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# 'Classical' Roots of Input–Output Analysis: a Short Account of its Long Prehistory

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ABSTRACT *This paper discusses the roots of input–output analysis in 'classical' economics. The authors considered include Petty and Cantillon; Quesnay, the physiocrats and their critic Isnard; Smith, Ricardo, Torrens and Dmitriev; Marx, von Bortkiewicz and von Charasoff; Leontief; and Remak. It is argued that, in terms of method and content, input–output analysis is akin to the classical approach to the theory of production, distribution and relative prices in that (i) it requires all magnitudes to be observable and (ii) starts essentially from the same set of data. It is shown that many important modern concepts have been anticipated by the earlier authors. The prehistory of input–output analysis is also meant to provide new perspectives on potential future developments of the field.*

KEYWORDS: *Circular flow; classical economics; reproduction; surplus product; value and distribution*

## 1. Introduction

According to Wassily Leontief, 'Input–output analysis is a practical extension of the classical theory of general interdependence which views the whole economy of a region, a country and even of the entire world as a single system and sets out to describe and to interpret its operation in terms of directly observable basic structural relationships' (Leontief, 1987, p. 860).

The key terms in this characterization are 'classical theory', 'general interdependence' and 'directly observable basic structural relationships'. In this overview of contributions, which can be said to have prepared the ground for input–output analysis proper, 'classical theory' will be interpreted to refer to the contributions of the early classical economists, from William Petty to David Ricardo; further elaborated by authors such as Karl Marx, Vladimir K. Dmitriev, Ladislaus von Bortkiewicz and Georg von Charasoff; and culminating in the works of John von Neumann and Piero Sraffa. 'General interdependence' will be taken to involve two intimately intertwined problems, which, in a first step of the analysis, may however be treated separately. First, there is the problem of *quantity*, for which a structure of the levels of operation of processes of production is needed, in order to guarantee the reproduction of the means of production that are used up in the course of production and the satisfaction of some 'final demand'; that is, the needs and

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wants of the different groups (or 'classes') of society, perhaps making allowance for the growth of the system. Secondly, there is the problem of *price*, for which a structure of exchange values of the different products or commodities is needed in order to guarantee a distribution of income between the different classes of income recipients that is consistent with the repetition of the productive process on a given (or increasing) level. It is a characteristic feature of input–output analysis that both the independent and the dependent variables are to be 'directly observable', at least in principle. The practical importance of this requirement is obvious, but there is also a theoretical motivation for it: the good of an economic analysis based on magnitudes that cannot be observed, counted and measured is necessarily uncertain.

In this paper, an attempt is made to locate input–output analysis within economics and to show which tradition in economic thought it belongs to. This necessitates tracing its roots to earlier economic theory. We shall see that input–output analysis can indeed look back at a formidable history prior to its own proper inception, which is often dated from the early writings of Wassily Leontief. These writings include his 1928 paper 'Die Wirtschaft als Kreislauf' (The economy as a circular flow) (Leontief, 1928) and his 1936 paper on 'Quantitative input–output relations in the economic system of the United States' (Leontief, 1936). Because of its applied character, the latter is occasionally considered 'the beginning of what has become a major branch of quantitative economics' (Rose & Miernyk, 1989, p. 229). The account of the prehistory of input–output analysis may also throw light on wider issues which played an important role in the past, but are commonly set aside in many, but not all, modern contributions to input–output analysis. This concerns, first and foremost, the subject of value and distribution. While in earlier authors, and also in Leontief (1928), that issue figured prominently, in modern contributions it is frequently set aside or dealt with in a cavalier way. This raises a problem, because production, distribution and relative prices are intimately intertwined and cannot, in principle, be tackled independently of one another. Scrutinizing the earlier literature shows why.

The historical point of view provides some new perspectives on the potentialities of input–output analysis. This is the main motivation for writing this paper. It goes without saying that only a very small selection of the relevant historical material can be reviewed. It is to be hoped, however, that the paper contains some useful hints of the origins and gradual development of certain concepts used in modern input–output analysis, which allow the reader to locate its place in the history of economics and to see whether and where this history is characterized by continuity, or otherwise. By way of contrast with earlier contributions, the paper may also contribute to a better understanding of the method, scope and content of contemporary input–output analysis, both its strengths and weaknesses, and its potential for further development. The present paper leads up to the material covered in the survey articles by Stone (1984) and Rose & Miernyk (1989).[1]

It is perhaps useful to specify more clearly right at the beginning of this paper what is meant by the *classical* approach to the theory of value and distribution and to contrast it with the alternative marginalist or *neoclassical* approach. In the theory of value and distribution, the elaborated versions of the former typically start from the following set of data:

(i) The set of technical alternatives from which cost-minimizing producers can choose. (In an extreme case, only one technique is taken to be available; that is, the problem of the choice of technique is set aside.)

(ii) The size and composition of the social product, reflecting the needs and wants of the members of the different classes of society and the requirements of reproduction and capital accumulation.
(iii) The ruling real wage rate(s) (or, alternatively, the general rate of profit).
(iv) The quantities of different qualities of land available and the known stocks of depletable resources, such as mineral deposits. (In an extreme case, natural resources are, for simplicity, set aside; that is, taken to be 'free goods'.)

In the analysis the emphasis is on free competition; that is, the absence of significant barriers to entry in and exit from markets. The treatment of wages (or, alternatively, the rate of profit) as an independent variable, and of the other distributive variables—the rate of profit (the wage rate) and the rents of land—as dependent residuals exhibits a fundamental *asymmetry* in the classical approach. Prices are considered to be the means of distributing the social surplus in the form of profits and rents (and possibly interest). It also deserves to be emphasized that these data, or independent variables, all satisfy Leontief's criterion of observability. Moreover, these data are sufficient to determine the unknowns, or dependent variables: the rate of profit (the wage rate), the rent rates, and the set of relative prices supporting the cost-minimizing system of producing the given levels of output. No other data, such as, for example, demand functions for commodities and factors of production are needed. The classical approach allows the consistent determination of the variables under consideration. It does so by separating the determination of income distribution and prices from that of quantities, taken as given in (ii) above. The latter were considered as determined in another part of the theory; that is, the analysis of capital accumulation, structural change and socio-economic development.

In contradistinction, the set of data in terms of which the neoclassical approach attempts to determine normal income distribution and relative prices exhibits some striking differences from the classical approach. First, it introduces independent variables, or explanatory factors, that are not directly observable, such as agents' preferences or utility functions. Second, it takes as given not only the amounts of natural resources available but also the economy's 'initial endowments' of labour and 'capital'. The data from which neoclassical theory typically begins its reasoning are:

(a) The set of technical alternatives from which cost-minimizing producers can choose.
(b) The preferences of consumers.
(c) The initial endowments of the economy with all 'factors of production', including 'capital', and the distribution of property rights among individual agents.

The basic novelty of marginalist theory consists of the following. While the received classical approach conceives of the real wage as determined prior to profits and rents, in the neoclassical approach all kinds of income are explained *symmetrically* in terms of supply and demand with regard to the services of the respective factors of production: labour, 'capital' and land. Supply and demand are conceptualized as functional relationships (or correspondences) between the price of a service (or good) and the quantity supplied or demanded. Here, there is no need to enter into a discussion of the marginalist long-period theory and its difficulties (see for

example, Kurz & Salvadori, 1995, ch. 14). Suffice it to say that while Leontief's characterization of input–output analysis, cited above, appears to be fully compatible with the classical approach, it is not obvious that it can be reconciled with the neoclassical one. This paper provides some evidence indicating why this is so.

The structure of the paper is the following. Section 2 deals briefly with William Petty and Richard Cantillon, to whom we owe clear statements of the concepts of production as a circular flow, reproduction and surplus product. Section 3 turns to the physiocrats, placing special emphasis on François Quesnay's *Tableau Économique*. Section 4 is devoted to a summary of ideas put forward by Achille-Nicolas Isnard, who was a critic of the narrow concept of productivity entertained by Quesnay and who stressed the role of prices in distributing the social surplus. Section 5 deals with the contribution of Robert Torrens, who anticipated, in embryonic form, the duality relationship between the quantity and the price system. Section 6 summarizes the contribution of Karl Marx, focusing attention on the schemes of reproduction in his theoretical construction. Section 7 has a look at the work of Vladimir K. Dmitriev who formalized Ricardo's approach to the theory of relative prices and income distribution, and the work of Ladislaus von Bortkiewicz who elaborated on Dmitriev's analysis in his criticism of Marx's labour value-based reasoning. Section 8 provides an overview of the contribution of Georg von Charasoff who analysed the duality between quantity and price system and anticipated the Leontief inverse. Section 9 turns to Wassily Leontief's early contributions; the emphasis is on his essay on the economy as a circular flow and his early input–output analysis. It is argued that Leontief's approach is firmly rooted in the classical tradition of economic thought and, setting aside some purely formal similarities, has little in common with Walras's general equilibrium model. Section 10 draws the attention to Robert Remak's contribution to establishing the existence of a unique non-negative solution to the relevant system of linear equations. Section 11 contains some concluding remarks.

