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Economic dynamics in a simple model with exhaustible resources and a given real wage rate☆

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Abstract

The paper elaborates a dynamic input–output model with exhaustible resources. Discoveries of new deposits and technical progress are set aside. It is assumed that there is a 'backstop technology' (based on solar energy), which implies that exhaustible resources are not indispensable in production. Given the real wage rate and the consumption pattern of profit and royalty recipients, it is then shown that the paths followed by the royalties paid to the owners of resources, the quantities produced of the different commodities, and their prices are determined once a sequence of nominal profit rates is given. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

For well-known reasons, an economic system using exhaustible resources, such as ores of coal, oil or metal, constitutes one of the most difficult objects of investigation in the theory of production (see, for example, Kurz and Salvadori, 1995, ch. 12; Kurz and Salvadori, 1997). In order to render the problem manageable, theorists frequently have recourse to strong simplifying assumptions. In much of the literature the problem is studied in a partial framework with a single kind of exhaustible resource: the prices of all commodities except the price of the resource are assumed to be given and constant over time. With natural resources that are used to produce energy, for example, this is clearly unsatisfactory, because it can safely be assumed that energy enters as an input in the production of most, if not all, commodities, which implies that a change in the price of energy has an impact on the prices of many, if not all, commodities. Hence, a general framework of the analysis is needed. Moreover, since with exhaustible resources both relative prices, income distribution and the quantities produced will generally change over time, in principle a dynamic analysis is required tracing the time paths of prices, quantities and the distributive variables.

Piero Sraffa, a pioneer of the modern 'classical' theory of production, distribution and value (see Sraffa, 1960; and Unpublished Papers and Correspondence, Trinity College Library, Cambridge, UK, as catalogued by Jonathan Smith), was perfectly aware of these difficulties already at an early stage of his work. As is well known, he adopted the concept of production as a *circular flow*, which he had encountered in the writings of the physiocrats and the classical economists, and also in Marx. However, he was clear that the assumption of self-replacement of an economic system, which is to be found in these authors and on which he based some of his analysis, was a bold one. In the following note dated 25 March 1946 from his hitherto unpublished papers[1] he first points out a difference between a *physical real cost* approach to the problem of value and distribution, which he endorsed, and the labour theory of value:

> The difference between the 'Physical real costs' and the Ricardo–Marxian theory of 'labour costs' is that the first does, and the latter does not, include in them the natural resources that are used up in the course of production (such as coal, iron, exhaustion of land) [Air, water, etc. are not used up: as there is an unlimited supply, no subtraction can be made from ∞]. This is fundamental because it does away with 'human energy' and such metaphysical things.

He added:

[1] The reference to the papers follows the catalogue prepared by Jonathan Smith, archivist. We should like to thank Pierangelo Garegnani, literary executor of Sraffa's papers and correspondence, for granting us permission to quote from them.

> But how are we going to replace these natural things? There are three cases: a) they can be reproduced by labour (land properties, with manures etc.); b) they can be substituted by labour (coal by hydroelectric plant: or by spending in research and discovery of new sources and new methods of economising); c) they cannot be either reproduced nor substituted[2] - and in this case they cannot find a place in a theory of *continuous* production and consumption: they are dynamical facts, i.e. a stock that is being gradually exhausted and cannot be renewed, and must ultimately lead to destruction of the society. But this case does not satisfy our conditions of a society that just manages to keep continuously alive. (Sraffa's papers, D3/12/42: 33; Sraffa's emphasis).

Obviously, any economic model is bound to distort reality in some way. Otherwise it would be identical with the 'seamless whole' and thus useless in interpreting aspects of the latter. In no way do we want to dispute the usefulness of Sraffa's approach in his 1960 book, which hardly needs to be justified, given the rich harvest of important findings it yielded. At the same time the 'dynamical facts' Sraffa speaks of cannot be ignored and ought to be studied.

