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Choice of technique in a model with fixed capital

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Abstract

The paper deals with the problem of the choice of technique in a model with fixed capital in which no more than one fixed capital item is utilized in the production of each commodity in each technique. In a first step single techniques are analysed, in a second step the cost-minimizing technique(s) is (are) determined. A general existence theorem and a uniqueness theorem regarding the prices of all commodities that are produced with cost-minimizing techniques are proved. Whereas previous uniqueness results referred to prices of finished goods only, the theorem put forward covers both finished goods and old machines. The paper thus makes good a lacuna in the conventional treatment of fixed capital in linear models of production.

Keywords: Economic analysis; Long period; Fixed capital; relative prices

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1. Introduction

Ever since the inception of systematic economic analysis in the time of the 'classical' political economists, fixed capital was seen to introduce additional

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complications into the theory of value and distribution, the most important of which appear to be the following two. First, while the circulating part of the capital advances contributes entirely to the annual output, i.e. 'disappears' from the scene, so to speak, the contribution of the durable part is less obvious and can only be imputed in correspondence with what may be considered the 'wear and tear' of fixed capital items. Second, with fixed capital there is always a problem of the choice of technique to be solved. This concerns both the choice of the mode of utilization of a durable capital good and the choice of the economic lifetime of such a good.

While the classical economists did not succeed in providing a general solution to the problems mentioned, they deserve the credit for having pointed out an analytical method by means of which fixed capital can be dealt with adequately. This method, which can be traced back to Robert Torrens (cf. Sraffa, 1960, pp. 94–95), consists in treating what is left of a fixed capital good at the end of the production period as an economically different good from the fixed capital good which entered the production process at the beginning of that period. This method seems to have fallen into oblivion in later times, but was reintroduced in economic analysis by von Neumann (1937, 1945) and explored in some detail by Sraffa (1960) in an explicit attempt to revive the 'classical' approach to the theory of value and distribution. [1]

As the literature following the publication of Sraffa's book shows, both complications mentioned above can be tackled in terms of the joint-products method. The more recent developments on the subject may be summarized as follows. In Chapter X of his book Sraffa (1960) provided details for the analysis of a single technique when no more than *one* old machine is used in each sector and the efficiency of machines is constant over time. It was particularly Schefold (1971, 1976, 1978, 1980b), Baldone (1974), and Varri (1974) who analysed the case in which efficiency is not constant. They remarked that if efficiency is decreasing, then the price of old machines may be negative even if the rate of profits is greater than zero but smaller than the maximum rate of growth. They argued also that if this is the case, then there is another technique which at the same rate of profits pays a larger wage rate at prices that are all positive. This further technique has been called a 'truncation' of the original technique since it can be interpreted as utilizing the same 'type of machine', but with a shorter economic life. Hence in these contributions the problem of the choice of technique is reduced to the determination of the optimal economic lifetime of machines, whereas the general problem of the choice of technique is left unsolved. Woods

[1] While we know of no evidence that von Neumann was aware of the writings of the classical authors on fixed capital, Sraffa assisted Champernowne in preparing his explanatory paper on the von Neumann model published together with the latter in the *Review of Economic Studies* (see Champernowne, 1945).

(1984, 1990) dealt with some additional problems concerning the choice of technique, but confined his investigation to the two-sectoral case. An analysis of the choice of technique in a general model that allows for the joint utilization of *several* machines [2] was provided by Salvadori (1988). In his paper Salvadori followed a complex procedure by means of which the cost-minimizing technique can directly be determined without studying techniques that are not cost-minimizing.

In this paper we consider once again the model in which no more than one used machine is employed in the production of each commodity in each technique. We shall prove: (i) the existence of a cost-minimizing technique; and (ii) the uniqueness of the price vector if more than one cost-minimizing technique exists. The procedure followed in this paper is similar to the traditional one (we will in fact investigate in separate steps a single technique and the determination of the cost-minimizing technique); it has the advantage of being much simpler than that followed by Salvadori (1988). The uniqueness theorem proved in this paper concerns the prices of *all* commodities (both finished goods and old machines) that are produced with cost-minimizing techniques. Previous uniqueness results were restricted to prices of finished goods only (cf. Salvadori (1988, Theorem 4); see also Stiglitz (1970), who uses a different framework). However, the procedure followed here cannot deal with the case of jointly utilized old machines, which was the main concern of Salvadori (1988). [3]

The structure of the paper is as follows. Section 2 presents the assumptions underlying the following analysis. In Section 3 the properties of a given technique utilizing durable capital goods are studied. Section 4 introduces the useful concept of the 'core processes'. Section 5 turns to a general discussion of the problem of the choice of technique and the determination of the cost-minimizing technique(s). Section 6 contains some remarks on *new* machines. Section 7 draws some conclusions.

2. Basic definitions

Let us first divide the commodities into two groups: commodities in the first group will be called *finished goods* and commodities in the second group will be

[2] The empirical importance of the joint utilization of machines was emphasized by authors from Babbage (1832) and Ure (1835) in the first half of the nineteenth century to Marris (1964) and Winston (1982) in our time.

[3] A simplified version of the model investigated here is to be found in Chapter 7 of Kurz and Salvadori (1994), where, for the sake of simplicity and contrary to what has been done here, all finished goods are assumed to be consumed (even new machines). It hardly needs to be pointed out that this is a very strong assumption, especially since it is combined with the other assumption that used machines are never consumed.

called *old machines*. Let m be the number of finished goods and $(n-m)$ the number of old machines. The following axioms hold:

(i) Old machines are never requested for consumption, i.e. they are requested only in order to be used as means of production.

(ii) Each process is assumed to produce one and only one finished good and, perhaps, an amount of one old machine.

(iii) Each process is also assumed to utilize as inputs only finished goods and, perhaps, an amount of one old machine.

(iv) No old machine can be utilized in the production of a finished good different from that alongside which it is produced. That is, old machines cannot be transferred among sectors; a *sector* being constituted by all the processes engaged in the production of a given finished good.

(v) All the processes in a sector can be divided in first degree processes, second degree processes, third degree processes, and so on. A *first degree process* is one which exclusively uses finished goods as inputs. A *second degree process* is one which uses finished goods and an amount of one old machine produced by a first degree process as inputs. A *third degree process* is one which uses finished goods and an amount of one old machine produced by a second degree process as inputs. And so on. A machine produced by a first degree process and utilized by a second degree process is said to be 'one year old'. A machine produced by a second degree process and utilized by a third degree process is said to be 'two years old'. And so on. Each process in sector i, $i = 1, 2, \ldots, m$, produces an amount of the finished good i and may or may not produce an amount of an old machine.

(vi) Any machine can at any time be worn out at a zero scrap value. This is obtained by assuming that for each process producing an old machine there is another process with the same inputs and the same outputs except for the old machine which is not produced. In other words, old machines can be disposed of freely.

If these assumptions hold, we say that the technology is a *fixed capital technology*. All the elements needed to understand this interpretation of the above axioms have been provided except one. In axioms (i)–(vi) each old machine is a commodity in itself, whereas in common language we speak of a machine at different ages as the *same* machine at different points in time. In order to avoid confusion, the expression '*type of machine*' will be used in the following sense. Two old machines one of which is among the inputs of a process and the other is among the outputs of the same process are (old) machines of the same *type*. More generally speaking, if there is a sequence of processes each of which (except the first one) uses as an input an old machine which has been produced as an output by the preceding process, then all machines involved constitute vintages of the same *type of machine*.

Axioms (ii)–(v) will be taken to hold good in regard to the positive elements of the material inputs and outputs of the existing processes. It will be assumed that there exist v_i, $v_i \geqslant 1$, processes to produce finished good i $(i = 1, 2, \ldots, m)$. In obvious notation, each process is referred to as

$$(\mathbf{a}^{(k)}, \mathbf{b}^{(k)}, l^{(k)}), \quad k = 1, 2, \ldots, \sum_{j=1}^{m} v_j$$

where $\mathbf{a}^{(k)}$ is the commodity input vector, $\mathbf{b}^{(k)}$ is the output vector, and $l^{(k)}$ is the direct labour input. Vectors $\mathbf{a}^{(k)}$ and $\mathbf{b}^{(k)}$ can be partitioned in the following way:

$$\mathbf{a}^{(k)} = \begin{bmatrix} \mathbf{a}_0^{(k)} \\ \mathbf{a}_1^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_m^{(k)} \end{bmatrix}, \quad \mathbf{b}^{(k)} \begin{bmatrix} \mathbf{b}_0^{(k)} \\ \mathbf{b}_1^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_m^{(k)} \end{bmatrix},$$

where subvectors $\mathbf{a}_0^{(k)}$ and $\mathbf{b}_0^{(k)}$ refer to finished goods and, obviously, have size m; subvectors $\mathbf{a}_i^{(k)}$ and $\mathbf{b}_i^{(k)}$ $(i = 1, 2, \ldots, m)$ refer to old machines utilized in the production of finished good i and, obviously, have size $t_{i1} + t_{i2} + \ldots + t_{iu_i}$, where u_i is the number of types of old machines existing in sector i and t_{ij} is the maximum life of the jth type of old machines existing in sector i. (The first element of subvectors $\mathbf{a}_i^{(k)}$ and $\mathbf{b}_i^{(k)}$ refers to the one year old machine of type 1, the second element refers to the two year old machine of type 1, and so on until the t_{i1}th element; the $(t_{i1} + 1)$th element refers to the one year old machine of type 2, and so on.)