## 2. Contributions Prior to the Writings of the Physiocrats: Petty and Cantillon

The importance of early contributions to the development of classical Political Economy lies first and foremost in the concepts and method put forward. Thus, the concepts of production as a circular flow, of productive interdependences between different sectors of the economy and of social surplus are clearly discernible in earlier authors. Scrutinizing their works, the attentive reader will come across some primitive conceptualizations of input–output systems designed to portray the relationships of production in the economy. These generally form the basis of an inquiry into the laws governing the production and distribution of the wealth of a nation. It is hardly an exaggeration to say that input–output analysis is an offspring of systematic economic analysis whose inception is in the seventeenth and eighteenth centuries. In this section this will be documented in terms of a few authors writing before the physiocrats.

While the notion of productive interdependence between different producers in a system characterized by the division of labour and that of the normal cost of production are already present in embryonic form in the doctrines of *justum pretium* (just price) in scholastic economic thought, an important author in the genealogy of input–output analysis is William Petty (1623–87). He coined the famous dictum: 'Labour is the Father and active principle of Wealth, as Lands are the Mother'

(Petty, 1986, p. 68). Marx considered him the founder of classical Political Economy (cf. Marx, 1954, p. 85, fn. 2). As early as the *Treatise of Taxes and Contributions*, his first economic work, published in 1662, Petty put forward a clear concept of *social surplus*. He expressed the agricultural surplus as corn output minus necessary corn input, including the subsistence of labourers measured in terms of corn, and identified it with the *rent* of land (Petty, 1986, p. 43).

Petty pointed out that, given the means of subsistence per person, the surplus can also be expressed in terms of the extra number of people that could be maintained by a certain number of labourers engaged in the production of necessaries, given the socio-technical condition of production. He regarded the cost of production of commodities as the main cause determining their true or 'natural value', which was seen to measure the difficulty of acquiring them. While the 'natural value' expresses the 'permanent Causes' governing the price of things, the 'accidental value' also reflects the 'contingent Causes' ruling in a particular situation (Petty, 1986, pp. 51 and 90). His main concern was, of course, with the 'natural' magnitudes. Hence, Petty saw the aspects of the production, distribution and disposal of the wealth of a nation as intimately intertwined, and the problem of value as reflecting the interrelationship among these aspects. There is no discussion of profits in Petty: since in his time most trades were in the hands of artisans, profits were not clearly distinguishable from wages. It is worth mentioning that Petty already introduced the principle of extensive (differential) rent in its simplest form: rent owing to the different distances of the plots of land on which corn is grown from, for example, the town, where most of the net output of corn is consumed (see Petty, 1986, p. 48). He was clear about the fact that larger amounts of corn may only be provided at rising unit cost.

Richard Cantillon (1697–1734), who was greatly influenced by Petty's work, distinguished between market price and 'intrinsic value' of a commodity. Of the latter he wrote in his *Essai sur la nature du commerce en général*, published posthumously in 1755, that it 'is the measure of the quantity of Land and of Labour entering into its production, having regard to the fertility or produce of the Land and to the quality of Labour' (Cantillon, 1931, p. 29; similarly p. 107). Market prices may deviate from natural prices or 'intrinsic values' due to a mismatch of demand and actual production. This deviation is reflected in differences in entrepreneurial rates of return, which will prompt producers to reallocate their capital. In this way market prices will tend to equality with 'intrinsic values', which themselves are taken to be invariant or only slowly changing (see Cantillon, 1931, p. 31). This foreshadows Adam Smith's idea of market prices oscillating around and gravitating towards natural prices.

Cantillon saw a tripartite distribution of the (gross) product between the proprietors of land, farmers and undertakers, and assistants and 'mechanicks', and had a very clear concept of *reproduction*. He emphasized that all members of society subsist on the basis of the produce of land. This seems to imply that, in his view, the source of any surplus can only be agriculture. However, there are passages in the *Essai* according to which a surplus can also arise in manufacturing as profits (see, for example, Cantillon, 1931, p. 203).

### 3. François Quesnay and the *Tableau Économique*

The view that only agriculture can generate a surplus, a *produit net*, was most clearly expressed by Quesnay (1694–1774) and his followers (INED, 1958). It was

around the concept of net product that Quesnay's entire economic analysis and not only the *Tableau Économique* was built: in particular, it was taken to hold the key to an explanation of the distribution of income in contemporary France. The *Tableau* contains a sophisticated two-sector expression of the production of commodities by means of commodities. Marx called the *Tableau* 'an extremely brilliant conception, incontestably the most brilliant for which political economy has up to then been responsible' (Marx, 1956, p. 344), and elaborated his schemes of reproduction taking it as a starting point. Leontief related his 1936 paper explicitly to the work of Quesnay when he wrote: 'The statistical study presented... may be best defined as an attempt to construct, on the basis of available statistical materials, a *Tableau Économique* of the United States for 1919 and 1929' (Leontief, 1936, p. 105).

The *Tableau*, the first version of which was published in 1758, was meant to portray the whole process of production, distribution and expenditure as a reproduction process, with the circulation of commodities and money as a part and parcel of this process. An important goal of the analysis was to lay bare the origin of revenue and thus the factors affecting its size—factors that can be manipulated by economic policy aimed at fostering national wealth and power.

According to their economic role in the reproduction process, Quesnay distinguished among the 'productive class' (*classe productive*), the 'sterile class' (*classe stérile*) and the class of proprietors of land and natural resources (*classe propriétaire*). The productive class, that is, those working in primary production, in particular, agriculture, are called 'productive' because the value of the commodities produced by them exceeds the incurred costs of production. The difference between total proceeds and total costs, where the latter include the upkeep of those employed in the primary sector, is distributed as rent to the propertied class. In contradistinction to the productive class, the sterile class, that is, those employed in manufacturing (and commerce), do not generate a revenue, or surplus: the prices of manufactures cover just costs of production, including, of course, the subsistence of artisans, tradesmen, etc. In the two-sector scheme put forward, neither sector can exist on its own. In addition to *intra*sectoral flows of commodities there are *inter*sectoral flows: agriculture receives produced means of production from industry, and industry receives raw materials and means of subsistence from agriculture. Indeed, both (composite) commodities enter directly or indirectly into the production of both commodities. Hence, the system of production underlying the *Tableau* can be represented by a matrix of material inputs (that is, means of production-cum-means of subsistence) that is indecomposable.

The characteristic features of the *Tableau* can be summarized as follows. First, the *Tableau* starts from the following set of data or independent variables: the system of production in use, defined in terms of (i) the (average) methods of production employed to produce (ii) given levels of (aggregate) output; and (iii) given real rates of remuneration of those employed in the two sectors of the economy; that is, essentially, wages.[2] The reference is to some 'normal' levels of output, defined in terms of some average of the conditions of production over a sequence of years (balancing good and bad harvests). Second, the *Tableau* distinguishes between capital of different durability, where all kinds of capital relate to productive capital only. The *avances annuelles* refer to yearly advances or circulating capital (raw materials, sustenance of workers etc.); the *avances primitives* to fixed capital (tools, buildings, machines, horses etc.); and the *avances foncières* to capital incorporated in the land (land melioration of all kinds etc.). Exclusively those parts

of capital that are used up during the process of production and have to be replaced periodically are taken into account in the table. This presupposes that the stocks of durable means of production employed in different branches of the economy, their modes of utilization and thus their patterns of wear and tear (and therefore depreciation) are known. Third, all shares of income other than wages are explained in terms of the surplus product (representing a certain surplus value), or residual, left after the means of subsistence in the support of workers (and masters) and what is necessary for the replacement of the used-up means of production has been deducted from the annual output. Hence, the distributive variables are treated *asymmetrically*: the wage rate is taken to be an *exogenous* variable, whereas the (rate of) rent is an *endogenous* variable. Fourth, and closely related to what has just been said, the physiocrats conceived of any surplus product that may exist as *generated* in the sphere of production and only *realized* in the sphere of circulation. Fifth, the process of circulation is assumed to work out smoothly. This involves, *inter alia*, the existence of a system of *relative* prices which support the process of reproduction, and a system of *absolute* prices compatible with the stock of money available in the economy and the going habits of payment. While in the *Tableau* the problem of accumulation of capital is set aside, it is well known that Quesnay was concerned with the sources of economic growth and stressed the role of accumulation (see Eltis, 1975).

Before we turn to the English classical economists, the work of one man must be mentioned, not least because it is hardly known and yet can be said to have anticipated important findings of the subsequent literature: Achille-Nicolas Isnard.

## 4. Achille-Nicolas Isnard

Isnard (1749–1803), a French engineer, was a critic of the physiocratic doctrine that only agriculture is productive. In his view, this doctrine was contradicted already by the fact that the *produit net* in the *Tableau Économique* consisted both of agricultural and manufactured products. More important, Isnard argued that whether a sector of the economy generates an income in excess of its costs of production cannot be decided independently of the exchange ratios between commodities, or *relative* prices. The latter do not only reflect the real physical costs of production of the various commodities, but, in addition, the rule according to which the surplus product is distributed between the propertied classes.[3]

In 1781, Isnard published, in two volumes, his *Traité des richesses* (Isnard, 1781); volume I is of particular interest to us. Isnard's analysis revolved around the concepts of production as a circular flow and of surplus, or 'disposable wealth'. He wrote: 'In the whole of the riches, and setting aside values, there are in reality two parts, one required in production, the other destined to enjoyments... The latter is the noble part of goods and the part which is nobly enjoyed by the proprietors' (Isnard, 1781, pp. 35–36).[4] Isnard added that they, or a part of them, may also be accumulated in order 'to increase the mass of productive wealth' (Isnard, 1781, p. 36). He emphasized that the magnitude of the surplus depends on the technical conditions of production and the 'exigence of nature' (Isnard, 1781, p. 37).

