SPACE-TIME L^∞- ESTIMATES FOR SOLUTIONS OF INFINITE HORIZON SEMILINEAR PARABOLIC EQUATIONS

EDUARDO CASAS AND KARL KUNISCH

Abstract. This paper is devoted to proving L^∞- estimates for the solution of semilinear parabolic equations. The uniform estimates are obtained on the infinite time interval under the assumption that the solution is square integrable. This setting is useful for stabilization problems formulated as optimal control problems. The inhomogenous forcing function are chosen as elements of anisotropic Lebesgue spaces. Different boundary conditions on bounded domains with a Lipschitz continuous boundary are investigated.

1. Introduction

The goal of this paper is to establish the existence, uniqueness, and $L^\infty(Q)$-estimates for the solution of the following infinite horizon problem

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial t} - \Delta u + au + d(x,t,u) = g \text{ in } Q = \Omega \times (0,\infty),\\ \partial_n u + b(x,t,u) = h \text{ on } \Sigma = \Gamma \times (0,\infty),\ u(0) = u_0 \text{ in } \Omega.\end{array}\right. \tag{1.1}$$

The motivation for this endeavour is two-fold. First, this is a question of intrinsic interest and second, such $L^\infty(Q)$ information is of importance for optimal control problems involving (1.1). The analysis of such control problems involves first and second order derivatives of the state u with respect to the control variable, where g or h represent the control variable, and this step is greatly facilitated, or even necessary, if the state u has $L^\infty(Q)$-regularity; see, for instance, [6, 16, 19]. Special attention is paid to obtain estimates over the infinite time horizon. This is motivated by stabilization problems, which can be treated by optimal control techniques, or by some applications in economy or biology. Here we focus on semilinear equations and semilinear Neumann boundary conditions. Dirichlet boundary conditions and more general elliptic operators, including the case of non-autonomous coefficients, are addressed in short sections at the end of the paper. The inhomogeneities g and h will be assumed to be elements of anisotropic Sobolev spaces, with the precise conditions given in the following section.

$L^\infty(Q)$-regularity of linear and semilinear parabolic equations has been investigated with much effort and a wide variety of techniques in earlier work. Let us first comment on results which allow for forcing functions in the state equation, which represent the controls in the control theoretic context. Hölder estimates on the state of linear parabolic equations on rough domains with mixed boundary conditions

2020 *Mathematics Subject Classification.* Primary: 35B65, 35K58; Secondary: 35K20.

Key words and phrases. Semilinear parabolic equation, L^∞-estimate, infinite time horizon.

The first author was supported by MCIN/ AEI/10.13039/501100011033/ under research project PID2020-114837GB-I00.

* Corresponding author: Eduardo Casas.

were obtained by maximum regularity results in [9], [14]. $L^\infty(Q)$-estimates have further been proved in [2], [7], [3, Theorem 17.2], and [17]. In [17] semilinear equations with Neumann boundary conditions are investigated and $L^\infty(Q)$-estimates are obtained by means of treating the nonlinear term by an appropriately chosen linearization and subsequently utilizing $L^\infty(Q)$-estimates obtained by semigroup theory and comparison principles for linear equations. Here the nonlinearities in the Neumann boundary problem are of class C^1 and monotone, and the analysis is performed for finite horizon problems. These conditions will be relaxed in our work. In [2], in the course of investigating control problems, $L^\infty(Q)$-estimates are obtained for semilinear equations with inhomogenous Dirichlet boundary condition and finite horizon as well. Again the proof technique for these estimates utilizes comparison principles. Due to the semigroup approach, the coefficients in the differential operator are autonomous. In these last two works the regularity of the boundary Γ is assumed to be of class $C^{2,\alpha}$ and the coefficients of the main part of the operator are of class $C^{1,\alpha}(\bar\Omega)$ for some $\alpha > 0$. In [2] and [17], the data were assumed to belong to isotropic Lebesgue spaces. However, in some control problems, the use of anisotropic Lebesgue spaces is necessary; see, for instance, [7]. Actually, $L^\infty(Q)$-estimates were proved in [7] for controls contained in $L^p(0,\infty;L^2(\Omega))$ with p large enough. In a classic result for L^∞ estimates, [3, Theorem 17.2], the time horizon T is finite, the nonlinear term of the equation is bounded with respect to the state u, and only the Dirichlet case is considered.

Next we exemplarily refer to [20] and [21] and the references therein for the study of semilinear parabolic equations, which do not allow for forcing functions, and with a different focus on the type of nonlinearity than in our work, since the focus lies on blow-up phenomena. There are also many contributions on semilinear parabolic systems; see, for instance, [1, 15]. The assumptions of these papers, when restricted to the scalar case, do not cover our results. Finally we mention [10] and [12], where the authors consider semilinear problems with a very particular classes of forcing terms and $\Omega = \mathbb{R}^n$, and concentrate on the analysis of blow or lack thereof.

In the present paper we investigate systematically the $L^\infty(Q)$-estimates for semilinear parabolic equations under minimal regularity assumptions on the boundary Γ and the coefficients of the elliptic operator. Here we use tools from [13, Chapter III, Sec.6-8] where $L^\infty(Q)$-estimates are obtained for linear parabolic equations and a Dirichlet boundary condition. More precisely, we establish $L^\infty(Q)$-estimates for semilinear parabolic equations with Dirichlet and Neumann conditions for infinite time horizon, with coefficients depending on (x,t) belonging to $L^\infty(Q)$, locally Lipschitz nonlinearities, with forcing functions in anisotropic Lebesgue spaces, under a Lipschitz regularity of boundary Γ.

The paper is organized in the following way. The precise problem statement and the main result are presented in Section 2. The proof of this result utilizes a splitting of the problem in two subproblems, with one of them involving a linear equation with an inhomogenous boundary condition, and a second one, involving the nonlinearities d and b, and a homogenous boundary condition. These two problems are investigated in Sections 3 and 4. In Section 5, we consider non-linearities under different assumptions than used earlier in the paper, as well as the cases of Dirichlet boundary conditions and more general second order operators than the Laplacian as in (1.1). There we also point out how certain conditions on the non-linearities can be modified, in case that (1.1) is considered on a finite time horizon $[0,T]$.

2. Assumptions and main theorem

We make the following assumptions on the problem data of (1.1). In case $n \geq 2$, Ω denotes an open bounded subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. For $n = 1$, $\Omega = (\alpha, \beta)$ with $-\infty < \alpha < \beta < \infty$ and $\Gamma = \{\alpha, \beta\}$. Throughout this paper we set $I = (0, \infty)$, $Q = \Omega \times I$, $\Sigma = \Gamma \times I$, $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \Gamma \times (0, T)$ for every $T \in (0, \infty)$.

For the partial differential equation we assume that $a \in L^\infty(Q)$, $0 \leq a \not\equiv 0$, $g \in L^2(Q) \cap L^r(I; L^p(\Omega))$ with $\frac{1}{r} + \frac{n}{2p} < 1$ and $p, r \in [1, \infty]$, and $d : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ denotes a Carathéodory function satisfying

$$d(x, t, 0) = 0, \tag{2.1}$$

$$\forall M\, \exists L_M : |d(x, t, u_2) - d(x, t, u_1)| \leq L_M |u_2 - u_1|, \forall |u_i| \leq M, i = 1, 2, \tag{2.2}$$

$$\exists R \text{ such that } d(x, t, u)u \geq 0 \quad \forall |u| \geq R, \tag{2.3}$$

for almost all $(x, t) \in Q$.

For the Neumann condition, in the case $n \geq 2$, we assume $h \in L^2(\Sigma) \cap L^s(I; L^q(\Gamma))$ with $\frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2}$ and $s, q \in (1, \infty]$, and $b : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying (2.1)–(2.3) with d and Q replaced by b and Σ.

In the case $n = 1$, we assume that $h = (h_\alpha, h_\beta)$ with $h_\alpha, h_\beta \in L^2(I) \cap L^s(I)$, $s > 2$, and $b = (b_\alpha, b_\beta)$ with $b_\alpha, b_\beta : I \times \mathbb{R} \longrightarrow \mathbb{R}$ Carathéodory functions satisfying (2.1)–(2.3) in the variables (t, u). Further, the boundary condition must be interpreted in the following sense

$$-\partial_x u(\alpha, t) + b(t, u(\alpha, t)) = h_\alpha(t) \text{ and } \partial_x u(\beta, t) + b(t, u(\beta, t)) = h_\beta(t). \tag{2.4}$$

Finally, we assume that the initial condition u_0 belongs to $L^\infty(\Omega)$.

As an immediate consequence of (2.1) and (2.2) we deduce

$$\forall M > 0 \;\; |d(x, t, u)| = |d(x, t, u) - d(x, t, 0)| \leq L_M |u| \leq L_M M \quad \forall |u| \leq M.$$

The same property is enjoyed by b.

Typical functions satisfying the conditions (2.1)-(2.3) are $d(x, t, u) = \alpha(x, t)[\mathrm{e}^u - 1]$ or $d(x, t, u) = \sum_{k=1}^{2m+1} \alpha_k(x, t) u^k$, where the coefficients α and α_k are functions of $L^\infty(Q)$, $\alpha(x, t) \geq \alpha_0$, and $\alpha_{2m+1}(x, t) \geq \alpha_0$ for some real constant $\alpha_0 > 0$. In particular, the Schlögl and Allen Cahn equations fit into the second example.

The following notation will be used in this paper. For $0 < T \leq \infty$ we consider the Hilbert space

$$W(0, T) = \{u \in L^2(0, T; H^1(\Omega)) : \frac{\partial u}{\partial t} \in L^2(0, T; H^1(\Omega)^*)\}$$

endowed with the norm $\|u\|_{W(0,T)} = \left(\|u\|^2_{L^2(0,T;H^1(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega)^*)}\right)^{\frac{1}{2}}$. The embedding $W(0, T) \subset C([0, T]; L^2(\Omega))$ is continuous for $T \leq \infty$ and $W(0, T)$ is compactly embedded in $L^2(0, T; L^2(\Omega))$ if $T < \infty$; see [18, page 106].

Definition 2.1. *We call u a solution to* (1.1) *if for every $T > 0$ the restriction of u to Q_T belongs to $W(0, T) \cap L^\infty(Q_T)$ and it satisfies the following equation in the variational sense*

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta u + au + d(x, t, u) = g \text{ in } Q_T, \\ \partial_n u + b(x, t, u) = h \text{ on } \Sigma_T, \;\; u(0) = u_0 \text{ in } \Omega, \end{array}\right. \tag{2.5}$$

thus $u(0) = u_0$ *and*

$$\int_0^T \langle \frac{\partial u}{\partial t}(t), v(t)\rangle_{H^1(\Omega)^*,H^1(\Omega)}\,\mathrm{d}t + \int_{Q_T} [\nabla u \nabla v + auv]\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} d(x,t,u)v\,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma_T} b(x,t,u)v\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{Q_T} gv\,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma_T} hv\,\mathrm{d}x\,\mathrm{d}t \quad \forall v \in W(0,T).$$

The following theorem is the main result of this paper.

Theorem 2.2. *Under the above assumptions, equation* (1.1) *has a unique solution* u. *Moreover, if* $u \in L^2(Q)$ *then* $u \in W(I) \cap L^\infty(Q)$ *and there exist constants* K_1 *and* K_2 *depending on* d, b, *and monotonically increasing on* $\|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))}$, *such that the following estimates hold*

$$\|u\|_Q \le K_1\Big(\|u\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|h\|_{L^2(\Sigma)}\Big), \tag{2.6}$$

$$\|u\|_{L^\infty(Q)} \le K_2\Big(\|u\|_{L^2(Q)} + \|u_0\|_{L^\infty(\Omega)}$$
$$+ \|g\|_{L^2(Q)} + \|g\|_{L^r(I;L^p(\Omega))} + \|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))} + R\Big), \tag{2.7}$$

$$\|d(\cdot,\cdot,u)\|_{L^\infty(Q)} + \|d(\cdot,\cdot,u)\|_{L^2(Q)} \le L_{K_\infty}\|u\|_{L^2(Q)}, \tag{2.8}$$

$$\|b(\cdot,\cdot,u)\|_{L^\infty(\Sigma)} + \|b(\cdot,\cdot,u)\|_{L^2(\Sigma)} \le L_{K_\infty}\|u\|_{L^2(Q)}, \tag{2.9}$$

where R *is given by* (2.3), L_{K_∞} *is the Lipschitz constant in* (2.2) *associated with* $M = K_\infty = \|u\|_{L^\infty(Q)}$, *and*

$$\|u\|_Q = \Big(\|u\|^2_{L^\infty(I;L^2(\Omega))} + \|u\|^2_{L^2(I;H^1(\Omega))}\Big)^{\frac{1}{2}}.$$

Further, if $u_0 \in C(\bar\Omega)$, *then the regularity* $u \in C(\bar\Omega \times [0,\infty))$ *is fulfilled.*

For $n = 1$ the norms $\|h\|_{L^2(\Sigma)}$ and $\|h\|_{L^s(I;L^q(\Gamma))}$ in the above theorem must be interpreted as follows

$$\|h\|_{L^2(\Sigma)} = \big(\|h_\alpha\|^2_{L^2(I)} + \|h_\beta\|^2_{L^2(I)}\big)^{\frac{1}{2}},\ \|h\|_{L^s(I;L^q(\Gamma))} = \big(\|h_\alpha\|^s_{L^s(I)} + \|h_\beta\|^s_{L^s(I)}\big)^{\frac{1}{s}}.$$

If $s = \infty$, then $\|h\|_{L^s(I;L^q(\Gamma))}$ must be replaced by $\|h_\alpha\|_{L^\infty(I)} + \|h_\beta\|_{L^\infty(I)}$.

