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On the Collaboration between Sraffa and Besicovitch: The 'Proof of Gradient'*

Heinz D. Kurz and Neri Salvadori

13.1 Introduction

As is well known, Chapters III–VI of *Production of Commodities by Means of Commodities* explore the results arrived at as regards the model defined in the second part of Chapter II. Whereas the main aim of Chapter III is to provide a 'preliminary survey' (p. 15) of price movements consequent upon changes in distribution without the help of any specific tool, the complete analysis of these movements is presented in Chapter VI, where Sraffa has recourse to the tool elaborated in Chapters IV and V, the Standard Commodity. In analyzing price movements, Sraffa maintains in section 49, i.e. the very last section of Chapter VI, that 'there is ... a restriction to the movement of the price of any product: if as a result of a rise in the rate of profits the price falls, its rate of fall cannot exceed the rate of fall of the wage.' This property is important because if the wage as a function of the rate of profits is decreasing in any numeraire, then it is decreasing in all numeraires. And since the Standard Commodity is a numeraire in which the wage as a function of the rate of profits is a decreasing straight line, then 'if the the wage is cut in terms of *any* commodity (no matter whether it is one that will consequently rise or fall relatively to the Standard) the rate of profit will rise; and vice versa for an increase of the wage.' (Sraffa, 1960, pp. 38 and 40; Sraffa's italics)

In this paper we analyse how Sraffa in 1944 elaborated this property and which role Sraffa's 'mathematical friend' Abram S. Besicovitch played in this. 1944 also the concept of Standard commodity matured[1] jointly with a number of other concepts.[2] All these developments were going on *pari passu*. However, in this paper

* We should like to thank Pierangelo Garegnani, literary executor of Sraffa's papers and correspondence, for granting us permission to quote from them. Citations beginning with the letter D followed by a series of numbers refer to Sraffa's papers at Trinity College Library, Cambridge; the format of the citations follows the catalogue prepared by Jonathan Smith, archivist. Unless otherwise stated, all emphases are in the original; we shall use italics when Sraffa used underlinings. He frequently abbreviated *and* by +; we shall use the word instead of the symbol. We are grateful to Jonathan Smith and the staff of Trinity College Library for continuous assistance while working on the Sraffa papers. The views contained in this article have not been discussed with all the participants in the project of preparing an edition of Sraffa's papers and correspondence and therefore do not implicate them.

we will not deal with the parallel evolution of all the other concepts, including the Standard commodity.

The composition of the paper is the following. In Section 2 we summarize briefly the relevant bits of the path Sraffa's work took in the first period devoted to what in December 1927 he called 'la mia teoria' and his 'libro' (D3/12/11: 55) which extended roughly from late 1927 to the end of 1931. The emphasis is on his early attempts to come to grips with the dependance of the rate of profits and relative prices on wages, given the system of production in use. Towards the end of this period Sraffa had essentially elaborated the analytical framework within which he then, upon the resumption of his work in 1942, began to study the various problems he faced in detail. It was clear from an early point onward that the problems were of a nature that required the use of mathematics, especially linear algebra . While Sraffa in the first period of his work had made a few attempts to acquaint himself with the relevant mathematical tools, his command of them was insufficient to have allowed him to effectively tackle the problems he encountered without the assistance of mathematicians. Sraffa was in the lucky position that with Abram S. Besicovitch he found another Trinity fellow that was not only able to provide the missing expertise (at least in the majority of cases in which Sraffa consulted him) but who also willingly did so. Section 3 turns to the collaboration between the two men, who soon got on friendly terms with one another, on what Sraffa called the 'proof of the gradient', an expression partially adopted from Besicovitch. Their respective collaboration took place in the last three months of 1944. Section 3 focuses on the path that led to the proof of the gradient with regard to single-products systems. Sraffa also attempted to generalize the proof to systems with joint production. While these attempts failed, his work was not entirely futile because he found other properties of the system he had not searched for. In particular, he found another proof of the gradient which we eventually encounter in section 49 of his book. Section 4 deals with a few further notes drafted by Sraffa in 1945 in which he pointed out some of the properties of the system of production that followed from the gradient argument. Section 5 contains some concluding remarks.

AQ: Please check whether the change from 'requested' to 'required' is acceptable.