The impression generated by the physiocrats that only agriculture is productive is closely related to the system of prices underlying their schema. These prices are such that the entire *produit net* is indeed appropriated by the landowners in the form of rent. Other rules of distribution would immediately reveal the peculiarity of the physiocratic doctrine. Isnard stressed: 'The values of the different products

determine the portions of total wealth allotted to the various producers; these portions change with the values of the objects which each producer has to acquire for production' (Isnard, 1781, p. xv; similarly p. 37). The first book of the *Traité* was designed to clarify, by way of a mathematical argument, the role of relative prices as the media to realize a given distribution of income.

Isnard started with a system of the division of labour with only two commodities. Each producer produces a certain amount of one commodity, a part of which he uses as a means of production and as a means of subsistence. He swaps the sectoral surplus for the other commodity he is in need of, but does not produce himself. Isnard put forward the following system of simultaneous equations (our notation):

$$(1-a)p_1 + bp_2 = p_1$$

$$ap_1 + (1-b)p_2 = p_2$$

where $a$ represents the surplus of the first commodity, $b$ that of the second, and $p_1$ and $p_2$ are the unit prices of commodities 1 and 2, respectively. He showed that the exchange rate that guarantees the repetition of the process of production and consumption is given by: $p_1/p_2 = b/a$.

He then turned to a system with three commodities and argued that the exchange ratios between the commodities can again be determined, provided we are given (i) the commodity surplus in each line of production and (ii) the way it is distributed between the two remaining sectors. Let $a$, $b$ and $c$ be the amounts of surplus in the three sectors. Each surplus is then divided in two parts, depending on the sector (or proprietor) they are designated for. Let $e$ be the share of the surplus of commodity 1 earmarked for sector 2; $(1-e)$ is, accordingly, the share that goes to sector 3. Let $f$ be the share of the surplus of commodity 2 earmarked for sector 3; $(1-f)$ is, accordingly, the share that goes to sector 1. And let $h$ be the share of the surplus of commodity 3 earmarked for sector 1; $(1-h)$ is, accordingly, the share that goes to sector 2. Isnard emphasized that a solution to the problem of relative prices can be found 'if there are as many equations as there are commodities' (Isnard, 1781, p. 19). The system of equations he put forward is

$$\left.\begin{aligned} (1-a)p_1 + (1-f)bp_2 + hcp_3 &= p_1 \\ eap_1 + (1-b)p_2 + (1-h)cp_3 &= p_2 \\ (1-e)ap_1 + fbp_2 + (1-c)p_3 &= p_3 \end{aligned}\right\} \qquad (1)$$

where $p_i$ is the price of commodity $i$, $i = 1, 2, 3$. This is a closed system in the sense that the above coefficients reflect both the amounts of the means of production plus the means of subsistence needed in the three sectors (per unit of output), that is, what the classical economists were to call 'productive consumption', and the consumption of the propertied classes, that is, 'unproductive consumption'.

Obviously, the sum of the quantities of any column is equal to the sum of the corresponding row. For example, the sum of the second column is $(1-f)b + (1-b) + fb$, which equals 1. This means that only two of the three equations are independent. Taking one of the commodities as a standard of value, or numeraire, as it was to be called later, system (1) allows one to determine the remaining two prices. In this view, prices reflect the dominant conditions of production and distribution. The prices of the *Tableau* represent but a special system of prices, which gives rise to the misconception that only agriculture is productive. If the producers in agriculture would have to pay more of their own

(composite) product per unit of the manufactured (composite) product, the situation would be different: the surplus of agriculture would be smaller or, in the extreme, nil, whereas the surplus of industry would be positive or, in the extreme, equal to the surplus of the system as a whole.

Isnard (1781, p. 36) even put forward a numerical example of two sectors of production which can be tabulated as follows:

$$10 \text{ qr. wheat} + 10 \text{ t. iron} \rightarrow 40 \text{ qr. wheat}$$

$$5 \text{ qr. wheat} + 10 \text{ t. iron} \rightarrow 60 \text{ t. iron}$$

The figures to the left of each arrow give total inputs in the sector, consisting of means of production and means of subsistence in the support of workers, whereas the figure to the right gives gross output. Accordingly, the system as a whole produces a net product consisting of $(40 - 15 =) 25$ qr. wheat and $(60 - 20 =) 40$ t. iron. The distribution of this net product between the two kinds of producers cannot be decided independently of the price of wheat relative to that of iron. It is also clear that if the (physical) net product of one of the commodities were nil, this need not imply that the producers of the respective sector would not get a share of the surplus: it all depends on which price ratio occurs. He concluded: 'When a production does not guarantee a producer a disposable income, one must not infer from this that his activity is not productive, because in reality he produces some of the things which are partly absorbed as costs and partly, via the exchanges, are passed on to the class of disposable riches.... Quesnay and *les économistes* were therefore wrong in asserting that industry is generally not productive' (Isnard, 1781, pp. 38–9).

## 5. Robert Torrens

The concepts of production as a circular flow and of the surplus product surfaced again in the writings of Adam Smith (1723–90), who also provided an analysis of the interdependence of the different sectors of the economy (Smith, 1976, Book V, Ch. V). The concepts are present in David Ricardo's (1772–1823) *Essay on the Influence of a low Price of Corn on the Profits of Stock* published in 1815 (cf. Ricardo, 1951–73, *Works* IV), and in his *Principles* (cf. Ricardo, 1951–73, *Works* I). However, the author who put these concepts again into sharp relief within an explicit input–output framework was Robert Torrens (1780–1864) in the second edition of his *Essay on the External Corn Trade* (cf. Torrens, 1820). In his formulation, the two problems identified above—that of relative quantities and the rate of growth and that of relative prices and the rate of profit—emerged with great clarity.

Torrens made clear that the concept of surplus provides the key to an explanation of shares of income other than wages and the rate of profit. In the *Essay* he determined the agricultural rate of profit in physical terms as the ratio between the net output of corn and corn input (corn as seed and food for the workers) and took the exchange value of manufactured goods relative to corn to be so adjusted that the same rate of profit obtains in manufacturing. This he called a 'general principle' (Torrens, 1820, p. 361) and acknowledged his indebtedness to Ricardo's 'original and profound inquiry into the laws by which the rate of profits is determined' (Torrens, 1820, p. xix).[5]

It was, of course, clear to the older authors that the capital advanced in a sector is never homogeneous with the sector's product. We encounter a first relaxation of

this bold assumption in Torrens's *Essay on the Production of Wealth*, published in 1821. There he put forward an example with two sectors, both of which use both products in the same proportions as inputs (see Torrens, 1821, pp. 372–73). He concluded that the rate of profit is given in terms of the surplus left after the amounts of the used up means of production and the means of subsistence in the support of labourers have been deducted from gross output. With the surplus and the social capital consisting of the same commodities in the same proportions, the general rate of profit can be determined without having recourse to the system of relative prices.

However, the physical schema is not only important for the determination of the rate of profit (and relative prices), it also provides the basis for assessing the potential for expansion of the economy. As Torrens stressed, 'this surplus, or profit of ten per cent they [i.e. the cultivators and manufacturers] might employ either in setting additional labourers to work, or in purchasing luxuries for immediate enjoyment' (Torrens, 1821, p. 373). If in each sector the entire surplus were to be used for accumulation purposes in the same sector, then the rates of expansion of the two sectors would be equal to one another and equal to the rate of profit. Champernowne in his commentary on von Neumann's growth model was later to call a constellation of equi-proportionate growth a 'quasi-stationary state' (Champernowne, 1945, p. 10).

The next author we have to turn to is Karl Marx. In his treatment of the aspect of quantities, Marx was concerned with studying under which conditions the system is capable of reproducing itself either on the same or an upward spiralling level, that is, the case of 'simple' and that of 'extended reproduction'.

## 6. Karl Marx

Marx (1818–83) was an attentive student of the writings of the physiocrats and praised Quesnay and his followers as 'the true fathers of modern political economy' (Marx, 1963, p. 44). We have already heard what he had to say about the *Tableau Économique*. The latter was of crucial importance in shaping his own ideas and constituted, in modified form, the backbone both of his theory of reproduction and his theory of value and distribution.[6]

According to Marx the linchpin of the classical approach to the theory of value and distribution is the concept of 'surplus product'—that is, all shares of income other than wages—and its relationship to the real wage. Taking the methods of production employed and thus the productivity of labour as given, the higher the real wage rate, the smaller is the surplus product, and *vice versa*. This idea also constituted the nucleus of the elaborate form of the classical argument in Ricardo with its emphasis on the inverse relationship between the *rate of profit* on the one hand and the real wage rate, or rather the total amount of labour needed to produce the wage commodities, on the other.