Let us number processes in such a way that the first v_1 processes produce finished good 1, the following v_2 processes produce finished good $2, \ldots,$ the last v_m processes produce finished good m. Therefore if $z_{i-1} < k \leqslant z_i$, where $z_0 = 0$ and

$$z_i = \sum_{j=1}^{i} v_j, \quad (i = 1, 2, \ldots, m),$$

then $\mathbf{b}_0^{(k)} = \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector (because of axiom (ii)); $\mathbf{a}_j^{(k)} = \mathbf{b}_j^{(k)} = 0$, $j \neq i$ (because of axiom (iv)); $\mathbf{e}_t^{\mathrm{T}} \mathbf{b}_i^{(k)} > 0$ if either $\mathbf{e}_{t-1}^{\mathrm{T}} \mathbf{a}_i^{(k)} > 0$ or $\mathbf{a}_i^{(k)} = 0$ and the tth element of the subvectors $\mathbf{a}_i^{(k)}$ and $\mathbf{b}_i^{(k)}$ refers to a one year old machine (because of axiom (v)); only one element of subvectors $\mathbf{a}_i^{(k)}$ and $\mathbf{b}_i^{(k)}$ may be positive, all the others being nought (because of axioms (ii) and (iii)).

Figs. 1a and 1b represent the material input matrix and the output matrix where it is assumed that the existing processes are arranged in the following way: the first five commodities are finished goods, all other commodities are old machines, and all finished goods except commodity 3 are produced by using old machines:

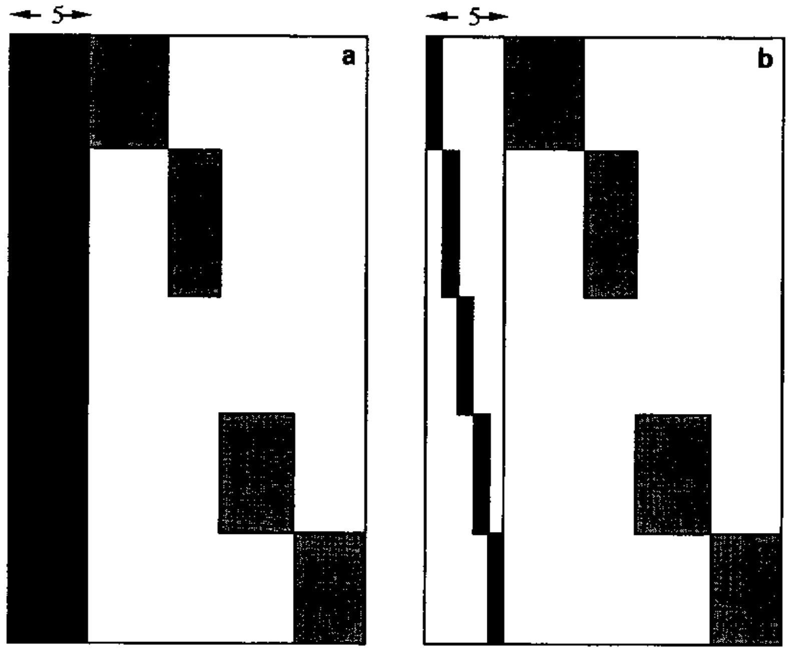

Fig. 1.

Grey areas (light and dark) represent nonnegative elements, white areas represent zero elements, and black areas represent positive elements. Moreover, the row vectors in the light grey areas are either zero vectors or vectors with one positive element only (all others being nought); they satisfy the following additional rules: if the positive element in a row vector in the material input matrix is in the jth column, then the corresponding vector in the output matrix has either a positive element in the $(j+1)$th column or has no positive element; if a row vector in the material input matrix is zero, then the corresponding vector in the output matrix is either a zero vector, or a vector with one positive element in a column referring to a one year old machine.

There are two further assumptions on the technology:

(vii) For each commodity there exists at least one process by means of which it can be produced.

(viii) It is not possible to produce a nonnegative net product without employing a positive amount of labour, i.e. labour enters directly or indirectly into the production of every commodity.

The assumptions relating to a fixed capital technology are now stated.

We shall begin with studying single techniques and then turn to cost-minimizing techniques. We first have to define these concepts. This presupposes some information about 'requirements for use', i.e. 'demand' (cf. Salvadori, 1985). It will be shown, however, that the only information about demand needed with a fixed capital technology is the set of commodities that are consumed. Let us then divide finished goods in two groups: commodities that are consumed and commodities that are not consumed (because of axiom (i) old machines are never consumed). Let us assume that the first h commodities are consumed, $1 \leqslant h \leqslant m$, and let c be a vector whose first h components are positive, all the others being nought. Let $(\mathbf{a}_i, \mathbf{b}_i, l_i)$, $i = 1, 2, \ldots, s$, be s processes, $h \leqslant s \leqslant n$. Then the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{l})$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_s^{\mathrm{T}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1^{\mathrm{T}} \\ \mathbf{b}_2^{\mathrm{T}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_s^{\mathrm{T}} \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ \cdot \\ \cdot \\ \cdot \\ l_s \end{bmatrix},$$

will be called a *technique*, or a *system of production*, if there exists a nonnegative scalar r and a positive vector $\mathbf{x}$ which is unique with respect to r such that

$$\mathbf{x}^{\mathrm{T}}[\mathbf{B} - (1 + r)\mathbf{A}] = \mathbf{c}^{\mathrm{T}}.$$

The existence condition requires that the number of processes in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ is larger or equal to the number of commodities involved: in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ for each process producing a machine there must exist another process using that machine as an input. That is, for each type of machine j used in the production of finished good i there must be at least $1 + t_j$ processes producing finished good i in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$, where t_j is the maximal age of the type of machine j in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ $(t_j \geqslant 0)$. The uniqueness condition requires that the rows of matrix $[\mathbf{B} - (1 + r)\mathbf{A}]$ are linearly independent so that the number of processes is not larger than the number of commodities involved; as a consequence no more than one type of old machine can be used in each sector in each technique. It will be proved later (cf. Corollary 1) that the magnitudes of those entries of vector $\mathbf{c}$ that are positive do not play any role in the definition of a technique.

The price vector $\mathbf{p}$ and the wage rate w of technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ are defined by the equations

$$\mathbf{Bp} = (1 + r)\mathbf{Ap} + w\mathbf{l}$$

$$\mathbf{c}^{\mathrm{T}}\mathbf{p} = 1,$$

where the numeraire has been fixed in such a way as to contain only commodities which are certainly produced. Note that prices of nonproduced commodities are not determined.

A *cost-minimizing technique* at rate of profits r is then defined as a technique at whose prices there is no set of known processes which, if operated, obtain extra-profits. Therefore, the technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ is *cost-minimizing at rate of profits* r if and only if

$$\left[\mathbf{B}_j - (1+r)\mathbf{A}_j\right]\mathbf{p}_{kj} \leqq w_k \mathbf{l}_j, \quad \text{each } \left(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j\right) \in J$$

where J is the set of all existing techniques and $(\mathbf{p}_{kj}, w_k)$ are determined by the following equations:

$$\left[\mathbf{B}_{kj} - (1+r)\mathbf{A}_{kj}\right]\mathbf{p}_{kj} = w_k \mathbf{l}_{kj},$$

$$\mathbf{c}^T\mathbf{p}_{kj} = 1,$$

where $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ is the set of processes made up by all the processes in $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ and by those processes in $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ producing commodities not produced by processes in $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$. Note that a process in technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ producing only a finished good enters into the set of processes $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ either if it is in technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ or if that finished good is not produced within technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$.

An example may help to understand why we need to refer to the set of processes $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ in defining cost-minimizing techniques. Let us assume that

(i) in technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ corn is produced with tractor 'K', lasting three years,
(ii) in technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ corn is produced with tractor 'J', lasting two years.

Hence technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ cannot determine the price of the one year old tractor J and perhaps also the price of the *new* tractor J. Similarly, technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ cannot determine the prices of the one and the two year old tractor K and perhaps also the price of the *new* tractor K. Yet a comparison between the two techniques is only possible if all involved prices are determined. In contradistinction, each of the sets of processes $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ and $(\mathbf{A}_{jk}, \mathbf{B}_{jk}, \mathbf{l}_{jk})$ take into account all the involved commodities.