Remark 2.3. *Let us point out that in the context of many optimal control problems the minimization of the* $L^2(Q)$*-distance between the solution* u *of the state equation and a desired target is the goal. Hence, we get the* $L^2(Q)$*-estimate for every admissible state in a natural way. Then, their* $L^\infty(Q)$*-estimates follow from the above theorem. This* $L^\infty(Q)$ *property of the states is essential in the subsequent analysis of optimality conditions, which involves the first and second derivatives of the control-to-state mapping.*

All along the proof of this theorem the following inequality will be used:

$$\exists C_a > 0 : C_a\|v\|^2_{H^1(\Omega)} \le \int_\Omega [|\nabla v|^2 + av^2]\,\mathrm{d}x \quad \forall v \in H^1(\Omega). \tag{2.10}$$

Proof of Uniqueness. Let us assume that u_1 and u_2 are two solutions of (1.1) and set $u = u_2 - u_1$. Given $T < \infty$, subtracting the equations satisfied by u_2 and u_1 and testing the resulting equation with u we get for $t \in (0,T)$

$$\begin{aligned}&\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_{Q_t} [|\nabla u|^2 + au^2]\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_{Q_t} [d(x,\tau,u_1) - d(x,\tau,u_2)]u\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Sigma_t} [b(x,\tau,u_1) - b(x,\tau,u_2)]u\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Taking $M = \max\{\|u_1\|_{L^\infty(Q_T)}, \|u_2\|_{L^\infty(Q_T)}\}$ in (2.2) and using the Lipschitz properties of d and b, and (2.10) we infer

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + C_a\|u\|^2_{L^2(0,t;H^1(\Omega))} \le L_M \int_0^t \left(\|u(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_{L^2(\Sigma)}\right)\mathrm{d}\tau.$$

We recall the inequality

$$\int_\Gamma v^2\,\mathrm{d}x \le K\Big(\varepsilon\int_\Omega |\nabla v|^2\,\mathrm{d}x + \varepsilon^{-1}\int_\Omega v^2\,\mathrm{d}x\Big) \quad \forall v \in H^1(\Omega) \tag{2.11}$$

for all $\varepsilon \in (0,1)$ and a constant K independent of ε; see [11, Theorem 1.5.1.10]. Choosing $\varepsilon = \min\left\{\frac{1}{2}, \frac{C_a}{2KL_M}\right\}$ we deduce from the above inequalities

$$\|u(t)\|^2_{L^2(\Omega)} + C_a\|u\|^2_{L^2(0,t;H^1(\Omega))} \le 2L_M(1+K\varepsilon^{-1})\int_0^t \|u(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau.$$

Then, Gronwall's inequality implies that $u = 0$. $\square$

To prove the existence of a solution we decompose equation (1.1) into two parts:

$$\left\{\begin{array}{l}\dfrac{\partial v}{\partial t} - \Delta v + av = 0 \text{ in } Q,\\ \partial_n v = h \text{ on } \Sigma,\ v(0) = 0 \text{ in } \Omega,\end{array}\right. \tag{2.12}$$

$$\left\{\begin{array}{l}\dfrac{\partial w}{\partial t} - \Delta w + aw + \hat{d}(x,t,w) = g \text{ in } Q,\\ \partial_n w + \hat{b}(x,t,w) = 0 \text{ on } \Sigma,\ w(0) = u_0 \text{ in } \Omega,\end{array}\right. \tag{2.13}$$

with $\hat{d}(x,t,w) = d(x,t,v(x,t)+w)$ and $\hat{b}(x,t,w) = b(x,t,v(x,t)+w)$. Let us observe that if v and w solve the equations (2.12) and (2.13), then $u = v + w$ is the solution of (1.1). The motivation for this decomposition is to deal with the different orders of integrability of g and h. The analysis of (2.12) and (2.13) is carried out in Sections 3 and 4, respectively.

For the analysis of these equations we will use the following lemmas.

Lemma 2.4. *Let $\hat{r}$ and $\hat{q}$ be real numbers satisfying*

$$\begin{array}{lll}\dfrac{1}{\hat{r}} + \dfrac{n}{2\hat{q}} = \dfrac{n}{4}, & & \\ \hat{r} \in [2,\infty], & \hat{q} \in [2, \frac{2n}{n-2}] & \textit{for } n > 2,\\ \hat{r} \in (2,\infty], & \hat{q} \in [2,\infty) & \textit{for } n = 2,\\ \hat{r} \in [4,\infty], & \hat{q} \in [2,\infty] & \textit{for } n = 1.\end{array}$$

Then, there exists a constant C only depending on $\hat{r}$, $\hat{q}$, and n, but independent of $T \in (0,\infty]$, such that

$$\|u\|_{L^{\hat{r}}(0,T;L^{\hat{q}}(\Omega))} \le C\|u\|_{Q_T} \quad \forall u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{2.14}$$

where

$$\|u\|_{Q_T} = \Big(\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\Big)^{\frac{1}{2}}.$$

Utilizing the Gagliardo inequality [4, page 173]

$$\|u\|_{L^{\hat q}(\Omega)} \le C\|u\|_{L^2(\Omega)}^{1+\frac{n}{\hat q}-\frac{n}{2}}\|u\|_{H^1(\Omega)}^{\frac{n}{2}-\frac{n}{\hat q}} \quad \forall u \in H^1(\Omega),$$

the proof of this lemma can be achieved by following the steps of the proof in [13, page 74-75] by changing $\|\nabla u\|_{L^2(\Omega)}$ for $\|u\|_{H^1(\Omega)}$.

Lemma 2.5. *Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of nonnegative real numbers satisfying the recursion relation*

$$\xi_{j+1} \le cb^j\xi_j^{1+\varepsilon} \quad \text{for } j = 0, 1, \dots, \tag{2.15}$$

for positive constants c, ε, and $b > 1$. If $\xi_0 \le c^{-\frac{1}{\varepsilon}}b^{\frac{-1}{\varepsilon^2}}$, then the inequality

$$\xi_j \le c^{-\frac{1}{\varepsilon}}b^{\frac{-1}{\varepsilon^2}}b^{-\frac{j}{\varepsilon}} \tag{2.16}$$

holds and, consequently, $\lim_{j\to\infty}\xi_j = 0$.

See [13, Lemma II-5.6] for the proof.

Remark 2.6. *In sections 3 and 4, the proofs will be carried out for finite values s, q, r, and p. In the case where one of these numbers is ∞, for instance $s = \infty$ and the condition $\frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2}$ holds, then we can select a number $\tilde s < \infty$ sufficiently large such that $\frac{1}{\tilde s} + \frac{n-1}{2q} < \frac{1}{2}$. Since $h \in L^2(\Sigma) \cap L^\infty(I; L^q(\Gamma))$, then we get that $h \in L^{\tilde s}(I; L^q(\Gamma))$ and the existence and $L^\infty(Q)$ estimates of the solution is established. Finally, it is enough to use that*

$$\begin{aligned}\|h\|_{L^{\tilde s}(I;L^q(\Gamma))} &\le \|h\|^{1-\frac{2}{\tilde s}}_{L^\infty(I;L^q(\Gamma))}\|h\|^{\frac{2}{\tilde s}}_{L^2(I;L^q(\Gamma))}\\ &\le \frac{\tilde s}{\tilde s - 2}\|h\|_{L^\infty(I;L^q(\Gamma))} + \frac{\tilde s}{2}\|h\|_{L^2(I;L^q(\Gamma))}.\end{aligned}$$

Hence, the estimate (2.7) follows. The same argument applies to the other values whenever one of them is ∞.

3. Analysis of the linear equation.

In this section we prove the following theorem.

Theorem 3.1. *Equation (2.12) has a unique solution v that belongs to $W(I) \cap C(\bar\Omega \times [0,\infty))$. Moreover, we have the estimates*

$$\|v\|_Q + \|v\|_{W(I)} \le M_1\|h\|_{L^2(\Sigma)}, \tag{3.1}$$

$$\|v\|_{L^\infty(Q)} \le M_2\big(\|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Omega))}\big) \tag{3.2}$$

with constants M_1 and M_2 independent of h.

Proof. From [18, Section III.2] we know the existence and uniqueness of a solution $v \in W(0,T)$. Testing the equation with v and integrating in Q_T we infer with (2.10)

$$\begin{aligned}
&\frac{1}{2}\|v(t)\|^2_{L^2(\Omega)} + C_a\|v\|^2_{L^2(0,T;H^1(\Omega))}\\
&\le \frac{1}{2}\|v(t)\|^2_{L^2(\Omega)} + \int_{Q_T}[|\nabla v|^2 + av^2]\,\mathrm{d}x\,\mathrm{d}t = \int_{\Sigma_T} hv\,\mathrm{d}x\,\mathrm{d}t\\
&\le \|h\|_{L^2(\Sigma)}\|v\|_{L^2(\Sigma_T)} \le C_\Omega\|h\|_{L^2(\Sigma)}\|v\|_{L^2(0,T;H^1(\Omega))}\\
&\le \frac{C_\Omega^2}{2C_a}\|h\|^2_{L^2(\Sigma)} + \frac{C_a}{2}\|v\|^2_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

where we used that $\|v\|_{L^2(\Gamma)} \le C_\Omega\|v\|_{H^1(\Omega)}$ for every $v \in H^1(\Omega)$ and some constant C_Ω depending on Ω. The above equation implies

$$\|v\|_{C([0,T];L^2(\Omega))} + \sqrt{C_a}\|v\|_{L^2(0,T;H^1(\Omega))} \le \sqrt{\frac{2}{C_a}}C_\Omega\|h\|_{L^2(\Sigma)}.$$

Taking the supremum in T we obtain the estimate for $\|v\|_Q$ in (3.1). Now, from the variational formulation of (2.12) we get $\frac{\partial v}{\partial t} \in L^2(I;H^1(\Omega)^*)$ and, hence, $v \in W(I)$ and the associated estimate follows.

Next, we prove the estimate (3.2). To this end we follow some ideas of [13, Theorem III-7.1]. First we analyze the case $n > 1$. Given $T < \infty$ arbitrary, we define for every real number $\rho > 0$ and $(x,t) \in Q_T$

$$v_\rho(x,t) = v(x,t) - \mathrm{Proj}_{[-\rho,+\rho]}(v(x,t)) \quad \text{and} \quad A_\rho(t) = \{x \in \Omega : |v(x,t)| > \rho\}.$$

Since $v \in W(0,T)$ we also have that $v_\rho \in W(0,T)$ and $\nabla v \cdot \nabla v_\rho = |\nabla v_\rho|^2$. The fact that $\partial_t v_\rho \in L^2(0,T;H^1(\Omega)^*)$ follows from the results in [18, pages 104–105]. Additionally we have

$$\int_0^t \big\langle \frac{\partial v(\tau)}{\partial t}, v_\rho(\tau)\big\rangle_{H^1(\Omega)^*,H^1(\Omega)}\,\mathrm{d}\tau = \frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} \quad \forall t \in (0,T).$$