### *6.1. The Schemes of Reproduction*

In Marx's view the *Tableau* had been unduly neglected by the English political economists so that an important achievement of economic analysis had been lost sight of for almost an entire century (cf. Marx, 1963, p. 344). He called the system of the physiocrats 'the first systematic conception of capitalistic production' (Marx, 1956, p. 363). The *Tableau* was the foil against which Marx developed his own

*schemes of reproduction* (see Marx, 1956, part III). The schemes are concerned with the distribution of labour among the different sectors of the economy. That distribution was envisaged by Marx to depend on the socially dominant techniques of production, the distribution of income between wages and profits, and the expenditures out of these incomes, especially whether or not parts of profits are accumulated. In principle, the quantity system could be studied without any recourse to the problem of valuation. Marx nevertheless chose to provide both a description of the requirements of reproduction in physical terms (use-values) and in value terms (labour values). Thus, he intended to show that the physical reproduction of capital and its value reproduction are two sides of a single coin.

An early version of the scheme of simple reproduction was elaborated in Marx's letter to Engels of 6 July 1863. Scrutiny shows that Marx's scheme shares all the features of Quesnay's *Tableau* enumerated above (cf. Section 3). Marx divided the economy into two 'classes' or 'categories': class I represents the production of the means of subsistence, class II that of the means of production, that is, commodities 'which enter as raw materials, machinery etc. in the process of production'; the latter commodities 'form the *constant capital*' (MEW 1956 *et seq.*). (In volume II of *Capital* the numbering of departments is reversed.) Marx emphasized that the two classes or departments represent productive *aggregates* in a special sense.[7] This becomes clear with regard to agriculture, in which 'a part of the same products (e.g. corn) forms means of subsistence, whereas another part (corn e.g.) enters again as a raw material in its natural form (e.g. as *seeds*) into the reproduction. This does not change things, since according to one characteristic these branches of production belong in class II and according to the other in class I' (MEW 30, p. 363; emphasis in the original).

Marx's numerical example can be rewritten in a form which became prominent with volume II of *Capital* (Marx, 1956, ch. XX), i.e.

$$\text{class I:} \quad 700 = 400_c + 100_v + 200_s$$

$$\text{class II:} \quad 933\tfrac{1}{3} = 533\tfrac{1}{3}_c + 133\tfrac{1}{3}_v + 266\tfrac{1}{3}_s$$

where the subscripts $c$, $v$ and $s$ stand for 'constant capital', 'variable capital' and 'surplus value', respectively. Simple reproduction requires that the constant capitals used up in both sectors ($400_c + 533\frac{1}{3}_c$) are equal to the total product of class II ($933\frac{1}{3}$); and that the variable capitals, or wages bills ($100_v + 133\frac{1}{3}_v$), plus the surplus values, or profits ($200_s + 266\frac{2}{3}_s$), of the whole system are equal to the total product of class I (700). Accordingly, simple reproduction involves (using again the notation employed in volume II of *Capital*)

$$\text{I}(400_c) = \text{II}(133\tfrac{1}{3}_v + 266\tfrac{2}{3}_s)$$

In contrast to Quesnay's *Tableau*, here the labour performed in both sectors is taken to be productive, that is, generating a surplus value. If a part of the surplus value is saved and invested, the system reproduces itself on an ever larger scale. This is dealt with in Marx's schemes of extended reproduction (cf. Marx, 1956, ch. XXI), which provide a theory of the relationship between quantities, or sectoral proportions, and the rate of growth of the economic system as a whole.

### 6.2. *Prices of Production*

However, Marx saw that the importance of the *Tableau* was not restricted to the problem of quantities and growth: it also provided a much needed *general* framework

to determine the general rate of profit consistently. While Ricardo had a clear view of the inverse relationship between the rate of profit and the real wage rate, in Marx's view he had failed to show how the level of the rate of profit was actually ascertained, given the real wage rate. Marx saw that the data on which Ricardo's argument was based were essentially the same as the data (i)–(iii) underlying the *Tableau* (see Section 3). There was a single important difference between the physiocratic and the classical scheme: the rule according to which the social surplus is distributed—as rent in the case of the physiocrats, and as rent and profits in the case of the classical economists from Smith to Ricardo. It was indeed the determination of the general rate of profit which became a major focus of classical analysis. The implicit question was whether Ricardo's labour-based approach could be integrated with an appropriately modified *Tableau*. This reformulation had to leave the basic structure of the approach defined in terms of the exogenous variables untouched. Marx's theory of the general rate of profit and prices of production in part II of volume III of *Capital* can indeed be interpreted as an amalgamation and elaboration of the insights Marx owed, first and foremost, to the physiocrats and Ricardo. There, the problem of the rent of land is set aside altogether. The entire surplus is assumed to accrue in the form of profits at a uniform rate.

Marx made clear that a determination of the rate of profit and relative prices presupposes taking into account the 'total social capital' and its distribution in the different 'spheres of production' (Marx, 1959, pp. 158 and 163). He proposed a two-step procedure which was aptly dubbed 'successivist', as opposed to 'simultaneous' (see von Bortkiewicz, 1906–7, I, p. 38). In a first step he specified the general rate of profit as the ratio between the (labour) value of the economy's surplus product, or surplus value, and the (labour) value of social capital, consisting of a constant capital (means of production) and a variable capital (wages). In a second step this (value) rate of profit was then used to calculate prices. We may illustrate his procedure as follows. Marx started from a description of the economic system divided into several sectors or spheres of production, each of which is represented by an equation giving the value of the sectoral output ($z_i$) as the sum of the sectoral constant capital ($c_i$), its variable capital ($v_i$) and the surplus value ($s_i$) generated in the sector (cf. Marx, 1959, ch. IX). This description involved given methods of production and a given real wage rate. Otherwise it would be impossible to derive the labour-value magnitudes. With a given and uniform real wage rate and a given and uniform length of the working day (reflecting free competition in the labour market), the rate of surplus value is uniform across sectors. The larger the real wage rate, the larger is the variable capital and the smaller is the sectoral surplus value. Assuming only two sectors in order to facilitate a comparison with the *Tableau* and setting aside the problem of fixed capital, we have

$$z_{\mathrm{I}} = c_{\mathrm{I}} + v_{\mathrm{I}} + s_{\mathrm{I}}$$

$$z_{\mathrm{II}} = c_{\mathrm{II}} + v_{\mathrm{II}} + s_{\mathrm{II}}$$

where sector I is now the sector that produces means of production and sector II means of subsistence. It was Marx's contention that from this system *alone*, reflecting the set of data specified above, both the general rate of profit, $\rho$, and prices of production can be determined. The former is given by

$$\rho = \frac{s_{\mathrm{I}} + s_{\mathrm{II}}}{c_{\mathrm{I}} + v_{\mathrm{I}} + c_{\mathrm{II}} + v_{\mathrm{II}}} = \frac{\Sigma_i s_i}{\Sigma_i (c_i + v_i)}$$

In Marx's view it is here that the labour theory of value is indispensable, because it allegedly allows the determination of the rate of profit *independently of, and prior to*, the determination of relative prices.

In a second step this 'value' rate of profit, $\rho$, as we may call it, is then used to discount forward sectoral costs of production, or 'cost prices', measured in terms of labour values (cf. Marx, 1959, p. 164). This is the (in)famous problem of the 'Transformation of Values of Commodities into Prices of Production' (Marx, 1959, part II). With $p_i$ as the value–price transformation coefficient applied to the product of department $i, i = \text{I, II}$, we have, following Marx's procedure,

$$\left.\begin{aligned} z_{\text{I}}p_{\text{I}} &= (1+\rho)(c_{\text{I}} + v_{\text{I}}) \\ z_{\text{I}}p_{\text{II}} &= (1+\rho)(c_{\text{II}} + v_{\text{II}}) \end{aligned}\right\} \tag{2}$$

Counting the number of equations and that of the unknowns, there are two equations with two unknowns: the value–price transformation coefficients $p_{\text{I}}$ and $p_{\text{II}}$. Hence, the 'prices of production' seem to be fully determined.

Marx's successivist procedure cannot be sustained. A first and obvious error concerns the fact that in the above price equations (2) the capitals ought to be expressed in price rather than in value terms. Marx was aware of this slip in his argument (cf. Marx, 1959, pp. 164–5 and 206–7), but apparently thought that it could easily be remedied without further consequences. He was wrong. Once the necessary corrections suggested by Marx himself are carried out, it becomes clear that it cannot generally be presumed that the 'transformation' of values into prices of production is relevant to single commodities only, while it is irrelevant to commodity aggregates, such as the surplus product or the social capital, the ratio of which gives the rate of profit. Since the rate of profit cannot be determined before knowing the prices of commodities, and since the prices cannot be determined before knowing the rate of profit, the rate of profit and prices have to be determined *simultaneously* rather than successively.

