gamma = (I, A, u)$ and $\Gamma' = (I, A, u')$ be two arbitrary symmetric two-player games with $u(a, b)$ the payoff to the player who plays pure action $a \in A$ against an opponent playing $b \in A$ for game Γ, and let $u'(a, b)$ be similarly defined for game Γ'. Let the set of feasible and individually rational payoff profiles be ordered such that $\mathcal{F}_{\Gamma'} \subset \mathcal{F}_{\Gamma}$. Then, for δ sufficiently close to 1, the symmetric equilibrium payoff profile sets are ordered such that $\mathcal{E}^s_{\Gamma'}(\delta) \subset \mathcal{E}^s_{\Gamma}(\delta)$.*

Proof. Let $\delta < 1$ be such that the set of all (possibly asymmetric) equilibria of the repeated game coincide with the set of feasible individually rational payoff profile for both games. That is, $\mathcal{E}_{\Gamma'}(\delta) = \mathcal{F}_{\Gamma'}$ and $\mathcal{E}_{\Gamma}(\delta) = \mathcal{F}_{\Gamma}$.

Recall that, as in the proof of Proposition 3, the set of symmetric equilibrium payoff profiles $\mathcal{E}^s_{\Gamma}(\delta)$ is given by the largest fixed point of some appropriate mapping f. In the proof of Proposition 3 we wrote this as a mapping from the set of all possible subsets of payoff profiles $\mathcal{G}$ to itself. We now need to emphasize how this mapping depends on the game. As before, any symmetric subgame perfect equilibrium of the whole game specifies two symmetric equilibria after HH and LL and one possibly asymmetric equilibrium after HL (and LH). These three equilibria then determine the randomization incentives for the players at stage 0 and, thus, determine the induced symmetric equilibrium of the whole game. Thus the set of symmetric equilibria is in fact the largest fixed point (in the first argument) of a mapping $f : \mathcal{G} \times \mathcal{E}_{\Gamma}(\delta) \to \mathcal{G}$, or for high enough discount factors, a mapping $f : \mathcal{G} \times \mathcal{F}_{\Gamma} \to \mathcal{G}$. Thus, the function f varies from game to game depending on the set of individually rational and feasible payoff profiles $\mathcal{F}_{\Gamma}$. Alternatively one can view this mapping as parameterized by $\mathcal{F}_{\Gamma}$.

Now suppose $\mathcal{F}_{\Gamma'} \subset \mathcal{F}_{\Gamma}$. Then, after HL or LH, under game Γ' there are fewer possible continuations than under game Γ. Thus this game has a smaller set of symmetric equilibrium payoff profiles. □

E.1. 2-player allocation games

Consider first 2-player allocation games. That is consider games given by

	H	L
H	$0, 0$	b, c
L	c, b	$0, 0$

,

where w.l.o.g. $b > c \geq 0$. Denote this game by $\Gamma(b, c)$.

Note that the set of feasible payoff profiles for game $\Gamma(b, c)$, which we denote by $\mathcal{F}_{\Gamma(b,c)}$, is a subset of $\mathcal{F}_{\Gamma(b+c,0)}$, the set of feasible payoff profiles for game $\Gamma(b + c, 0)$, which in turn is just $(b + c)$ times the set of feasible payoff profiles for game $\Gamma(1, 0)$.

By Proposition 3 we have that no symmetric equilibrium of the repeated game $\Gamma(b + c, 0)$ can give payoffs of more than $\frac{3}{4}(b + c)$. Thus, by Lemma 3, if $b > \frac{3}{4}(b + c)$ then also in game $\Gamma(b, c)$ the payoff of b (or even $b - \epsilon$) is not achievable in any symmetric equilibrium of the repeated game. This condition is met if $b > 3c$. This is true, for instance, for the game $\Gamma(4, 1)$.

E.2. 2-player games of the prisoners' dilemma variety

Consider now 2-player games of the prisoners' dilemma variety. That is games given by

	C	D
C	a, a	b, c
D	c, b	$0, 0$

,

where the payoff of $0, 0$ for D, D is w.l.o.g. and where $a < c$ and $b < 0$ in order for the game to have a dominant strategy D.

Now consider the following four cases.

Case 1: Let $a < 0$ and $b + c < 0$. Then the set of feasible individually rational payoff-pairs is the singleton $(0, 0)$ and is thus the only equilibrium payoff profile. Symmetry does not impose any additional restrictions on equilibrium payoff profiles.

Case 2: Let $a < 0$ and $b + c > 0$. Then there is an allocation game $\Gamma(\tilde{b}, 0)$ such that the set of feasible payoff profiles for the prisoners' dilemma is the same as that for $\Gamma(\tilde{b}, 0)$. Thus, for this game, Proposition 3, by force of Lemma 3, applies directly. Symmetric equilibrium poses exactly the same restriction for this prisoners' dilemma game as it does for the allocation game $\Gamma(\tilde{b}, 0)$.

Case 3: Let $a > 0$ and $a < \frac{b+c}{2}$. This case is equivalent to Case 2. Again, symmetric equilibrium poses exactly the same restriction for this prisoners' dilemma game as it does for the allocation game $\Gamma(\tilde{b}, 0)$.

Case 4: Let $a > 0$ and $a \geq \frac{b+c}{2}$. Then if, in addition, $c > \frac{3a-b}{2}$ (as is the case, for instance, for $a = 1$, $b = -1$, and c between 2 and 3), Proposition 3, by force of Lemma 3, implies that some feasible, highly asymmetric payoff profiles are not possible in a symmetric equilibrium of the repeated prisoners' dilemma game. The conditions derive from two observations. First, the set of individually rational and feasible payoff profiles for this prisoners' dilemma game is a subset of the set of feasible payoff profiles of the allocation game $\Gamma(2a, 0)$. Second, the highest feasible payoff to a player, while the other player receives a payoff of zero, is given by $a\frac{c-b}{a-b}$, which exceeds $\frac{3}{4}$ of $2a$ if $c > \frac{3a-b}{2}$. Thus, no payoff profile close to $(a\frac{c-b}{a-b}, 0)$ is possible in a symmetric equilibrium of the repeated game.

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