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Effects of background risks on cautiousness with an application to a portfolio choice problem ☆

Chiaki Hara [a,*], James Huang [b], Christoph Kuzmics [c]

[a] *Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan*
[b] *Department of Accounting and Management, Lancaster University Management School, United Kingdom*
[c] *MEDS, Kellogg School of Management, Northwestern University, United States*

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Abstract

We provide necessary and sufficient conditions on an individual's expected utility function under which any zero-mean idiosyncratic risk increases cautiousness (the derivative of the reciprocal of the absolute risk aversion), which is the key determinant for this individual's demand for options and portfolio insurance.

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* Corresponding author.
E-mail addresses: hara@kier.kyoto-u.ac.jp (C. Hara), james.huang@lancaster.ac.uk (J. Huang), c-kuzmics@kellogg.northwestern.edu (C. Kuzmics).

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1. Introduction

Portfolio insurance is a financial contract whose return depends on the payoff of a portfolio of assets in such a way that while a minimum level of return is guaranteed, the return is increasingly more aligned with the payoff of the underlying portfolio as the value of the latter goes up.

The *absolute risk tolerance* of a utility function is defined as the reciprocal of the Arrow–Pratt measure of absolute risk aversion. Its derivative is called the *cautiousness*. Cautiousness, first defined in [13], was shown in [11][1] to be the single key determinant in an individual's choice of whether or not to buy portfolio insurance by means of (put and call) options. Thus, the determinants of cautiousness are the determinants of the demand for options. In this paper we are interested in how a non-hedgeable zero-mean background risk affects an individual's cautiousness, and, thus, her demand for portfolio insurance.

In an economy in which all consumers have constant and equal cautiousness,[2] the *two-fund separation* holds: at every Pareto-efficient allocation, each consumer's consumption can be implemented by a portfolio consisting only of the bond and the market portfolio (the portfolio paying off the aggregate consumption). There is, in particular, no need to hold any portfolio insurance or other derivatives at any Pareto-efficient allocation, including the equilibrium allocations. This is somewhat puzzling in view of the recent surge in derivatives markets.

In [4] a solution to this puzzle was proposed. They showed (Theorem 3 of [4]) that even when all consumers have constant and equal cautiousness, if they also have heterogeneous background risks (risks that they themselves have to bear regardless of asset positions), then they may need portfolio insurance or other derivatives to implement their consumptions. More specifically, they showed that if a consumer has constant cautiousness, then the presence of a background risk increases his cautiousness; and then that in an economy in which there is only one consumer facing no background risk, the consumer sells portfolio insurance.

In this paper, we extend the result of [4] to the case where cautiousness need not be constant. We shall provide several versions of this extension, and the one that is easiest to grasp is perhaps inequality (15) in Theorem 2, which shows roughly that an essentially equivalent condition for any small background risk to increase cautiousness of a utility function v is that v is less risk-vulnerable than $-v'$ in the sense of [6, Eq. (10)]. Theorem 1 gives a more general equivalent condition in terms of higher order derivatives of v. Theorem 3 gives a sufficient condition for any risk, small or not, to increase cautiousness.

The main part of the paper then proceeds as follows. Section 2 formulates our problem. Section 3 provides a necessary and sufficient condition on the utility function to exhibit increased cautiousness under any small zero-mean risk. In this section we also discuss what it exactly means for a risk to be small. This condition obviously also serves as a necessary condition for increased cautiousness under all zero-mean background risks, small or not. Section 4 provides a sufficient condition on the utility function that ensures that cautiousness is increased under all zero-mean background risks, small or not. Section 5 provides a numerical example of how cautiousness and the demand for portfolio insurance is affected.

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[1] See also [2,3,7].

[2] Constant cautiousness is equivalent to linear absolute risk tolerance, and also to hyperbolic absolute risk aversion (HARA for short).

2. Setup

Let $v : (\underline{c}, \infty) \to \boldsymbol{R}$ be a von-Neumann Morgenstern utility function (also known as Bernoulli utility function), where $\underline{c} > -\infty$. Note that the domain is assumed to be bounded from below but not from above. Let v be at least twice continuously differentiable[3] and satisfy $v'(x) > 0$ and $v''(x) < 0$ for every $x > \underline{c}$, as well as the Inada conditions, $v'(x) \to \infty$ as $x \to \underline{c}$ and $v'(x) \to 0$ as $x \to \infty$.

We assume that there is a consumer with this utility function v, who in addition to being exposed to tradeable macroeconomic risk (which we do not explicitly model before Section 5), also faces an independent background risk ξ, which is assumed to be stochastically independent of the macroeconomic risk, and which the consumer cannot trade, i.e. is forced to absorb. This risk is described by a probability measure space $(\Theta, \mathcal{G}, Q)$, for which the expectation operator is denoted by E^Q or just E. The cumulative distribution function of ξ is denoted by $G : \boldsymbol{R} \to [0, 1]$. For simplicity, we use the following assumptions throughout the paper. First, the support of the distribution of G is bounded: there are two numbers $\underline{e}$ and $\bar{e}$ such that $G(\underline{e}) = 0$ and $G(\bar{e}) = 1$. Second, ξ has zero mean: $\int_{\underline{e}}^{\bar{e}} y \, dG(y) = 0$. The first assumption guarantees that all the expected values that we consider in the subsequent analysis are well defined and Leibnitz's rule is applicable, so that the order of integration and differentiation for smooth functions can be swapped. The second is a normalization and implies that $\underline{e} \leqslant 0$ and $\bar{e} \geqslant 0$.