Does Marx's blunder also falsify his intuition that, starting from the set of data (i)–(iii), which he had discerned in the *Tableau* and Ricardo, relative prices and the rate of profit can be determined in a logically coherent way? An answer to this question was provided by Vladimir K. Dmitriev and Ladislaus von Bortkiewicz.

## 7. Vladimir Karpovich Dmitriev and Ladislaus von Bortkiewicz

In 1898, the Russian mathematical economist Dmitriev (1868–1913) published, in Russian, 'An attempt at a rigorous analysis' of Ricardo's theory of value and distribution (Dmitriev, 1974). Dmitriev investigated first what is meant by the total amount of labour expended in the production of a commodity and how this amount can be ascertained. In particular, are we in need of a 'historical regress' in order to determine the indirect labour, that is, the one contained in the capital goods used up and thus transferred to the commodity in the course of its production? Dmitriev disposed of this misconception by showing that it is from a knowledge of the current conditions of production of the different commodities alone that one can determine the quantities of labour embodied (see Dmitriev, 1974, p. 44). Assuming single production, that is, setting aside joint production, and using matrix notation, the problem amounts to solving the following system of simultaneous equations:

$$\mathbf{z}^{\text{T}} = \mathbf{z}^{\text{T}}\mathbf{A} + \mathbf{l}^{\text{T}}$$

where $\mathbf{A}$ is the $n \times n$ matrix of material inputs, $\mathbf{l}$ is the $n$-vector of direct (homogeneous) labour inputs and $\mathbf{z}$ is the $n$-vector of quantities of labour embodied in the different commodities, or labour values. ($T$ is the sign for transpose.) Replacing repeatedly the $\mathbf{z}$ on the right-hand side of the equation by the right-hand side gives

$$\mathbf{z}^{\mathrm{T}} = \mathbf{l}^{\mathrm{T}} + \mathbf{l}^{\mathrm{T}}\mathbf{A} + \mathbf{l}^{\mathrm{T}}\mathbf{A}^2 + \mathbf{l}^{\mathrm{T}}\mathbf{A}^3 + \ldots, \tag{3}$$

where (3) is known as the 'reduction to dated quantities of labour'. In the single-products case contemplated by Dmitriev there are as many series of dated quantities of labour as there are products, and thus there are as many equations as unknowns.

Next, Dmitriev turned to an analysis of the rate of profit and 'natural' prices. He praised Ricardo, who had clearly specified the factors determining the general rate of profit, that is, (i) the real wage rate and (ii) the technical conditions of production in the wage goods industries: 'Ricardo's immortal contribution was his brilliant solution of this seemingly insoluble problem' (Dmitriev, 1974, p. 58). Prices are explained in terms of a reduction to (a finite stream of) dated wage payments, properly discounted forward. With $\mathbf{p}$ as the $n$-vector of prices, $w$ as the nominal wage rate and $r$ as the competitive rate of profit, and taking wages as paid *ante factum*, we get from (3)

$$\mathbf{p}^{\mathrm{T}} = w[(1+r)\mathbf{l}^{\mathrm{T}} + (1+r)^2\mathbf{l}^{\mathrm{T}}\mathbf{A} + (1+r)^3\mathbf{l}^{\mathrm{T}}\mathbf{A}^2 + \ldots] \tag{4}$$

Dmitriev also confirmed Ricardo's finding that relative prices are proportional to relative quantities of labour embodied in two special cases only: (i) when the reduction series are linearly dependent pairwise; and (ii) when the rate of profit is zero.

Ricardo's concept of the inverse relationship between the rate of profit and the real wage rate, given the technical conditions of production, or *wage-profit relationship*, was rendered precise in Dmitriev's flow-input point-output framework. Assume that the commodity content of real wages is proportional to the $n$-vector $\mathbf{b}$, $\mathbf{b} \geqslant \mathbf{0}$. Let $\omega$ designate the number of units of the elementary real wage basket. Then we have

$$w = \omega \mathbf{p}^{\mathrm{T}}\mathbf{b} \tag{5}$$

With the basket $\mathbf{b}$ as the standard of value,

$$\mathbf{p}^{\mathrm{T}}\mathbf{b} = 1 \tag{6}$$

and inserting (5) in (4), multiplying both sides by $\mathbf{b}$, and taking into account (6), we get

$$1 = \omega[(1+r)\mathbf{l}^{\mathrm{T}} + (1+r)^2\mathbf{l}^{\mathrm{T}}\mathbf{A} + (1+r)^3\mathbf{l}^{\mathrm{T}}\mathbf{A}^2 + \ldots]\mathbf{b} \tag{7}$$

which, for a given $\omega$, is one equation to determine the only unknown: $r$. With a $\omega$ that is low enough, equation (7) has a unique positive solution.[8] Equation (7) also demonstrates the correctness of Ricardo's dictum that the rate of profit depends exclusively on the conditions of production in the industries that produce wage goods and in those industries that directly or indirectly provide the former with means of production.[9]

The concept of production as a circular flow and that of the surplus product was further developed by Ladislaus von Bortkiewicz (1868–1931), who was born in St Petersburg into a family of Polish descent. From 1901 he taught economics and statistics at the University of Berlin, the same university which, in the late 1920s, also had Leontief, von Neumann and Robert Remak among its members.

In 1906, Bortkiewicz published the first part of his three-part treatise 'Wertrechnung und Preisrechnung im Marxschen System'; the remaining two parts followed in the subsequent year (von Bortkiewicz, 1906–7, I, II, and III). (Parts II and III were translated into English as 'Value and price in the Marxian system'; see von Bortkiewicz, 1952.) In 1907 there followed his paper 'Zur Berichtigung der grundlegenden theoretischen Konstruktion von Marx im dritten Band des "Kapital"' (von Bortkiewicz, 1907) ('On the correction of Marx's fundamental theoretical construction in the third volume of "Capital"'; see von Bortkiewicz, 1952.) A major source of inspiration for von Bortkiewicz was Dmitriev's treatment of Ricardo's theory of distribution and 'natural' prices.

The main objects of von Bortkiewicz's contributions can be summarized as follows. First, he wanted to demonstrate that Marx's construction of necessity failed. Second, he was concerned with showing that value analysis is not an indispensable step on the way to a consistent theory of the rate of profit and prices of production. Third, and notwithstanding what has just been said, he wanted to show that prices and the profit rate can be related to value and surplus value magnitudes in a logically consistent way. Fourth, this made him reject the then dominant critique of Marx which erroneously took the value-based reasoning in itself, rather than Marx's mistaken use of it, as the source of various misconceptions. Finally, and perhaps most importantly, von Bortkiewicz attempted to show that Ricardo's doctrine is superior to Marx's in almost every respect. His treatise is indeed as much about Ricardo as it is about Marx. He accused Marx of retrogressing in various ways to opinions that had already been shown to be defective by Ricardo.

Von Bortkiewicz pointed out that the *data* from which the classical approach to the theory of value and distribution starts are sufficient to determine the rate of profit and relative prices; no additional data are needed to determine these variables. He developed his argument both in terms of an approach in which it is assumed that commodities are obtained by a finite stream of labour inputs, that is, production is 'linear' (von Bortkiewicz, 1906–7), and one in which production is 'circular' (von Bortkiewicz, 1907). Following Dmitriev, von Bortkiewicz cast his argument in algebraic form. Considering the set of price equations associated with a given system of production with $n$ commodities, it is recognized that the number of unknowns exceeds the number of equations by two: there are $n + 2$ unknowns ($n$ prices, the nominal wage rate, and the rate of profit) and $n$ equations. With the real wage rate given from outside the system and fixing a standard of value or numeraire, one gets two additional equations (and no extra unknown) and the system can be solved for the rate of profit and prices in terms of the numeraire. Von Bortkiewicz, among other things, generalized the approach to cover fixed capital.

As we have seen, von Bortkiewicz was predominantly concerned with the price and distribution aspect, while the quantity and growth aspect was given little attention by him. It was Georg von Charasoff (1877–?) who pointed out a fundamental *duality* between the two.

## 8. Georg von Charasoff

Von Charasoff was born in Tiflis. He wrote his PhD thesis in mathematics at the University of Heidelberg. He published two books in 1909 and 1910, respectively, both in German, the second of which, *Das System des Marxismus. Darstellung und Kritik*, is of particular interest to us (see von Charasoff, 1910). In it, von Charasoff anticipated several results of modern reformulations of the classical approach and

of input–output analysis. Because of his highly condensed and abstract argument, which is mathematical without making use of formal language, his contribution was largely ignored at the time of its publication and has only recently been rediscovered (see Egidi & Gilibert, 1984).

Von Charasoff developed his argument within the framework of an interdependent model of (single) production, which exhibits all the properties of the later input–output model. The central concept of his analysis is that of a 'series of production': it consists of a sequence, starting with any (semipositive) net output vector (where net output is defined exclusive of wage goods), followed by the vector of the means of production and means of subsistence in the support of workers needed to produce this net output vector, then the vector of the means of production and means of subsistence needed to produce the previous vector of inputs, and so on. He called the first input vector 'capital of the first degree', the second input vector 'capital of the second degree', etc. This series 'has the remarkable property that each element of it is both the product of the following and the capital of the preceding element; its investigation is indispensable to the study of all the theoretical questions in political economy' (von Charasoff, 1910, p. 120).