Another example may also be useful. Let us assume that

(i) in technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ corn is produced with tractor 'Z', lasting four years,
(ii) in technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ corn is produced with the same tractor 'Z', but lasting two years.

Hence technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$ cannot determine the prices of the two and the three year old tractor Z. In contradistinction, the set of processes $(\mathbf{A}_{jk}, \mathbf{B}_{jk}, \mathbf{l}_{jk})$ takes into account all the involved prices, whereas the set of processes $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ coincides with the set of processes in technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$. The use of the sets of processes $(\mathbf{A}_{kj}, \mathbf{B}_{kj}, \mathbf{l}_{kj})$ and $(\mathbf{A}_{jk}, \mathbf{B}_{jk}, \mathbf{l}_{jk})$ allows us to understand better that the alternative processes are (i) the process utilizing the one year old tractor and producing no tractor for technique $(\mathbf{A}_j, \mathbf{B}_j, \mathbf{l}_j)$, and (ii) the process utilizing the three year old tractor and producing no old tractor for technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$.

It is also useful to define vector $\mathbf{x}_{kj}$:

$$\mathbf{x}_{kj}^{\mathrm{T}}\left[\mathbf{B}_{kj} - (1+r)\mathbf{A}_{kj}\right] = \mathbf{c}^{\mathrm{T}}.$$

Since

$$1 = \mathbf{c}^{\mathrm{T}}\mathbf{p}_{kj} = \mathbf{x}_{kj}^{\mathrm{T}}\left[\mathbf{B}_{kj} - (1+r)\mathbf{A}_{kj}\right]\mathbf{p}_{kj} = w_k \mathbf{x}_{kj}^{\mathrm{T}}\mathbf{l}_{kj}$$

we have

$$w_k = \frac{1}{\mathbf{x}_{kj}^{\mathrm{T}}\mathbf{l}_{kj}}.$$

3. A single technique

Let $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ be a technique. We begin by analysing first the properties of a technique and then the properties of cost-minimizing techniques. Matrices $\mathbf{A}$ and $\mathbf{B}$ are $s \times n$ matrices with $s \leqslant n$. If $s < n$, then there are $n - s$ columns that are nought in both matrices $\mathbf{A}$ and $\mathbf{B}$. Let us delete these columns to obtain matrices $\mathbf{C}$ and $\mathbf{D}$, respectively. In the following, we will refer to the triplet $(\mathbf{C}, \mathbf{D}, \mathbf{l})$ as a technique. This abuse of language must not confuse the reader. By appropriate reordering of commodities and processes, matrices $\mathbf{C}$ and $\mathbf{D}$ can be partitioned in the following way:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{C}_{21} & \mathbf{0} & \mathbf{C}_{23} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{C}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{C}_{34} & \dots & \mathbf{0} \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ \mathbf{C}_{f1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{C}_{ff+l} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{0} & \mathbf{D}_{23} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{D}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{34} & \dots & \mathbf{0} \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ \mathbf{D}_{f1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{D}_{ff+1} \end{bmatrix},$$

where f is the number of finished goods that are produced within technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$; each row of the $s_i \times f$ matrix $\mathbf{C}_{i1}$ is semipositive; the ith column of the

$s_i \times f$ matrix $\mathbf{D}_{i1}$ equals the sum vector, the other columns being nought; the $s_i \times (s_i - 1)$ matrix $\mathbf{D}_{ii+1}$ has the form

$$\begin{bmatrix} \mathbf{S} \\ \mathbf{0} \end{bmatrix},$$

where sub-matrix $\boldsymbol{S}$ is diagonal with a strictly positive main diagonal and $\mathbf{0}$ is a zero row vector; the $s_i \times (s_i - 1)$ matrix $\mathbf{C}_{ii+1}$ has the form

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{T} \end{bmatrix},$$

where sub-matrix $\boldsymbol{T}$ is diagonal with a strictly positive main diagonal and $\mathbf{0}$ is a zero row vector; $\sum_{i=1}^{s} s_i = s$.

The interpretation is the following. The first s_1 ($\geqslant 1$) processes produce the finished good 1 using, if $s_1 > 1$, a type of old machine that in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ lasts $s_1 - 1$ years; the next s_2 ($\geqslant 1$) processes produce the finished good 2 using, if $s_2 > 1$, a type of old machine that in technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ lasts $s_2 - 1$ years; and so on.

Let $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ be a fixed capital technique and let

$$\mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})} = \{ r \in \mathbb{R} \mid r \geqslant 0, \exists \mathbf{x} > 0 : \mathbf{x}^{\mathrm{T}}[\mathbf{B} - (1+r)\mathbf{A}] = \mathbf{c}^{\mathrm{T}} \}.$$

The set $\mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$ is not empty because of the definition of a technique. Moreover, we have

Theorem 1. *For each $r \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$*

(i) *matrix $[\mathbf{D} - (1+r)\mathbf{C}]$ is invertible and the first f rows of $[\mathbf{D} - (1+r)\mathbf{C}]^{-1}$ are semipositive;*

(ii) *prices of finished goods are defined and nonnegative.*

A proof of this Theorem on the assumption that $h = f = m$ can be found in Salvadori and Steedman (1988). The assumption that $h = f = m$ is very strong indeed, since 'new machines' are finished goods: it is not sensible to assume that they are consumed while old machines are not. The generalization provided here will be divided in three lemmas.

Lemma 1. *Let $r \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$, then there is a positive vector $\mathbf{z} \in \mathbb{R}^f$ and a semipositive vector $\mathbf{u} \in \mathbb{R}^s$ such that $\mathbf{u}^{\mathrm{T}}[\mathbf{D} - (1+r)\mathbf{C}] = (\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}})$.*

Proof. There is a positive vector $\boldsymbol{x}$ such that $\mathbf{x}^{\mathrm{T}}[\mathbf{B} - (1+r)\mathbf{A}] = \mathbf{c}^{\mathrm{T}}$, where the first h elements of vector $\boldsymbol{c}$ are positive, the others being nought. If $h = f$, then $\mathbf{u} = \mathbf{x}$ and $(\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}) = \mathbf{c}^{\mathrm{T}}$ satisfy the lemma. If $h < f$, let $\hat{\mathbf{x}}_1 \in \mathbb{R}^s$ be such that

$$\hat{\mathbf{x}}_1^{\mathrm{T}}\mathbf{e}_j = \begin{cases} = \mathbf{x}^{\mathrm{T}}\mathbf{e}_j & \text{if the } j\text{th process produces finished good } i \text{ and } \mathbf{c}^{\mathrm{T}}\mathbf{e}_i = 0, \\ = 0 & \text{otherwise,} \end{cases}$$

and let

$$\mathbf{b}_1^{\mathrm{T}} := \hat{\mathbf{x}}_1^{\mathrm{T}}[\mathbf{D} - (1+r)\mathbf{C}].$$

Since assumption (iv) of Section 2 holds, all elements of vector $\mathbf{b}_1$ corresponding to old machines are nought. Moreover, the elements of the same vector corresponding to the non consumed finished goods which enter *directly* into the production of the consumed finished goods are positive. Let h_1 be their number. Finally, the only elements of vector $\mathbf{b}_1$ which can be negative correspond to the consumed finished goods. Hence there is a positive scalar θ_1 such that $h + h_1$ among the first f elements of vector $\mathbf{b}_1 + \theta_1\mathbf{c}$ are positive, all other elements of the same vector being nought. If $h + h_1 = f$, then vectors $\mathbf{u} = \hat{\mathbf{x}}_1 + \theta_1\mathbf{x}$ and $(\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}) = \mathbf{b}_1 + \theta_1\mathbf{c}$ satisfy the Lemma. If not, we can iterate the same procedure with $\hat{\mathbf{x}}_1 + \theta_1\mathbf{x}$ and $\mathbf{b}_1 + \theta_1\mathbf{c}$ instead of $\mathbf{x}$ and $\mathbf{c}$, respectively. By doing so, we find $\hat{\mathbf{x}}_2$, $\mathbf{b}_2$ and θ_2. Let h_2 be the number of the positive elements of vector $\mathbf{b}_2$. (These elements correspond to the non- consumed finished goods which do not enter directly into the production of the consumed finished goods, but enter *directly* into the production of the finished goods which enter *directly* into the production of the consumed finished goods.) If $h + h_1 + h_2 = f$, then vectors $\mathbf{u} = \hat{\mathbf{x}}_2 + \theta_2\hat{\mathbf{x}}_1 + \theta_2\theta_1\mathbf{x}$ and $(\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}) = \mathbf{b}_2 + \theta_2\mathbf{b}_1 + \theta_2\theta_1\mathbf{c}$ satisfy the Lemma. If not, the same procedure can be iterated. Since the number of finished goods is finite, we can be sure that after a number of iterations vectors $\boldsymbol{u}$ and $\mathbf{z}$ satisfying the Lemma will be found. Q.E.D.