In this paper we shall make a further probing step in this direction. Our aim is very modest, though. In two previous contributions we studied the problem of exhaustible resources in a multisectoral framework, using a dynamic input–output model. In this paper we shall propose a significant modification of our previous formalizations, which, it is to be hoped, sheds some of their weaknesses. Compared with the earlier conceptualization, the new one exhibits the following features. While previously we started from a given nominal wage rate and a constant nominal rate of interest, we shall now assume a given and constant *real* wage rate, specified in terms of some given bundle of wage goods. Treating one of the distributive variables as given from outside the system of production (or treating it as independently variable) and taking the other variables (rate of profits and royalties) as endogenously determined is much more 'classical' in spirit than the previous premises. In particular, the classical concept of the 'surplus' product, and its sharing out between capitalists and resource owners as profits and royalties, is given a clear physical meaning. Further, we shall assume that all realised nonwage incomes, profits and royalties, will be spent on consumption; for simplicity it is assumed that this part of consumption will be proportional to a given vector of consumption goods, which does not change over time. We shall set aside technical progress both in the methods of production extracting and in those using resources. Discoveries of new deposits (or resources) are excluded; existing stocks of resources are taken to be known with certainty at any given moment of time. To avoid the implication mentioned by Sraffa — the 'destruction of society' — we shall assume that there is a 'backstop technology', which allows one to produce the given vector of consumption goods without using any of the exhaustible resources. The example

[2] This is Sraffa's formulation, which we left as it is.

given in our previous contributions was solar or geothermal energy which could replace other forms of energy.

The composition of the paper is as follows. Section 2 states the main assumptions that underlie the argument and presents the dynamic input–output model. Section 3 contains some preliminary result. Section 4 presents the complete analysis and the main results. Section 5 contains some concluding remarks.

2. The model and its assumptions

The formalization of the problem suggested in this paper is based on the following simplifying assumptions. A finite number n of different commodities, which are fully divisible, are produced in the economy and a finite number m ($>n$) of constant returns to scale processes are known to produce them. Let p_t be the vector of prices of commodities available at time $t\in\mathbb{N}_0$ and let x_t be the vector of the intensities of operation of processes at time $t\in\mathbb{N}$. A process or method of production is defined by a quadruplet (a, b, c, l), where $a\in\mathbb{R}^n$ is the commodity input vector, $b\in\mathbb{R}^n$ is the output vector, $c\in\mathbb{R}^s$ is the exhaustible resources input vector, and l is the labour input, a scalar; of course $a\geqslant 0$, $b\geqslant 0$, $c\geqq 0$, $l\geqslant 0$. The production period is uniform across all processes. It is important to remark that the inputs referred to in vector c are inputs of the resources *as they are provided by nature*; for example, extracted oil is *not* contained in c, but in b, if (a, b, c, l) is an extraction process, or in a, if (a, b, c, l) is a process that uses it, unless the extraction costs are nil. The m existing processes are defined by quadruplets

$$(a_j,\ b_j,\ c_j,\ l_j) \quad j=1, 2,\ldots, m$$

Then define matrices A, B, C and (now) vector l as follows:[3]

$$A=\begin{bmatrix} a_1^T \\ a_2^T \\ \cdot \\ \cdot \\ \cdot \\ a_m^T \end{bmatrix} \qquad B=\begin{bmatrix} b_1^T \\ b_2^T \\ \cdot \\ \cdot \\ \cdot \\ b_m^T \end{bmatrix} \qquad C=\begin{bmatrix} c_1^T \\ c_2^T \\ \cdot \\ \cdot \\ \cdot \\ c_m^T \end{bmatrix} \qquad l=\begin{bmatrix} l_1 \\ l_2 \\ \cdot \\ \cdot \\ \cdot \\ l_m \end{bmatrix}$$

Assume that the annual consumption of commodities by profit and royalty recipients is proportional to a vector d, which, for simplicity, is assumed to be given and constant over time, that is, independent of prices and quantities, including the quantities of the exhaustible resources left over at the end of each production period. More specifically, assume that the total amounts actually consumed by capitalists and the proprietors of deposits of exhaustible resources are constant over time and equal to γ units of vector d, $\gamma\geqslant 0$, where γ depends on the resources

[3] Transposition of a vector or a matrix is denoted by superscript T.