The series under consideration is closely related to the expanded Leontief inverse. Let $\mathbf{y}$ denote the $n$-dimensional vector of net outputs and $\mathbf{A}$ the $n \times n$-matrix of 'augmented' input coefficients; each coefficient represents the sum of the respective material and wage-good input per unit of output, since von Charasoff, like the classical economists and Marx, reckoned wage payments among capital advances.[10] Then the series is given by

$$\mathbf{y}, \mathbf{A}\mathbf{y}, \mathbf{A}^2\mathbf{y}, \ldots, \mathbf{A}^k\mathbf{y}, \ldots$$

With circular production this series is infinite. Tracing it backwards: first, all commodities that are 'luxury goods' disappear from the picture; next, all commodities that are specific means of production needed to produce the luxury goods disappear; then the specific means of production needed in the production of these means of production disappear, etc. On the assumption that none of the commodities mentioned so far enters in its own production, 'it is clear that from a certain finite point onwards no further exclusions have to be made, and all the remaining elements of the series of production will always be made up of the self-same means of production, which in the final instance are indispensable in the production of all the different products and which therefore will be called *basic products*.' He stressed: 'The whole problem of price boils down ... to the determination of the prices of these basic products' (von Charasoff, 1910, pp. 120–1).

A further property of the series of production deserves to be stressed: the capital of the second degree ($\mathbf{A}^2\mathbf{y}$) is obtained by multiplying the capital of the first degree ($\mathbf{A}\mathbf{y}$) by $\mathbf{A}$. 'Yet since the physical composition of a sum of capitals is obviously always a medium between the physical composition of the summands, it follows that capitals of the second degree deviate from one another to a smaller extent than is the case with capitals of the first degree' (von Charasoff, 1910, p. 123). The farther one goes back, the more equal the compositions of the capitals become; that is, capitals of a sufficiently high degree 'may practically be seen as different quantities of one and the same capital: the *original* or *prime capital*'. This finding is of the utmost importance for determining the rate of profit and the maximum rate of growth of the system. For it turns out that 'this original type, to which all capitals of lower degree converge, possesses the property of growing in the course

of the process of production without any qualitative change, and that the rate of its growth gives the general rate of profit' (von Charasoff, 1910, p. 124).

The rate of profit can thus be ascertained in terms of a comparison of two quantities of the same composite commodity: the 'original capital'. Let $\mathbf{u}$ designate the $n$-dimensional vector of an elementary unit of the original capital, $\mathbf{u} \geqslant \mathbf{0}$, then $\mathbf{Au}$ is the (original) capital corresponding to $\mathbf{u}$, and we have

$$\mathbf{u} = (1 + r)\mathbf{Au}$$

with $r$ as the general rate of profit. Von Charasoff emphasized: 'The original capital expresses the idea of a surplus-value yielding, growing capital in its purest form, and the rate of its growth appears in fact as the general capitalist profit rate' (von Charasoff, 1910, p. 112). And: 'The original capital is nothing else than the basic production, whose branches are taken in particular dimensions. As regards these dimensions the requirement is decisive that gross profits of the basic production ... are of the same type as its total capital' (von Charasoff, 1910, p. 126). This finding can be said to generalize Torrens's 'general principle' referred to above: it relies neither on the existence of a single sector whose capital is physically homogeneous with its product and whose product is used by all sectors as an input nor on the special case in which all sectors exhibit the same input proportions.[11]

These considerations provide the key to a solution of the problem of price. For, if the various capitals can be conceived of 'as different amounts of the selfsame capital..., then prices must be proportional to the dimensions of these, and the problem of price thus finds its solution in this relationship based on law' (von Charasoff, 1910, p. 123). Let $\mathbf{p}$ designate the $n$-dimensional vector of prices, $\mathbf{p} \geqslant \mathbf{0}$, then we have the following price system

$$\mathbf{p}^{\mathrm{T}} = (1 + r)\mathbf{p}^{\mathrm{T}}\mathbf{A}$$

Thus, while $\mathbf{u}$ equals the right-hand eigenvector of $\mathbf{A}$, $\mathbf{p}$ equals the left-hand eigenvector; $1/(1 + r)$ equals the dominant eigenvalue of matrix $\mathbf{A}$. The solution to the price problem can therefore be cast in a form in which 'the concept of labour is almost entirely bypassed' (von Charasoff, 1910, p. 112). Implicit in this reasoning is the abandonment of the labour theory of value as a basis for the theory of relative prices and the rate of profit.

With von Neumann (1937) von Charasoff shared a concern with the possibility of equi-proportionate growth. In the hypothetical case in which all profits are accumulated, the proportions of the different sectors equal the proportions of the original capital. In this case the actual rate of growth equals the rate of profit: the system expands along a von Neumann ray. Von Charasoff was perhaps the first author to note clearly what von Neumann more than two decades later was to call 'the remarkable duality (symmetry) of the monetary variables (prices $p_j$, interest factor $\beta$) and the technical variables (intensities of production, $q_i$, coefficient of expansion of the economy $\alpha$)' (von Neumann, 1945, p. 1).

## 9. Wassily Leontief

Leontief (1905–1999) was born in St Petersburg. After his studies at the university of his home town, then Leningrad, he went to Berlin to work on his doctorate under the supervision of von Bortkiewicz. In 1928 he published a part of his thesis entitled 'Die Wirtschaft als Kreislauf'.[12] In it Leontief put forward a two-sectoral input–output system that was designed to describe the production, distribution

and consumption aspects of an economy as a single process. In 1932 he joined the faculty at Harvard University and began the construction of the first input–output tables of the American economy. These tables, together with the corresponding mathematical model, were published in 1936 and 1937 (see Leontief, 1941; see also Leontief, 1987). In this section we shall first deal with Leontief's 1928 article and then provide a summary statement of the closed and open input–output model. We shall see that Leontief's 1928 approach bears a close resemblance to Isnard's, dealt with in Section 4.

### *9.1. The Economy as a Circular Flow*

In his thesis, Leontief advocated the view that economics should start from 'the ground of what is objectively given' (Leontief, 1928, p. 583); economic concepts are meaningless and potentially misleading unless they can be observed and measured. He adopted a 'naturalistic' perspective (Leontief, 1928, p. 622; the English translation [p. 211] speaks of a 'material' perspective). The starting point of the marginalist approach, the *homo oeconomicus*, is considered inappropriate because it gives too much room to imagination and too little to facts (Leontief, 1928, pp. 619–20). Economic analysis should rather focus on the concept of circular flow, which expresses one of the fundamental 'objective' features of economic life. A careful investigation of its 'technological' aspects is said to be an indispensable prerequisite to any economic reasoning.

Leontief distinguished between 'cost goods' and 'revenue goods', that is, inputs and goods satisfying final demand. Throughout his investigation he assumed single production and constant returns to scale; scarce natural resources are mentioned only in passing. The argument is developed within the confines of what was to become known as the *Non-substitution Theorem* (see Koopmans, 1951; Samuelson, 1951). In much of the analysis it is also assumed that the system of production (and consumption) is indecomposable. Leontief suggested (1928, p. 585) that the process of production should be described in terms of three sets of 'technical coefficients': (i) 'cost coefficients'; that is, the proportion in which two cost goods $h$ and $k$ participate in the production of good $j$ (in familiar notation: $a_{hj}/a_{kj}$); (ii) 'productivity coefficients'; that is, the total quantity produced of good $j$ in relation to the total quantity used up of the $i$th input (in familiar notation: $1/a_{ij}$); (iii) 'distribution coefficients'; that is, the proportion of the total output of a certain good allotted to a particular point (or pole) in the scheme of circular flow; as is explained later in the paper, such a point may represent a particular group of property income receivers. A major concern of Leontief's was with a stationary system characterized by constant technical coefficients; in addition he discussed cases in which one or several coefficients change, thereby necessitating adjustments of the system as a whole. Here we shall set aside the second problem.

Starting from a physically specified system of production-cum-distribution, Leontief is to be credited with having provided a clear idea of the concept of *vertical integration* (Leontief, 1928, p. 589). As regards the reduction to dated quantities of labour (Leontief, 1928, pp. 596 and 621–2), he pointed out that because of the circular character of production 'a complete elimination of a factor of production from the given system is in principle impossible. Of course, the size of the "capital factor" can be reduced to any chosen level by referring back to even earlier periods of production' (Leontief, 1928, p. 622 [p. 211]). This reduction has nothing to do with an historical regress (Leontief, 1928, p. 596, fn. 6 [p. 192 fn]).

Next, Leontief addressed exchange relationships. The emphasis is on 'the general conditions which must be fulfilled within the framework of a circular flow' (Leontief, 1928, p. 598 [p. 193]). The concept of 'value' adopted is explicitly qualified as one that has nothing to do with any intrinsic property of goods, such as utility; it rather refers to the 'exchange relation deduced from all the relationships... analysed so far' (Leontief, 1928, p. 598 [p. 193]). In the case of a model with two goods, the 'relations of reproduction' are expressed as follows:

$$\left.\begin{aligned} aA + bB &\rightarrow A \\ (1-a)A + (1-b)B &\rightarrow B \end{aligned}\right\} \tag{8}$$

where $A$ and $B$ give the total quantities produced of two, possibly composite, commodities, and $a$ and $b$ [$(1-a)$ and $(1-b)$] give the shares of those commodities used up as means of production and means of subsistence in the first (second) sector. It should be stressed that the system, albeit stationary, generates a surplus.