Lemma 2. *If* $r > -1$,

$$\left(\mathbf{u}^{\mathrm{T}}[\mathbf{D} - (1+r)\mathbf{C}] = (\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}),\ \mathbf{z} \in \mathbb{R}^f,\ \mathbf{z} > \mathbf{0},\ \mathbf{u} \geqslant \mathbf{0}\right) \Rightarrow \mathbf{u} > \mathbf{0}.$$

Proof. It is immediately obtained that

$$\mathbf{u}_i^{\mathrm{T}}[\mathbf{D}_{ii+1} - (1+r)\mathbf{C}_{ii+1}] = \mathbf{0}^{\mathrm{T}} \tag{1.i}$$

$$\mathbf{u}_1^{\mathrm{T}}\mathbf{D}_{11} + \mathbf{u}_2^{\mathrm{T}}\mathbf{D}_{21} + \ldots + \mathbf{u}_f^{\mathrm{T}}\mathbf{D}_{f1} > \mathbf{0}^{\mathrm{T}}, \tag{2}$$

where $(\mathbf{u}_1^{\mathrm{T}}, \mathbf{u}_2^{\mathrm{T}}, \ldots, \mathbf{u}_f^{\mathrm{T}}) = \mathbf{u}^{\mathrm{T}}$. Because of the form of matrices $\mathbf{D}_{ii+1}$ and $\mathbf{C}_{ii+1}$ $(i = 1, 2, \ldots, f)$, Eq. (1.i) has non-trivial solutions for each r, and each non-trivial solution to it for $r > -1$ is either a positive vector or a negative one.[4] Then,

[4] Let

$$\mathbf{v}^{\mathrm{T}}\left\{\begin{bmatrix}\mathbf{S}\\ \mathbf{0}\end{bmatrix} - (1+r)\begin{bmatrix}\mathbf{0}\\ \mathbf{T}\end{bmatrix}\right\} = \mathbf{0}^{\mathrm{T}},$$

where sub-matrices $\mathbf{S}$ and $\mathbf{T}$ are diagonal with a strictly positive main diagonal and $\mathbf{0}$ is a zero row vector. Then, for $r \neq -1$,

$$v_i s_i = v_{i+1}(1+r)t_i, \quad i = 1, 2, \ldots, s-1$$

i.e.

$$v_{i+1} = (1+r)^{-1}\frac{s_i}{t_i}v_i = v_1(1+r)^{-i}\prod_{h=1}^{i}\frac{s_h}{t_h}, \quad i = 1, 2, \ldots, s-1.$$

because of the form of matrices $\mathbf{D}_{i1}$, Eqs. (1.1) – (1.f) and inequality (2) have a solution only if $\mathbf{u}_i > \mathbf{0}$, each $i = 1, 2, \ldots, f$. Q.E.D.

Lemma 3. *Let $\mathbf{a} \in \mathbb{R}^s$ be a nonnegative vector whose last $s - f$ entries are all nought and let $r \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$. Then any vector $\mathbf{v}$ such that*

$$\mathbf{v}^{\mathrm{T}}[\mathbf{D} - (1 + r)\mathbf{C}] = \mathbf{a}^{\mathrm{T}}$$

is nonnegative.

Proof. Since Lemma 1 holds, there exists a positive vector $\mathbf{u}$ such that

$$\mathbf{u}^{\mathrm{T}}[\mathbf{D} - (1 + r)\mathbf{C}] = (\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}),$$

where $\mathbf{z} \in \mathbb{R}^f$, $z > \mathbf{0}$. Assume now that there exists a vector $\mathbf{v}$ with at least one negative element such that

$$\mathbf{v}^{\mathrm{T}}[\mathbf{D} - (1 + r)\mathbf{C}] = \mathbf{a}^{\mathrm{T}}.$$

Let i be such that

$$0 > \frac{\mathbf{v}^{\mathrm{T}}\mathbf{e}_i}{\mathbf{u}^{\mathrm{T}}\mathbf{e}_i} \leqslant \frac{\mathbf{v}^{\mathrm{T}}\mathbf{e}_j}{\mathbf{u}^{\mathrm{T}}\mathbf{e}_j} \quad \text{each } j.$$

Then

$$\mathbf{w} := \left[\mathbf{v} - \frac{\mathbf{v}^{\mathrm{T}}\mathbf{e}_i}{\mathbf{u}^{\mathrm{T}}\mathbf{e}_i}\mathbf{u}\right] \geqslant \mathbf{0}$$

and

$$\mathbf{w}^{\mathrm{T}}[\mathbf{D} - (1 + r)\mathbf{C}] = \mathbf{a}^{\mathrm{T}} - \frac{\mathbf{v}^{\mathrm{T}}\mathbf{e}_i}{\mathbf{u}^{\mathrm{T}}\mathbf{e}_i}(\mathbf{z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}) := \mathbf{b}^{\mathrm{T}},$$

where $\mathbf{b}$ is a vector whose first f entries are all positive and the last $s - f$ are all nought. Then Lemma 2 applies and $\mathbf{w} > \mathbf{0}$. Hence we have a contradiction, since the ith element of vector $\mathbf{w}$ cannot be positive. And this proves the Lemma. Q.E.D.

Proof of Theorem 1. Let $r \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$. If $\mathbf{y}$ is a solution to the equation

$$\mathbf{y}^{\mathrm{T}}[\mathbf{D}(1 + r)\mathbf{C}] = \mathbf{0}^{\mathrm{T}},$$

then also $-\mathbf{y}$ is one, and, by Lemma 3, both $\mathbf{y}$ and $-\mathbf{y}$ are nonnegative. Therefore $\mathbf{y} = \mathbf{0}$ and $\det([\mathbf{D} - (1 + r)\mathbf{C}]) \neq 0$. As a consequence, there exist semipositive vectors $\mathbf{z}_i$ such that $\mathbf{z}_i^{\mathrm{T}}[\mathbf{D} - (1 + r)\mathbf{C}] = \mathbf{e}_i^{\mathrm{T}}$ ($i = 1, 2, \ldots, f$). That is, the first f rows of the matrix $[\mathbf{D} - (1 + r)\mathbf{C}]^{-1}$ are semipositive. Statement (i) is thus proved. Statement (ii) is a direct consequence of statement (i) since the vector of prices of produced commodities, $\hat{\mathbf{p}}$, is determined by the equation

$$\hat{\mathbf{p}} = w[\mathbf{D} - (1 + r)\mathbf{C}]^{-1}\mathbf{l}. \quad \text{Q.E.D.}$$

Corollary 1. *The magnitudes of the positive entries of vector* $\mathbf{c}$ *play no role in the definition of a technique.*

In this section it has been proved that for each technique there is a set of values for the rate of profits such that if the going rate of profits is in that set, then the prices of finished goods are nonnegative. Section 5 is devoted to a discussion of the choice of techniques. It will be proved that if there exists a technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ such that $r^* \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$, where r^* is any given rate of profits, then there is a cost-minimizing technique at the rate of profits r which can sustain a growth rate equal to g, where $0 \leqslant g \leqslant r \leqslant r^*$. Moreover, all prices will be nonnegative in cost-minimizing techniques. Before studying the choice of techniques problem a useful device will be examined in Section 4: the 'core processes'. This device allows us to grasp the single product nature of the fixed capital models investigated in this paper.

4. A useful device: The 'core processes'

Let $(\mathbf{C}, \mathbf{D}, \mathbf{l})$ be a technique. Let matrices $\mathbf{C}$ and $\mathbf{D}$ be partitioned as in the previous section. Let $\mathbf{x}_i^{\mathrm{T}}(r)$ be a positive vector such that

$$\mathbf{x}_i^{\mathrm{T}}(r)\left[\mathbf{D}_{ii+1} - (1+r)\mathbf{C}_{ii+1}\right] = \mathbf{0}^{\mathrm{T}}$$

$$\mathbf{x}_i^{\mathrm{T}}(r)\mathbf{D}_{i1} = \mathbf{e}_i^{\mathrm{T}}.$$

Vector $\mathbf{x}_i^{\mathrm{T}}(r)$ exists for each $r > -1$ and each i (see footnote 4). Let us define the triplet

$$\left(\mathbf{x}_i^{\mathrm{T}}(r)\mathbf{C}_{i1}, \mathbf{e}_i, \mathbf{x}_i^{\mathrm{T}}(r)\mathbf{l}_i\right)$$

a *core-process of finished good i.* A core-process looks like a single product process with material and labour inputs as functions of r. The concept of core processes can be traced back to Sraffa (1960, § 76), who, however, did not give the concept a name; it was extensively used by Schefold (1971, 1976, 1978), who called them 'centre processes'; see also Baldone (1974) and Varri (1974). The core processes of technique $(\mathbf{C}, \mathbf{D}, \mathbf{l})$ determine the prices of finished goods independently of the prices of old machines (if the numeraire consists of finished goods only). This can be easily seen. Let the $f \times s$ matrix $\mathbf{X}(r)$ be defined as