available at time zero. In addition, the real wage rate, defined by a commodity vector w, is taken to be given and constant over time. Technical innovations of any kind are set aside. All exhaustible resources are private property. In conditions of free competition there will be a (tendency towards a) uniform nominal rate of profits r_t across all production activities in the economy. This implies that, for each time $t \in \mathbb{N}_0$, the following inequalities and equations are to be satisfied:

$$Bp_{t+1} \leqq (1+r_t)(Ap_t + Cy_t) + lw^T p_{t+1} \tag{1a}$$

$$x^T_{t+1} Bp_{t+1} = x^T_{t+1}[(1+r_t)(Ap_t + Cy_t) + lw^T p_{t+1}] \tag{1b}$$

$$y_{t+1} \leqq (1+r_t) y_t \tag{1c}$$

$$z^T_{t+1} y_{t+1} = (1+r_t) z^T_{t+1} y_t \tag{1d}$$

$$x^T_{t+1}(B - lw^T) \geqq x^T_{t+2} A + \gamma d^T \tag{1e}$$

$$x^T_{t+1}(B - lw^T) p_{t+1} = (x^T_{t+2} A + \gamma d^T) p_{t+1} \tag{1f}$$

$$z^T_t \geqq x^T_{t+1} C + z^T_{t+1} \tag{1g}$$

$$z^T_t y_t = (x^T_{t+1} C + z^T_{t+1}) y_t \tag{1h}$$

$$\gamma > 0,\ p_t \geqq 0,\ y_t \geqq 0,\ z_t \geqq 0,\ x_{t+1} \geqq 0 \tag{1i}$$

Inequality (1a) means that nobody can get extra profits by producing commodities available at time $t+1$. Eq. (1b) implies, because of inequalities (1a) and (1i), that commodities available at time $t+1$ will only be produced if the ruling nominal rate of interest is obtained. Inequality (1c) means that nobody can get extra profits by storing exhaustible resources from time t to time $t+1$. Eq. (1d) implies, because of inequalities (1c) and (1i), that exhaustible resources will be stored from time t to time $t+1$ only if the ruling nominal rate of interest will be obtained by this storage activity. Inequality (1e) implies that the amounts of commodities produced are not smaller than the amounts of commodities required, and Eq. (1f) implies that if an amount is larger, then the price of that commodity is zero. Inequality (1g) implies that the amounts of exhaustible resources available at time t are not smaller than the amounts of exhaustible resources available at time $t+1$ plus the amounts of exhaustible resources utilized to produce commodities available at time $t+1$, and Eq. (1h) implies that if an amount is larger, then the price of that exhaustible resource is zero. The meaning of inequalities (1i) is obvious.

Model (1) is not yet complete, because some initial conditions are needed. A first obvious initial condition is that the amounts of exhaustible resources available at time 0 are given, that is,

$$z_0 = \bar{z} \tag{1j}$$

A second initial condition, which is perhaps less obvious, is that the amounts of commodities available at time 0 are given. This can be stated as

$$v^T \geqq x^T_1 A + \gamma d^T \tag{1k}$$

$$v^T p_0 = (x^T_1 A + \gamma d^T) p_0 \tag{1l}$$

where v is a given positive vector.

It is easily checked that the given sequence $\{r_t\}$ plays no role in determining the relative actualized prices in the sense that if the sequences $\{p_t\}$, $\{y_t\}$, $\{z_t\}$, $\{x_{t+1}\}$ are a solution to system (1a)–(1l) for the given sequence $\{r_t\}$, then the sequences $\{q_t\}$, $\{u_t\}$, $\{z_t\}$, $\{x_{t+1}\}$ are a solution to the same system for the given sequence $\{\rho_t\}$, provided that

$$q_t = \prod_{\tau=0}^{t-1} \frac{1+\rho_\tau}{1+r_\tau} p_t$$

$$u_t = \prod_{\tau=0}^{t-1} \frac{1+\rho_\tau}{1+r_\tau} y_t$$

This is so because r_t is the *nominal* rate of interest and thus incorporates also the rate of inflation, so that a change in r_t leaves unchanged the *real* rate of profits and involves only a change in the rate of inflation.

It is also easily checked that the above model can determine only the relative actualized prices in the sense that if the sequences $\{p_t\}$, $\{y_t\}$, $\{z_t\}$, $\{x_{t+1}\}$ are a solution to system (1a)–(1l), then the sequences $\{\eta p_t\}$, $\{\eta y_t\}$, $\{z_t\}$, $\{x_{t+1}\}$ are also a solution, where η is a positive scalar. In order to fix the numeraire and to preserve the property that a change in the nominal rates of profit does not affect relative prices, the numeraire is set in terms of actualized prices, that is, we add the following equation

$$\sum_{t=0}^{\infty} \frac{u_t^T p_t}{\prod_{\tau=0}^{t-1}(1+r_\tau)} = 1 \tag{1m}$$

where $\{u_t\}$ is a sequence of known nonnegative vectors (at least one of which is semipositive).