Following [8] and [12], we can define the consumer's *induced utility function* by

$$u(x) = E\big(v(x + \xi)\big).$$

This is the utility function the consumer takes to the market, i.e., he makes decisions about what assets to buy on the basis of the induced utility function.

In this reformulation, the realized consumption level, inclusive of the realized background risk, must of course be in the domain $(\underline{c}, \infty)$ almost surely. To guarantee this, we concentrate on the consumption levels $x > \underline{c} - \underline{e}$. Denoting $\underline{d} = \underline{c} - \underline{e}$, the domain of the induced utility function u becomes $(\underline{d}, \infty)$. Define the consumer's original (absolute) risk tolerance, $s : (\underline{c}, \infty) \to \boldsymbol{R}_{++}$, by

$$s(x) = -\frac{v'(x)}{v''(x)}.$$

This is just the reciprocal of the consumer's Arrow–Pratt coefficient of absolute risk aversion $a(x) = -v''(x)/v'(x)$. The (absolute) risk tolerance of the corresponding induced utility function u shall be denoted by $t : (\underline{d}, \infty) \to \boldsymbol{R}_{++}$. By Leibnitz's rule,

$$t(x) = -\frac{E(v'(x + \xi))}{E(v''(x + \xi))}.$$

Following the terminology coined in [13] the derivative of risk tolerance shall be called (absolute) *cautiousness*. The consumer's original cautiousness is therefore given by $s'(x)$, while the consumer's induced (absolute) cautiousness is given by $t'(x)$ for x in the respective domain. Denote by $\psi(x)$ the prudence of v, as defined in [9]:

$$\psi(x) = -\frac{v'''(x)}{v''(x)}.$$

[3] The degree of continuous differentiability necessary for each of the subsequent results will be made clear in its proof.

Similarly denoting by φ the prudence of the induced utility function u, we have

$$\varphi(x) = -\frac{E(v'''(x+\xi))}{E(v''(x+\xi))}.$$

The following relationship among risk tolerance, prudence, and cautiousness is easy to prove and yet useful.

Lemma 1.

1. *For every* $x > \underline{c}$, $s'(x) = s(x)\psi(x) - 1$.
2. *For every* $x > \underline{d}$, $t'(x) = t(x)\varphi(x) - 1$.

Gollier and Pratt [6, Propositions 2 and 3] gave sufficient conditions under which, if ξ has a positive variance, then $t(x) \leqslant s(x)$, that is, the background risk makes the consumer less risk tolerant (more risk averse). They called utility functions having this property *risk-vulnerable*.

3. Necessary and sufficient conditions for small risks

The objective of this paper is to provide necessary as well as sufficient conditions for the cautiousness to increase under any zero-mean background risk. In symbols, this means that $t'(x) \geqslant s'(x)$ for every $x > \underline{d}$. This is equivalent to[4]

$$\frac{E(v'(x+\xi))E(v'''(x+\xi))}{(E(v''(x+\xi)))^2} \geqslant \frac{v'(x)v'''(x)}{(v''(x))^2}, \tag{1}$$

and also to

$$\big(v''(x)\big)^2 E\big(v'(x+\xi)\big)E\big(v'''(x+\xi)\big) - v'(x)v'''(x)\big(E\big(v''(x+\xi)\big)\big)^2 \geqslant 0. \tag{2}$$

Gollier and Pratt [6, Eq. (12)] provide necessary and sufficient conditions for risk aversion to increase under any *small* zero-mean background risk. Their proof is based on Taylor series expansions on relevant derivatives of the utility function v. We can similarly provide an analogous result of this necessary and sufficient condition for small background risks for the question at hand. The utility function v is *real analytic* if for each point in the domain of v, there exists a neighborhood of the point on which v coincides with an absolutely convergent power series. Every real analytic utility function is infinitely many times differentiable but the converse does not hold. Yet the class of real analytic utility functions is broad enough to contain all utility functions exhibiting hyperbolic absolute risk aversion.

Theorem 1. *Let* $(\xi_m)_m$ *be a sequence of random variables with mean zero and support in the interval* $[-b_m, b_m]$ *for every m, where* $b_m \to 0$ *as* $m \to \infty$*. Assume that v is real analytic and denote by* t_m *the induced risk tolerance of the consumer having the background risk* ξ_m*. Then, for*

[4] In this paper we, thus, are interested in conditions on the utility function v under which $\frac{d}{dx}(-\frac{E(v'(x+\xi))}{E(v''(x+\xi))}) \geqslant \frac{d}{dx}(-\frac{v'(x)}{v''(x)})$ for all (or all small) zero-mean background risks ξ. The same analysis can be used to identify analogous conditions under which $\frac{d}{dx}(-\frac{E(v^{(n-1)}(x+\xi))}{E(v^{(n)}(x+\xi))}) \geqslant \frac{d}{dx}(-\frac{v^{(n-1)}(x)}{v^{(n)}(x)})$ for any $n \geqslant 2$. We are grateful for an anonymous referee for pointing this out.