The present paper is the third in a row in which we investigate Sraffa's collaboration with his 'mathematical friends' (see als Kurz and Salvadori, 2001, 2004). The evidence provided shows that there can be no doubt about the great importance of this collaboration, and especially the one with Besicovitch, for the success of Sraffa's intellectual project. Not for nothing did Sraffa, in the Preface of his book, emphasize: 'My greatest debt is to Professor A. S. Besicovitch for invaluable mathematical help over many years.' (Sraffa, 1960, p. vi)

13.2 Sraffa's work before autumn 1944

In November 1927 Sraffa began to elaborate his systems of equations. His attention focused first on systems without a surplus. Almost in parallel he studied systems with a surplus and wages fixed in terms of a given inventory of commodities. Some of the mathematical properties of such systems he was able to establish in the first half of 1928, assisted by Frank Ramsey (see Kurz and Salvadori, 2001, pp. 262–4).

Following Ricardo's lead, Sraffa in early 1928 also began to study the implications for the rate of profits and relative prices in the case in which workers' participated in the sharing out of the surplus product. He did this first in terms of a redistribution of the surplus product away from profits and toward wages in proportion to the given vector of the surplus product (see, for example, D3/12/7: 93). This allowed him to conceive of the redistribution of the surplus in straightforward physical terms and yet advocate a share concept of surplus wages that is independent of relative prices. A share concept had already been introduced by Ricardo who had referred to the portion of any given annual value of the net product (in terms of labour values) paid to the labouring class (see Ricardo, *Works*, Vol. I, pp. 49–50). Sraffa demonstrated that an increase in surplus wages implied a decrease in the rate of interest and in general a change in relative prices. Investigating the properties of a system with only two industries, one producing 'consumer goods', the other 'intermediate goods', Sraffa in a document composed in the late 1920s stressed: 'È chiaro come il sole che se cambiano i salari cambiano anche i valori' (D3/12/7: 95).[3] He worked out a number of numerical examples that illustrated the phenomenon.

However, having to solve a set of simultaneous equations for each and every level of wages in order to find the corresponding level of the rate of interest and relative prices was cumbersome and the results not very transparent. Sraffa was therefore on the lookout for a second method that would provide a more direct grasp of the essential features of the economic system. It was apparently this concern that made him study, around the turn of 1928, among other things, Eugen von Böhm-Bawerk's *Positive Theory of Capital* (Böhm-Bawerk, 1889, 1959) (see, for example, D3/12/7: 89). Böhm-Bawerk in his book had put forward a concept of production according to which any product could be envisaged as the result of a flow of dated inputs of labour (and other 'original' factors of production) of finite length. Sraffa had come across the idea of reduction already in the writings of Adam Smith and other classical authors, and, in taking up a suggestion of William Petty to reduce the values of commodities to 'food', had elaborated himself some such reduction with regard to his first equations. He now applied the method of reduction to workers's means of subsistence to his with-surplus equations: 'It is necessary therefore to reduce each commodity ultimately to two things, surplus {the rate of interest} and any one commodity (or group of commodities, which are in constant proportion, such as subsistence of workers).' (D3/12/7: 126)

This was basically the stage at which Sraffa had to leave the matter in order to focus attention on the edition of the works and correspondence of David Ricardo, a task to which the Royal Economic Society had appointed him at the beginning of 1930.

When in the summer of 1942 Sraffa resumed work on his constructive task he quickly advanced in a variety of directions. This was not least due to his collaboration with Besicovitch, which appears to have started in June of that year (D3/12/2: 15). In October 1942 Sraffa, upon a suggestion of Besicovitch, adopted the notation we eventually encounter in his 1960 book (see D3/12/23: 1(1–4), 2 and 4). At the time he worked with great dedication and untiring energy on several problems, including the problem of reduction. The most important steps taken by him as seen from the perspective of the present paper were finding the maximum rate of profits of a given system of single production, *R*, and the identification of the Standard Commodity associated with the system. He knew, in fact, that for a system of *k* commodities, there existed *k* solutions for the maximum rate of profits, the set of prices and the multipliers of the Standard system. Sraffa in fact, in 1944, considered each eigenvector (of the input matrix of the means of production) as a 'Standard commodity' and indicated them with letter U (U', U'', ...); see, for instance, D3/12/40: 3–18. These analytical steps mentioned above Sraffa was able to take at the beginning of 1944 (see folder D3/12/36). The Standard Commodity's characteristic feature was foreshadowed in several earlier notes and papers, some composed already in the first period. For example, in a paper he had begun to write in February 1931 he contemplated the case in which 'the value of total capital in terms of total goods produced cannot vary [as income distribution changes], since the goods are composed exactly in the same proportions as the capitals which have produced them.' (D3/12/7: 157(3)) He was sure that in any real economy this condition was never actually met, but it might perhaps be approximately true. Some twelve years later, in a note composed a few weeks before the discovery of the Standard Commodity, he clarified that his earlier proposition was based on the 'statistical compensation of large numbers' (D3/12/35: 28). Henceforth he called the assumption that the value of social capital relative to the of social product does not change with a change in distribution 'My Hypothesis'. He had encountered a similar hypothesis in Marx at the beginning of the 1940s, if not earlier, who had characterised a given system of production in terms of a given 'organic composition of capital', that is, the ratio of the labour embodied in the means of production and of living labour expended during a year. The inverse of the organic composition gave the maximum rate of profits compatible with the system. Sraffa had understood that no actual system could ever be expected to satisfy the hypothesis. He therefore had to construct an artificial system that did so. He constructed it out of the equations representing the actual system and was thus able to preserve all the salient features of the latter.