To establish this identity we take a sequence $\{v_k\}_{k=1}^\infty \subset C^\infty(\bar{Q}_T)$ such that $v_k \to v$ in $W(0,T)$. This convergence implies that $v_k \to v$ in $C([0,T];L^2(\Omega))$. Now we set $v_{k,\rho}(x,t) = v_k - \mathrm{Proj}_{[-\rho,+\rho]}(v_k(x,t))$. From the Lipschitz property of the projection it is immediate to check that $v_{k,\rho} \to v_\rho$ in $L^2(0,T;H^1(\Omega))$ and in $C([0,T];L^2(\Omega))$. Then, we have for every $t \in (0,T)$

$$\begin{aligned}
&\int_0^t \big\langle \frac{\partial v(\tau)}{\partial t}, v_\rho(\tau)\big\rangle_{H^1(\Omega)^*,H^1(\Omega)}\,\mathrm{d}\tau = \lim_{k\to\infty}\int_0^t \big\langle \frac{\partial v_k(\tau)}{\partial t}, v_{k,\rho}(\tau)\big\rangle_{H^1(\Omega)^*,H^1(\Omega)}\,\mathrm{d}\tau\\
&= \lim_{k\to\infty}\int_0^t\int_\Omega \frac{\partial v_{k,\rho}}{\partial t}v_{k,\rho}\,\mathrm{d}x\,\mathrm{d}\tau = \lim_{k\to\infty}\frac{1}{2}\int_0^t \frac{d}{dt}\|v_{k,\rho}(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\
&= \lim_{k\to\infty}\frac{1}{2}\|v_{k,\rho}(t)\|^2_{L^2(\Omega)} = \frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Taking $\sigma = \frac{qn}{qn-n+1} \in [1,2)$, we have that $q' = \frac{(n-1)\sigma}{n-\sigma}$ and, hence, the trace mapping $\gamma : W^{1,\sigma}(\Omega) \longrightarrow L^{q'}(\Gamma)$ is well defined with an embedding constant C_γ. Testing equation (2.12) with v_ρ, using the identity established above, and (2.10) we get

$$
\begin{aligned}
&\frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} + C_a\|v_\rho\|^2_{L^2(0,T;H^1(\Omega))}\\
&\le \frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega [|\nabla v_\rho|^2 + av_\rho^2]\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t\int_\Gamma h\,v_\rho\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le \int_0^t \|h\|_{L^q(\Gamma)}\|v_\rho(\tau)\|_{L^{q'}(\Gamma)}\,\mathrm{d}\tau \le C_\gamma \int_0^t \|h\|_{L^q(\Gamma)}\|v_\rho(\tau)\|_{W^{1,\sigma}(A_\rho(\tau))}\,\mathrm{d}\tau\\
&\le C_\gamma \int_0^t \|h\|_{L^q(\Gamma)}\|v_\rho(\tau)\|_{H^1(A_\rho(\tau))}|A_\rho(\tau)|^{\frac{2-\sigma}{2\sigma}}\,\mathrm{d}\tau\\
&\le C_\gamma\|h\|_{L^s(I;L^q(\Gamma))}\|v_\rho\|_{L^2(0,T;H^1(\Omega))}\Big(\int_0^t |A_\rho(\tau)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}\tau\Big)^{\frac{s-2}{2s}}.
\end{aligned}
$$

For the last inequality we used Hölder's inequality with exponents s, 2, and $\frac{2s}{s-2}$. With Young's inequality we obtain for a constant C_1 depending only on C_a and C_γ

$$
\|v_\rho\|_{Q_T} \le C_1\|h\|_{L^s(I;L^q(\Gamma))}\Big(\int_0^T |A_\rho(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t\Big)^{\frac{s-2}{2s}} \quad \forall\rho > 0. \tag{3.3}
$$

Let us select $\hat r$ and $\hat q$ satisfying the system

$$
\left\{\begin{aligned}
&\frac{1}{\hat r} + \frac{n}{2\hat q} = \frac{n}{4}\\
&\frac{\hat r}{\hat q} = \frac{s(qn-2n+2)}{(s-2)qn}.
\end{aligned}\right.
$$

This results in

$$
\hat r = \frac{4}{n} + \frac{2s(qn-2n+2)}{(s-2)qn} \quad\text{and}\quad \hat q = 2 + \frac{4(s-2)q}{s(qn-2n+2)}.
$$

It is easy to check that $(\hat r, \hat q)$ satisfies the assumptions of Lemma 2.4, consequently the inequality (2.14) holds. The motivation for the second equation in the above system will get transparent from the estimates further below.

For every $j = 0, 1, 2, \ldots$ we set $\delta_j = \rho(2 - 2^{-j})$. We observe that $\rho \le \delta_j \le 2\rho$, $A_\rho(t) = A_{\delta_0}(t) \supset A_{\delta_1}(t) \supset A_{\delta_2}(t) \supset \ldots$, and $A_{\delta_j}(t) \supset A_{2\rho}(t)$ for every $j \ge 0$. Then, we have with (2.14)

$$
\begin{aligned}
&C\|v_{\delta_j}\|_{Q_T} \ge \|v_{\delta_j}\|_{L^{\hat r}(0,T;L^{\hat q}(\Omega))} = \left(\int_0^T \|v_{\delta_j}\|^{\hat r}_{L^{\hat q}(A_{\delta_j}(t))}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}\\
&\ge \left(\int_0^T \|v_{\delta_j}\|^{\hat r}_{L^{\hat q}(A_{\delta_{j+1}}(t))}\,\mathrm{d}t\right)^{\frac{1}{\hat r}} \ge (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{\hat r}{\hat q}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}\\
&= (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}.
\end{aligned}
$$

Taking $\rho = \delta_j$ in (3.3) and using the last estimate we obtain

$$\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t \le \frac{C_2}{(\delta_{j+1}-\delta_j)^{\hat r}}\|h\|^{\hat r}_{L^s(I;L^q(\Gamma))}\Big(\int_0^T |A_{\delta_j}(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t\Big)^{\frac{\hat r(s-2)}{2s}}$$

with $C_2 = (CC_1)^{\hat r}$. Denoting

$$\xi_j = \int_0^T |A_{\delta_j}(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t,\ j\ge 0,\ \ c = C_2\Big[\frac{2}{\rho}\|h\|_{L^s(I;L^q(\Gamma))}\Big]^{\hat r},\ \ b = 2^{\hat r},$$

and using that $\delta_{j+1} - \delta_j = \rho 2^{-(j+1)}$, we infer from the above inequality

$$\xi_{j+1} \le c b^j \xi_j^{\frac{\hat r(s-2)}{2s}} \quad \text{for all } j \ge 0. \tag{3.4}$$

In order to apply Lemma 2.5, we have to check that $b > 1$, what is obvious, $\frac{\hat r(s-2)}{2s} > 1$, and $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, where $\varepsilon = \frac{\hat r(s-2)}{2s} - 1$. From the definition of $\hat r$ the second inequality is true if and only if

$$\frac{\hat r(s-2)}{2s} = \frac{2(s-2)}{sn} + \frac{qn-2n+2}{qn} = \frac{2}{n} - \frac{4}{sn} + 1 - \frac{2}{q} + \frac{2}{qn} > 1$$
$$\iff 1 - \frac{2}{s} - \frac{n}{q} + \frac{1}{q} > 0 \iff \frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2},$$

which is exactly our assumption on (s,q). To check the condition of ξ_0 we distinguish two cases. First we assume that $s \le \frac{qn}{n-1}$, which is equivalent to $\frac{2s(qn-2n+2)}{(s-2)qn} \ge 2$. Hence, using that $A_{\delta_0}(t) = A_\rho(t)$ and $|A_\rho(t)| \le \frac{1}{\rho^2}\|v(t)\|^2_{L^2(A_\rho(t))}$, we get with (3.1)

$$\xi_0 \le \frac{1}{\rho^{\frac{2s(qn-2n+2)}{(s-2)qn}}}\int_0^T \|v(t)\|_{L^2(\Omega)}^{\frac{2s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t \le \frac{1}{\rho^{\frac{2s(qn-2n+2)}{(s-2)qn}}}\|v\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{2s(qn-2n+2)}{(s-2)qn}-2}\|v\|^2_{L^2(Q_T)}$$
$$\le \Big(\frac{1}{\rho}\|v\|_{Q_T}\Big)^{\frac{2s(qn-2n+2)}{(s-2)qn}} \le \Big(\frac{M_1}{\rho}\|h\|_{L^2(\Sigma)}\Big)^{\frac{2s(qn-2n+2)}{(s-2)qn}}.$$

Setting $\eta = \frac{2s(qn-2n+2)}{(s-2)qn}$ and

$$\rho = 4\max\{1, C_2^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon \hat r}}\}\Big(\|h\|_{L^s(I;L^q(\Gamma))} + M_1\|h\|_{L^2(\Sigma)}\Big),$$

we obtain

$$c^{\frac{1}{\varepsilon}} b^{\frac{1}{\varepsilon^2}} \xi_0 \le C_2^{\frac{1}{\varepsilon}} 2^{\frac{\hat r}{\varepsilon}} \frac{1}{\rho^{\frac{\hat r}{\varepsilon}}}\|h\|^{\frac{\hat r}{\varepsilon}}_{L^s(I;L^q(\Gamma))} b^{\frac{1}{\varepsilon^2}} \frac{1}{\rho^\eta} M_1^\eta \|h\|^\eta_{L^2(\Sigma)} \le \frac{1}{2^{\frac{\hat r}{\varepsilon}}} < 1.$$

Hence, Lemma 2.5 implies that

$$\int_0^T |A_{2\rho}(t)|^{\frac{s(qn-2n+2)}{(s-2)qn}}\,\mathrm{d}t \le \lim_{j\to\infty}\xi_j = 0,$$

which implies that $|A_{2\rho}(t)| = 0$ for almost every $t \in (0,\infty)$ and, consequently, $|v(x,t)| \le 2\rho$ a.e. in Q and the estimate (3.2) holds.

Now, we assume that $s > \frac{qn}{n-1}$, hence $\frac{2s(qn-2n+2)}{(s-2)qn} < 2$. Then, we have that $2 < \frac{2(s-2)qn}{s(qn-2n+2)} < \frac{2n}{n-2}$. To check the second inequality we observe that it is equivalent to $\frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2} + \frac{n}{2s}$, which holds due to our assumption on (s,q).

The second inequality implies the continuous embedding $H^1(\Omega) \subset L^{\frac{2(s-2)qn}{s(qn-2n+2)}}(\Omega)$. Hence we can estimate

$$\xi_0 \le \frac{1}{\rho^2}\int_0^T \|v(t)\|^2_{L^{\frac{2(s-2)qn}{s(qn-2n+2)}}(\Omega)}\,\mathrm{d}t \le \frac{C_3}{\rho^2}\int_0^T \|v(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t$$
$$\le C_3\big(\frac{1}{\rho}\|v\|_{Q_T}\big)^2 \le \big(\frac{\sqrt{C_3}M_1}{\rho}\|h\|_{L^2(\Sigma)}\big)^2.$$

Then, taking

$$\rho = 4\max\{1, C_2^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon \hat r}}\}\Big(\|h\|_{L^s(I;L^q(\Gamma))} + \sqrt{C_3}M_1\|h\|_{L^2(\Sigma)}\Big),$$

and arguing as before we deduce again the estimate (3.2).

Now, we address the case $n = 1$. Starting as we did for the case $n > 1$ and recalling that $s > 2$ we get

$$\frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} + C_a\|v_\rho\|^2_{L^2(0,T;H^1(\Omega))} \le \int_{\Sigma_T} h v_\rho \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^t [h_\alpha(\tau)v_\rho(\alpha,\tau) + h_\beta(\tau)v_\rho(\beta,\tau)]\,\mathrm{d}\tau$$
$$\le C_4\int_0^t [|h_\alpha(\tau)| + |h_\beta(\tau)|]\|v_\rho(\tau)\|_{W^{1,1}(A_\rho(\tau))}\,\mathrm{d}\tau$$
$$\le C_4\int_0^t [|h_\alpha(\tau)| + |h_\beta(\tau)|]\|v_\rho(\tau)\|_{H^1(A_\rho(\tau))}|A_\rho(\tau)|^{\frac{1}{2}}\,\mathrm{d}\tau$$
$$\le C_4(\|h_\alpha\|_{L^s(I)} + \|h_\beta\|_{L^s(I)})\|v_\rho\|_{L^2(0,T;H^1(\Omega))}\Big(\int_0^T |A_\rho(\tau)|^{\frac{s}{s-2}}\,\mathrm{d}\tau\Big)^{\frac{s-2}{2s}},$$

where we used that $W^{1,1}(\Omega) = W^{1,1}(\alpha,\beta) \subset C[\alpha,\beta]$ and Hölder's inequality with exponents s, 2, and $\frac{s-2}{2s}$.