Leontief, in fact, assumed that a part of the product of each sector is appropriated by a so-called ownership group: 'In the general circular flow scheme, income from ownership is of course considered alongside other cost items without the slightest direct reference to how it originates (the phenomenon of ownership). It is the task of the theory of interest [profit] to investigate these fundamental relationships' (Leontief, 1928, p. 600 [p. 196]). His argument resulted in setting up price equations which reflect the going rule that fixes the distribution of income. Counting unknowns and equations, Leontief found that the number of variables exceeds the number of equations by one. He concluded: 'No clear resolution of this problem is possible. One may vary at will the exchange proportions and consequently the distribution relationships of the goods without affecting the circular flow of the economy in any way' (Leontief, 1928, pp. 598–9 [p. 194]). In other words, the same quantity system is assumed to be compatible with different price systems reflecting different distributions of income. He added: 'The sense of the surplus theory is represented by the classical school (e.g. even by Ricardo) and... is best understood if one enquires into the use of this "free" income. The answer is: it either accumulates or is used up unproductively' (Leontief, 1928, p. 619 [p. 209]). Hence, the exchange ratios of goods reflect not only 'natural', that is, essentially technological, factors, but also 'social causes'. Given the rate of profit together with the system of production, relative prices can be determined. 'But this is the "law of value" of the so-called objective value theory' (Leontief, 1928, p. 601 [p. 196]), Leontief concluded. The reader will notice a striking similarity between Leontief's considerations and those of Isnard.

Before we turn briefly to Leontief's contributions to input–output analysis, more narrowly defined, it should be recalled that in the late 1920s he was a member of a research group at the University of Kiel, Germany. The group was led by Adolf Löwe (later Adolph Lowe) (1893–1995), and included Fritz (later Fred) Burchardt (1902–58) and Alfred Kähler (1900–81), among others. One of the main issues tackled by this group was the displacement of workers by technical progress and their absorption, or lack thereof, through capital accumulation. To enable them to take into account both the direct and indirect effects of technical progress, they developed multisectoral analyses. In two instalments in the *Weltwirtschaftliches Archiv*, Burchardt in 1931 and 1932 published an essay in which he attempted to cross-breed Marx's scheme of reproduction and Eugen von Böhm-Bawerk's temporal view of production (Burchardt, 1931–2). Alfred Kähler in his

PhD thesis of 1933 entitled *Die Theorie der Arbeiterfreisetzung durch die Maschine* (The theory of labour displacement by machinery) put forward a sophisticated argument which entailed a static input–output model and the way different forms of technical progress affect the coefficients of production of the different sectors and how these effects yield secondary effects etc. (Kähler, 1933; see also the paper by C. Gehrke in this volume). He also tried to calculate the change in the price system made necessary by technical change, assuming that any improvement is eventually passed on to workers in the form of a higher wage rate.

### *9.2. Input–Output Analysis*

While Leontief conceived of his early contribution as firmly rooted in the classical tradition, he called his input–output method developed in the 1930s and 1940s 'an adaptation of the neo-classical theory of general equilibrium to the empirical study of the quantitative interdependence between interrelated economic activities' (Leontief, 1966, p. 134). Scrutiny shows, however, that in his input–output analysis he preserved the concept of circular flow and did not, as is maintained by some interpreters, adopt the Walras–Cassel view of production.[13] In the second edition of *The Structure of American Economy*, published in 1951, he even explicitly rejected the view of production as a one-way avenue that leads from the services of the 'original' factors of production: land, labour and capital—the 'venerable trinity'—to final goods (Leontief *et al.*, 1951, p. 112). Unlike the theories of Walras and Cassel, in Leontief there are no given initial endowments of these factors. We shall refrain from speculating about the reasons for the change in Leontief's characterization of his own approach, which seems to have occurred after his move from Europe to the United States.[14]

Input–output analysis is meant to provide a detailed (that is, disaggregated) quantitative description of the structural characteristics of all component parts of a given economic system. The interdependence among the different sectors of a given system is described by a set of linear equations; the numerical magnitudes of the coefficients of these equations reflect the system's structural properties. The values of the coefficients are ascertained empirically; they are commonly derived from statistical input–output tables, which describe the flow of goods and services between the different sectors of a national economy over a given period of time, usually a year. In static input–output analysis the input coefficients are generally assumed to be constant, that is, independent of the overall level and composition of final demand. The problem of the choice of technique, which plays an important role in classical and neoclassical analysis, is often given only slight attention.

*(i) The closed Leontief model.* When all sales and purchases are taken to be endogenous, the input–output system is called 'closed'. In this case, final demand is treated as if it were an ordinary industry: the row associated with it represents the 'inputs' it receives from the various industries, and the corresponding column giving the value added in the various industries is assumed to represent its 'output' allocated to these industries. With $\mathbf{A}$ as the non-negative 'structural matrix' of an economy giving both material input requirements and final demand, and $\mathbf{x}$ as the $n$-vector of gross outputs, the closed input–output model is given by the linear homogeneous system

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

that is,

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

This model was discussed in Leontief (1941). In order for the system of equations to have non-negative solutions, the largest real eigenvalue of matrix **A** must be unity.[15] The price system which is dual to the above quantity system is

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{A}$$

that is,

$$\mathbf{p}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}) = \mathbf{0}^{\mathrm{T}} \tag{9}$$

The problem of the existence of a (non-negative) solution of system (9) was first investigated by Remak (1929) (see Section 10).

*(ii) The open Leontief model.* In the second edition of Leontief (1941), which was published a decade later, Leontief elaborated the 'open' input–output model which treats the technological and the final demand aspects separately. Now **A** represents exclusively the matrix of interindustry coefficients and **y** the vector of final demand, which is given from outside the system. The matrix of input coefficients is then used to determine the sectoral gross outputs as well as the necessary intersectoral transactions that enable the system to meet final demand and reproduce all used up means of production. The equation describing the relationship between **x** and **y** is

$$\mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{x}$$

that is,

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y}$$

On the assumption that the inverse of matrix $(\mathbf{I} - \mathbf{A})$ exists, we get as the general solution of the open input–output model

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}$$

The 'Leontief inverse matrix' $(\mathbf{I} - \mathbf{A})^{-1}$ is semipositive if the largest real eigenvalue of matrix **A** is smaller than unity (cf. Hawkins & Simon, 1949).

As to the determination of prices in the open input–output model, Leontief proposed a set of 'value-added price equations'. The price each productive sector is assumed to receive per unit of output equals the total outlays incurred in the course of its production. These outlays comprise the payments for material inputs purchased from the same or another productive sectors plus the *given* 'value added'. Assuming a closed economy without a government, the latter represents payments to the owners of productive factors: wages, rents, interest and profits. The price system, which is dual to the above quantity system, is given by

$$\mathbf{p}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}) = \mathbf{v}^{\mathrm{T}}$$

where **p** is the $n$-vector of prices and **v** is the $n$-vector of values added per unit of output. Solving for **p** gives

$$\mathbf{p}^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}(\mathbf{I} - \mathbf{A})^{-1}$$

The main problem with this approach is that the magnitudes of value added per unit of output in the different sectors cannot generally be determined prior to, and independently of, the system of prices. Another way of putting it is that in this formulation two things are lost from sight: the constraint binding changes in the distributive variables, and the dependence of relative prices on income distribution—facts rightly stressed by Leontief in his 1928 paper.

### *9.3. Input–Output Analysis and Walrasian General Equilibrium Theory*

In the literature on input–output analysis, one frequently encounters the view that the Leontief-system is an offspring of the general equilibrium model put forward by Léon Walras (1834–1910) in his *Eléments d'économie politique pure* (Walras, 1874). Leontief at times has himself expressed the opinion that his analysis and that of Walras are compatible with one another. Here we shall, on the contrary, draw the reader's attention to some aspects of the two approaches that appear to be difficult to reconcile.