$$\mathbf{X}(r) = \begin{bmatrix} \mathbf{x}_1^{\mathrm{T}}(r) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2^{\mathrm{T}}(r) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_3^{\mathrm{T}}(r) & \dots & \mathbf{0} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{x}_f^{\mathrm{T}}(r) \end{bmatrix},$$

then the $f+1$ equations

$$\mathbf{X}(r)\mathbf{B}\mathbf{p} = (1+r)\mathbf{X}(r)\mathbf{A}\mathbf{p} + w\mathbf{X}(r)\mathbf{l}$$
$$\mathbf{c}^{\mathrm{T}}\mathbf{p} = 1$$

determine the wage rate and the prices of the finished goods since the columns of matrix $\mathbf{X}(r)[\mathbf{B}-(1+r)\mathbf{A}]$ relative to old machines have vanished. Moreover, by deleting all zero columns, i.e. all the columns referring to old machines and to finished goods which are not produced, and by deleting the corresponding elements of vectors $\mathbf{p}$ and $\mathbf{c}$ we obtain a single product technique with material and labour inputs as functions of r, the output matrix being the usual matrix $\mathbf{I}$. Such a 'single product technique' will be denoted as $(\mathbf{F}(r), \mathbf{X}(r)\mathbf{l})$, where $\mathbf{F}(r)$ is a square matrix obtained by the first f columns of matrix $\mathbf{X}(r)\mathbf{C}$. We will refer to the pair $(\mathbf{F}(r), \mathbf{X}(r)\mathbf{l})$ as to the core-format of technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$.

Once the prices of finished goods have been ascertained, the prices of produced old machines can be determined from equation

$$\hat{\mathbf{D}}\hat{\mathbf{p}} = (1+r)\hat{\mathbf{C}}\hat{\mathbf{p}} + w\hat{\mathbf{l}}, \tag{3}$$

where $\hat{\mathbf{D}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{l}}$ are obtained from $\mathbf{D}$, $\mathbf{C}$, and $\mathbf{l}$, respectively, by deleting the f rows relative to processes not producing old machines, and $\hat{\mathbf{p}}$ is obtained from $\mathbf{p}$ by deleting the elements corresponding to commodities which are not produced. Eq. (3) can be stated as

$$\left[\hat{\mathbf{D}}_2 - (1+r)\hat{\mathbf{C}}_2\right]\hat{\mathbf{p}}_2 = w\hat{\mathbf{l}} - \left[\hat{\mathbf{D}}_1 - (1+r)\hat{\mathbf{C}}_1\right]\hat{\mathbf{p}}_1, \tag{4}$$

where $\hat{\mathbf{D}} = [\hat{\mathbf{D}}_1, \hat{\mathbf{D}}_2]$, $\hat{\mathbf{C}} = [\hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2]$, $\hat{\mathbf{p}}^{\mathrm{T}} = [\hat{\mathbf{p}}_1^{\mathrm{T}}, \hat{\mathbf{p}}_2^{\mathrm{T}}]$, and

$$\hat{\mathbf{C}}_2 = \begin{bmatrix} \hat{\mathbf{C}}_{12} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{C}}_{23} & \dots & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdot \\ \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \mathbf{0} & \mathbf{0} & \dots & \hat{\mathbf{C}}_{ff+1} \end{bmatrix},$$

$$\hat{\mathbf{D}}_2 = \begin{bmatrix} \hat{\mathbf{D}}_{12} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{D}}_{23} & \dots & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdot \\ \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \mathbf{0} & \mathbf{0} & \dots & \hat{\mathbf{D}}_{ff+1} \end{bmatrix},$$

where $\hat{\mathbf{C}}_{ii+1}$ and $\hat{\mathbf{D}}_{ii+1}$ are obtained from $\mathbf{C}_{ii+1}$ and $\mathbf{D}_{ii+1}$ by deleting the last row. Hence $\hat{\mathbf{C}}_{ii+1}$ is a nonnegative square matrix with positive elements only on the diagonal just below the main diagonal, and $\hat{\mathbf{D}}_{ii+1}$ is a square diagonal matrix with elements on the main diagonal all positive. Thus, matrix $[\hat{\mathbf{D}}_2 - (1+r)\hat{\mathbf{C}}_2]$ is a square matrix with elements on the main diagonal all positive and some negative elements on the diagonal below the main one, all other elements being nought. This is enough to assert that matrix $[\hat{\mathbf{D}}_2 - (1+r)\hat{\mathbf{C}}_2]$ is invertible and its inverse is semipositive. This is easily obtained by applying the Perron–Frobenius Theorem to the nonnegative matrix $\hat{\mathbf{D}}_2^{-1}\hat{\mathbf{C}}_2$. Since

$$\det\left(\lambda \mathbf{I} - \hat{\mathbf{D}}_2^{-1}\hat{\mathbf{C}}_2\right) = \lambda^{s-f},$$

all eigenvalues of matrix $\hat{\mathbf{D}}_2^{-1}\hat{\mathbf{C}}_2$ equal zero. Finally, since $(1+r)^{-1} > 0$ is larger than the largest real eigenvalue of matrix $\hat{\mathbf{D}}_2^{-1}\hat{\mathbf{C}}_2$,

$$\left[\frac{1}{1+r}\mathbf{I} - \hat{\mathbf{D}}_2^{-1}\hat{\mathbf{C}}_2\right]^{-1} \geqslant \mathbf{0}.$$

The fact that matrix $[\hat{\mathbf{D}}_2 - (1+r)\hat{\mathbf{C}}_2]$ is invertible and its inverse is semipositive does not imply that prices of old machines are nonnegative, since the vector on the RHS of Eq. (4) may have negative elements.

We conclude this section with a further application of the concept of 'core processes'. Let us consider first the following example (the first two commodities are finished goods, the other two are old machines).

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 212 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 55 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 216 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is easily obtained by calculation that

$$\mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})} = \{ r \in \mathbb{R} \mid \text{either } 0 \leqslant r < 1 \text{ or } 2 < r < 3 \}.$$

Therefore $\mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$ can consist of a number of disconnected segments. Let us then prove that if this happens to be the case, only the segment containing the zero is relevant from an economic point of view, since if r is in any of the other segments, then prices of old machines cannot be all nonnegative. [5] This result is an obvious consequence of the following theorem.

Theorem 2. *Let* $(\mathbf{A},\mathbf{B},\mathbf{l})$ *be a fixed capital technique. If there exists a nonnegative vector* $\mathbf{p}$ *and a positive scalar* w *such that*

$$[\mathbf{B} - (1+r)\mathbf{A}]\mathbf{p} = w\mathbf{l},$$

[5] Schefold (1978) limited his analysis without further explanation to the segment containing the zero. (It may be conjectured that he followed a suggestion by Sraffa (1960, § 64).) The above argument proves that Schefold's procedure is correct and that it is appropriate to call the upper limit of this segment the 'maximum rate of profits'.

then there is a semipositive vector $\mathbf{z}$ *such that*

$$\mathbf{z}^{\mathrm{T}}[\mathbf{B}-(1+g)\mathbf{A}]=\mathbf{c}^{\mathrm{T}},$$

where $0 \leqslant g \leqslant r$ *and* $\mathbf{c}$ *is any semipositive vector whose last* $s-f$ *entries are nought.*

Proof. Since

$$[[\mathbf{I}-(1+g)\mathbf{F}(g)],\mathbf{0}]\mathbf{p}=\mathbf{X}(g)[\mathbf{B}-(1+g)\mathbf{A}]\mathbf{p}$$
$$=\mathbf{X}(g)[w\mathbf{l}+(r-g)\mathbf{A}\mathbf{p}]:=\mathbf{m}\geqslant\mathbf{0},$$

where $\mathbf{F}(g)$ is the square matrix consisting of the columns of matrix $\mathbf{X}(g)\mathbf{A}$ which refers to produced finished goods,

$$\mathbf{p}_1=\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{p}_1,$$

where

$$\mathbf{p}=\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\end{bmatrix},$$

and $\mathbf{p}_1$ refers to produced finished goods. By following a well known procedure,

$$\mathbf{p}_1=\mathbf{m}+(1+g)\mathbf{F}(g)[\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{p}_1]$$
$$=\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{m}+[(1+g)\mathbf{F}(g)]^2\mathbf{p}_1.$$

By iterating the same procedure,

$$\mathbf{p}_1=\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{m}+\ldots+[(1+g)\mathbf{F}(g)]^t\mathbf{m}+\ldots.$$

The above series is increasing and bounded, hence it is convergent. As a consequence

$$\lim_{t\to\infty}[(1+g)\mathbf{F}(g)]^t\mathbf{m}=\mathbf{0}. \tag{5}$$

If vector $\mathbf{m}$ is positive, then equality (5) implies that

$$\lim_{t\to\infty}[(1+g)\mathbf{F}(g)]^t=\mathbf{0}. \tag{6}$$

If vector $\mathbf{m}$ has some zero element, a deeper analysis is necessary. Since labour is required directly or indirectly in the production of every finished good (cf. Axiom (viii) in Section 2), there exists a natural number $z\leqslant f$ such that

$$\mathbf{X}(g)\mathbf{l}+\mathbf{F}(g)\mathbf{X}(g)\mathbf{l}+\ldots+[\mathbf{F}(g)]^z\mathbf{X}(g)\mathbf{l}>\mathbf{0}.$$