In the following we will assume that $\{r_t\}$ is a constant sequence and that $r_t = 0$. A change to a more appropriate sequence of nominal rates of profit can be effected at any time, as indicated above. In the following it will also be assumed that $u_t = d$ in Eq. (1m). Then system (1) is more simply stated as

$$(B - lw^T)p_{t+1} \leqq (Ap_t + Cy_t) \tag{2a}$$

$$x_{t+1}^T(B - lw^T)p_{t+1} = x_{t+1}^T(Ap_t + Cy_t) \tag{2b}$$

$$y_{t+1} \leqq y_t \tag{2c}$$

$$z_{t+1}^T y_{t+1} = z_{t+1}^T y_t \tag{2d}$$

$$v^T \geqq x_1^T A + \gamma d^T \tag{2e}$$

$$v^T p_0 = (x_1^T A + \gamma d^T)p_0 \tag{2f}$$

$$x_{t+1}^T(B - lw^T) \geqq x_{t+2}^T A + \gamma d^T \tag{2g}$$

$$x_{t+1}^T(B - lw^T)p_{t+1} = (x_{t+2}^T A + \gamma d^T)p_{t+1} \tag{2h}$$

$$z_t^T \geqq x_{t+1}^T C + z_{t+1}^T \tag{2i}$$

$$z_t^T y_t = (x_{t+1}^T C + z_{t+1}^T) y_t \tag{2j}$$

$$z_0 = \bar{z} \tag{2k}$$

$$\sum_{t=0}^{\infty} d^T p_t = 1 \tag{2l}$$

$$\gamma > 0,\ p_t \geqq 0,\ y_t \geqq 0,\ z_t \geqq 0,\ x_{t+1} \geqq 0 \tag{2m}$$

Each of the exhaustible resources is assumed to provide directly or indirectly[4] services that are useful in production. However, it is assumed that the same kind of services can also be produced by solar energy, the source of which does not risk exhaustion in any relevant time-frame. More specifically, we shall assume that the commodities annually required for consumption, defined in terms of vector d, can be produced without using exhaustible resources. Hence, there is a 'backstop technology'. The processes defining that backstop technology ($\bar{A}$, $\bar{B}$, 0, $\bar{l}$) are obtained from (A, B, C, l) by deleting all the processes using directly some natural resource (i.e. process ($e_i^T A$, $e_i^T B$, $e_i^T C$, $e_i^T l$) is in the set of processes ($\bar{A}$, $\bar{B}$, 0, $\bar{l}$) if and only if $e_i^T C = 0^T$). We may conveniently summarize what has just been said in the following

Assumption 1. There is a scalar r^* and there are vectors x^* and p^* which solve the system

$$x^T(\bar{B} - \bar{A} - \bar{l}w^T) \geqq d^T \tag{3a}$$

$$x^T(\bar{B} - \bar{A} - \bar{l}w^T)p = d^T p \tag{3b}$$

$$\bar{B}p \leqq [(1+r)\bar{A} + \bar{l}w^T]p \tag{3c}$$

$$x^T\bar{B}p = x^T[(1+r)\bar{A} + \bar{l}w^T]p \tag{3d}$$

$$x \geqq 0,\ p \geqq 0,\ d^T p = 1 \tag{3e}$$

In the following discussion we will refer to the processes operated at the intensity vector $\bar{x}$, obtained by augmenting vector x^* with zeros, as the 'cost-minimizing backstop processes', and we will denote these processes by the quadruplet ($\hat{A}$, $\hat{B}$, 0, $\hat{l}$).

The assumption that there is a backstop technology (i.e. Assumption 1) is necessary in order to avoid the 'end of the world' scenario, on which there is nothing to be said. This is the case because we excluded discoveries of new deposits (or resources) and innovations. Seen from this perspective, Assumption 1 may be considered as a sort of simple corrective device to counterbalance the bold premises that underlie our analysis. The following assumptions characterize the backstop technology and the cost-minimizing backstop processes.