every $x > \underline{c}$, if $t'_m(x) \geqslant s'(x)$ for infinitely many m, then each (and, hence, all) of the following three equivalent inequalities holds:

$$v''(x)\big(v'''(x)\big)^2 + v'(x)v''(x)v^{(5)}(x) - 2v'(x)v'''(x)v^{(4)}(x) \leqslant 0, \tag{3}$$

$$\big(s(x)\big)^2 s'''(x) - 2s(x)\big(1 + 2s'(x)\big)s''(x) + 2\big(s'(x)\big)^2\big(1 + s'(x)\big) \geqslant 0, \tag{4}$$

$$a(x)a'''(x) - 2\big(a'(x) + \big(a(x)\big)^2\big)a''(x) + 2a(x)\big(a'(x)\big)^2 \leqslant 0. \tag{5}$$

Conversely if any (and, hence, all) of the above three equivalent inequalities holds as a strict inequality, then $t'_m(x) > s'(x)$ for every sufficiently large m.

The proof of this theorem, given in detail below, has three parts. In part 1 we show that, given our definition of small risks, it is sufficient to consider the first three terms of the Taylor series expansions of $v^{(n)}(x+\xi)$ for all relevant n. Note that this is not a trivial exercise. The Taylor series approximations are only valid if, in addition to all moments converging to zero ($E(\xi_m^k) \to 0$), we also have that the higher moments do so faster than the second moment ($E(\xi_m^k)/E(\xi_m^2) \to 0$). This is guaranteed under the assumptions we make about what constitutes a small risk, as we show in part 1 of the proof. In general, however, the convergence of all moments does not always imply the faster convergence of higher moments. For example, if ξ_m takes values 1 and -1 with probability $1/m$ each, and 0 with probability $1 - 2/m$, then, for every even k, $E(\xi_m^k) = 2/m \to 0$ and yet $E(\xi_m^k)/E(\xi_m^2) = 1$ for every m. For such cases the Taylor series approximation of order three is *not* a valid approximation. In such cases higher ($\geqslant 4$) order terms have essentially the same importance as the third order term. Indeed, the theorem does not hold for general risks that are "small" only in the sense of all higher order moments converging to zero. To eliminate such cases, we assumed in Theorem 1 that the sequence of supports converges to $\{0\}$ ($b_m \to 0$). Although the meaning of being small in "small risks" should be carefully specified, the need for convergence in supports has, as far as we know, not been explicitly stated in the relevant literature, such as [6].

In part 2 of the proof we use the Taylor series approximations of order three to prove the statement of the theorem for the first inequality (3). This part is brief. Finally, the longest, and somewhat tedious, part of the proof is part 3, in which we demonstrate the equivalence of the three inequalities (3), (4), and (5).

Proof of Theorem 1. Part 1 (our definition of small risks implies we can use Taylor series approximations of order three): Let $x \in (\underline{c}, \infty)$ and $b \in (0, x - \underline{c})$ be such that v is an absolutely convergent power series on $(x - b, x + b)$. Then, for every $n \geqslant 1$, $v^{(n)}$ is also an absolutely convergent power series on $(x - b, x + b)$ and, more specifically, can be written as

$$v^{(n)}(x+z) = \sum_{k=0}^{\infty} \frac{v^{(n+k)}(x)}{k!} z^k$$

for every $z \in (-b, b)$.

We can assume without loss of generality that $E(\xi_m^2) > 0$ and $0 < b_m < \min\{b, 1\}$ for every m. Then $|\xi_m|^k \leqslant b_m |\xi_m|^{k-1}$ and hence $E(|\xi_m|^k) \leqslant b_m E(|\xi_m|^{k-1})$. Thus $|E(\xi_m^k)| \leqslant b_m^k < b^k$ for every $k \geqslant 1$ and the infinite series

$$\sum_{k=0}^{\infty} \frac{v^{(n+k)}(x)}{k!} E\big(\xi_m^k\big)$$

is absolutely convergent. Moreover, as $m \to \infty$,

$$E\big(\xi_m^k\big) \to 0 \quad \text{for every } k \geqslant 2 \quad \text{and} \quad \frac{E(\xi_m^k)}{E(\xi_m^2)} \to 0 \quad \text{for every } k \geqslant 3. \tag{6}$$

Write

$$\sigma_m^2 = E\big(\xi_m^2\big) \quad \text{and} \quad \gamma_m^n = \frac{1}{\sigma_m^2}\sum_{k=3}^{\infty}\frac{v^{(n+k)}(x)}{k!}E\big(\xi_m^k\big). \tag{7}$$

By (6), as $m \to \infty$, $\sigma_m^2 \to 0$ and, since $E(\xi_m^k)/\sigma_m^2 \to 0$ for every $k \geqslant 3$, $\gamma_m^n \to 0$ for every n. By the dominated convergence theorem,

$$E\big(v^{(n)}(x+\xi_m)\big) = \sum_{k=0}^{\infty}\frac{v^{(n+k)}(x)}{k!}E\big(\xi_m^k\big) = v^{(n)}(x) + \big(v^{(n+2)}(x) + \gamma_m^n\big)\sigma_m^2 \tag{8}$$

for every n and every m.