The ground is now prepared for entering into the main theme of this paper which concerns Sraffa's collaboration with Besicovitch on the 'proof of gradient' and which for the most part took place from October to December 1944.

13.3 Sraffa's collaboration with Besicovitch on the 'proof of gradient'

If the rate of profits is given, the price equations and the equation fixing the numeraire are linear and therefore one solution exists for prices and the wage rate. If, on the contrary, the wage rate is given, then the system is not linear and *k* solutions for the prices and the rate of profits exist, where *k* is the number of commodities involved. Some of these solutions are real, others are not, and some real solutions have negative values. We know, now, that only one real solution has non-negative prices and wage rate, provided the given wage rate is low enough. This was not known to Sraffa at the time and actually he was looking to find a solution to this problem of multiplicity. When working on this problem on 3 October 1944 he put to himself a different exercise. Instead of giving the wage rate he took the wage as an unknown and considered the equation

$$A_1 p_a = Lw \tag{1}$$

(D3/12/40: 12). Yet this did not really change the problem at hand, since this equation is equivalent to giving the wage using commodity *a* as numeraire. On 10 October Besicovitch replied to some other questions contained in the same document, but not to this one. A reply to the latter was provided on 12 October: '*k* different Systems of Solutions' (D3/12/40: 16). On the same day Sraffa wrote a note in Italian in which he mentions Besicovitch's answer, but with details which are not in the written answer by his friend. There is every reason to think that the reference is to an oral answer accompanying the written one:

> In sostanza, l'unica obbiezione sensata fatta da B., è che se, invece di dare un valore arbitrario a w, introduco l'equazione $A_1 p_a = Lw$ lasciando w variabile, il risultato è: non posso più ottenere l'equazione ... $r = R(1 - w)$, e quindi anche prendendo la Standard Commodity per unità, la soluzione non è più unica; ci sono quindi k valori di w.[4]

Two days later he added

> *N.B.* La sostanza dell'obbiezione di B. sembra questa. Per qualsiasi U Lw è una funzione lineare di r. Ma il prezzo della merce a, e quindi $A_1 p_a$ *non* è funzione lineare di r: quindi oscilla col variare di r, e la linea del prezzo $A_1 p_a$ *interseca in k punti* (in k valori di r) la retta che rappresenta Lw: in ciascuno di questi punti il *dato* salario reale è uguale a un *diverso* salario proporzionale (w).[5]

Exploring the unpublished papers we see that after this event Sraffa went back to some of the results he had obtained earlier, annotating them. For instance, on the same day, 14 October, he jotted down a few remarks on the document containing a proof of the existence of 'at most *one* value of *R* to which correspond positive prices and positive multipliers' (D3/12/39: 3), obtained in May of the same year. He also formulated further conjectures about answers to the objection by Besicovitch. However, two weeks later the situation changed dramatically and Sraffa wrote:

> 31 Ottobre 44. Oggi, dopo dieci mesi di resistenza, Besicovich {sic} ha finalmente ceduto, ammettendo che il sistema ha una soluzione unica {insertion later in pencil:} [i]nel caso che ogni processo dà un solo prodotto[i]. L'argomento decisivo è stato che la curva del prezzo della merce A non può tagliare più di una volta la retta del salario; perchè il prezzo non può mai cadere (in consequenza di una caduta dell salario) in proporzione maggiore del salario; e ciò perchè il prezzo può essere espresso in termini di una serie (v. sopra). Il caso in questione è quello

> in cui una data Merce Standard è presa per unità dei prezzi e una merce arbitraria per unità del salario. (D3/12/40: 28)[6]

Following his hint ('see above'), we find the equation

$$p_a = L_0 w + L_1 w(1+r) + L_2 w(1+r)^2 + \cdots \tag{2}$$

equation (1), and a figure that comes close to Fig. 4 in Sraffa's book (1969, p. 39).