[4] Assume, for instance, that electric energy can be produced from oil which is extracted from the ground. The unextracted oil is the resource, whereas the extracted oil is a commodity produced by means of that resource. Then we say that the resource produces electric energy *indirectly*.

Assumption 2. The backstop technology is such that it converges to the processes $(\hat{A}, \hat{B}, 0, \hat{l})$. In other words, the backstop processes $(\bar{A}, \bar{B}, 0, \bar{l})$ are such that the system made up by equations and inequalities (2a), (2b), (2e), (2f), (2g), (2h) and (2l) and the first two and the fifth of inequalities (2m), with $A = \bar{A}$, $B = \bar{B}$, $C = 0$, $l = \bar{l}$, is such that, for each of its solutions (if there is one), there is a natural number θ^* such that, for each $t \geqslant \theta^*$, only the processes $(\hat{A}, \hat{B}, 0, \hat{l})$ are operated.

Assumption 3. The number of cost-minimizing backstop processes is exactly n (the number of commodities); the matrix $[\hat{B} - \hat{l}w^T]$ is invertible; the matrix $[\hat{B} - \hat{l}w^T]^{-1}\hat{A}$ is non-negative; and the eigenvalue of maximum modulus of matrix $[\hat{B} - \hat{l}w^T]^{-1}\hat{A}$ is smaller than unity.

Assumption 3 certainly holds if there is no joint production and if the real wage rate is such that, for each commodity, no more than one process producing it can be operated in the long run. In fact, in this case, we can order processes $(\hat{A}, \hat{B}, 0, \hat{l})$ in such a way that $\hat{B}$ is diagonal, with the elements on the main diagonal all positive; finally, the other properties mean just that the backstop technology can support the given real wage rate w. This assumption implies also that r^* as determined in system (3) is positive, since the eigenvalue of maximum modulus of the matrix $[\hat{B} - \hat{l}w^T]^{-1}\hat{A}$ equals $(1 + r^*)^{-1}$, which has been assumed to be smaller than 1.

3. A preliminary result

Assume that system (2) has a solution. Call the set of processes operated at time t in such a solution the *position at time t*. Because the number of processes is finite, the number of possible positions is also finite. Hence, at least one position is replicated for an infinite number of times. Because the amounts of exhaustible resources available at time 0 are finite, and because the vector of the amounts of resources utilized in a position employing exhaustible resources is bounded from below (recall that vector γd is constant over time), with regard to any position which is replicated an infinite number of times we have: either it does not use exhaustible resources at all; or, if it uses them, it includes processes which can be operated in order to produce the consumption vector γd without using exhaustible resources, which means that the intensities of operation of the processes in the position under consideration can be changed from time t to time $t+1$ in order to reduce the amounts of natural resources utilized continuously. Hence, we can divide the period from time 0 to infinity into two subperiods: a finite subperiod from time 0 to time τ' and an infinite subperiod from time $\tau' + 1$ to infinity, on the condition that, in the second subperiod, only the backstop processes $(\bar{A}, \bar{B}, 0, \bar{l})$ concur in determining the dynamics of the prices of producible commodities. Moreover, if Assumptions 2 and 3 hold, we can divide the period from time $\tau' + 1$ to infinity into two subperiods: a finite subperiod from time $\tau' + 1$ to time τ'' and an infinite subperiod from time $\tau'' + 1$ to infinity, on the condition that, in the second subperiod,

$$p_t = A^{*t-\tau''} p_{\tau''}$$

$$y_t = y_{\tau''}$$

where $A^* = [\hat{B} - \hat{l}w^T]^{-1}\hat{A}$. If process (a_j, b_j, c_j, l_j) is a process in a position replicated for an infinite number of times, such that $c_j \geqslant 0$, then

$$(b_j - l_j w)^T A^{*t+1-\tau''} p_{\tau''} = (a_j^T A^{*t-\tau''} p_{\tau''} + c_j^T y_{\tau''}) \quad \text{each } t \geq \tau''$$

that is,

$$[(b_j - l_j w)^T A^* - a_j^T] A^{*t-\tau''} p_{\tau''} = c_j^T y_{\tau''} \quad \text{each } t \geq \tau''$$

Hence, Assumption 3 implies that

$$c_j^T y_{\tau''} = 0$$

$$(b_j - l_j w)^T A^* = a_j^T \tag{4}$$

In other words, the exhaustible resources eventually used in the position replicated for an infinite number of times have a zero price and the input–output conditions relative to producible commodities of any process using exhaustible resources in the position replicated for an infinite number of times satisfy the proportionality condition (4). A process (a_j, b_j, c_j, l_j) such that $c_j \geqslant 0$ and Eq. (4) holds is certainly a dominated process, because there is a combination of some other processes which require exactly the same inputs except the exhaustible resources c_j, which are not needed, and produce the same outputs. Hence, there appears to be no harm in adopting the following

Assumption 4. There is no process (a_j, b_j, c_j, l_j) such that $c_j \geqslant 0$ and Eq. (4) holds.