Furthermore, since $\mathbf{m}\geqslant w\mathbf{X}(g)\mathbf{l}$ and $g\geqslant 0$,

$$\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{m}+\ldots+[(1+g)\mathbf{F}(g)]^z\mathbf{m}>\mathbf{0}.$$

Finally, we obtain from equality (5) that

$$\lim_{t\to\infty}[(1+g)\mathbf{F}(g)]^t\{\mathbf{m}+(1+g)\mathbf{F}(g)\mathbf{m}+\ldots+[(1+g)\mathbf{F}(g)]^z\mathbf{m}\}$$
$$=(z+1)\lim_{t\to\infty}[(1+g)\mathbf{F}(g)]^t\mathbf{m}=\mathbf{0},$$

which implies equality (6). The latter is enough to sustain that matrix $[\mathbf{I}-(1+g)\mathbf{F}(g)]$ is invertible and its inverse is semipositive. Finally, let $\hat{\mathbf{c}}\in\mathbb{R}^f$ be the vector consisting of the elements of vector $\boldsymbol{c}$ corresponding to produced finished goods. Then

$$\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I}-(1+g)\mathbf{F}(g)]^{-1}\mathbf{X}(g)[\mathbf{B}-(1+g)\mathbf{A}]=\mathbf{c}^{\mathrm{T}},$$

which proves the Theorem since

$$\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I}-(1+g)\mathbf{F}(g)]^{-1}\mathbf{X}(g)\geqslant\mathbf{0}^{\mathrm{T}}.\quad \text{Q.E.D.}$$

5. The choice of technique

In this section the choice of technique problem in models with fixed capital will be dealt with [6].

Proposition 1. *Let* $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$ *be a technique, let* $r\in\mathbb{R}^*_{(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)}$, *let* $w_i>0$ *at rate of profits* r, *and let* $(\mathbf{a},\mathbf{b},l)$ *be a process producing the finished good* i *in technique* $(\mathbf{A}_j,\mathbf{B}_j,\mathbf{l}_j)$ *such that at rate of profits* r

$$[\mathbf{b}-(1+r)\mathbf{a}]^{\mathrm{T}}\mathbf{p}_{ij}>w_i l.$$

Then there exists a technique $(\mathbf{A}_k,\mathbf{B}_k,\mathbf{l}_k)$ *such that* $r\in\mathbb{R}^*_{(\mathbf{A}_k,\mathbf{B}_k,\mathbf{l}_k)}$ *and, at rate of profits* r, $w_k>w_i$.

Proof. Let $(\mathbf{M},\mathbf{N},\mathbf{q})$ be a set of processes obtained from the set of processes $(\mathbf{A}_{ij},\mathbf{B}_{ij},\mathbf{l}_{ij})$ by substituting the process producing finished good i which uses the same machine as process $(\mathbf{a}, \mathbf{b}, l)$, if it exists, or any other process which is not in the set of processes $(\mathbf{A}_{ji}, \mathbf{B}_{ji}, \mathbf{l}_{ji})$, otherwise, with process $(\mathbf{a},\mathbf{b},l)$ itself. Obviously,

$$[\mathbf{M}-(1+r)\mathbf{N}]\mathbf{p}_{ij}\geqslant w_i\mathbf{q}.$$

[6] As was indicated in the introductory section, in the case of fixed capital the choice of technique problem concerns several aspects including the choice of the modes of utilization of durable capital goods and the choice of the economic lifetimes of such goods. All these aspects can be dealt with in terms of the above model or some version of it which is adapted to the particular case under consideration. Here we shall restrict the investigation to cases which do not necessitate any modification in the set of assumptions underlying the above fixed capital model. For a treatment of a special case of capital utilization, i.e. shift work, which requires, inter alia, to distinguish between the wage rate paid per unit of labour employed during the day shift and the wage rate paid during the night shift, see Kurz (1990, chs. 5 and 9).

The processes not producing finished good i in the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ are obviously able to determine the corresponding core processes. With respect to the processes producing finished good i we need to distinguish three cases.

Case 1: finished good i is produced within the two techniques $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$ and $(\mathbf{A}_j,\mathbf{B}_j,\mathbf{l}_j)$ either with different types of machines or without utilizing old machines in at least one of the two techniques.

Case 2: commodity i is produced with the same type of machine in the two techniques and either process $(\mathbf{a},\mathbf{b},l)$ or the process of the same degree in technique $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$ does not produce an old machine.

Case 3: commodity i is produced with the same type of machine in the two techniques and process $(\mathbf{a},\mathbf{b},l)$ produces the same old machine produced by the process of the same degree in technique $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$.

In case 1, the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ includes all processes of technique $(\mathbf{A}_j,\mathbf{B}_j,\mathbf{l}_j)$ producing finished good i. In case 2 the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ includes the processes of technique $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$ producing finished good i which are of a degree lower than that of process $(\mathbf{a},\mathbf{b},l)$, and the processes producing finished good i in technique $(\mathbf{A}_j,\mathbf{B}_j,\mathbf{l}_j)$ which are of a degree larger than or equal to that of process $(\mathbf{a},\mathbf{b},l)$. In case 3, the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ includes process $(\mathbf{a},\mathbf{b},l)$ and all processes of technique $(\mathbf{A}_i,\mathbf{B}_i,\mathbf{l}_i)$ except that producing finished good i which is of the same degree as process $(\mathbf{a},\mathbf{b},l)$. In all the three cases, the mentioned processes producing finished good i in the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ are able to determine a core process.

Let $(\mathbf{F}(r),\mathbf{X}(r)\mathbf{q})$ be the set of core processes derived from the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$, as indicated above. Moreover, let $\hat{\mathbf{p}}$ and $\hat{\mathbf{c}}$ be obtained from $\mathbf{p}_{ij}$ and $\mathbf{c}$, respectively, by deleting the entries corresponding to the columns of $\mathbf{X}(r)\mathbf{M}$ deleted in order to obtain $\mathbf{F}(r)$. Hence

$$[\mathbf{I}-(1+r)\mathbf{F}(r)]\hat{\mathbf{p}} \geqslant w_i\mathbf{X}(r)\mathbf{q} := \mathbf{m}. \tag{7}$$

By following a procedure analogous to that utilized to prove Theorem 2 we obtain that matrix $[\mathbf{I}-(1+r)\mathbf{F}(r)]$ is invertible and its inverse is semipositive. Hence

$$\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I}-(1+r)\mathbf{F}(r)]^{-1} \geqslant \mathbf{0}^{\mathrm{T}}. \tag{8}$$

By taking those processes of the set of processes $(\mathbf{M},\mathbf{N},\mathbf{q})$ which produce finished goods corresponding to positive elements of vector (8), we obtain the set of processes $(\mathbf{A}_k,\mathbf{B}_k,\mathbf{l}_k)$, which is a technique since

$$\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I}-(1+r)\mathbf{F}(r)]^{-1}\hat{\mathbf{X}}(r) > \mathbf{0}^{\mathrm{T}}.$$

and

$$\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I}-(1+r)\mathbf{F}(r)]^{-1}\hat{\mathbf{X}}(r)[\mathbf{B}_k-(1+r)\mathbf{A}_k] = \mathbf{c}^{\mathrm{T}},$$

where $\hat{\mathbf{X}}(r)$ is obtained from $\mathbf{X}(r)$ by substituting the row vectors corresponding to zero elements of vector (8) with zero row vectors and then deleting all columns consisting only of zeros.

Inequality (7) implies also that

$$1 = \hat{\mathbf{c}}^{\mathrm{T}}\hat{\mathbf{p}} > w_i\hat{\mathbf{c}}^{\mathrm{T}}[\mathbf{I} - (1+r)\mathbf{F}(r)]^{-1}\mathbf{X}(r)\mathbf{q} = \frac{w_i}{w_k},$$

where w_k is the wage rate of technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$. And this proves the proposition. Q.E.D.

Proposition 2. *Let $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$ and $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ be two techniques such that at rate of profits r*

$$0 < w_i < w_k.$$

Then there is a process in technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ which pays extra profits at rate of profits r, prices $\mathbf{p}_{ik}$, and wage rate w_i.

Proof. Assume that the Proposition does not hold. Then

$$[\mathbf{B}_{ki} - (1+r)\mathbf{A}_{ki}]\mathbf{p}_{ik} \leqq w_i\mathbf{l}_{ki}.$$

Therefore,

$$1 = \mathbf{c}^{\mathrm{T}}\mathbf{p}_{ik} = \mathbf{x}_{ki}^{\mathrm{T}}[\mathbf{B}_{ki} - (1+r)\mathbf{A}_{ki}]\mathbf{p}_{ik} \leqslant w_i\mathbf{x}_{ki}^{\mathrm{T}}\mathbf{l}_{ki} = \frac{w_i}{w_k}.$$

Hence a contradiction. Q.E.D.