First, there is the problem of method. Leontief opted for a 'naturalistic' or 'material' point of view. He insisted that the investigation should focus on 'directly observable basic structural relationships' (Leontief, 1987, p. 860) and not, like Walras's general equilibrium theory, on utility, demand functions etc., that is, things that are not directly observable. Second, there is the content of the theory. Some observers may be inclined to base the hypothesis of close similarity between the analyses of Leontief and Walras on the observation that the systems of price equations elaborated by Leontief in his 1928 paper, starting from schema (8), and those of Walras in his models of pure exchange in parts II and III of the *Eléments* are formally similar. Essentially the same formal similarity appears to have prompted some interpreters to consider that the analyses of Walras and Isnard belong to the same tradition in the theory of value and distribution.[16] However, it has to be pointed out that Isnard's argument, as well as Leontief's, does not refer to a pure exchange economy, but to an economy in which both capital and consumption goods are produced and reproduced.[17] Additionally, in Isnard as well as in Leontief, the parameters that determine relative prices are technological and institutional data, whereas in Walras's case of the pure exchange economy the 'effective demands' are ultimately rooted in the agent's utility maximizing disposition. There is a real and close similarity between the contributions of Leontief and Isnard, whereas there is only a questionable one between those of Leontief and Walras. Finally, as regards systems with production, in Isnard and Leontief the problem of distribution is not approached in terms of relative 'scarcities' of the respective factors of production, that is, in terms of the set of data (a)–(c) of Section 1 of this paper. In Leontief, the rate of interest is not conceived of as a scarcity index of a given endowment of capital. Walras's theory on the other hand starts from a given vector of capital goods and attempts to determine the 'rate of net income' (rate of profit) in terms of the demand for and the supply of capital (see Kurz & Salvadori, 1995, pp. 22–6). We may conclude that, setting aside purely formal similarities, the analyses of Leontief and Walras have little in common.

## 10. Robert Remak

We now turn to the contribution of Robert Remak (1888–1942). He studied mathematics and, in 1929, acquired the *venia legendi* at the University of Berlin

and was a *Privatdozent* there until 1933. According to the information gathered by Wittmann, from some of Remak's former friends and colleagues, Remak was in all probability stimulated by a group of economists around Bortkiewicz to study the problem of the conditions under which positive solutions of systems of linear equations obtain (cf. Wittmann, 1967, p. 401). As we have seen, Leontief's 1928 analysis was, for the most part, limited to the two-commodity case. One year later, Remak published a paper entitled 'Kann die Volkswirtschaftslehre eine exakte Wissenschaft werden?' (Can economics become an exact science?), generalizing the system to the $n$-commodity case, $n \geqslant 2$ (Remak, 1929).

Remak's paper begins with a definition of what is meant by an exact science, which bears a striking resemblance to Leontief's point of view: an exact science regards as 'exactly correct' only what can be ascertained by physical observation, counting or calculation (Remak, 1929, p. 703). Conventional economics, which Remak tended to equate with Marshallian demand and supply analysis, is said not to allow 'quantitative calculations that can also be carried out practically' (Remak, 1929, pp. 712). The alternative are 'superposed' or 'reasonable' prices: 'A superposed price system has nothing to do with values. It only satisfies the condition that each price covers the costs of the things required in production, and the consumption of the producer on the assumption that it is both just and feasible' (Remak, 1929, p. 712). Its calculation requires a detailed knowledge of the socio-technical relations of production, that is, the methods of production in use and the needs and wants of producers (Remak, 1929, pp. 712–13).

Remak then constructs 'superposed prices' for an economic system in stationary conditions in which there are as many single-product processes of production as there are products, and each process or product is represented by a different 'person' or rather activity or industry.[18] The amounts of the different commodities acquired by a person over a certain period of time in exchange for his or her own product are of course the amounts needed as means of production to produce this product and the amounts of consumption goods in support of the person (and his or her family), given the levels of sustenance. With an appropriate choice of units, the resulting system of 'superposed prices' can be written as

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{C} \tag{10}$$

where $\mathbf{C}$ is the augmented matrix of inputs per unit of output, and $\mathbf{p}$ is the vector of exchange ratios. Discussing system (10) Remak arrived at the conclusion that there is a solution to it, which is semipositive and unique except for a scale factor. The system refers to a kind of ideal economy with independent producers, no wage labour and hence no profits. However, in Remak's view it can also be interpreted as a socialist economic system.

## 11. Concluding Remarks

This paper contains a short account of some of the most important contributions to the long prehistory of input–output analysis. It has been shown that the latter is an offspring of classical economics with its emphasis on production as a circular flow and the capacity of the economy to create a surplus over and above the physical real costs of production, including the necessary means of subsistence in the support of workers. The physical scheme of production was considered as crucial for an understanding both of the problem of growth and that of the distribution of income and relative prices.

The theoretical efforts just surveyed bore two major fruits. On the one hand they laid the foundation to Leontief's empirical work, his input–output analysis, which turned out to be an indispensable tool in applied economics. On the other hand they stimulated further developments in the theory of value, distribution and growth. Two contributions are of particular importance in this regard: John von Neumann's famous growth model[19] and Piero Sraffa's 1960 book, which was explicitly designed to resurrect the 'classical' approach. A discussion of these contributions is, however, beyond the scope of this paper.

## Notes

1. In preparing this paper we have made extensive use of the material contained in Kurz & Salvadori (1995, 1998). See also the 'Introduction to Part I: Foundations of Input–Output Analysis' in Kurz *et al.* (1998, vol. I, pp. xix–xxxviii).
2. Notice the close similarity to the data describing the classical approach in the Introduction.
3. For the following see also Jaffé (1969) and Gilibert (1981).
4. Translations of sources of which no English version was available are ours.
5. Torrens's 'general principle' is the same thing as the 'basic principle' referred to by Sraffa in his discussion of Ricardo's early theory of profits (cf. Sraffa, 1951, p. xxxi).
6. For the following, see, in particular, Bródy (1970) and Gehrke & Kurz (1995).
7. As in the *Tableau* the concept of an 'industry', 'sector' or 'department' is an analytical one. Yet while in Quesnay the dividing line between the two departments is whether a line of production is 'productive' or not, in Marx the dividing line is whether it produces means of production or means of consumption.
8. It is necessary and sufficient that

$$\omega < 1/(\mathbf{l}^{\mathrm{T}} + \mathbf{l}^{\mathrm{T}}\mathbf{A} + \mathbf{l}^{\mathrm{T}}\mathbf{A}^2 + \ldots)\mathbf{b}$$

9. Dmitriev deserves the credit for having demonstrated that starting from the data of Ricardo's approach, relative prices and the rate of profit can be determined simultaneously. The system is complete and not underdetermined, as Walras (1954, Lesson 40) had objected. Walras's further criticism that Ricardo's 'cost of production explanation of prices' is circular, 'defining prices from prices', while based on a correct observation, is beside the point: prices and the rate of profit are fully determined in terms of the given technical conditions of production and the given real wage rate.
10. If a technique is defined in terms of the material input matrix $\mathbf{A}^*$ and the vector of direct labour inputs $\mathbf{l}$, and if $\omega\mathbf{b}$ is the vector of commodities consumed per unit of labour employed, then $\mathbf{A} = \mathbf{A}^* + \omega\mathbf{b}\mathbf{l}^{\mathrm{T}}$.
11. Von Charasoff's construction also bears a close resemblance to Sraffa's device of the Standard system in which the rate of profit 'appears as a ratio between quantities of commodities irrespective of their prices' (Sraffa, 1960, p. 22).
12. An English translation entitled 'The economy as a circular flow' which, unfortunately, omits certain passages, was published in 1991; see Leontief (1991). In what follows, the English version will be used whenever this is possible. Page numbers in square brackets refer to the latter.
13. For a characterization of the Walras–Cassel point of view, see, for example, Kurz & Salvadori (1995, chapter 13, subsection 7.1).
14. For a comparison of Leontief's approach and that of Walras, and the different traditions to which the two belong, see below; see also Gilibert (1981, 1991).
15. This does not mean that the economy is unable to produce a surplus. In fact, if $(\mathbf{A}^*,\mathbf{l})$ is a technique, where $\mathbf{A}^*$ gives the material input matrix and $\mathbf{l}$ the vector of direct labour inputs per unit of output in the different sectors of the economy, then

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^* & \mathbf{v} \\ \mathbf{l}^{\mathrm{T}} & h \end{bmatrix}$$

where $\mathbf{v}$ is the vector of values added per unit of output, and $h$ is the input of labour in households per unit of labour employed. Therefore, if the largest eigenvalue of matrix $\mathbf{A}^*$ is not larger than unity, then the definitions of $\mathbf{v}$ and $h$ imply that the largest eigenvalue of matrix $\mathbf{A}$ equals unity.

16. Thus, Schumpeter contended: 'The first to attempt a (primitive) mathematical definition of equilibrium and a (also primitive) mathematical proof of that proposition was Isnard, who has as yet to conquer the position in the history of economic theory that is due him as a precursor of Léon Walras' (Schumpeter, 1954, p. 217). And: 'In his not otherwise remarkable book there is an elementary system of equations that... describes the interdependence within the universe of prices in a way suggestive of Walras' (Schumpeter, 1954, p. 307; see also p. 242).
17. Hence, the appropriate point of reference would be Walras's developed theory including the production of consumption goods and the reproduction of capital goods proper. For a comparison of that theory with the 'classical' theory, see Kurz & Salvadori (1995, pp. 23–6).
18. The somewhat unfortunate phrasing of the problem by Remak may have been the source of the misconception that his concern was with a pure exchange economy; for this interpretation, see Gale (1960, p. 290).
19. We have argued elsewhere (see Kurz & Salvadori, 1993) that John von Neumann's famous paper on equi-proportionate growth (Neumann, 1937) can be interpreted as containing an implicit comment on Remak. In his paper, von Neumann put forward a general linear analysis of production, distribution and economic expansion, allowing for joint production, fixed capital and a choice of technique.

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