Assumptions 1–4 ensure that processes $(\hat{A}, \hat{B}, 0, \hat{l})$ constitute the unique position which can be replicated for an infinite number of times. This fact suggests the following problem, the study of which is a preliminary step to an analysis of system (2). Let θ be a positive natural number and let us investigate the following system (5).

$$(B - lw^T)p_{t+1} \leqq (Ap_t + Cy_t) \quad 0 \leq t \leq \theta - 1 \tag{5a}$$

$$x_{t+1}^T(B - lw^T)p_{t+1} = x_{t+1}^T(Ap_t + Cy_t) \quad 0 \leq t \leq \theta - 1 \tag{5b}$$

$$y_{t+1} \leqq y_t \quad 0 \leq t \leq \theta - 1 \tag{5c}$$

$$z_{t+1}^T y_{t+1} = z_{t+1}^T y_t \tag{5d}$$

$$v^T \geqq x_1^T A + \gamma d^T \tag{5e}$$

$$v^T p_0 = (x_1^T A + \gamma d^T) p_0 \tag{5f}$$

$$x_t^T(B - lw^T) \geqq x_{t+1}^T A + \gamma d^T \quad 1 \leq t \leq \theta - 1 \tag{5g}$$

$$x_t^T(B - lw^T)p_t = (x_{t+1}^T A + \gamma d^T)p_t \quad 1 \leq t \leq \theta - 1 \tag{5h}$$

$$x_\theta^T(B - lw^T) \geqq \gamma d^T + \gamma d^T(I - A^*)^{-1}A^* \tag{5i}$$

$$x_\theta^T(B - lw^T)p_\theta = [\gamma d^T + \gamma d^T(I - A^*)^{-1}A^*]p_\theta \tag{5j}$$

$$\bar{z}^T \geqq x_1^T C + z_1^T \tag{5k}$$

$$\bar{z}^T y_0 = (x_1^T C + z_1^T) y_0 \tag{5l}$$

$$z_t^T \geqq x_{t+1}^T C + z_{t+1}^T \quad 1 \leq t \leq \theta - 1 \tag{5m}$$

$$z_t^T y_t = (x_{t+1}^T C + z_{t+1}^T) y_t \quad 1 \leq t \leq \theta - 1 \tag{5n}$$

$$\sum_{t=0}^{\theta-1} d^T p_t + \sum_{t=\theta}^{\infty} d^T A^{*t-\theta} p_\theta = 1 \tag{5o}$$

$$p_t \geqq 0, y_t \geqq 0, \quad 0 \leq t \leq \theta \tag{5p}$$

$$z_t \geqq 0, \ x_t \geqq 0, \quad 1 \leq t \leq \theta \tag{5q}$$

$$\gamma > 0 \tag{5r}$$

System (5) can be considered as consisting of the first θ steps of system (2), on the assumption that $x_{\theta+1} = \gamma \bar{x}$ (and therefore $x_{\theta+1}^T A = \gamma d^T (I - A^*)^{-1} A^*$), that is, on the assumption that, at time $\theta + 1$, the cost-minimizing backstop processes are operated and are operated at the cost-minimizing backstop intensities to produce γ times the consumption vector, and, as a consequence, the price vectors for $t > \theta$ mentioned in the equation fixing the numeraire are

$$p_t = A^{*t-\theta} p_\theta$$

Because of the equilibrium theorem of linear programming, system (5a)–(5q) is equivalent to each of the following two linear programming problems, which are dual to each other:

(primal):