Part 2 (using Taylor series approximations of order three, we prove the first statement of the theorem): Suppose that $t_m'(x) \geqslant s'(x)$ for infinitely many m. By taking a subsequence if necessary, we can assume that $t_m'(x) \geqslant s'(x)$ for every m. Thus, by (2),

$$\big(v''(x)\big)^2 E\big(v'(x+\xi_m)\big)E\big(v'''(x+\xi_m)\big) - v'(x)v'''(x)\big(E\big(v''(x+\xi_m)\big)\big)^2 \geqslant 0 \tag{9}$$

for every m. By (8), the left-hand side of (9) is equal to

$$\begin{aligned}&\big(v''(x)\big)^2\left(v'(x) + \left(\frac{v'''(x)}{2} + \gamma_m^1\right)\sigma_m^2\right)\left(v'''(x) + \left(\frac{v^{(5)}(x)}{2} + \gamma_m^3\right)\sigma_m^2\right)\\ &\quad - v'(x)v'''(x)\left(v''(x) + \left(\frac{v^{(4)}(x)}{2} + \gamma_m^2\right)\sigma_m^2\right)^2.\end{aligned}$$

Rearranging these terms and dividing them by σ_m^2, we obtain

$$\begin{aligned}&\big(v''(x)\big)^2 v'(x)\left(\frac{v^{(5)}(x)}{2} + \gamma_m^3\right) + \big(v''(x)\big)^2 v'''(x)\left(\frac{v'''(x)}{2} + \gamma_m^1\right)\\ &\quad - v'(x)v'''(x)2v''(x)\left(\frac{v^{(4)}(x)}{2} + \gamma_m^2\right)\\ &\quad + \left(\big(v''(x)\big)^2\left(\frac{v'''(x)}{2} + \gamma_m^1\right)\left(\frac{v^{(5)}(x)}{2} + \gamma_m^3\right) - \left(\frac{v^{(4)}(x)}{2} + \gamma_m^2\right)^2\right)\sigma_m^2 \geqslant 0\end{aligned} \tag{10}$$

for every m. As $m \to \infty$, $\sigma_m^2 \to 0$ and $\gamma_m^n \to 0$ for every n. Thus the left-hand side of (10) converges to

$$\big(v''(x)\big)^2 v'(x)\frac{v^{(5)}(x)}{2} + \big(v''(x)\big)^2 v'''(x)\frac{v'''(x)}{2} - v'(x)v'''(x)2v''(x)\frac{v^{(4)}(x)}{2}, \tag{11}$$

and this is non-negative, because it is the limit of a sequence of non-negative numbers. By multiplying $2/v''(x)$, which is negative, we obtain (3).

Part 3 (the three inequalities (3), (4), and (5) are equivalent): We shall now rewrite (3) in terms of $s(x)$ and its higher order derivatives to arrive at (4). Since dividing expression (3) by $(v'(x))^3$ does not change its sign,

$$\frac{v''(x)}{v'(x)}\left(\frac{v'''(x)}{v''(x)}\right)^2+\frac{v''(x)}{v'(x)}\frac{v^{(5)}(x)}{v'(x)}-2\frac{v'''(x)}{v'(x)}\frac{v^{(4)}(x)}{v'(x)}\leqslant 0. \tag{12}$$

By definition,

$$\frac{v''(x)}{v'(x)}=-\frac{1}{s(x)}.$$

Differentiating both sides of $-s(x)v''(x)=v'(x)$ with respect to x and using the above equality to eliminate $v''(x)$ from the expression, we obtain

$$\frac{v'''(x)}{v'(x)}=\frac{1+s'(x)}{(s(x))^2}.$$

We can similarly obtain

$$\begin{aligned}\frac{v^{(4)}(x)}{v'(x)}&=\frac{1}{(s(x))^2}\left(s''(x)-\frac{1}{s(x)}\big(1+s'(x)\big)\big(1+2s'(x)\big)\right),\\ \frac{v^{(5)}(x)}{v'(x)}&=\frac{1}{(s(x))^2}\left(s'''(x)-\frac{2}{s(x)}\big(2+3s'(x)\big)s''(x)\right.\\ &\qquad\left.+\frac{1}{(s(x))^2}\big(1+s'(x)\big)\big(1+2s'(x)\big)\big(1+3s'(x)\big)\right).\end{aligned}$$

Using these terms expression (12) is equal to

$$\begin{aligned}&\big(1+s'(x)\big)^2\\ &\quad+\Big(\big(s(x)\big)^2s'''(x)-2s(x)\big(2+3s'(x)\big)s''(x)+\big(1+s'(x)\big)\big(1+2s'(x)\big)\big(1+3s'(x)\big)\Big)\\ &\quad+2\big(1+s'(x)\big)\Big(s(x)s''(x)-\big(1+s'(x)\big)\big(1+2s'(x)\big)\Big)\end{aligned}$$

divided by $-(s(x))^5<0$. Then (3) is equivalent to this expression being non-negative. Rearranging the terms, we obtain (4).

As for (5), we can analogously show that

$$\begin{aligned}\frac{v''(x)}{v'(x)}&=-a(x),\\ \frac{v'''(x)}{v'(x)}&=\big(a(x)\big)^2-a'(x),\\ \frac{v^{(4)}(x)}{v'(x)}&=-\big(a(x)\big)^3+3a(x)a'(x)-a''(x),\\ \frac{v^{(5)}(x)}{v'(x)}&=\big(a(x)\big)^4-6\big(a(x)\big)^2a'(x)+3\big(a'(x)\big)^2+4a(x)a''(x)-a'''(x).\end{aligned}$$

Plugging these terms into expression (12) and rearranging the terms, we establish (5).