To get a lower bound by using the inequality (2.14), we select $\hat r$ and $\hat q$ satisfying the equations

$$\begin{cases} \dfrac{1}{\hat r} + \dfrac{1}{2\hat q} = \dfrac{1}{4} \\[2mm] \dfrac{\hat r}{\hat q} = \dfrac{s}{s-2}, \end{cases}$$

which gives $\hat r = 4 + \frac{2s}{s-2} \in [6,\infty)$ and $\hat q = 2 + \frac{4(s-2)}{s} \in (2,6]$. Then, arguing as for $n > 1$ we get

$$C\|v_{\delta_j}\|_{Q_T} \ge (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{\hat r}{\hat q}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}$$
$$= (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{s}{s-2}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}.$$

The two estimates lead to

$$\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{s}{s-2}}\,\mathrm{d}t$$
$$\le \frac{C_5}{(\delta_{j+1} - \delta_j)^{\hat r}}(\|h_\alpha\|_{L^s(I)} + \|h_\beta\|_{L^s(I)})^{\hat r}\Big(\int_0^T |A_{\delta_j}(t)|^{\frac{s}{s-2}}\,\mathrm{d}t\Big)^{\frac{\hat r(s-2)}{2s}}.$$

Now, we get an estimate for ξ_0:

$$\xi_0 = \int_0^T |A_\rho(t)|^{\frac{s}{s-2}}\,\mathrm{d}t \le \frac{1}{\rho^{\frac{2s}{s-2}}} \int_\Omega \|v(t)\|^{\frac{2s}{s-2}}_{L^2(\Omega)} \le \Big(\frac{1}{\rho}\|v\|_{Q_T}\Big)^{\frac{2s}{s-2}}$$
$$\le \Big(\frac{M_1}{\rho}(\|h_\alpha\|_{L^2(I)} + \|h_\beta\|_{L^2(I)})\Big)^{\frac{2s}{s-2}}.$$

We also have that $\frac{\hat r(s-2)}{2s} = 1 + \frac{2(s-2)}{2} > 1$. Hence, taking $\varepsilon = \frac{2(s-2)}{2}$ and

$$\rho = 4\max\{1, C_5^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon \hat r}}\}\Big(\|h_\alpha\|_{L^s(I)} + \|h_\beta\|_{L^s(I)} + M_1(\|h_\alpha\|_{L^2(I)} + \|h_\beta\|_{L^2(I)})\Big),$$

applying Lemma 2.5, and arguing similarly as we did for $n > 1$ we get (3.2).

Finally, we prove the continuity of v in $\bar\Omega \times [0,\infty)$. To this end we first assume that h is a continuous function. Then, from [8] we deduce that $v \in C(\bar Q_T)$ for every $T < \infty$. Combining this and the fact $v \in L^\infty(Q)$ we infer that $v \in C(\bar\Omega \times [0,\infty))$. If h is not continuous, we take a sequence of continuous functions $\{h_k\}_{k=1}^\infty$ such that $h_k \to h$ in $L^2(\Sigma) \cap L^s(I; L^q(\Gamma))$. For every k we obtain a solution $v_k \in C(\bar\Omega \times [0,\infty))$. Taking into account (3.2) we deduce

$$\|v - v_k\|_{L^\infty(Q)} \le M_2\big(\|h - h_k\|_{L^2(\Sigma)} + \|h - h_k\|_{L^s(I;L^q(\Gamma))}\big) \to 0 \text{ as } k \to \infty.$$

Thus, we also have that $v \in C(\bar\Omega \times [0,\infty))$. ∎

Remark 3.2. *(i) If equation* (2.12) *is considered in a finite horizon T, obviously the proof given above establishes the $L^\infty(Q_T)$ regularity of v and the estimates* (3.1) *and* (3.2) *with Q, Σ, and I replaced by Q_T, Σ_T, and $(0,T)$, respectively.*

(ii) Suppose that the Neumann condition $\partial_n v = h$ is replaced by the Dirichlet condition $v = h$ on Σ with h being the trace on Σ of a function belonging to $W(I)$ and such that $h \in L^\infty(\Sigma)$. Then, the Dirichlet problem has a unique solution $v \in W(I)$ satisfying the estimate $\|v\|_{L^\infty(Q)} \le \|h\|_{L^\infty(\Sigma)}$, which is well known for finite horizon. This estimate can be checked by choosing $\rho = \|h\|_{L^\infty(\Sigma)}$ and setting $v_\rho(x,t) = v(x,t) - \mathrm{Proj}\,[-\rho,+\rho](v(x,t))$, we have that $v_\rho = 0$ on Σ and then for every $T < \infty$ and $t \in (0,T)$

$$\frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} + C_a\|v_\rho\|^2_{L^2(0,T;H^1(\Omega))}$$
$$\le \frac{1}{2}\|v_\rho(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega [|\nabla v_\rho|^2 + a v_\rho^2]\,\mathrm{d}x\,\mathrm{d}\tau = 0.$$

Hence, $v_\rho \equiv 0$ on Q_T for all T and the stated estimate follows.

4. Analysis of the nonlinear equation.

This section is dedicated to the analysis of equation (2.13).

Theorem 4.1. *Equation* (2.13) *has a unique solution w. Moreover, if $w \in L^2(Q)$ then it belongs to $W(I) \cap L^\infty(Q)$ and we have the estimates*

$$\|w\|_Q \le M_3\Big(\|w\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|h\|_{L^2(\Sigma)}\Big), \tag{4.1}$$

$$\|w\|_{L^\infty(Q)} \le M_4\big(\|w\|_{L^2(Q)} + \|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(Q)} + \|g\|_{L^r(I;L^p(Q))} + \|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))} + R\big), \tag{4.2}$$

where the constants M_3 and M_4 depend on d, b, and monotonically increasing on $\|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))}$. Further, if $u_0 \in C(\bar\Omega)$, then the regularity $w \in C(\bar\Omega \times [0,\infty))$ is fulfilled.

Proof. The proof is divided into several steps.

I - Preliminaries. First, we observe that $v \in L^\infty(Q)$ implies that $\hat d$ and $\hat b$ satisfy the local Lipschitz assumption (2.2). Hence, the uniqueness of a solution of (2.13) follows from the uniqueness already established for equation (1.1). Next, we prove the existence of a solution in Q_T for every fixed $T < \infty$. For every integer $k \geq 1$ we define the functions

$$\hat d_k(x,t,\xi) = \hat d(x,t,\mathrm{Proj}_{[-k,+k]}(\xi)) = d(x,t,v(x,t) + \mathrm{Proj}_{[-k,+k]}(\xi)),$$
$$\hat b_k(x,t,\xi) = \hat b(x,t,\mathrm{Proj}_{[-k,+k]}(\xi)) = b(x,t,v(x,t) + \mathrm{Proj}_{[-k,+k]}(\xi)).$$

From (2.1)–(2.2) we deduce for $M = \|v\|_{L^\infty(Q)}$ and $M_k = \|v\|_{L^\infty(Q)} + k$

$$\begin{aligned}|\hat d_k(x,t,\xi)| &\leq |\hat d_k(x,t,\xi) - \hat d_k(x,t,0)| + |\hat d_k(x,t,0)|\\ &\leq L_{M_k}k + |d(x,t,v(x,t)) - d(x,t,0)| \leq L_{M_k}k + L_M M \quad \forall \xi \in \mathbb{R}.\end{aligned}$$

The same estimate is obtained for $\hat b_k$. Associated with $\hat d_k$ and $\hat b_k$ we consider the equation

$$\left\{\begin{array}{l}\dfrac{\partial w_k}{\partial t} - \Delta w_k + a w_k + \hat d_k(x,t,w_k) = g \text{ in } Q_T,\\ \partial_n w_k + \hat b_k(x,t,w_k) = 0 \text{ on } \Sigma_T,\ \ w_k(0) = u_0 \text{ in } \Omega.\end{array}\right. \tag{4.3}$$

II - Existence of solution for (4.3). We define the function $F_k : L^2(Q_T) \times L^2(\Sigma_T) \longrightarrow L^2(Q_T) \times L^2(\Sigma_T)$ by $F_k(\varphi,\psi) = (w, w_{|\Sigma_T})$, where $w \in W(0,T)$ is the solution of the linear equation

$$\left\{\begin{array}{l}\dfrac{\partial w}{\partial t} - \Delta w + a w = g - \hat d_k(x,t,\varphi) \text{ in } Q_T,\\ \partial_n w = -\hat b_k(x,t,\psi) \text{ on } \Sigma_T,\ \ w(0) = u_0 \text{ in } \Omega.\end{array}\right. \tag{4.4}$$

Using the boundedness of $\hat d_k$ and $\hat b_k$ by a constant independent of (φ,ψ) the continuity of F_k follows. Furthermore, we have an estimate $\|w\|_{W(0,T)} \leq C_k$ uniform for all $(\varphi,\psi) \in L^2(Q_T) \times L^2(\Sigma_T)$. Applying the Aubin-Lions Theorem [18, Proposition III.1.3] first with $B_0 = H^1(\Omega)$, $B_1 = L^2(\Omega)$, and $B_2 = H^1(\Omega)^*$ we get the compactness of the embedding $W(0,T) \subset L^2(Q_T)$. Later taking $B_0 = H^1(\Omega)$, $B_1 = H^{\frac34}(\Omega)$, and $B_2 = H^1(\Omega)^*$, the compactness of the embedding $W(0,T) \subset L^2(0,T;H^{\frac34}(\Omega))$ follows. Finally, the continuity of the trace $w \to w_{|\Sigma_T}$ from $L^2(0,T;H^{\frac34}(\Omega))$ to $L^2(\Sigma_T)$ implies that the image of F_k is relatively compact in $L^2(Q_T) \times L^2(\Sigma_T)$. Then, Schauder's fixed point theorem implies the existence of a solution $w_k \in W(0,T)$ for equation (4.3).

Next we prove that for k large enough $w_k = w \in W(0,T) \cap L^\infty(Q_T)$ is independent of k and solves the equation (2.13) in Q_T.

III - Estimate for $\|w_k\|_{Q_T}$. We assume that $k \geq M_{R,v} = R + \|v\|_{L^\infty(Q)}$. Then, we have that

$$\hat d_k(x,t,\xi)\xi \geq 0 \quad \forall |\xi| \geq M_{R,v}. \tag{4.5}$$

Indeed, if $\xi \geq M_{R,v}$, then $v(x,t) + \mathrm{Proj}_{[-k,+k]}(\xi) \geq R$ holds. Hence (2.3) implies that

$$\hat{d}_k(x,t,\xi) = d(x,t,v(x,t) + \mathrm{Proj}_{[-k,+k]}(\xi)) \geq 0 \text{ and then } \hat{d}_k(x,t,\xi)\xi \geq 0.$$

Analogously we can argue for $\xi \leq -M_{R,v}$. If $|\xi| < M_{R,v}$ then assumptions (2.1) and (2.2) lead to

$$\begin{aligned} |\hat{d}_k(x,t,\xi)| &= |d(x,t,v(x,t) + \mathrm{Proj}_{[-k,+k]}(\xi)) - d(x,t,0)| \\ &\leq L_M(|\xi| + |v(x,t)|) \end{aligned} \tag{4.6}$$

with $M = M_{R,v} + \|v\|_{L^\infty(Q)}$. Henceforth we denote $C_{M_{R,v}} = L_M$ to recall the dependence of the Lipschitz constant L_M on R and $\|v\|_{L^\infty(Q)}$. The same properties hold for $\hat{b}_k$. Setting $\Omega_{M_{R,v}(\tau)} = \{x \in \Omega : |w_k(x,\tau)| < M_{R,v}\}$, $\Gamma_{M_{R,v}(\tau)} = \{x \in \Gamma : |w_k(x,t)| < M_{R,v}\}$, and testing equation (4.3) with w_k, integrating in $(0,t)$ with $t \in (0,T)$, using (4.5), (4.6), and (2.11) with $\varepsilon = \min\left\{1, \frac{C_a}{4KC_{M_{R,v}}}\right\}$, we get with Young's inequality

$$\begin{aligned} &\frac{1}{2}\|w_k(t)\|^2_{L^2(\Omega)} + C_a\|w_k\|^2_{L^2(0,t;H^1(\Omega))} \\ &\leq \frac{1}{2}\|w_k(t)\|^2_{L^2(\Omega)} + \int_{Q_t}[|\nabla w_k|^2 + aw_k^2]\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_{Q_t} g\,w_k\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_0^t\int_{\Omega_{M_{R,v}}(\tau)}|\hat{d}_k(x,\tau,w_k)||w_k|\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{\Gamma_{M_{R,v}}(\tau)}|\hat{b}_k(x,\tau,w_k)||w_k|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t\Big(\|g(\tau)\|_{L^2(\Omega)} + C_{M_{R,v}}\|v(\tau)\|_{L^2(\Omega)}\Big)\|w_k(\tau)\|_{L^2(\Omega)}\,\mathrm{d}\tau \\ &+ C_{M_{R,v}}\int_0^t\|v(\tau)\|_{L^2(\Gamma)}\|w_k\|_{L^2(\Gamma)}\,\mathrm{d}\tau \\ &+ C_{M_{R,v}}\int_0^t\Big[\|w_k(\tau)\|^2_{L^2(\Omega)} + \|w_k(\tau)\|^2_{L^2(\Gamma)}\Big]\,\mathrm{d}\tau \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\ &+ \frac{2}{C_a}\Big(\|g\|_{L^2(Q_T)} + C_{M_{R,v}}\|v\|_{L^2(Q_T)}\Big)^2 + \frac{C_a}{8}\int_0^t\|w_k(\tau)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau \\ &+ \frac{2C_\Omega^4C^2_{M_{R,v}}}{C_a}\|v\|^2_{L^2(0,T;H^1(\Omega))} + \frac{C_a}{8}\int_0^t\|w_k(\tau)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau \\ &+ C_{M_{R,v}}\Big(1 + \frac{4K^2C_{M_{R,v}}}{C_a}\Big)\int_0^t\|w_k(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \frac{C_a}{4}\int_0^t\|\nabla w_k(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\Big]. \end{aligned}$$