Proposition 3. *Let $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$ and $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ be two techniques which are both cost-minimizing at rate of profits r where $w_k > 0$ and $w_i > 0$, then $w_k = w_i$ and $\mathbf{p}_{ki} = \mathbf{p}_{ik}$.*

Proof. It is an immediate consequence of Proposition 2 that $w_k = w_i (= w)$. Then by assumption

$$[\mathbf{B}_{ik} - (1+r)\mathbf{A}_{ik}(r)]\mathbf{p}_{ik} = w\mathbf{l}_{ik}, \tag{9a}$$

$$[\mathbf{B}_{ki} - (1+r)\mathbf{A}_{ki}(r)]\mathbf{p}_{ki} = w\mathbf{l}_{ki}, \tag{9b}$$

$$\mathbf{c}^{\mathrm{T}}\mathbf{p}_{ik} = \mathbf{c}^{\mathrm{T}}\mathbf{p}_{ki} = 1, \tag{9c}$$

$$[\mathbf{B}_{ik} - (1+r)\mathbf{A}_{ik}(r)]\mathbf{p}_{ki} \leqq w\mathbf{l}_{ik}, \tag{9d}$$

$$[\mathbf{B}_{ki} - (1+r)\mathbf{A}_{ki}(r)]\mathbf{p}_{ik} \leqq w\mathbf{l}_{ki}. \tag{9e}$$

Let $(\mathbf{F}_{ik}(r), \mathbf{X}_{ik}(r)\mathbf{l}_{ik})$ be a set of core-processes. If finished good j is produced within technique $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$, then the set of processes $(\mathbf{A}_{ik}, \mathbf{B}_{ik}, \mathbf{l}_{ik})$ includes all processes producing finished good j within technique $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$ and perhaps some processes producing finished good j within technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$. On the contrary, if finished good j is not produced within technique $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$, then the set of processes $(\mathbf{A}_{ik}, \mathbf{B}_{ik}, \mathbf{l}_{ik})$ includes all processes producing finished good j within technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ and, obviously, no process producing finished good j within technique $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$. In the first case the core-process in the set $(\mathbf{F}_{ik}(r), \mathbf{X}_{ik}(r)\mathbf{l}_{ik})$ relative to finished good j is determined by the processes producing finished good j within technique $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{l}_i)$. In the second case the core-process in the set $(\mathbf{F}_{ik}(r), \mathbf{X}_{ik}(r)\mathbf{l}_{ik})$ relative to finished good j is determined by the processes producing finished good j within technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$. In a similar way the set of core-processes $(\mathbf{F}_{ki}(r), \mathbf{X}_{ki}(r)\mathbf{l}_{ki})$ is obtained. Then equations and inequalities (9) imply

$$[\mathbf{I} - (1+r)\mathbf{F}_{ik}(r)]\hat{\mathbf{p}}_{ik} = w\mathbf{X}_{ik}(r)\mathbf{l}_{ik}, \tag{10a}$$

$$[\mathbf{I} - (1+r)\mathbf{F}_{ki}(r)]\hat{\mathbf{p}}_{ki} = w\mathbf{X}_{ki}(r)\mathbf{l}_{ki}, \tag{10b}$$

$$\hat{\mathbf{c}}^{\mathrm{T}}\hat{\mathbf{p}}_{ik} = \hat{\mathbf{c}}^{\mathrm{T}}\hat{\mathbf{p}}_{ki} = 1, \tag{10c}$$

$$[\mathbf{I} - (1+r)\mathbf{F}_{ik}(r)]\hat{\mathbf{p}}_{ki} \leqq w\mathbf{X}_{ik}(r)\mathbf{l}_{ik}, \tag{10d}$$

$$[\mathbf{I} - (1+r)\mathbf{F}_{ki}(r)]\hat{\mathbf{p}}_{ik} \leqq w\mathbf{X}_{ki}(r)\mathbf{l}_{ki}, \tag{10e}$$

where $\hat{\mathbf{p}}_{ik}$, $\hat{\mathbf{p}}_{ki}$, and $\hat{\mathbf{c}}$ are vectors obtained from vectors $\mathbf{p}_{ik}$, $\mathbf{p}_{ki}$, and $\mathbf{c}$, respectively, by deleting the entries corresponding to the columns of matrix $\mathbf{X}_{ik}(r)\mathbf{A}_{ik}$ deleted in order to obtain $\mathbf{F}_{ik}(r)$. Then the prices of finished goods which are produced in at least one of the two techniques are proved to be the same in both techniques. In fact by following a procedure adopted in proving Theorem 2 it is obtained from (10a) and (10b) that matrices $[\mathbf{I} - (1+r)\mathbf{F}_{ik}(r)]$ and $[\mathbf{I} - (1+r)\mathbf{F}_{ki}(r)]$ are invertible and their inverses are semipositive. As a consequence it is obtained from inequalities (10d) and (10e) that

$$\hat{\mathbf{p}}_{ki} \leqq w[\mathbf{I} - (1+r)\mathbf{F}_{ik}(r)]^{-1}\mathbf{X}_{ik}(r)\mathbf{l}_{ik} = \hat{\mathbf{p}}_{ik},$$

$$\hat{\mathbf{p}}_{ik} \leqq w[\mathbf{I} - (1+r)\mathbf{F}_{ki}(r)]^{-1}\mathbf{X}_{ki}(r)\mathbf{l}_{ki} = \hat{\mathbf{p}}_{ki},$$

which is enough to assert that $\hat{\mathbf{p}}_{ki} = \hat{\mathbf{p}}_{ik}$. In order to prove that also the prices of old machines which are produced in at least one of the two techniques are equal, let us first obtain in an obvious way from equations and inequalities (9) that

$$[\hat{\mathbf{D}}_{ik2} - (1+r)\hat{\mathbf{C}}_{ik2}]\overline{\mathbf{p}}_{ik} = w\hat{\mathbf{l}}_{ik} - [\hat{\mathbf{D}}_{ik1} - (1+r)\hat{\mathbf{C}}_{ik1}]\hat{\mathbf{p}}_{ik}, \tag{11a}$$

$$[\hat{\mathbf{D}}_{ki2} - (1+r)\hat{\mathbf{C}}_{ki2}]\overline{\mathbf{p}}_{ki} = w\hat{\mathbf{l}}_{ki} - [\hat{\mathbf{D}}_{ki1} - (1+r)\hat{\mathbf{C}}_{ki1}]\hat{\mathbf{p}}_{ki}, \tag{11b}$$

$$[\hat{\mathbf{D}}_{ki2} - (1+r)\hat{\mathbf{C}}_{ki2}]\overline{\mathbf{p}}_{ik} \leqslant w\hat{\mathbf{l}}_{ki} - [\hat{\mathbf{D}}_{ki1} - (1+r)\hat{\mathbf{C}}_{ki1}]\hat{\mathbf{p}}_{ik}, \tag{11c}$$

$$[\hat{\mathbf{D}}_{ik2} - (1+r)\hat{\mathbf{C}}_{ik2}]\overline{\mathbf{p}}_{ki} \leqslant w\hat{\mathbf{l}}_{ik} - [\hat{\mathbf{D}}_{ik1} - (1+r)\hat{\mathbf{C}}_{ik1}]\hat{\mathbf{p}}_{ki}, \tag{11d}$$

where $\overline{\mathbf{p}}_{ik}$ and $\overline{\mathbf{p}}_{ki}$ are obtained from $\mathbf{p}_{ik}$ and $\mathbf{p}_{ki}$ by eliminating the prices of finished goods and the prices of old machines which are not produced in both techniques, the other terms obviously being obtained in the same way as in Section 3. In particular, vectors $\hat{\mathbf{l}}_{ik}$ and $\hat{\mathbf{l}}_{ki}$ are obtained from vectors $\mathbf{l}_{ik}$ and $\mathbf{l}_{ki}$, respectively, by deleting the entries corresponding to processes producing only finished goods. Similarly, matrices $\hat{\mathbf{D}}_{ik2}$, $\hat{\mathbf{D}}_{ki2}$, $\hat{\mathbf{C}}_{ik2}$, and $\hat{\mathbf{C}}_{ki2}$ are obtained from matrices $\mathbf{B}_{ik}$, $\mathbf{B}_{ki}$, $\mathbf{A}_{ik}$, and $\mathbf{A}_{ki}$, respectively, by deleting (i) the row vectors corresponding to processes producing only finished goods, (ii) the column vectors corresponding to finished goods, and (iii) the column vectors corresponding to old machines which are not produced within both techniques. As a consequence matrices $\hat{\mathbf{D}}_{ik2}$ and $\hat{\mathbf{D}}_{ki2}$ are diagonal matrices with elements on the main diagonal all positive and matrices $\hat{\mathbf{C}}_{ik2}$ and $\hat{\mathbf{C}}_{ki2}$ are semipositive matrices with nonzero elements only on the diagonal just below the main one. Finally, since $\hat{\mathbf{p}}_{ik} = \hat{\mathbf{p}}_{ki}$, we obtain from equations and inequalities (11) that