Suppose conversely that (3) holds as a strict inequality. Then (11) is strictly positive. Thus the right-hand side of (10) is strictly positive for every sufficiently large m. Since it is equal to the left-hand side of (9) divided by $\sigma_m^2>0$, this implies that the left-hand side of (9) is strictly positive. The proof is thus completed.[5] □

[5] The proof of the necessity part of this theorem (that is, if the cautiousness is increased by the background risk ξ_m for infinitely many m, then $v''(x)(v'''(x))^2+v'(x)v''(x)v^{(5)}(x)-2v'(x)v'''(x)v^{(4)}(x)\leqslant 0$) can be simplified by

To facilitate a comparison with the necessary and sufficient conditions for risk-vulnerability for small risks as given in [6, inequality (12)], we define, for each $n \in \{1, 2, 3, 4\}$, $a_n(x) = -v^{(n+1)}(x)/v^{(n)}(x)$ whenever $v^{(n)}(x) \neq 0$. Then $a_1(x)$ is nothing but the absolute risk aversion $a(x)$, $a_2(x)$ is the absolute prudence $\psi(x)$, and $a_3(x)$ is what is termed *temperance* in [10]. Then the necessary condition for risk-vulnerability for small risks, derived from [6, Eq. (10)], is that

$$a_2(x)\big(a_3(x) - a_1(x)\big) \geqslant 0, \tag{13}$$

while the sufficient condition is obtained by replacing the weak inequality by a strict inequality in (13).

The following theorem restates the necessary and sufficient conditions of Theorem 1 in terms of the $a_n(x)$ when the consumer is prudent.

Theorem 2. *If* $v'''(x) > 0$, *then* (3) *is equivalent to*

$$\frac{v'''(x)}{v'(x)} + \frac{v^{(5)}(x)}{v'''(x)} - 2\frac{v^{(4)}(x)}{v''(x)} \geqslant 0. \tag{14}$$

If, in addition, $v^{(4)}(x) \neq 0$, *then* (14) *is equivalent to each of the following three inequalities*:

$$a_3(x)\big(a_4(x) - a_2(x)\big) \geqslant a_2(x)\big(a_3(x) - a_1(x)\big), \tag{15}$$

$$2a_2(x)a_3(x) \leqslant a_1(x)a_2(x) + a_3(x)a_4(x), \tag{16}$$

$$\frac{d}{dx}\left(\frac{a_2(x)}{a_1(x)}\big(a_3(x) - a_1(x)\big)\right) \leqslant 0. \tag{17}$$

The equivalence remains to hold when the weak inequalities in (14)–(16)*, and* (17) *are all replaced by strict inequalities.*

Proof of Theorem 2. Inequality (14) follows from inequality (3) by dividing both sides by $v'(x)v''(x)v'''(x)$, which is strictly negative. Further rearranging yields

$$\frac{v^{(5)}(x)}{v'''(x)} - \frac{v^{(4)}(x)}{v''(x)} \geqslant \frac{v^{(4)}(x)}{v''(x)} - \frac{v'''(x)}{v'(x)}$$

and hence, if $v^{(4)}(x) \neq 0$, then

$$-\frac{v^{(4)}(x)}{v'''(x)}\left(-\frac{v^{(5)}(x)}{v^{(4)}(x)} + \frac{v'''(x)}{v''(x)}\right) \geqslant -\frac{v'''(x)}{v''(x)}\left(-\frac{v^{(4)}(x)}{v'''(x)} + \frac{v''(x)}{v'(x)}\right).$$

By the definition of the $a_n(x)$, the last inequality is equivalent to inequality (15), which after rearranging delivers inequality (16). (17) is equivalent to (16), because

$$\frac{d}{dx}\left(\frac{a_2(x)}{a_1(x)}\big(a_3(x) - a_1(x)\big)\right) = \frac{a_2(x)}{a_1(x)}\big(2a_2(x)a_3(x) - a_1(x)a_2(x) - a_3(x)a_4(x)\big). \qquad \square$$

Note that the right-hand side of (15) is exactly the same as the left-hand side of (13). Note moreover that the left-hand side of (15) is the same as its right-hand side except that every

considering $\xi_m = (1/m)\hat{\xi}$ for any given $\hat{\xi}$ with $E(\hat{\xi}) = 0$ and $E(\hat{\xi}^2) = 1$. We are grateful to an anonymous referee for pointing this out.

expression is of exactly one degree higher. This has a couple of easy-to-grasp implications. Suppose that v is risk-vulnerable for small risks, i.e., the right-hand side of inequality (15) is positive. Then a necessary condition for v to exhibit increased cautiousness under small risks is that the utility function $-v'$ is also risk-vulnerable for small risks. For another implication, recall that [6, inequality (10)] shows that the percentage increase in absolute risk aversion due to a small zero-mean background risk is proportional to $a_2(x)(a_3(x) - a_1(x))$. Hence the expression $a_2(x)(a_3(x) - a_1(x))$ can be interpreted as a measure of risk-vulnerability for small risks. (15) says that v exhibits increased cautiousness for small risks if and only if $-v'$ is more risk-vulnerable than v. Finally, (17) states that the Gollier–Pratt measure of risk-vulnerability divided by the Arrow–Pratt measure of absolute risk aversion is a decreasing function of consumption levels x if and only if v exhibits increasing cautiousness, and also if and only if $-v'$ is more risk-vulnerable than v. This means that once we know the ratio between the Gollier–Pratt and Arrow–Pratt measures as a function of consumption levels, we can tell whether v exhibits increased cautiousness for small risks.