Using (3.1), this leads to

$$\begin{aligned} &\|w_k(t)\|^2_{L^2(\Omega)} + C_a\|w_k\|^2_{L^2(0,t;H^1(\Omega))} \leq \|u_0\|^2_{L^2(\Omega)} \\ &+ \frac{1}{C_a}\Big(\|g\|_{L^2(Q_T)} + C_{M_{R,v}}M_1\|h\|_{L^2(\Sigma_T)}\Big)^2 + \frac{4C_\Omega^4C^2_{M_{R,v}}}{C_a}M_1^2\|h\|^2_{L^2(\Sigma_T)} \\ &+ 2C_{M_{R,v}}\Big(1 + 4K^2C_{M_{R,v}}C_a\Big)\int_0^t\|w_k(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leq \hat{C}_1\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)^2 + \hat{C}_2\int_0^t\|w_k(\tau)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.7}$$

1 Using Gronwall's inequality, we get from above for every $t \in [0, T]$

$$\|w_k(t)\|_{L^2(\Omega)} \leq \sqrt{\hat{C}_1}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big) \exp\big(\frac{1}{2}\hat{C}_2 T\big). \tag{4.8}$$

2 Combining (4.7) and (4.8) we infer

$$\|w_k\|_{Q_T} \leq C_T\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big) \quad \forall k \geq M_{R,v}. \tag{4.9}$$

3 We observe that constant C_T depends on T and monotonically increasing on $C_{M_{R,v}}$.

IV - Estimate for $\|w_k\|_{L^\infty(Q_T)}$. For $\rho \geq \max\{\|u_0\|_{L^\infty(\Omega)}, M_{R,v}\}$, we define $w_{k,\rho}(x,t) = w_k(x,t) - \mathrm{Proj}_{[-\rho,+\rho]}(w_k(x,t))$ for every $(x,t) \in Q_T$ and $A_\rho(t) = \{x \in \Omega : |w_k(x,t)| > \rho\}$. Similarly as in the proof of Theorem 3.1, we test equation (4.3) with $w_{k,\rho}$ and use (4.5) to obtain

$$\begin{aligned}
&\frac{1}{2}\|w_{k,\rho}(t)\|^2_{L^2(\Omega)} + C_a\|w_{k,\rho}\|^2_{L^2(0,T;H^1(\Omega))}\\
&\leq \frac{1}{2}\|w_{k,\rho}(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega [|\nabla w_{k,\rho}|^2 + a w_{k,\rho}^2]\,\mathrm{d}x\,\mathrm{d}\tau \leq \int_0^t\int_\Omega g\, w_{k,\rho}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq \|g\|_{L^r(0,T;L^p(\Omega))}\Big(\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^{p'}(\Omega)}\,\mathrm{d}\tau\Big)^{\frac{1}{r'}}.
\end{aligned} \tag{4.10}$$

4 We distinguish several cases depending on n and r.

Case 1: $n \geq 2$ and $r > 2$. Observe that the condition $\frac{1}{r} + \frac{n}{2p} < 1$ implies that $p > \frac{n}{2}$. If $n = 2$ we select σ satisfying $\frac{1}{r} + \frac{1}{p} < \frac{1}{\sigma} < 1$. We set $\sigma = \frac{2n}{n+2}$ otherwise. Then, we have

$$\begin{aligned}
&\Big(\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^{p'}(\Omega)}\,\mathrm{d}\tau\Big)^{\frac{1}{r'}} \leq \Big(\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^{\sigma'}(\Omega)}|A_\rho(\tau)|^{\frac{r'(\sigma'-p')}{\sigma' p'}}\,\mathrm{d}\tau\Big)^{\frac{1}{r'}}\\
&\leq C_1\|w_{k,\rho}\|_{L^2(0,t;H^1(\Omega))}\Big(\int_0^t |A_\rho(\tau)|^{\frac{2r(p-\sigma)}{(r-2)p\sigma}}\,\mathrm{d}\tau\Big)^{\frac{r-2}{2r}}\\
&\leq C_1\|w_{k,\rho}\|_{L^2(0,T;H^1(\Omega))}\Big(\int_0^T |A_\rho(\tau)|^{\frac{2r(p-\sigma)}{(r-2)p\sigma}}\,\mathrm{d}\tau\Big)^{\frac{r-2}{2r}}.
\end{aligned}$$

5 This inequality and (4.10) imply

$$\|w_{k,\rho}\|_{Q_T} \leq C_2\|g\|_{L^r(0,T;L^p(\Omega))}\Big(\int_0^T |A_\rho(t)|^{\frac{2r(p-\sigma)}{(r-2)p\sigma}}\,\mathrm{d}t\Big)^{\frac{r-2}{2r}}. \tag{4.11}$$

6 Let us denote by $\hat{r}$ and $\hat{q}$ the solutions of the system

$$\begin{cases} \dfrac{1}{\hat{r}} + \dfrac{n}{2\hat{q}} = \dfrac{n}{4}\\[2mm] \dfrac{\hat{r}}{\hat{q}} = \dfrac{2r(p-\sigma)}{(r-2)p\sigma}. \end{cases}$$

7 Then, we have

$$\hat{r} = \frac{4}{n} + \frac{4r(p-\sigma)}{(r-2)p\sigma} \quad \text{and} \quad \hat{q} = 2 + \frac{2(r-2)p\sigma}{nr(p-\sigma)}.$$

It is easy to check that $(\hat{r}, \hat{q})$ satisfies the assumptions of Lemma 2.4, hence the inequality (2.14) holds. Again, as in the proof of Theorem 3.1, we set $\delta_j = \rho(2-2^{-j})$

for every $j = 0, 1, 2, \dots$ and obtain

$$\begin{aligned} C\|w_{k,\delta_j}\|_{Q_T} &\geq \|w_{k,\delta_j}\|_{L^{\hat r}(0,T;L^{\hat q}(\Omega))} = \left(\int_0^T \|w_{k,\delta_j}\|^{\hat r}_{L^{\hat q}(A_{\delta_j}(t))}\,\mathrm{d}t\right)^{\frac{1}{\hat r}} \\ &\geq (\delta_{j+1}-\delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{2r(p-\sigma)}{(r-2)p\sigma}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}. \end{aligned}$$

Setting $b = 2^{\hat r}$, $C_3 = (CC_2)^{\hat r}$,

$$\xi_j = \int_0^T |A_{\delta_j}(t)|^{\frac{2r(p-\sigma)}{(r-2)p\sigma}}\,\mathrm{d}t, \quad j \geq 0,\ c = C_3\Big(\frac{2}{\rho}\|g\|_{L^r(0,T;L^p(\Omega))}\Big)^{\hat r},$$

and taking into account that $\delta_{j+1} - \delta_j = \rho 2^{-(j+1)}$, we deduce from the above inequality and (4.11) that $\xi_{j+1} \leq c b^j \xi_j^{\frac{\hat r(r-2)}{2r}}$. In order to apply Lemma 2.5 we have to check that $b > 1$, which is obvious, $\frac{\hat r(r-2)}{2r} = 1 + \varepsilon$ for some $\varepsilon > 0$, and $\xi_0 \leq c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$. To check that $\varepsilon > 0$ we insert the value of $\hat r$ and obtain

$$\varepsilon > 0 \Leftrightarrow \frac{\hat r(r-2)}{2r} > 1 \Leftrightarrow \frac{2(r-2)}{rn} + \frac{2(p-\sigma)}{p\sigma} > 1. \tag{4.12}$$

In the case $n \geq 3$ the last inequality is equivalent to $1 - \frac{1}{r} - \frac{n}{2p} > 0$. This is precisely the inequality assumed for (r, p), and hence $\varepsilon > 0$. In the case $n = 2$ inequality (2.12) is equivalent to $\frac{1}{\sigma} - \frac{1}{r} - \frac{1}{p} > 0$, which is the assumption on the triple (p, r, σ) that was made above.

Next, we select ρ big enough such that $\xi_0 \leq c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$ holds. We consider two different cases:

Case A: $\frac{2r(p-\sigma)}{(r-2)p\sigma} \geq 1$. In this case we use that $|A_\rho(t)| \leq \frac{1}{\rho^2}\|w_k(t)\|^2_{L^2(\Omega)}$ and (4.9) to deduce

$$\begin{aligned} \xi_0 &\leq \frac{1}{\rho^{\frac{4r(p-\sigma)}{(r-2)p\sigma}}}\int_0^T \|w_k(t)\|^{\frac{4r(p-\sigma)}{(r-2)p\sigma}}_{L^2(\Omega)}\,\mathrm{d}t \leq \Big(\frac{1}{\rho}\|w_k\|_{Q_T}\Big)^{\frac{4r(p-\sigma)}{(r-2)p\sigma}} \\ &\leq \Big[\frac{C_T}{\rho}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)\Big]^{\frac{4r(p-\sigma)}{(r-2)p\sigma}}. \end{aligned}$$

Taking

$$\begin{aligned} \rho = 4\max\{1, C_3^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon\hat r}}\}\big(&\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))} \\ &+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big), \end{aligned}$$

we get that $\xi_0 \leq c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$.

Case B: $\frac{2r(p-\sigma)}{(r-2)p\sigma} < 1$. In this case we have that $2 < \frac{(r-2)p\sigma}{r(p-\sigma)}$. Moreover, if $n > 2$ then we have that $\frac{(r-2)p\sigma}{r(p-\sigma)} < \frac{2n}{n-2}$. Indeed, this inequality is equivalent to

$$\frac{1}{r} + \frac{n}{2p} < 1 + \frac{n}{2r},$$

which follows from our assumption on (r,p). Then, we get the estimate for ξ_0 as follows

$$\xi_0 \le \frac{1}{\rho^2}\int_0^T \|w_k(t)\|^2_{L^{\frac{(r-2)p\sigma}{r(p-\sigma)}}(\Omega)}\,\mathrm{d}t \le \frac{C_4}{\rho^2}\int_0^T \|w_k(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t$$
$$\le C_4\Big(\frac{1}{\rho}\|w_k\|_{Q_T}\Big)^2 \le \Big[\frac{\sqrt{C_4}C_T}{\rho}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)\Big]^2.$$

This time we set

$$\rho = 4\max\{1, C_4^{\frac{1}{2\hat r}} b^{\frac{1}{\varepsilon\hat r}}\}\big(\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))}$$
$$+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big)$$

and $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$ is fulfilled.

Applying Lemma 2.5 we get $|A_{2\rho}(t)| \le \lim_{j\to\infty}\xi_j = 0$ and, consequently, we infer

$$\|w_k\|_{L^\infty(Q_T)} \le K\big(\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))}$$
$$+ 2C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big), \tag{4.13}$$

where $K = 8\max\{1, C_4^{\frac{1}{2\hat r}} b^{\frac{1}{\varepsilon\hat r}}\}$ is independent of T.

Case 2: $n \ge 2$ and $r \le 2$. From $\frac{1}{2} + \frac{n}{2p} \le \frac{1}{r} + \frac{n}{2p} < 1$ we deduce that $p > n$ and, consequently, $p' < \frac{n}{n-1} \le 2$. Using Hölder's inequality we infer

$$\Big(\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^{p'}(\Omega)}\,\mathrm{d}\tau\Big)^{\frac{1}{r'}} \le \Big(\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^2(\Omega)}|A_\rho(\tau)|^{\frac{r'(2-p')}{2p'}}\,\mathrm{d}\tau\Big)^{\frac{1}{r'}}$$
$$\le \|w_{k,\rho}\|_{L^\infty(0,T;L^2(\Omega))}\Big(\int_0^T |A_\rho(\tau)|^{\frac{r(p-2)}{2p(r-1)}}\,\mathrm{d}\tau\Big)^{\frac{r-1}{r}}.$$

Inserting this inequality in (4.10) we deduce

$$\|w_{k,\rho}\|_{Q_T} \le C_5\|g\|_{L^r(0,T;L^p(\Omega))}\Big(\int_0^T |A_\rho(t)|^{\frac{r(p-2)}{2(r-1)}}\,\mathrm{d}t\Big)^{\frac{r-1}{r}}. \tag{4.14}$$

Now we take $\hat r$ and $\hat q$ as the solutions of the system

$$\begin{cases} \dfrac{1}{\hat r} + \dfrac{n}{2\hat q} = \dfrac{n}{4} \\[2mm] \dfrac{\hat r}{\hat q} = \dfrac{r(p-2)}{2p(r-1)}. \end{cases}$$

Then, we have

$$\hat r = \frac{4}{n} + \frac{r(p-2)}{p(r-1)} \quad \text{and} \quad \hat q = 2 + \frac{8p(r-1)}{nr(p-2)}.$$

Once again it is immediate to check that $(\hat r, \hat q)$ satisfies the assumptions of Lemma 2.4. Hence, arguing as in Case 1, we obtain

$$C\|w_{k,\delta_j}\|_{Q_T} \ge (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{r(p-2)}{2p(r-1)}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}.$$

Taking $b = 2^{\hat r}$, $C_6 = (CC_5)^{\hat r}$,

$$\xi_j = \int_0^T |A_{\delta_j}(t)|^{\frac{r(p-2)}{2p(r-1)}}\,\mathrm{d}t, \quad j \ge 0, c = C_6\Big(\frac{2}{\rho}\|g\|_{L^r(0,T;L^p(\Omega))}\Big)^{\hat r},$$

we deduce from the above inequality and (4.14) that $\xi_{j+1} \le cb^j \xi_j^{\frac{\hat r(r-1)}{r}}$. It is easy to check that $\frac{\hat r(r-1)}{r} = 1+\varepsilon$ with $\varepsilon > 0$. As in the previous case, we have to select ρ in such a way that $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$. To this end, we also distinguish two cases.