$$\left[\hat{\mathbf{D}}_{ki2} - (1+r)\hat{\mathbf{C}}_{ki2}\right]\bar{\mathbf{p}}_{ik} \leqq \left[\hat{\mathbf{D}}_{ki2} - (1+r)\hat{\mathbf{C}}_{ki2}\right]\bar{\mathbf{p}}_{ki},$$

$$\left[\hat{\mathbf{D}}_{ik2} - (1+r)\hat{\mathbf{C}}_{ik2}\right]\bar{\mathbf{p}}_{ki} \leqq \left[\hat{\mathbf{D}}_{ik2} - (1+r)\hat{\mathbf{C}}_{ik2}\right]\bar{\mathbf{p}}_{ik},$$

and since $[\hat{\mathbf{D}}_{ki2} - (1+r)\hat{\mathbf{C}}_{ki2}]$ and $[\hat{\mathbf{D}}_{ik2} - (1+r)\hat{\mathbf{C}}_{ik2}]$ are invertible with non-negative inverse (cf. Section 3), $\bar{\mathbf{p}}_{ik} \leqq \bar{\mathbf{p}}_{ki} \leqq \bar{\mathbf{p}}_{ik}$.

Theorem 3.

(a) *If there exists a technique* $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ *such that* $r^* \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$, *then there exists a cost-minimizing technique at rate of profits r which can sustain the growth rate g, where* $0 \le g \le r \le r^*$.

(b) *If there exists a technique* $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ *such that* $r^* \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$, *then there exists a cost-minimizing technique at rate of profits* r^*.

(c) *A technique is cost-minimizing if and only if it is able to pay the largest wage rate when the numeraire consists of a positive amount of each consumed commodity.*

(d) *If there exists more than one cost-minimizing technique at rate of profits* r^*, *then they all share the same wage rate and the same prices of commodities which are produced in at least one of them.*

(e) *Price vectors of cost-minimizing techniques are semipositive.*

(f) *If only finished goods enter into the numeraire, then the* $w - r$ *relationship of a technique which is cost-minimizing at rate of profits* r^* *is decreasing at* r^*.

Proof. The 'only if' part of statement (c) is a direct consequence of Proposition 2. The 'if' part is a consequence of Proposition 1: if there exists a process able to pay extra profits, then there is a technique which can pay a larger wage rate; hence a contradiction. Statement (d) is equivalent to Proposition 3. Statement (b) is a direct consequence of statement (c) since the number of processes is finite. [7] If the price of old machine j is negative for technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$, then there is a

[7] If the number of processes is not finite, statement (b) is to be changed in the following way: 'If there exists a technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ such that $r^* \in \mathbb{R}^*_{(\mathbf{A},\mathbf{B},\mathbf{l})}$ and no technique can pay a wage rate larger than that paid by technique $(\mathbf{A}, \mathbf{B}, \mathbf{l})$ at rate of profits r^*, then there exists a cost-minimizing technique at rate of profits r^*'. If the number of processes is not finite, statement (a) does not need to hold.

process paying extra profits at prices of technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ since axiom (vi) of Section 1 holds. This proves statement (e). Statement (a) is a consequence of statements (b) and (e) because of Theorem 2. By differentiating totally with respect to **r** the equations defining the price vector $\mathbf{p}_k$ and the wage rate w_k of technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$, which is cost-minimizing at rate of profits r^*, we obtain

$$\mathbf{D}_k \dot{\mathbf{p}}_k = \mathbf{C}_k \mathbf{p}_k + (1 + r^*)\mathbf{C}_k \dot{\mathbf{p}}_k + \dot{w}_k \mathbf{l}_k, \tag{12a}$$

$$\mathbf{c}^{\mathrm{T}} \dot{\mathbf{p}}_k = 0, \tag{12b}$$

where $\dot{\mathbf{p}}_k$ and $\dot{w}_k$ are the vector of derivatives of prices of produced commodities with respect to r and the derivative of the wage rate with respect to r, respectively, and elements of vectors $\mathbf{p}_k$ and $\mathbf{c}$ corresponding to commodities which are not produced within technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ have been dropped. From (12a) we obtain

$$\left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right] \dot{\mathbf{p}}_k = \mathbf{C}_k \mathbf{p}_k + \dot{w}_k \mathbf{l}_k,$$

i.e.

$$\dot{\mathbf{p}}_k = \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{C}_k \mathbf{p}_k + \dot{w}_k \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{l}_k.$$

Then, because of Eq. (12b),

$$0 = \mathbf{c}^{\mathrm{T}} \dot{\mathbf{p}}_k = \mathbf{c}^{\mathrm{T}} \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{C}_k \mathbf{p}_k + \dot{w}_k \mathbf{c}^{\mathrm{T}} \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{l}_k,$$

i.e.

$$\dot{w}_k = -\frac{\mathbf{c}^{\mathrm{T}} \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{C}_k \mathbf{p}_k}{\mathbf{c}^{\mathrm{T}} \left[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k\right]^{-1} \mathbf{l}_k},$$

which is negative since $\mathbf{c}^{\mathrm{T}}[\mathbf{D}_k - (1 + r^*)\mathbf{C}_k]^{-1} \geqslant \mathbf{0}^{\mathrm{T}}$, because the last $s - f$ elements of vector $\boldsymbol{c}$ are nought, and $\mathbf{p}_k \geqslant \mathbf{0}$ because of statement (e) since technique $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{l}_k)$ is cost-minimizing at rate of profits r^*. And this proves statement (f).
Q.E.D.

6. On *new* machines

Up to now no definition of a 'new machine' has been provided. There is, however, no reason for not regarding an input of a process of the first degree, necessarily a finished good, as a 'new machine'. This can be considered just a convention and it does not change in any way the model built up so far. [8] The

[8] Up to now also the physical measure units have not been discussed. In the present section the physical units cannot be arbitrary: they need to be consistent for all machines, old and new, of the same type.

introduction of a 'new machine' allows us to deal with depreciation. In fact, the change in the price of a type of machine over one production period is, by definition, the *depreciation* of that type of machine for that period. Of course, the depreciation per period will in general depend on the rate of profits. Even for a given rate of profits it depends on the age of the machine and is in general not equal to a constant fraction of the value of the machine at the beginning of its productive life. [9] Moreover, the analysis of depreciation allows to define variable efficiency as Schefold (1978, p. 425) does. [10]

In this paper it has been assumed that *old* machines are not utilized jointly. However, there is no reason which would prevent the investigation of the case in which *new* machines are utilized jointly within the analytical framework of the present paper. One could, indeed, with regard to a process of the first degree, which by definition uses finished goods only, take more than one input as representing 'new machine*s*'. These inputs could then be assembled to constitute a *plant*, so that even if there is only *one old machine*, it is, in fact, consisting of more than one 'machine' (quotation marks are required when common language rather than the technical language of the rest of the paper is used).

This assumption is, of course, a strong one, since the cost-minimizing management of the fixed capital stock might well involve separating previously jointly utilized machines and reassembling them differently. There may indeed be cases in which the costs of separating and reassembling components of a plant are sufficiently large as to ensure that the cost-minimizing solution will not involve any such dismantling and reassembling; in these cases the concept of the 'plant' is useful. Nevertheless, the 'plant' is not and cannot be a device to avoid a general analysis of jointly utilized (old) machines. [11]

7. Conclusion

The present paper has dealt with the problem of the choice of technique in a model with fixed capital. The approach chosen shares the following features of the conventional post-Sraffian contributions to the theory of fixed capital: (i) it is assumed that no more than one used fixed capital item is employed in the production of each commodity in each technique; (ii) a two-step procedure is adopted: in a first step single techniques are analysed, while in a second step the cost-minimizing technique(s) is (are) determined. In this framework a general

[9] The idea that fixed capital does not pose particular analytical difficulties and that the depreciation charge could be taken to be proportional to the price of the new durable instrument of production was advocated, for example, by Walras (1954, lesson 23).

[10] This is the definition referred to in Section 1 above.

[11] For further remarks on this issue, see Salvadori (1987).

existence theorem and a uniqueness theorem regarding the prices of all commodities that are produced with cost-minimizing techniques are proved: whereas previous uniqueness results referred to prices of finished goods only, the theorem put forward here covers both finished goods and old machines. The paper thus makes good a lacuna in the conventional treatment of fixed capital in the tradition of Sraffa's analysis. It is also pointed out that the procedure followed in this approach cannot deal with the important case in which old machines are utilized jointly.

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