4. Sufficient condition for increased cautiousness

In this section, we present a sufficient condition for the cautiousness to be increased by all zero-mean background risks, small or not.

Theorem 3. *If $s'(x) \geqslant 0$, $s''(x) \leqslant 0$, and $s'''(x) \geqslant 0$ for every $x > \underline{c}$, then for every zero-mean background risk ξ and for every $x > \underline{d}$, $t'(x) \geqslant s'(x)$. The inequality is strict if, in addition, $s'(x) \neq 0$ for every $x > \underline{c}$.*

This theorem says that at any given consumption level $x > \underline{d}$, the cautiousness $t'(x)$ of the induced utility function u is not exceeded by the cautiousness $s'(x)$ of the original utility function v if the cautiousness s' is a non-negative, non-increasing, and convex function of consumption levels. The first sign condition is nothing but non-increasing absolute risk aversion (DARA). The second sign condition is that the risk tolerance s be concave, which implies that the absolute risk aversion a is convex.[6] It is easy to see that the three sign conditions imply the necessary and sufficient conditions ((4) and (5)) in Theorem 1 for the cautiousness to be increased by small risks, as well as the sufficient condition for risk-vulnerability of [6, Corollary 1]. Thus $t(x) \leqslant s(x)$. By Lemma 1, $t'(x) \geqslant s'(x)$ if and only if $\varphi(x)t(x) \geqslant \psi(x)s(x)$. Hence $\varphi(x) \geqslant \psi(x)$, that is, the prudence is also increased by the background risk.

Proof of Theorem 3.[7] Let $x > \underline{d}$. Proving that $t'(x) \geqslant s'(x)$ is equivalent to proving $t(x)\varphi(x) \geqslant s(x)\psi(x)$ by Lemma 1. Note that

$$t(x) = \frac{E(v'(x+\xi))}{E(-v''(x+\xi))} = E\left(\frac{v'(x+\xi)}{-v''(x+\xi)} \frac{-v''(x+\xi)}{E(-v''(x+\xi))}\right).$$

[6] The converse, however, does not hold. Even when the absolute risk aversion is convex, the absolute risk tolerance may not be concave. An undesirable implication of concave absolute risk tolerance, which is not implied by convex absolute risk aversion, is increasing relative risk aversion: Let $\underline{c} = 0$, then, by the Inada condition, $s(x) \to 0$ as $x \to 0$. Thus the concavity of s implies that its elasticity is not greater than one; and it is strictly less than one beyond any point at which s'' is strictly negative. But it can be shown that the elasticity is strictly less than one if and only if the first derivative of the relative risk aversion is strictly positive.

[7] There are many ways to prove this theorem (some included in the previous versions of this paper). The proof given here is perhaps the most elegant and was pointed out to us by an anonymous referee.

The random variable $-v''(x+\xi)/E(-v''(x+\xi))$ has the property of a Radon–Nikodym derivative on $(\Theta, \mathcal{G}, Q)$. So we let $\hat{E}$ be the expectation operator of the probability measure for which it is the Radon–Nikodym derivative. Then $t(x) = \hat{E}(s(x+\xi))$. Similarly, $\varphi(x) = \hat{E}(\psi(x+\xi))$. Thus, we need to show that

$$\hat{E}\big(s(x+\xi)\big)\hat{E}\big(\psi(x+\xi)\big) \geqslant s(x)\psi(x).$$

Since $\psi(x+\xi) = (s'(x+\xi)+1)/s(x+\xi)$ and $s(x)\psi(x) = s'(x)+1$ by Lemma 1, we need to prove that

$$\hat{E}\big(s(x+\xi)\big)\hat{E}\left(\frac{s'(x+\xi)+1}{s(x+\xi)}\right) \geqslant s'(x)+1. \tag{18}$$

Since $s' \geqslant 0$ and $s'' \leqslant 0$, $s(x+z)$ is non-decreasing in z and $(s'(x+z)+1)/s(x)$ is non-increasing in z. Thus, by [5, Proposition 15 in Section 6.4],

$$\hat{E}\big(s(x+\xi)\big)\hat{E}\left(\frac{s'(x+\xi)+1}{s(x+\xi)}\right) \geqslant \hat{E}\left(s(x+\xi)\frac{s'(x+\xi)+1}{s(x+\xi)}\right) = \hat{E}\big(s'(x+\xi)\big)+1. \tag{19}$$

This inequality is strict if $s'(x+\xi) > 0$.