Min $v^T p_0 + \bar{z}^T y_0$

s.to $\quad Ap_t - (B - lw^T) p_{t+1} + C y_t \geqq 0 \quad 0 \leq t \leq \theta - 1$

$$y_t - y_{t+1} \geqq 0 \quad 0 \leq t \leq \theta - 1$$

$$\sum_{t=0}^{\theta-1} d^T p_t + d^T (I - A^*)^{-1} p_\theta = 1$$

$$p_t \geqq 0, \ y_t \geqq 0 \quad 0 \leq t \leq \theta$$

(dual):

Max γ

s.to $\quad x_1^T A + \gamma d^T \leqq v^T \quad 0 \leq t \leq \theta - 1 \qquad (6a)$

$$-x_t^T (B - lw^T) + x_{t+1}^T A + \gamma d^T \leqq 0^T \quad 1 \leq t \leq \theta - 1 \tag{6b}$$

$$-x_\theta^T (B - lw^T) + \gamma d^T (I - A^*)^{-1} \leqq 0^T \tag{6c}$$

$$x_1^T C + z_1^T \leqq \bar{z}^T \tag{6d}$$

$$x^T_{t+1}C - z^T_t + z^T_{t+1} \leqq 0^T \quad 1 \leq t \leq \theta - 1 \tag{6e}$$

$$z_t \geqq 0,\ x_t \geqq 0 \quad 1 \leq t \leq \theta \tag{6f}$$

where γ does not need to be nonnegative. The following proposition gives an if and only if condition of the existence of a solution to system (5).

Proposition 1. If there is a backstop technology, system (5) has a solution for $\theta = \theta'$, if and only if the following Assumption 5 holds.

Assumption 5. There are two finite sequences x_t and z_t ($t = 1, 2,\ldots, \theta'$) and a positive real number γ such that system (6) holds for $\theta = \theta'$.

Proof. It is easily checked that there is a real number $\sigma > 0$ so large that the two finite sequences

$$p_t = \frac{r^*}{(1 + r^*)^{t+1}} p^* \quad (t = 0, 1,\ldots, \theta')$$

$$y_t = \sigma e \quad (t = 0, 1\ldots,\theta'$$

are feasible solutions to the primal; then both the primal and the dual have optimal solutions with a positive optimal value of γ if and only if Assumption 5 holds, because of the duality theorem of linear programming. Q.E.D.

4. The complete analysis and the main results

The following proposition provides an information about the solutions to system (5) for *different* θs.

Proposition 2. If system (5) has a solution for $\theta = \theta'$, then it has a solution for $\theta = \theta''$, each $\theta'' \geqslant \theta'$.

Proof. If the two finite sequences x'_t, z'_t ($t = 1, 2,\ldots, \theta'$) and the real number γ' satisfy system (6) for $\theta = \theta'$, then the two finite sequences x''_t, z''_t ($t = 1, 2,\ldots, \theta''$) with $x''_t = x'_t$ and $z''_t = z'_t$ for $t = 1, 2,..., \theta'$, and $x''_t = \gamma'\bar{x}$ and $z''_t = z'_\theta$ for $t = \theta' + 1, \theta' + 2,..., \theta''$, and the real number γ' satisfy system (5) for $\theta = \theta''$. Q.E.D.

Assume now that there is a natural number θ' such that Assumption 5 holds. Then, because of Proposition 2, for each $\theta \geqslant \theta'$, the maximum value of the dual (exists and) is positive; we will call it γ_θ. Moreover, for each $\theta \geqslant \theta'$, four infinite sequences $\{x_{t\theta}\}$, $\{z_{t\theta}\}$, $\{p_{t\theta}\}$ and $\{y_{t\theta}\}$ are defined, where, for $t \leq \theta$, $p_{t\theta}$ and $y_{t\theta}$ equal the corresponding elements of the optimal solution of the primal, and $x_{t\theta}$ and $z_{t\theta}$ equal the corresponding elements of the optimal solution of the dual and, for $t \geqslant \theta$, we have

$$p_{t\theta} = (A^*)^{t-\theta} p_{\theta\theta}$$

$$y_{t\theta} = y_{\theta\theta}$$

$$x_{t\theta} = \gamma_\theta \bar{x}$$

$$z_{t\theta} = z_{\theta\theta}$$

where matrix $A^* = [\hat{B} - \hat{1}w^T]^{-1}\hat{A}$ has the properties mentioned in Assumption 3. The following remarks are immediately checked:

Remark 1. For each $t \geqslant 0$ and for each $\theta \geqslant \max\,(t+1,\ \theta')$, $p_{t\theta}$, $p_{t+1,\theta}$, and $y_{t\theta}$ satisfy inequality (2a).