Case A: $\frac{r(p-2)}{2p(r-1)} \ge 1$. Using that $|A_\rho(t)| \le \frac{1}{\rho^2}\|w_k(t)\|^2_{L^2(\Omega)}$ and (4.9) we get

$$\begin{aligned}\xi_0 &\le \frac{1}{\rho^{\frac{r(p-2)}{p(r-1)}}}\int_0^T \|w_k(t)\|_{L^2(\Omega)}^{\frac{r(p-2)}{p(r-1}}\,\mathrm{d}t \le \Big(\frac{1}{\rho}\|w_k\|_{Q_T}\Big)^{\frac{r(p-2)}{p(r-1)}}\\ &\le \Big[\frac{C_T}{\rho}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)\Big]^{\frac{r(p-2)}{p(r-1)}}.\end{aligned}$$

Taking

$$\begin{aligned}\rho = 4\max\{1, C_6^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon\hat r}}\}\big(&\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))}\\ &+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big),\end{aligned}$$

we get that $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$.

Case B: $\frac{r(p-2)}{2p(r-1)} < 1$. In this case we have that $\frac{4p(r-1)}{r(p-2)} > 2$. Then we have

$$\begin{aligned}\xi_0 &\le \frac{1}{\rho^2}\int_0^T \|w_k(t)\|^2_{L^{\frac{4p(r-1)}{r(p-2)\sigma}}(\Omega)}\,\mathrm{d}t \le \frac{C_7}{\rho^2}\int_0^T \|w_k(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t\\ &\le C_7\big(\frac{1}{\rho}\|w_k\|_{Q_T}\big)^2 \le \big[\frac{\sqrt{C_7}C_T}{\rho}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)\big]^2.\end{aligned}$$

This time we set

$$\begin{aligned}\rho = 4\max\{1, C_7^{\frac{1}{2\hat r}} b^{\frac{1}{\varepsilon\hat r}}\}\big(&\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))}\\ &+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big)\end{aligned}$$

and $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$ is fulfilled. Therefore, applying Lemma 2.5 we get $|A_{2\rho}(t)| \le \lim_{j\to\infty}\xi_j = 0$ and (4.13) holds for a constant K independent of T.

Case 3: $n = 1$. First we consider the situation where $r \in [1,2]$. Under this assumption we have that $\frac{1}{2} + \frac{1}{2p} \le \frac{1}{r} + \frac{1}{2p} < 1$. Hence, we get that $p > 1$, $p' < \infty$, and $\frac{1}{2p} < 1 - \frac{1}{r} = \frac{1}{r'}$, which is equivalent to $\frac{r'}{p} < 2$. To deal with this case we also use the following Gagliardo inequality [4, page 173]

$$\|\varphi\|_{L^{p'}(\Omega)} \le C\|\varphi\|_{L^1(\Omega)}^{\frac{1}{p'}}\|\varphi\|_{W^{1,1}(\Omega)}^{\frac{1}{p}} \quad \forall\varphi\in W^{1,1}(\Omega).$$

Using this inequality and Hölder and Young inequalities we infer

$$\int_0^t \|w_{k,\rho}(\tau)\|^{r'}_{L^{p'}(\Omega)}\,\mathrm{d}\tau \le C^{r'} \int_0^t \|w_{k,\rho}(\tau)\|^{\frac{r'}{p'}}_{L^1(\Omega)} \|w_{k,\rho}(\tau)\|^{\frac{r'}{p}}_{W^{1,1}(\Omega)}\,\mathrm{d}\tau$$

$$\le C^{r'} \int_0^t \|w_{k,\rho}(\tau)\|^{\frac{r'}{p'}}_{L^2(\Omega)} |A_\rho(\tau)|^{\frac{r'}{2p'}} \|w_{k,\rho}(\tau)\|^{\frac{r'}{p}}_{H^1(\Omega)} |A_\rho(\tau)|^{\frac{r'}{2p}}\,\mathrm{d}\tau$$

$$\le C^{r'} \|w_{k,\rho}\|^{\frac{r'}{p'}}_{L^\infty(0,T;L^2(\Omega))} \int_0^T \|w_{k,\rho}(\tau)\|^{\frac{r'}{p}}_{H^1(\Omega)} |A_\rho(\tau)|^{\frac{r'}{2}}\,\mathrm{d}\tau$$

$$\le C^{r'} \|w_{k,\rho}\|^{\frac{r'}{p'}}_{L^\infty(0,T;L^2(\Omega))} \|w_{k,\rho}\|^{\frac{r'}{p}}_{L^2(0,T;H^1(\Omega))} \Big(\int_0^T |A_\rho(\tau)|^{\frac{pr'}{2p-r'}}\,\mathrm{d}\tau\Big)^{\frac{2p-r'}{2p}}$$

$$\le C^{r'} \Big(\frac{1}{p'}\|w_{k,\rho}\|_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{p}\|w_{k,\rho}\|_{L^2(0,T;H^1(\Omega))}\Big)^{r'}$$

$$\times \Big(\int_0^T |A_\rho(\tau)|^{\frac{pr'}{2p-r'}}\,\mathrm{d}\tau\Big)^{\frac{2p-r'}{2p}}.$$

Inserting this estimate in (4.10) we obtain

$$\|w_{k,\rho}\|_{Q_T} \le C_8 \|g\|_{L^r(0,T;L^p(\Omega))} \Big(\int_0^T |A_\rho(t)|^{\frac{pr'}{2p-r'}}\,\mathrm{d}t\Big)^{\frac{2p-r'}{2pr'}}. \tag{4.15}$$

Now, we define $(\hat r, \hat q)$ as the solution of the system

$$\begin{cases} \dfrac{1}{\hat r} + \dfrac{1}{2\hat q} = \dfrac{1}{4} \\[2mm] \dfrac{\hat r}{\hat q} = \dfrac{pr'}{2p - r'}, \end{cases}$$

which gives

$$\hat r = 4 + \frac{2pr'}{2p - r'} \quad \text{and} \quad \hat q = 2 + \frac{4(2p - r')}{pr'}.$$

Obviously $(\hat r, \hat q)$ satisfies the conditions of Lemma 2.4. Then, arguing as in the previous two cases we deduce

$$C\|w_{k,\delta_j}\|_{Q_T} \ge (\delta_{j+1} - \delta_j) \left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{pr'}{2p-r'}}\,\mathrm{d}t\right)^{\frac{1}{\hat r}}.$$

Taking $b = 2^{\hat r}$, $C_9 = (CC_8)^{\hat r}$,

$$\xi_j = \int_0^T |A_{\delta_j}(t)|^{\frac{pr'}{2p-r'}}\,\mathrm{d}t, \quad j \ge 0, \quad c = C_9\Big(\frac{2}{\rho}\|g\|_{L^r(0,T;L^p(\Omega))}\Big)^{\hat r},$$

we deduce from the above inequality and (4.15) that $\xi_{j+1} \le c b^j \xi_j^{\hat r \frac{2p-r'}{2pr'}}$. It is easy to check that $\hat r \frac{2p-r'}{2pr'} = 1 + \varepsilon$ with $\varepsilon > 0$. As in the previous case, we have to select ρ in such a way that $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$. First, we observe that $r \le 2$ and $r' < 2p$ imply that $\frac{pr'}{2p-r'} > 1$ holds. Then, using again that $|A_\rho(t)| \le \frac{1}{\rho^2}\|w_k(t)\|^2_{L^2(\Omega)}$ and (4.9) we get

$$\xi_0 \le \frac{1}{\rho^{\frac{2pr'}{2p-r'}}} \int_0^T \|w_k(t)\|_{L^2(\Omega)}^{\frac{2pr'}{2p-r'}} \,\mathrm{d}t \le \Big(\frac{1}{\rho}\|w_k\|_{Q_T}\Big)^{\frac{2pr'}{2p-r'}}$$
$$\le \Big[\frac{C_T}{\rho}\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\Big)\Big]^{\frac{2pr'}{2p-r'}}.$$

Taking

$$\rho = 4\max\{1, C_9^{\frac{1}{\hat r}} b^{\frac{1}{\varepsilon \hat r}}\}\big(\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))}$$
$$+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big),$$

we get that $\xi_0 \le c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$. Once again Lemma 2.5 implies that $|A_{2\rho}(t)| = 0$ and (4.13) holds for a constant K independent of T.

To conclude the proof of Case 3, we have to address the situation where $r \in (2,\infty)$. First of all, we observe that when $p = 1$ the estimates for w_k can be deduced from the estimates for $p > 1$. Indeed, from the interpolation inequality [5, page 93]

$$\|\varphi\|_{L^{\frac{4}{3}}(\Omega)} \le \|\varphi\|_{L^1(\Omega)}^{\frac{1}{2}} \|\varphi\|_{L^2(\Omega)}^{\frac{1}{2}},$$

we infer for every $T \le \infty$

$$\int_0^T \|g(t)\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4r}{2+r}} \,\mathrm{d}t \le \int_0^T \|g(t)\|_{L^1(\Omega)}^{\frac{2r}{2+r}} \|g(t)\|_{L^2(\Omega)}^{\frac{2r}{2+r}} \,\mathrm{d}t \le \|g\|_{L^r(0,T;L^1(\Omega))}^{\frac{2r}{2+r}} \|g\|_{L^2(Q_T)}^{\frac{2r}{2+r}}.$$

This leads to

$$\|g\|_{L^{\frac{4r}{2+r}}(0,T;L^{\frac{4}{3}}(\Omega))} \le \|g\|_{L^r(0,T;L^1(\Omega))}^{\frac{1}{2}} \|g\|_{L^2(Q_T)}^{\frac{1}{2}} \le \|g\|_{L^r(0,T;L^1(\Omega))} + \|g\|_{L^2(Q_T)}.$$

Thus, if $g \in L^r(0,T;L^1(\Omega)) \cap L^2(Q_T)$, then it belongs to $L^{\frac{4r}{2+r}}(0,T;L^{\frac{4}{3}}(\Omega))$ and the estimate of w_k in terms of $\|g\|_{L^{\frac{4r}{2+r}}(0,T;L^{\frac{4}{3}}(\Omega))}$ yields the estimate in terms of $\|g\|_{L^r(0,T;L^1(\Omega))} + \|g\|_{L^2(Q_T)}$. Therefore, we assume that $r \in (2,\infty)$ and $p \in (1,\infty)$. Then, using Hölder's inequality we infer

$$\Big(\int_0^t \|w_{k,\rho}(\tau)\|_{L^{p'}(\Omega)}^{r'} \,\mathrm{d}\tau\Big)^{\frac{1}{r'}} \le C_{10}\Big(\int_0^t \|w_{k,\rho}(\tau)\|_{W^{1,1}(\Omega)}^{r'} \,\mathrm{d}\tau\Big)^{\frac{1}{r'}}$$
$$\le C_{10}\Big(\int_0^T \|w_{k,\rho}(\tau)\|_{H^1(\Omega)}^{r'} |A_\rho(\tau)|^{\frac{r'}{2}} \,\mathrm{d}\tau\Big)^{\frac{1}{r'}}$$
$$\le C_{10}\|w_k\|_{L^2(0,T;H^1(\Omega))}\Big(\int_0^T |A_\rho(\tau)|^{\frac{r'}{2-r'}} \,\mathrm{d}\tau\Big)^{\frac{2-r'}{2r'}}.$$