Next, since $s'' \leqslant 0$, $s'(x+z)$ is non-increasing in z. Since

$$s' = \left(-\frac{v'}{v''}\right)' = -\frac{(v'')^2 - v'v'''}{(v'')^2} \geqslant 0,$$

$v''' \geqslant 0$ and $-v''(x+z)$ is non-increasing in z. Thus, again by [5, Proposition 15 in Section 6.4],

$$\hat{E}\big(s'(x+\xi)\big) = \frac{E(s'(x+\xi)(-v''(x+\xi)))}{E(-v''(x+\xi))} \geqslant E\big(s'(x+\xi)\big). \tag{20}$$

Finally, since $s''' \geqslant 0$, $s'(x+z)$ is convex in z and, by Jensen's inequality,

$$E\big(s'(x+\xi)\big) \geqslant s'(x). \tag{21}$$

Combining Eqs. (19), (20), and (21) yields inequality (18). □

Although we assumed in this theorem that $s'(x) \geqslant 0$ for every $x > \underline{c}$, this assumption can be derived from the assumption that $s''(x) \leqslant 0$ for every $x > \underline{c}$. Indeed, if $s''(x) \leqslant 0$ for every $x > \underline{c}$ and yet there were an $x_0 > \underline{c}$ such that $s'(x_0) < 0$, then s would be a concave and strictly decreasing function on $[x_0, \infty)$, and hence $s(x) < 0$ for every sufficiently large $x > \underline{c}$, which is a contradiction. Thus, if $s''(x) \leqslant 0$ for every $x > \underline{c}$, then $s'(x) \geqslant 0$ for every $x > \underline{c}$.

Although we imposed the three sign conditions over the entire domain $[\underline{c}, \infty)$ of consumption levels, the conclusion $t'(x) \geqslant s'(x)$ still holds as long as they are met on the interval $[x+\underline{e}, x+\bar{e}]$ of realizable consumption levels. Bear in mind, however, that s' need no longer be non-negative even if s'' is non-positive on $[x+\underline{e}, x+\bar{e}]$.

The conditions for Theorem 3 are satisfied by every HARA utility function, with the second and third derivatives of the risk tolerance s being always zero.

Corollary 1. *If there exist a $\tau \in \boldsymbol{R}$ and a $\gamma \in \boldsymbol{R}_{++}$ such that $s(x) = \tau + \gamma x$ for every $x > \underline{c}$, and if* $\operatorname{Var}(\xi) > 0$*, then $t'(x) > s'(y)$ for every $x > \underline{d}$ and every $y > \underline{c}$.*

Corollary 1 shows that if the cautiousness s' of the original utility function v is constant, then we have $t'(x) > s'(y)$ regardless of the choice of $x > \underline{d}$ and $y > \underline{c}$. Note that this corollary, together with [3, Theorem 2] or [7, Proposition 3], could be used to provide an alternative and simple proof of [4, Theorem 3], which asserts that in an economy in which all consumers have constant and equal cautiousness and only one consumer has no background risk, the consumer sells portfolio insurance.

5. Consequences for portfolio insurance

In this section we provide numerical examples on how a consumer's portfolio choice is affected by the presence of a non-hedgeable background risk. Specifically, we give examples of efficient risk-sharing rules[8,9] in a two-consumer economy, in which both consumers share the same original utility function, but in which one consumer faces a zero-mean background risk, while the other does not.

Somewhat formally, the efficient risk-sharing rules can be defined as follows. The aggregate (tradeable) endowment of the economy is given by a random variable ζ. Each consumer $i = 1, 2$ has an original utility function $v_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$ and a (possibly zero) background risk ξ_i. His induced utility function u_i is defined by $u_i(x) = E(v_i(x + \xi_i))$. Denote the risk tolerance for u_i by t_i. Assume that ξ_i is stochastically independent of ζ.

At any efficient allocation of the tradeable aggregate endowment ζ, each consumer's consumption level can be written as a function of aggregate endowments. That is, for each Pareto-efficient allocation (η_1, η_2) ($\eta_1 + \eta_2 = \zeta$ and, for each i, $\eta_i + \xi_i > 0$ almost surely), there exists a pair of functions (f_1, f_2), called *efficient risk-sharing rules*, such that $\eta_i = f_i(\zeta)$ for each i. According to part a) of [3, Theorem 2],[10] there is a function t, interpreted as the representative consumer's risk tolerance, such that

$$\frac{f_i''(x)}{f_i'(x)} = \frac{1}{t(x)}\bigl(t_i'\bigl(f(x)\bigr) - t'(x)\bigr) \tag{22}$$

for each i and x. Thus, if $t_i'(f_i(x)) > t'(x)$, then $f_i''(x) > 0$. In other words, if consumer i is more cautious than the representative consumer, then his efficient risk-sharing rule is convex and he buys portfolio insurance. As stated in [7, Lemma 1], the representative consumer's cautiousness is a weighted average of the individual consumers' counterparts. We can conclude, therefore, that in a two-consumer economy, if one consumer is more cautious than the other, then he buys portfolio insurance, and if one is less cautious than the other, then he sells portfolio insurance.

By [4, Theorem 3], if the two consumers have the same original utility function in the HARA class, then the consumer with background risk will buy portfolio insurance, while the other consumer sells it. A special case of such a situation is graphically represented in the left panels of Fig. 1. The common original utility function has the constant coefficient 2 of relative risk aversion and only consumer 2 has a background risk, which takes value 1 or -1 with probability $1/2$ each. Consumer 2 is more cautious than consumer 1 and buys portfolio insurance.

[8] By the second welfare theorem this is also a general equilibrium in which the consumers have appropriate endowments.

[9] The study of efficient risk-sharing rules goes back at least to [1] and [13]. See, for example, [5] for a textbook treatment.