Remark 2. For each $t \geqslant 0$ and for each $\theta \geqslant \theta' + 1$, $p_{t\theta}$, $p_{t+1,\theta}$, $y_{t\theta}$, $y_{t+1,\theta}$, $x_{t+1,\theta}$, $x_{t+2,\theta}$, $z_{t\theta}$, $z_{t+1,\theta}$, and γ_θ satisfy inequalities and equations (2b)–(2m).

As a consequence,

Proposition 3. The sequences $\{p_t^*\}$, $\{y_t^*\}$, $\{x_t^*\}$, $\{z_t^*\}$, and the real number γ^* defined as

$$p_t^* = \lim_{\theta \to \infty} p_{t\theta} \tag{7a}$$

$$y_t^* = \lim_{\theta \to \infty} y_{t\theta} \tag{7b}$$

$$x_t^* = \lim_{\theta \to \infty} x_{t\theta} \tag{7c}$$

$$z_t^* = \lim_{\theta \to \infty} z_{t\theta} \tag{7d}$$

$$\gamma^* = \lim_{\theta \to \infty} \gamma_\theta \tag{7e}$$

constitute a solution to system (2).

Proof. It is easily checked that if there are the limits (7), and if they are finite, then the sequences $\{p_t^*\}$, $\{y_t^*\}$, $\{x_t^*\}$, $\{z_t^*\}$, and the real number γ^* constitute a solution to system (2). In fact, if p_t^*, p_{t+1}^*, y_t^* do not satisfy inequality (2a) for some t, then there is a $\tau \geqslant \max\,(t+1,\ \theta')$ such that for that τ and for each $\theta \geqslant \tau$ Remark 1 is contradicted. Similarly, if p_t^*, p_{t+1}^*, y_t^*, y_{t+1}^*, x_{t+1}^*, x_{t+2}^*, z_t^*, z_{t+1}^*, γ^* do not satisfy any of inequalities or equations (2b)–(2m) for some t, then there is a $\tau \geqslant \max \theta' + 1$ such that for that τ and for each $\theta \geqslant \tau$ Remark 2 is contradicted. In order to show that limits (7) do exist, it is enough to check that, because of Remark 2, $\gamma_{\theta+1} \geqslant \gamma_\theta$. Hence, the sequence $\{\gamma_\theta\}$ is increasing and, because it is bounded (it must satisfy inequality (5e)), it is convergent. Since γ_θ is the maximum value of the dual linear programme above, this is enough to assert that all the mentioned limits exist. This proves also that the limit (7e) is finite. To show that limits (7a) and (7b) are finite, it is enough to remark that

$$0 \leqq p_{t+1}^* \leqq A^* p_t^*, \quad 0 \leqq y_{t+1}^* \leqq y_t^*, \quad \text{and} \quad \bar{z}^T y_0^* + v^T p_0^* = \gamma^*$$

The fact that limits (7c) and (7d) are finite is an obvious consequence of inequalities (2e), (2g), (2i) and Eq. (2k). Q.E.D.

5. Concluding remarks

In this paper a dynamic input–output model has been developed which is able to deal with exhaustible resources based on a number of simplifying assumptions. In particular, each resource is taken to be available in a quantity which, at time 0, is known with certainty. Discoveries of new resources (or deposits of known resources) are excluded. Technical progress in the industries extracting or utilizing the resources is set aside. It is assumed that there is a ‘backstop technology’, which implies that exhaustible resources are useful but not indispensable in the production and reproduction of commodities. The real wage rate is given and constant. The annual consumption of commodities by profit and royalty recipients is assumed to be proportional to a given vector of commodities which is constant over time. On the basis of these assumptions the paths followed by the endogenous variables — especially the royalties paid to the owners of the exhaustible resources, the quantities produced of the different commodities and their prices — are determined once a sequence of nominal profit rates is given. A change in such a sequence does not affect the quantities produced or the relative royalties and prices actualized at any time. One aspect of the solution of the model is the structural change of the economy over time, that is, the change in the methods of production adopted to satisfy effectual demand and the intensities with which the processes are operated, the overall level and composition of employment, etc.

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