Inserting this inequality in (4.10) we deduce

$$\|w_{k,\rho}\|_{Q_T} \le C_{11}\|g\|_{L^r(0,T;L^p(\Omega))}\Big(\int_0^T |A_\rho(t)|^{\frac{r'}{2-r'}} \,\mathrm{d}t\Big)^{\frac{2-r'}{2r'}}. \tag{4.16}$$

Now we take $\hat r$ and $\hat q$ as the solutions of the system

$$\begin{cases} \dfrac{1}{\hat r} + \dfrac{1}{2\hat q} = \dfrac{1}{4} \\[2mm] \dfrac{\hat r}{\hat q} = \dfrac{r'}{2-r'}. \end{cases}$$

Then, we have

$$\hat{r} = 4 + \frac{2r'}{2-r'} \quad \text{and} \quad \hat{q} = 2 + \frac{4(2-r')}{r'}.$$

Since $(\hat{r}, \hat{q})$ satisfies the assumptions of Lemma 2.4, arguing as in Case 1, we obtain

$$C\|w_{k,\delta_j}\|_{Q_T} \geq (\delta_{j+1} - \delta_j)\left(\int_0^T |A_{\delta_{j+1}}(t)|^{\frac{r'}{2-r'}}\,\mathrm{d}t\right)^{\frac{1}{\hat{r}}}.$$

Taking $b = 2^{\hat{r}}$, $C_{12} = (CC_{11})^{\hat{r}}$,

$$\xi_j = \int_0^T |A_{\delta_j}(t)|^{\frac{r'}{2-r'}}\,\mathrm{d}t, \quad j \geq 0, c = C_{12}\Big(\frac{2}{\rho}\|g\|_{L^r(0,T;L^p(\Omega))}\Big)^{\hat{r}},$$

we deduce from the above inequality and (4.16) that $\xi_{j+1} \leq cb^j \xi_j^{\frac{\hat{r}(2-r')}{2r'}}$. It is easy to check that $\frac{\hat{r}(2-r')}{2r'} = 1 + \varepsilon$ with $\varepsilon > 0$. Since $\frac{r'}{2-r'} \geq 1$, we can proceed as in the case $r \in [1,2]$ and get that $\xi_0 \leq c^{-\frac{1}{\varepsilon}} b^{\frac{-1}{\varepsilon^2}}$ for

$$\begin{aligned}\rho = 4\max\{1, C_{12}^{\frac{1}{\hat{r}}} b^{\frac{1}{\varepsilon \hat{r}}}\}\big(&\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))} \\ &+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big).\end{aligned}$$

Hence, Lemma 2.5 implies that $|A_{2\rho}(t)| = 0$ and (4.13) holds for a constant K independent of T.

V - End of proof. We observe that the equalities

$$\hat{d}_k(x,t,w_k(x,t)) = \hat{d}(x,t,w_k(x,t)) \quad \text{and} \quad \hat{b}_k(x,t,w_k(x,t)) = \hat{b}(x,t,w_k(x,t))$$

hold for every k satisfying

$$\begin{aligned}k \geq K\big(&\|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^r(0,T;L^p(\Omega))} \\ &+ C_T\big[\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} + \|h\|_{L^2(\Sigma_T)}\big] + M_{R,v}\big).\end{aligned}$$

Hence, w_k is the unique solution of (2.13) and, consequently, it is independent of k for k large enough. Thus, we can drop the index k from the notation w_k. In addition, if $w \in L^2(Q)$ then (4.1) follows from (4.7). To prove (4.2) we proceed as follows. In the established estimates of ξ_0 we replace $\|w_k\|_{Q_T}$ by $\|w\|_Q$ and use (4.1) instead of (4.9) in the definition of ρ, which leads to (4.2). Finally, if $u_0 \in C(\bar{\Omega})$, once again using [8] we deduce that $w \in C(\bar{Q}_T)$ for every $T < \infty$. Combining this and the fact $w \in L^\infty(Q)$ we infer that $w \in C(\bar{\Omega} \times [0,\infty))$. ∎

End of proof of Theorem 2.2. Setting $u = v + w$, we deduce from Theorems 3.1 and 4.1 that $u \in W(0,T) \cap L^\infty(Q_T)$ for every $T < \infty$ and u is the unique solution of (1.1). In addition, if $u \in L^2(Q)$, recalling that $v \in W(I) \cap L^\infty(Q)$, we deduce that $w \in L^2(Q)$ as well. Then (2.6) and (2.7) are consequence of (3.1), (3.2), (4.1), and (4.2) along with the following inequality

$$\|w\|_{L^2(Q)} \leq \|u\|_{L^2(Q)} + \|v\|_{L^2(Q)} \leq \|u\|_{L^2(Q)} + M_1\|h\|_{L^2(\Sigma)}.$$

The inequalities (2.8) and (2.9) are an immediate consequence of (2.1) and (2.2). If $u_0 \in C(\bar{\Omega})$, the property $u \in C(\bar{\Omega} \times [0,\infty))$ is consequence of the regularity $v, w \in C(\bar{\Omega} \times [0,\infty))$ established in Theorems 3.1 and 4.1. ∎

Remark 4.2. *In the case where* b *and* d *are globally Lipschitz, then the constant* M_3 *in* (4.1) *is independent of* h*. Indeed, in the proof of the estimate of* $\|w_k\|_{Q_T}$ *carried out in Step III of the previous demonstration, we have that* (4.6) *is replaced by* $|\hat{d}_k(x,t,\xi)| \le L(|\xi|+|v(x,t)|)$*, where* L *is the Lipschitz constant of* d*, hence* $C_{M_{R,v}} = L$*. The same applies to* b*. Then, in the proof of* (4.7) $\Omega_{M_{R,v}}(\tau)$ *and* $\Gamma_{M_{R,v}}(\tau)$ *are simply* Ω *and* Γ*, respectively. Then the constants in* (4.7)–(4.9) *are independent of* v *and, hence, of* h*. Therefore, the constant* M_3 *in* (4.1) *is independent of* h*. As a consequence,* K_1 *in* (2.6) *is also independent of* h*.*

5. Extensions and discussion

In this section we discuss special cases and extensions of the main Theorem 2.2.

5.1. **Global sign conditions on d and b.** Here we assume that d and b satisfy (2.1)-(2.2) and (2.3) with $R = 0$. For g and h we assume the same hypotheses made in section 2. Hence, we know that there exists a unique solution $u \in W(0,T) \cap L^\infty(Q_T)$ of (2.5) for every $T < \infty$. Testing (2.5) with u we obtain

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + C_a\|u\|^2_{L^2(0,t;H^1(\Omega))}\,\mathrm{d}\tau \le \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_{Q_t}[|\nabla u|^2 + au^2]\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_{Q_t} d(x,\tau,u)u\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Sigma_t} b(x,\tau,u)u\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_{Q_t} gu\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Sigma_t} hu\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \Big(\|g\|_{L^2(Q)} + C_\Omega\|h\|_{L^2(\Sigma)}\Big)\|u\|_{L^2(0,t;H^1(\Omega))}
\end{aligned}$$

From here we infer that

$$\|u\|_Q = \sup_{T>0}\|u\|_{Q_T} \le C_Q\Big(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} + \|h\|_{L^2(\Sigma)}\Big) \quad \forall T < \infty \tag{5.1}$$

with C_Q independent of (u_0, g, h) and (d, b). Then, $u \in L^2(Q)$ and estimate (2.7) can be replaced by

$$\begin{aligned}
\|u\|_{L^\infty(Q)} \le C_\infty\Big(&\|u_0\|_{L^\infty(\Omega)} \\
&+ \|g\|_{L^2(Q)} + \|g\|_{L^r(I;L^p(\Omega))} + \|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))}\Big),
\end{aligned} \tag{5.2}$$

where C_∞ depends on d, b, and monotonically increasing on $\|h\|_{L^2(\Sigma)} + \|h\|_{L^s(I;L^q(\Gamma))}$.

5.2. **Mixed boundary conditions.** Here we assume that Assume that Γ_D is a measurable subset of Γ with strictly positive measure and $\Gamma_N = \Gamma\setminus\Gamma_D$. We address the problem

$$\left\{\begin{array}{l}
\dfrac{\partial u}{\partial t} - \Delta u + d(x,t,u) = g \text{ in } Q, \\
u = h_D \text{ on } \Sigma_D,\ \partial_n u + b(x,t,u) = h_N \text{ on } \Sigma_N,\ u(0) = u_0 \text{ in } \Omega,
\end{array}\right. \tag{5.3}$$

where $\Sigma_D = \Gamma_D \times I$ and $\Sigma_N = \Gamma_N \times I$, $h_D \in \tilde{W}(I) = L^2(I; H^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ and $h_{D|_{\Sigma_D}} \in L^\infty(\Sigma_D)$, $h_N \in L^2(\Sigma_N) \cap L^s(I; L^q(\Gamma_N))$ with $\frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2}$, $g \in L^2(Q) \cap L^r(I; L^p(\Omega))$ with $\frac{1}{r} + \frac{n}{2p} < 1$, and $u_0 \in L^\infty(\Omega)$. For the nonlinear terms

d and b we assume the conditions (2.1)–(2.3). This equation can be analyzed by performing the decomposition into two problems

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = 0 \text{ in } Q, \\ v = h_D \text{ on } \Sigma_D,\ \partial_n v = h_N \text{ on } \Sigma_N,\ v(0) = 0 \text{ in } \Omega, \end{cases} \tag{5.4}$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + \hat{d}(x,t,w) = g \text{ in } Q, \\ w = 0 \text{ on } \Sigma_D,\ \partial_n w + \hat{b}(x,t,w) = 0 \text{ on } \Sigma_N,\ w(0) = u_0 \text{ in } \Omega, \end{cases} \tag{5.5}$$

with $\hat{d}(x,t,w) = d(x,t,v(x,t)+w)$ and $\hat{b}(x,t,w) = b(x,t,v(x,t)+w)$. The estimate for $\|v\|_{L^\infty(Q)}$ can be proved exactly as in Theorem 3.1 simply by selecting $\rho \geq \|h_D\|_{L^\infty(\Sigma_D)}$ and replacing the norms $\|h\|_{L^s(I;L^q(\Gamma))}$ and $\|h\|_{L^2(\Sigma)}$ by $\|h_N\|_{L^s(I;L^q(\Gamma_N))}$ and $\|h_N\|_{L^2(\Sigma_N)}$. In the particular case where $h_N = 0$, we have the estimate $\|v\|_{L^\infty(Q)} \leq \|h_D\|_{L^\infty(\Sigma_D)}$. Indeed, taking $\rho = \|h_D\|_{L^\infty(\Sigma)}$ and testing (5.4) with $v_\rho(x,t) = v(x,t) - \mathrm{Proj}_{[-\rho,+\rho]}(v(x,t))$ we get

$$\frac{1}{2}\|v_\rho(T)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega |\nabla v_\rho|^2 \,\mathrm{d}x\,\mathrm{d}t = 0,$$

where we have utilized that $v_\rho = 0$ on Σ_D. Therefore, $v_\rho \equiv 0$ and the estimate follows.

The proof of the estimate for the solution of (5.5) is the same as for the proof of Theorem 4.1. Finally, arguing as in the proof of Theorem 2.2 we get the estimate

$$\begin{aligned} \|u\|_{L^\infty(Q)} \leq K_2\Big(&\|u\|_{L^2(Q)} + \|u_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(Q)} + \|g\|_{L^r(I;L^p(\Omega))} \\ &+ \|h_D\|_{L^\infty(\Sigma_D)} + \|h_N\|_{L^2(\Sigma_N)} + \|h_N\|_{L^s(I;L^q(\Gamma_N))} + R\Big). \end{aligned}$$

We point out that the linear term au in the equation was necessary for the Neumann problem (1.1), but this is not the case if the measure of Γ_D is positive. However, it could be included in the equation by considering it as part of the definition of the nonlinear term d.

5.3. **General elliptic operators.** Let us consider the following operator in Q

$$Au = -\sum_{i,j=1}^n \partial_{x_j}(a_{ij}\partial_{x_i}u) + \sum_{j=1}^n a_j\partial_{x_j}u + au$$

with measurable coefficients depending on $(x,t) \in Q$, $a(x,t) \geq 0$ in Q, $a \not\equiv 0$, and

$$a_{i,j} \in L^\infty(Q_T),\ a, a_j^2 \in L^{\hat{r}}(0,T;L^{\hat{p}}(\Omega)) \text{ for } \frac{1}{\hat{r}} + \frac{n}{2\hat{p}} \leq 1 \quad \forall T < \infty.$$

Hence, we have that $A : W(0,T) \longrightarrow W(0,T)^*$ defines a continuous operator by the expression

$$\begin{aligned} \int_0^T \langle Au(t), v(t)\rangle_{H^1(\Omega)^*,H^1(\Omega)} \,\mathrm{d}t &= \sum_{i,j=1}^n \int_{Q_T} a_{ij}(x,t)\partial_{x_i}u\partial_{x_j}v \,\mathrm{d}x\,\mathrm{d}t \\ &+ \sum_{j=1}^n \int_{Q_T} a_j(x,t)\partial_{x_j}uv \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} a(x,t)uv \,\mathrm{d}x\,\mathrm{d}t; \end{aligned}$$

see [13, page 137].