[10] The result was also given in [7, Proposition 3].

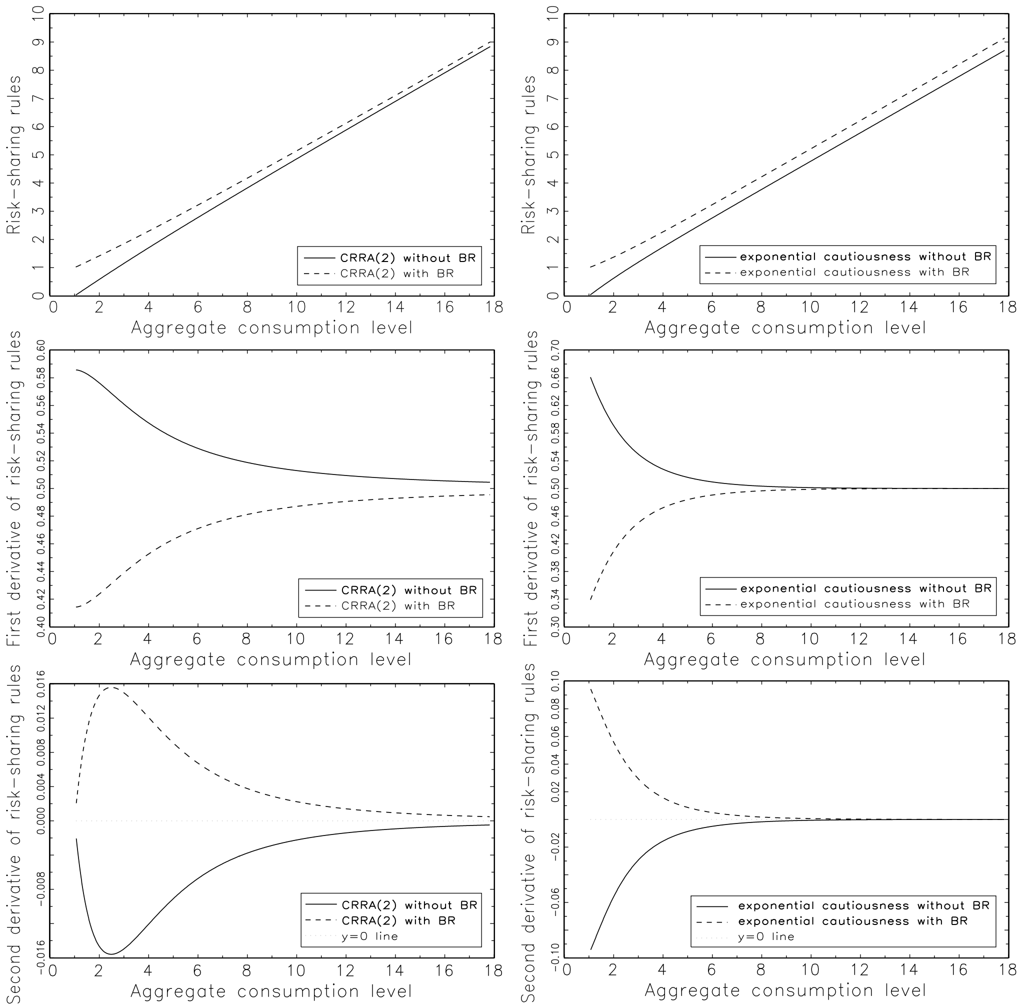

Fig. 1. Efficient risk-sharing rules, and their first and second derivatives. On the left both consumers have a common utility function with constant coefficient 2 of relative risk aversion. On the right both consumers have a utility function with risk tolerance $s_i(x_i) = 1 - e^{-x_i}$, and hence exhibit exponentially decreasing cautiousness $s_i'(x_i) = e^{-x_i}$. In each economy, consumer 1 has no background risk (solid curve), while consumer 2 has a two-point background risk with probability $1/2$ on 1 and -1 each (dashed curve).

In the panels on the right of Fig. 1 we have a case, not covered by the result in [4], but covered by Theorem 3 of this paper, which guarantees that any background risk increases cautiousness. The original utility function both consumers share has the risk tolerance $s_i(x_i) = 1 - e^{-x_i}$ and hence the exponentially decreasing cautiousness $s_i'(x_i) = e^{-x_i}$. It satisfies the conditions of Theorem 3. As in the previous example, only consumer 2 has a background risk, which takes value 1 or -1 with probability $1/2$ each. Very much as in the HARA case, it is again consumer 2 who buys portfolio insurance.

6. Conclusion

The consumer's cautiousness (the derivative of the reciprocal of the Arrow–Pratt measure of absolute risk aversion) determines her demand for options (portfolio insurance) relative to the risky asset. We have investigated how this cautiousness for macroeconomic risks is affected by the presence of idiosyncratic (background) risks and were interested in conditions under which cautiousness, and, hence, the demand for options relative to the risky asset, is increased in the presence of any background risk.

We gave a necessary and sufficient condition on a consumer's utility function to exhibit increased cautiousness under small background risks (Theorem 1), which, of course, is also a necessary condition for a consumer's utility function to exhibit increased cautiousness under any background risk. We also provided a sufficient condition (Theorem 3) on the original utility function under which the cautiousness, at any given level of consumption, is higher in the presence of a background risk than in its absence.

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