Probably Sraffa was too emphatic. The 'ten months of resistance' were *not* on the issue under discussion, but on the uniqueness of the Standard ratio and the uniqueness of the Standard commodity. The solution was to be found in another direction than Sraffa had originally thought, a fact that was actually close at hand since May 1944: uniqueness could not be found by manipulating the equations,[7] but by specifying more carefully the desired properties of the sought standard. This Sraffa eventually did by considering the *unique* 'Standard Commodity'[8] with nonnegative multipliers (all positive for processes producing basics and zero for the others). What Besicovitch recognised on 31 October was that there was a single solution to equation (1). Yet this statement is partially wrong: equation (1) has actually n solutions, some real, others not, and among the real solutions some are positive, others not. What *is* unique is a positive solution with a positive wage and non-negative prices. The proof apparently given orally on 31 October was then provided in written form on 2 November. Before the written proof was available to Sraffa, the two met anew on 1 November. Sraffa, noted on the back of document D3/12/40: 28:

> 1.11.44 Bes. ammette che la conclusione rimane valida se, invece della Standard Commodity, si prende come unità di prezzi la merce A: poichè si tratta solo di un cambiamento di unità.[9]

The document containing the proof is D3/12/63: 6; it is dated, as mentioned, 2 November 1944. It has a part emphasized by red lines around it which reads:

> [Dictated by Besicovich]
> w and r vary always in the opposite direction. p_a being capable of being expressed in the form
>
> $$p_a = L_0 w + L_1 w(1+r) + L_2 w(1+r)^2 + \cdots$$
>
> where Ls are all non-negative, the gradient of p_a is always less than the gradient of w. And therefore the equation
>
> $$A_1 p_a = Lw$$
>
> cannot have more than one real root for r

As it is stated, the sentence is not fully correct: p_a is capable of being expressed in the form (2) only for

$$-1 \leq r < \frac{1-\lambda}{\lambda},$$

where λ is the eigenvalue of maximum modulus of the matrix of inputs, and therefore what has been proved is not that equation (1) 'cannot have more than one real root for r': other real solutions are possible, but they are smaller than -1 or larger than $(1-\lambda)/\lambda$.

Another remark concerns the fact that the proof is based on a difference in the slopes of w and p_a. To indicate such slopes Besicovitch used the mathematical term 'gradient'. Sraffa used the term when characterising the proof under consideration in his unpublished writings, but he did not use it in his book. This may be seen as justifying the title of the present paper.

Immediately after this issue had been settled with respect to systems with single production, Sraffa, on 3 November, tried to generalize the result to joint production. He started to write a long document (D3/12/40: 19–27) which he continued on November 4, 5, 7, 9, and 10. This was just the first of a number of attempts. Eventually he understood that the result cannot be carried over to joint production (cf. Sraffa, 1960, pp. 61–2). The earlier parts of the document were also shown to Besicovitch on 4 November and Sraffa noted in the margin of D3/12/40: 22: '4.11.44 Bes. says this is *wrong'*.

On 12 November Sraffa wrote a paper (D3/12/40: 33–36) in which he put down in clean form the whole result with regard to single production. He drew also a diagram in which commodity a serves as numeraire and one in which commodity a serves as numeraire, but the equation is $B_1 p_b = Lw$ (D3/12/40: 35), and extends the same argument to the case in which wages are paid *ante factum* (D3/12/40: 36). On the same and the following days he tried to extend the result. He asked: What happens if $A_1 = 0$? After some deliberation he arrived at the following remark (D3/12/40: 40) on 14 November:

> In the System
>
> $$\left.\begin{array}{c} (A_a p_a + \cdots + K_a p_k)(1+r) + L_a w = A p_a \\ \cdots \\ (A_k p_a + \cdots + K_k p_k)(1+r) + L_k w = K p_k \end{array}\right\}$$
>
> make $\quad p_b = 1$
> and $\quad A_1 p_a = Lw$
> Now suppose we make $A_1 = 0$, does it follow that $w = 0$? No, because p_a might be ∞. And this is the case, e.g. when b is one of the 'other' Standard Commodities, U'', U''' etc.

This is actually the right answer when a is a basic commodity and the eigenvalues of basics coincide with the eigenvalues of the whole system, as is usually (and implicitly) assumed by Sraffa. In this note he gets very close to recognizing the reason on the basis of which the appropriate R was to be chosen, but this result was effectively obtained only later.

In another note (D3/12/40: 39) dated 12 November he asks himself 'Is labour itself such a commodity?' He answers:

> Suppose we take labour as unit of prices, i.e. make $w = 1$, and we make real wages $= A_1$. Then in the equation $A_1 p_a = Lw$ we have $wL = 1$ and therefore $A_1 = \frac{1}{p_a}$ so that at $A_1 = 0$ it must be $p_a = \infty$.