We also assume that there exists a constant $C_a > 0$ independent of T such that

$$\int_0^T \langle Au(t), u(t)\rangle_{H^1(\Omega)^*, H^1(\Omega)}\,\mathrm{d}t \geq C_a \int_0^T \|u(t)\|^2_{H^1(\Omega)}\,\mathrm{d}t \quad \forall u \in W(0,T).$$

In this situation the Neumann condition in (1.1) is replaced by

$$\partial_{n_A} u = \sum_{i,j=1}^n \sum_{i=1}^n a_{ij}\partial_{x_i} u n_j.$$

Under these assumptions Theorem 2.2 again holds. Indeed, the reader can simply refer back to the proof of that theorem and observe that

$$\langle Au, u_\rho\rangle_{H^1(\Omega)^*, H^1(\Omega)} \geq \langle Au_\rho, u_\rho\rangle_{H^1(\Omega)^*, H^1(\Omega)}$$

for $u_\rho(x,t) = u(x,t) - \mathrm{Proj}_{[-\rho,+\rho]}(u(x,t))$.

5.4. **Finite horizon problems.** Looking at the proof of Theorem 2.2, we observe that the existence and uniqueness of a solution $u \in W(0,T) \cap L^\infty(Q_T)$ of (2.5) has been established for data $u_0 \in L^2(\Omega)$, $g \in L^r(0,T; L^p(\Omega))$, and $h \in L^s(0,T; L^q(\Gamma))$ with $\frac{1}{r} + \frac{n}{2p} < 1$ and $\frac{1}{s} + \frac{n-1}{2q} < \frac{1}{2}$. In this section we prove that the same result holds if we exchange the assumption (2.3) for the following one: there exists $\Lambda \geq 0$ such that

$$(d(x,t,u_2) - d(x,t,u_1))(u_2 - u_1) \geq -\Lambda(u_2-u_1)^2 \quad \forall u_1, u_2 \in \mathbb{R}. \tag{5.6}$$

This assumption is also imposed on b. The existence and uniqueness of a solution of (2.5) as well as the corresponding estimates can also be obtained if we assume (2.3) on d and (5.6) on b, or vice-reverse. However, here we assume that d and b satisfy (5.6) to show how this assumption is used. We take $\lambda = 1+\frac{1}{2}(1+K^2)(2\Lambda+1)^2$, where K is given in (2.11), and perform the usual change of variables $\tilde{u}(x,t) = \mathrm{e}^{-\lambda t}u(x,t)$ to transform equation (1.1) into

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \Delta\tilde{u} + (a+\lambda)\tilde{u} + \tilde{d}(x,t,\tilde{u}) = \tilde{g} \text{ in } Q_T, \\ \partial_n \tilde{u} + \tilde{b}(x,t,\tilde{u}) = \tilde{h} \text{ on } \Sigma_T,\ v(0) = u_0 \text{ in } \Omega, \end{cases} \tag{5.7}$$

where $\tilde{g} = \mathrm{e}^{-\lambda t}g$, $\tilde{h} = \mathrm{e}^{-\lambda t}h$,

$$\tilde{d}(x,t,\tilde{u}) = \mathrm{e}^{-\lambda t}d(x,t,\mathrm{e}^{\lambda t}\tilde{u}) \text{ and } \tilde{b}(x,t,\tilde{u}) = \mathrm{e}^{-\lambda t}b(x,t,\mathrm{e}^{\lambda t}\tilde{u}).$$

To analyze (5.7) we proceed as we did for (1.1) by decomposing the equation into two parts

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + (a+\lambda)v = 0 \text{ in } Q_T, \\ \partial_n v = \tilde{h} \text{ on } \Sigma_T,\ v(0) = 0 \text{ in } \Omega, \end{cases} \tag{5.8}$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + (a+\lambda)w + \hat{d}(x,t,w) = \tilde{g} \text{ in } Q_T, \\ \partial_n w + \hat{b}(x,t,w) = 0 \text{ on } \Sigma_T,\ w(0) = u_0 \text{ in } \Omega, \end{cases} \tag{5.9}$$

where $\hat{d}(x,t,w) = \tilde{d}(x,t,v(x,t)+w)$ and $\hat{b}(x,t,w) = \tilde{b}(x,t,v(x,t)+w)$, and obtain the solution to (5.7) as $\tilde{u} = v + w$.

The parameter λ will be fixed below. For the moment we simply assume that $\lambda > 0$. Hence, we can apply Theorem 3.1 to equation (5.8), even if $a = 0$, to deduce the existence of a unique solution $v \in W(0,T) \cap L^\infty(Q_T)$ with estimates depending only on $\tilde h$; see (3.1) and (3.2).

For the analysis of equation (5.9) we set

$$\hat d_k(x,t,w) = \hat d(x,t,\mathrm{Proj}_{[-k,+k]}(w)), \quad \hat b_k(x,t,w) = \hat b(x,t,\mathrm{Proj}_{[-k,+k]}(w)).$$

Using (2.2) and (5.6) we get for every $w \in \mathbb{R}$

$$\begin{aligned}
&\hat d_k(x,t,w)w = \mathrm{e}^{-\lambda t} d(x,t,\mathrm{e}^{\lambda t}[v(x,t) + \mathrm{Proj}_{[-k,+k]}(w)])w\\
&= \mathrm{e}^{-\lambda t}\Big(d(x,t,\mathrm{e}^{\lambda t}[v(x,t) + \mathrm{Proj}_{[-k,+k]}(w)]) - d(x,t,\mathrm{e}^{\lambda t}v(x,t))\Big)w\\
&+ \mathrm{e}^{-\lambda t} d(x,t,\mathrm{e}^{\lambda t}v(x,t))w\\
&\ge -\Lambda w^2 - L_v|v(x,t)||w| \ge -\frac{L_v^2}{2}v(x,t)^2 - (\Lambda + \frac{1}{2})w^2,
\end{aligned} \tag{5.10}$$

where we have applied (2.2) with $M = \mathrm{e}^{\lambda T}\|v\|_{L^\infty(Q_T)}$ and $L_v = L_M$ to get the above inequality. The same inequality holds for $\hat b_k$. As in the proof of Theorem 4.1 we deduce the existence and uniqueness of a solution $w_k \in W(0,T)$ of the equation

$$\left\{\begin{array}{l}
\dfrac{\partial w}{\partial t} - \Delta w + (a+\lambda)w + \hat d_k(x,t,w) = \tilde g \text{ in } Q_T,\\
\partial_n w + \hat b_k(x,t,w) = 0 \text{ on } \Sigma_T,\ w(0) = u_0 \text{ in } \Omega,
\end{array}\right. \tag{5.11}$$

Using (2.11) with $\varepsilon = \frac{1}{K(2\Lambda+1)}$ we get

$$\begin{aligned}
&\int_0^t\int_\Gamma \hat b_k(x,t,w_k)w_k \,\mathrm{d}x\,\mathrm{d}\tau\\
&\ge -\frac{L_v^2}{2}\int_0^t\int_\Gamma v^2\,\mathrm{d}x\,\mathrm{d}\tau - (\Lambda + \frac{1}{2})\int_0^t\int_\Gamma w_k^2\,\mathrm{d}x\,\mathrm{d}\tau \ge -\frac{L_v^2}{2}\int_0^t\int_\Gamma v^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&-\frac{1}{2}\int_0^t\int_\Omega|\nabla w_k|^2\,\mathrm{d}x\,\mathrm{d}\tau - \frac{1}{2}K^2(2\Lambda+1)^2\int_0^t\int_\Omega w_k^2\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Recalling the definition of λ, using (5.7) and the above inequality, and testing the equation (5.11) with w_k we infer

$$\begin{aligned}
&\frac{1}{2}\|w_k(t)\|^2_{L^2(\Omega)} + \int_0^t\|w_k(\tau)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau - \frac{L_v^2}{2}\int_0^t\left(\|v(\tau)\|^2_{L^2(\Omega)} + \|v(\tau)\|^2_{L^2(\Gamma)}\right)\mathrm{d}\tau\\
&\le \frac{1}{2}\|w_k(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t\int_\Omega[|\nabla w_k\|^2 + (a+\lambda)w_k^2]\,\mathrm{d}x\,\mathrm{d}\tau\\
&+ \int_0^t\int_\Omega \hat d_k(x,\tau,w_k)w_k\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Gamma \hat b_k(x,\tau,w_k)w_k\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \tilde g w_k\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

This yields with (3.1)

$$\begin{aligned}
&\|w_k(t)\|^2_{L^2(\Omega)} + 2\int_0^t\|w_k(\tau)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau\\
&\le \|u_0\|^2_{L^2(\Omega)} + \mathrm{e}^{2\lambda T}\Big(\|g\|^2_{L^2(Q_T)} + CL_v^2\|h\|^2_{L^2(\Sigma_T)}\Big) + \int_0^t\|w_k(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau.
\end{aligned}$$

Arguing as in the proof of Theorem 4.1 we infer the estimate (4.9) from the above inequality as well as the $L^\infty(Q)$ estimate independent of k. Thus $w = w_k$ is the solution of (5.9) for k large enough and $\tilde{u} = v + w$ solves (5.7). Finally, $u = \mathrm{e}^{\lambda t}\tilde{u} \in W(0,T) \cap L^\infty(Q)$ and the associated estimates hold.

References

1. N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations **4** (1979), no. 8, 827–868. MR 537465
2. N. Arada and J.-P. Raymond, *Dirichlet boundary control of semilinear parabolic equations part 1: Problems with no state constraints*, Applied Mathematics and Optimization **45** (2002), no. 2, 125–43.
3. E. Di Benedetto, *Degenerate parabolic equations*, Universitext, Springer–Verlag, New York, 1993.
4. F. Boyer and P. Fabrie, *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, Springer, New York, 2013.
5. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.
6. E. Casas, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control Optim. **35** (1997), no. 4, 1297–1327.
7. E. Casas and K. Kunisch, *Infinite horizon optimal control for a general class of semilinear parabolic equations*, Appl. Math. Optim. **88** (2023), Paper No. 47, 36.
8. E. Di Benedetto, *On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), no. 3, 487–535.
9. K. Disser, A. F. M. ter Elst, and J. Rehberg, *Hölder estimates for parabolic operators on domains with rough boundary*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 1, 65–79. MR 3676040
10. A.Z. Fino, M. Jleli, and B. Samet, *Blow-up and global existence for semilinear parabolic systems with space-time forcing terms*, Chaos Solitons Fractals **147** (2021), Paper No. 110982, 8. MR 4256158
11. P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston-London-Melbourne, 1985.
12. M. Jleli, T. Kawakami, and B. Samet, *Critical behavior for a semilinear parabolic equation with forcing term depending on time and space*, J. Math. Anal. Appl. **486** (2020), no. 2, 123931, 16.
13. O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, 1988.
14. H. Meinlschmidt and J. Rehberg, *Hölder estimates for non-autonomous parabolic problems with rough data*, Evol. Equ. Control Theory **5** (2016), no. 1, 147–184.
15. M. Pierre, *Global existence in reaction-diffusion systems with control of mass: a survey*, Milan J. Math. **78** (2010), no. 2, 417–455. MR 2781847
16. J. P. Raymond and H. Zidani, *Pontryagin's principle for state-constrained control problems governed by parabolic equations with unbounded controls*, SIAM Journal on Control and Optimization **36** (1998), no. 6, 1853–1879.
17. ______, *Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations*, Appl. Math. Optim. **39** (1999), no. 2, 143–177. MR 1665668
18. R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Math. Surv. and Monogr., vol. 49, American Mathematical Society, Providence, RI, 1997.
19. F. Tröltzsch, *Optimal control of partial differential equations. Theory, methods and Applications*, vol. 112, American Math. Society, Providence, 2010.
20. J. Zhang, V. Rădulescu, M. Yang, and Z. Jiazheng, *Global existence and blow-up solutions for a parabolic equation with critical nonlocal interactions*, Journal of Dynamics and Differential Equations (2023), 1–39.
21. J. Zhou, *Gobal existence and L^∞ estimates of solutions for a quasilinear parabolic system*, J. Nonlinear Sci. Appl. **3** (2010), no. 4, 245–255.

Department of Applied Mathematics and Computer Science, Universidad de Cantabria, 39005 Santander, Spain
Email address: eduardo.casas@unican.es

Institute for Mathematiques and Scientific Computing, University of Graz, A-8010 Graz, Austria
Email address: karl.kunisch@uni-graz.at