Once again he found a nice result and his comment on it may clarify why he was not favourable to using labour as standard: 'Note the disastrous effects of taking the '*price* of labour' ... as standard'. He continued:

> Therefore, whenever we can guarantee that in the equation $A_1 p_a = Lw$ at $A_1 = 0$ there cannot correspond $p_a = \infty$ then at $A_1 = 0$ there *must* correspond $w = 0$.

And then, marked with three red lines in the margin, indicating the great importance attached by him to the statement, followed:

> But the most simple way to obtain this result is to take *the same* commodity A as unit of ps and w as well as component of the real wage.

The note concludes:

> For in that case $p_a = 1$ at all values of r; and when in the equation $A_1 p_a = A_1 = Lw$ we make $A_1 = 0$ w *must* be $= 0$.

Here, too, he was close to recognizing the reason for choosing the appropriate R. Many other attempts he undertook in order to extend the argument were in the direction of joint production, but, as we know, no solution could be found. However, as by-products, Sraffa was able to establish other results concerning joint production. For instance, he recognized that negative prices reflected the presence of 'uneconomical' processes[10] and that

> The remedy would be to give up the unec. process, to use only the other, and give away free the 'neg. price' commodity – all through: in other words, drop it from among the 'commodities' and treat the surviving equation as producing only the other. (D3/12/63: 65)

This is substantially stating the rule of free goods, which he did not adopt in his book, but which was employed by many of his followers.[11] Finally, on 16 November, he recognised that 'The "Reduction to Labour and Gradient" proof cannot be extended from one-one case to many-one case.' (D3/12/63: 45) However he tried to find other proofs for the single production ('one-one') case which could perhaps be extended to the joint production ('many-one') case. His respective work turned out to be fruitful, but not because it fulfilled the expectations Sraffa appears to have had when embarking on it: While he did not find what he sought to find, he found something else that was useful. In the same document dated 16 November we find some elements of the other proof we encounter in section 49 of *Production of Commodities*:

> There are three elements [i]variables[i] constituting the price of a product (i.e. on left side):
> 1) Lw, rises with w
> 2) $1 + r$, falls with rise of w
> 3) price of Const. Cap., may rise or fall with rise of w.
> For the price to fall more than in proportion with w it is necessary for (3) to fall *much more* than in proportion with w, so as to annull [i]cancel entirely[i] the contrary effect of (2) and leave something over.
>
> But (3) is the price of a group of other commodities.
>
> Therefore for p_a to fall proportionately more than w, it is necessary for the price of some other commod. to fall more than p_a.
>
> Thus we can arrange the commodities in the order of their gradient, and there will be one (or several) with the maximum gradient. But this impossible, since it could only be due to other commodities (its means of production) having a gradient greater than the maximum.

On 4 December Besicovitch went back to what he had dictated on 2 November and Sraffa noted on the respective document D3/12/63: 6:

> 4.12.44 This holds for $r \geqq -1$. I.e. the function is monotone, therefore not more than one root > -1. ...
> N.B. – B. had a doubt whether when the equations are of even degree there could be only one root: for in that case they must have two or none. But this was dispelled by the consideration that, although the polynomial expressing p_a is of kth degree (and k may be even) the above expression is an infinite series of infinite degree, and that makes it OK.

Thus Besicovitch introduced a first correction, but he did not observe that there is also another boundary, namely that within which the series is convergent. Further, the doubt of the 'N.B.' adjunct is inappropriate even if the number of the terms of the sum were to be finite and this is so exactly for the reason that there is the interval $r < -1$. In fact, because of Descartes' Theorem, the equation in $1 + r$ has only one positive solution for $1 + r$, whereas all other real solutions are negative and all complex solutions have a negative real part since there is only one variation.

On the same day Besicovitch expressed also some further doubts. On a piece of recycled paper (D3/12/63: 5) Besicovitch wrote

$$A_1\{L_0 + L_1(1+r) + L_2(1+r)^2 + \cdots\} = L$$
$$\text{when } A_1, L_0, L_1, \ldots, L \geq 0$$

In order to remind himself of what Besicovitch had said to him, Sraffa added before the formulas by Besicovitch 'The equation $A_1 p_a = Lw$ is equivalent to' and after them '(where *w* is eliminated, as it occurs on both sides. Therefore the condition '*w* and *r* vary always in the opposite direction' is unnecessary. [says Bes.])'. Sraffa added also the date and in the margin the fact that the reference is to the document of 2.11.1944 mentioned above (D3/12/63: 6). It is clear what had happened. Besicovitch had got doubts as to the correctness of what he had dictated about one month earlier: since there is no constraint on *w*, why should this expression make it necessary that '*w* and *r* vary always in the opposite direction'? Also this doubt by Besicovitch was inappropriate. He probably had forgotten that the necessity of the statement '*w* and *r* vary always in the opposite direction' was a consequence of the existence of a single real solution – in the relevant range – of exactly that equation.

On 12 December (D3/12/63: 5) Sraffa summarized the debate with Besicovitch on this point in a *nota bene*, where the N.B. is in red pencil:

> N.B. Notare che, in un primo tempo (31.10. e 2.11.44) B. aveva accettato la 'prova del gradient' incondizionatamente. Gli si era affacciato il dubbio che, se l'equazione era di grado pari, doveva avere almeno due soluzioni; e l'aveva risolto dicendo che la Reduction equation è di grado infinito.
>
> Poi, il 4.12.44, (senza ricordarsi l'ultimo punto) ha aggiunto la restrizione che r $\geqq -1$.[12]

This makes us think that Besicovitch's other doubt was not considered relevant or that further discussions with no written trace have taken place. This summary was written in order to introduce the following conjecture.

> Ora, mi sembra, questa restrizione può essere eliminata se si pone $e^\rho = (1+r)$, dove $\rho = \log(1+r)$. Perchè in tal caso a $r = -1$ corrisponde $\rho = -\infty$, e ad $r < -1$ non corrisponde nessun valore *reale* di ρ. In altri termini, alla proposizione 'vi è una sola radice reale $\sigma \geqq -1$ per *r*' si sostituisce l'equivalente 'vi è una sola radice *reale* per ρ.[13]

Obviously, the passage to continuous time does not change things and Sraffa was certainly wrong with his conjecture, but it shows his interest in eliminating a problem that from an economic point of view is totally irrelevant: if negative levels of the rate of profits are irrelevant in a long-period analysis, rates of profits lower than -1 are even more so in this and in other contexts.

Besicovitch's doubts probably prompted Sraffa to try to find an alternative proof such as the one he started to compose on 16 November. On 20 December the proof is clearly laid out in the following document. It is noteworthy that the first line is underlined in bold red, once again indicating the importance of the element at hand.

> Other proof of gradient ('commodity' point of view)
>
> in one-one case
>
> Consider the 'elements' composing the cost of production of a commodity.
> (a) the labour-term, which falls in proportion to *w*
> (b) the 'means of production' terms. Each of these contains two variables, *p* and *r*. Since *r* rises, if the price of the product has to fall more than *w*, the price of the means of production must fall more than *w*, to make up: since they are multiplied by $1 + r$ (which is rising) they can only contribute to the rate of decrease of the price of the product less than their own rate of decrease.
>
> Now consider, among the *k* products, the one that has (over a given interval) the greatest rate of decrease: its means of production's price has a smaller rate of decrease, and it is further reduced (as it is passed on in the product) through being multiplied by $1 + r$ (increasing). Therefore this product can only obtain its higher rate of decrease from its labour term: but this cannot contribute more than it has, i.e. *w*, therefore its rate of decrease must be < rate of *w*. (D3/12/63: 7)

This gets very close to the wording used in the published version. We may conclude by saying that both proofs we encounter in section 49 were produced in 1944. Besicovitch played an important part in finding the former, but none in finding the latter. As a matter of fact there is no trace that Sraffa even showed the latter to Besicovitch. In the case that he did, which we think can safely be assumed, Besicovitch could be expected to have swiftly approved of it.

13.4 The afterglow

While the main issue was settled, Sraffa nevertheless went back to the discussion of the gradient a few times shortly afterwards in attempts to round off the argument by exploring some of its implications. The following document entitled 'Implications of the "Gradient" argument', dated 21 January 1945, is of particular interest, because in it he relates his findings to David Ricardo's analysis which, as we have seen in Section 2 above, was the starting point of his own probing into the problem under consideration more than fifteen years earlier:

> 1) In *whatever* commodity wages may be fixed, they will always fall with the rise of the rate of profit. or
> 2) If labour is taken as measure of prices, the prices of *all* commodities fall [this is clearly a typo; it should obviously read 'rise'] with the rise of rate of profit. or

> 3) With the rise of the rate of profit some commodities will rise and some will fall – depending on which commodity is taken as standard of prices; but the prices of labour will fall in any case, *independently* of the standard chosen.
>
> Cp. Ricardo, Princ., 1st ed., my ed. p. 63 'no commodities whatever are raised in absolute price [in ed 2 "exchangeable value"] merely because wages rise.' In this argument he is supposing 'money to be of an invariable value; in other words to be always the produce of the same quantity of *unassisted labour*.' (ib., next para; my italics).

The first proposition simply reiterates the fact that there is an inverse relation between the rate of profits and the wage rate, independent of the standard in which wages are expressed. If expressed in terms of national income, then *w* represents the share of wages in national income. With a rise in the rate of profits, given the system of production in use, proportional wages, i.e. the wage share, must of necessity fall. This is a restatement of Ricardo's fundamental theorem concerning income distribution within the context of a circular flow. The second proposition states that prices measured in units of 'labour commanded' are bound to rise as the rate of profits rises. Clearly, the smaller the wage rate, the more labour a given commodity can 'command' or is exchanged for. The third proposition implies that with a rise in the wage rate (a fall in the rate of profits), the exchangeable values of all commodities would fall, expressed in a standard that is produced without any means of production, that is, by 'unassisted labour'. This contradicts a view that was widespread at the time of Ricardo and which Adam Smith had elevated to the level of a scientific truth, namely, that a rise in wages would entail a rise in the prices of all other commodities. Obviously, this view is untenable.

On 21 May 1945 Sraffa put forward the following

> *Proposition*: Total profits must *always* increase with the rise of *r* and fall of *w*, in whatever commodity they are measured.

This proposition cannot be sustained: While profits divided by wages are a rising function of the rate of profits, profits taken by themselves are not unless we use the social product as standard and wages are paid in units of the social product. In this way the value of the social product is fixed, the wage part is decreasing when *r* rises and therefore profits are increasing. Sraffa continued:

> Proof. We have seen that the price of no commodity (in terms of any other [or of the Standard Commodity?]) can fall as fast as *w*, if only circ. cap. is used; and it cannot do so without negative prices, if also fixed capital is used.

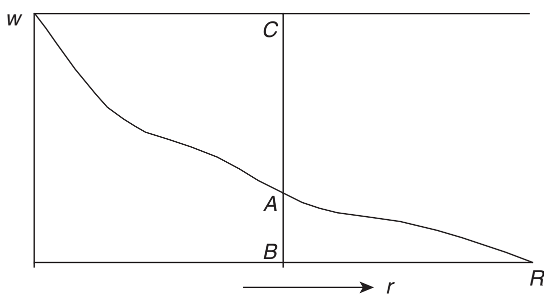
We were not able to find in Sraffa papers the proof that the 'proof of gradient' holds when fixed capital is used, and prices are non-negative. However the proposition is trueSraffa's final proposition reads:

AQ: Please clarify whether the question mark at the beginning of the sentense is to be retained or change it as quotations mark.

> But r, being a linear function of w, rises as fast as w falls.
> Therefore, 'the price of any commodity, simple or composite, multiplied by r' must always rise with the fall of w: And total profits are 'the price of the aggregate means of production multiplied by r'.
> [?] This would be so if r were = $1/w$. But is it with $r = R(1 \times w)$?]
>
> (D3/12/40: 185)

This proposition is certainly true for sufficiently low levels of the rate of profits, but does not need to be true in general. The sentence in square brackets bears testimony to the fact that Sraffa saw the difficulty, which is certainly solved if $rw = 1$, which, however, is generally not the case.

On 2 June 1945 Sraffa added some further 'Propositions' which he illustrated with the help of a diagram:



> From 'Gradient' it follows that the *quantity* of commodity in Real Wage always *falls* with the rise of r, whatever the commodity
>
> – Therefore the aggr. income of workers (*however* measured) falls with rise of r.
> – Therefore the aggreg. income of capitalists *always rises* with rise of r (however measured), being the complement of worker's income CA, i.e. $(1 - w)$
> – Therefore the price of the aggregate Const. Cap. of any SubSystem relatively to its product cannot fall as much as the reciprocal of r, with the rise of r. Since at an increased r the price of that Const. Cap. multiplied by r must be equal to a larger quantity of the product.
>
> (D3/12/40: 182)

AQ: Does the underline exist in the original?

In the margin of the second proposition Sraffa put a question mark, and for good reason, because the proposition is true, as mentioned above, only if the standard of value is national income (the social product) and workers are actually paid in terms of this standard. In the diagram Sraffa apparently plotted the social product as a line parallel to the abscissa at the level of the maximum level of wages compatible with a nonnegative rate of profits. This might be interpreted as implying that in this way he indicated that wages were actually taken to be measured in terms of national income. However, in the rest of the document there is nothing that would remove the uncertainty as to the interpretation of the diagram. To be sure, in order to avoid such uncertainty it would have to be explicitly stated in terms of which commodity (or standard) workers are actually taken to be paid. This problem was probably not seen by Sraffa at the time. In the margin of the third proposition Sraffa put a straight line, indicating approval. However Sraffa did not insert this result in his book.

The above quotations document well Sraffa's analytical concerns in the 1940s and his vacillations as regards the correctness or otherwise of some of his propositions concerning the properties of the economic system under consideration. It suffices to note that none of the errors contained in the above manuscripts survived and made their way into the book. Sraffa's untiring effort to get his argument right bore fruit.

13.5 Concluding remarks

Prices, Sraffa in Chapter III of his book (1960, pp. 12 and 15) insisted, generally do not follow a 'simple rule' and that 'the pattern of price-variations arising from a change in distribution' may be rather 'complex'. In Chapter VI he qualified these statements by pointing out: 'There is however a restriction to the movement of the price of any product: if as a result of a rise in the rate of profits the price falls, its rate of fall cannot exceed the rate of fall of the wage rate' (p. 38). This holds true (in single production) independently of the standard in terms of which prices and the wage rate are expressed.

This result was established by Sraffa who was assisted by Besicovitch towards the end of 1944. A first proof of the 'gradient' was provided by Besicovitch at the beginning of November 1944, a different proof was apparently independently elaborated by Sraffa in the second half of December of the same year. We encounter both again in section 49 of Sraffa's book.

Notes

1. On 31 January 1955 Sraffa wrote: 'The Standard Commodity is first identified in the packet of small sheets of College notepaper dated 27.1.44 and headed "Hypothesis"'; see D3/12/36: 91, with a clear reference to D3/12/36: 61–67.
2. These include the distinction between basics and non-basics, subsystems, and the further elaboration of the reduction to dated quantities of labour.
3. 'It is as clear as sunlight that if wages change also values change.'
4. 'In substance, the only objection I felt B. was making was that if, instead of attributing an arbitrary value to w, I introduce equation $A_1 p_a = Lw$, leaving w variable, the result is: I can no longer obtain the equation ... $r = R(1 - w)$, and therefore even if I were to take the Standard Commodity as unity, the solution is no longer unique; there are k values of w.'
5. '*N.B.* The substance of B.'s objection seems to be the following. For whichever U Lw is a linear function or r. But the price of commodity a, and therefore $A_1 p_a$, is not a linear function of r: therefore it oscillates as r varies, and the curve of the price $A_1 p_a$ *intersects in k points* (k values of r) the straight line that represents Lw: in each one of these points the *given* real wage is equal to a different proportional wage (w).'

6. '31 October 44. Today, after ten months of resistance, Besicovich [sic] has finally given in, admitting that the system has a unique solution in the case in which each process gives only a single product. The decisive argument is that the curve of the price of commodity A cannot cut the wage line more than once; because the price can never fall (in consequence of a fall in the wage) in a larger proportion than the wage; and this is so because the price can be expressed by means of a series (see above). The case under consideration is the one in which a given Standard Commodity is taken as the unit of prices and an arbitrary commodity as wage unit.'
7. Besicovitch insisted with Sraffa as to this aspect on 10 October 1944: 'anyhow by manipulating with your equations only you cannot single out a value of *r*' (D3/12/40: 13).
8. The inverted commas are required since we are using the language Sraffa used at the time and not the one used in the published book: In 1944 a 'Standard commodity' was any composite commodity made up in a proportion to an eigenvector; in 1960, the 'Standard commodity' is unique since it is the composite commodity made up in a proportion to the unique nonnegative eigenvector normalized by the condition that the direct and indirect labour required to produce them equals unity.
9. '1.11.44 Bes. admits that the conclusion remains valid when instead of the Standard Commodity one takes as unit of prices commodity *A*: because what is at issue is only a change of unit.'
10. In modern parlance we would say 'dominated processes'; see Filippini and Filippini (1982).
11. The rule of free goods is alluded to in a note on joint production contained in a folder (D3/12/11: 25) dated November 1927.
12. 'N.B. Note that B. had first (31.10. and 2.11.44) unconditionally accepted the 'proof of the gradient'. Then faced the doubt that when the equation was of even degree, it had to have at least two solutions; and this he solved by saying that the Reduction equation is of infinite degree.

    Then, on 4.12.44, (without remembering the last point) he added the restriction that r $\geqq -1$.'
13. 'It seems to me now that this restriction could be eliminated if one puts $e^{\rho} = (1+r)$, where $\rho = \log(1+r)$. Because in this case there corresponds to $r = -1$:$\rho = -\infty$, and to $r < -1$ there corresponds no *real* value of ρ. Put differently, to the proposition 'there is only a single real root $\sigma \geqq -1$ for *r*' one substitutes the equivalent 'there is only a single *real* root for ρ.'

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