FEEDBACK STABILIZATION TO NON-STATIONARY SOLUTIONS
OF A CLASS OF REACTION DIFFUSION EQUATIONS OF
FITZHUGH-NAGUMO TYPE

TOBIAS BREITEN†, KARL KUNISCH‡, AND SÉRGIO S. RODRIGUES‡

Abstract. Stabilization to a trajectory for the monodomain equations, a coupled nonlinear
PDE-ODE system, is investigated. The results rely on stabilization of linear first-order in time
nonautonomous evolution equations combined with stabilizability results for the linearized mon-
odomain equations and a fixed point argument to treat local stabilizability of the nonlinear system.
Numerical experiments for feedback stabilization of reentry phenomena are included.

Key words. feedback stabilization, finite dimensional control, differential Riccati equation,
reaction diffusion equation.

AMS subject classifications. 35K57, 93D15, 93B52.

1. Introduction. Let Ω ⊂ ℝ^n, n ∈ {2, 3}, denote a bounded domain with
smooth boundary Γ = ∂Ω. Consider the following controlled coupled reaction-diffusion
system

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v, w) + f + Bu, \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial w}{\partial t} &= \gamma v - \delta w, \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \Gamma \times (0, \infty), \\
v(x, 0) &= v_0(x) \quad \text{and} \quad w(x, 0) = w_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(1.1)

where \( f = f(x,t) \) is an external forcing term, \( I_{\text{ion}}(v, w) \) is a non-monotone nonlinear
function, \( u = u(t) \in L^2((0, +\infty); \mathbb{R}^m) \) denotes a finite dimensional control and \( \nu \) is
the unit outward normal vector to \( \Gamma \). In electrophysiology, system (1.1) is known as
the monodomain equations, see e.g. [10, Section 12.3.3]. In this context, the variable
\( v = v(x, t) \) models the transmembrane electric potential of the human heart and
\( w = w(x, t) \) is a so-called gating variable. Some typical models for the ionic current
include the FitzHugh-Nagumo model

\[
I_{\text{ion}}^{\text{FN}}(v, w) = av^3 - bv^2 + cv + dw,
\]

(1.2)
as well as the Rogers-McCulloch model

\[
I_{\text{ion}}^{\text{RM}}(v, w) = av^3 - bv^2 + cv + dw,\]

(1.3)

where \( a, b, c, d \) are positive real constants. Besides leading to different linearizations
(see below), distinct dynamical behaviors can be observed for these two models. In
particular, a typical solution waveform of the FHN system includes negative values
for the potential \( v \), see [20]. This unphysiological undershoot does not appear for the
bilinear coupling used in the Rogers-McCulloch model.

†Institute for Mathematics and Scientific Computing, Karl-Franzens-Universität, Heinrichstr. 36,
8010 Graz, Austria (tobias.breiten@uni-graz.at).
‡Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian
Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria (karl.kunisch@uni-graz.at,
sergio.rodrigues@oeaw.ac.at)
Our interest in studying (optimal) control problems for the monodomain equations has several reasons. The specific PDE-ODE structure of \((1.1)\) poses a significant mathematical challenge on its own right. To some extent, this is due to rather unexpected phenomena such as reentry waves, where wave phenomena are usually attributed to hyperbolic equations. A further notable property concerns the linearized version of \((1.1)\). As shown in [4], in contrast to other parabolic equations, the spectrum is no longer discrete and, as a consequence, the system is not exactly null controllable. Also from a practical point of view, the monodomain equations are of interest since \((1.1)\) allows to model fibrillation processes of the human heart. The control \(u(t)\) here can be interpreted as an external stimulus resembling a defibrillation process, see [13,17].

With this in mind, assume that a desired heart rhythm is given as the solution of the uncontrolled system

\[
\begin{align*}
\frac{\partial \bar{v}}{\partial t} &= \Delta \bar{v} - I_{\text{ion}}(\bar{v}, \bar{w}) + f, \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial \bar{w}}{\partial t} &= \gamma \bar{v} - \delta \bar{w}, \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial \bar{v}}{\partial \nu} &= 0, \quad \text{on } \Gamma \times (0, \infty), \\
\bar{v}(x,0) &= \bar{v}_0(x) \text{ and } \bar{w}(x,0) = \bar{w}_0(x), \quad \text{in } \Omega.
\end{align*}
\]

The goal of this paper is to design a feedback control law of the form

\[
u(t) = k(v - \bar{v}, w - \bar{w})
\]

such that the solution \((v, w)\) of \((1.1)\) converges exponentially to the solution \((\bar{v}, \bar{w})\) of \((1.4)\) provided that \(\| (v_0, w_0) - (\bar{v}_0, \bar{w}_0) \|\) is small enough. For this, we consider the difference of \((1.1)\) and \((1.4)\) as an infinite dimensional time varying control system of the form

\[
\dot{\tilde{z}}(t) = A(t) \tilde{z}(t) + F(\tilde{z}) + Bu(t), \quad \tilde{z}(0) = \tilde{z}_0,
\]

where

\[
\tilde{z} := (z_v, z_w) = (v - \bar{v}, w - \bar{w}).
\]

For the sake of illustration, let us consider the Rogers-McCulloch model \((1.3)\). We obtain

\[
\begin{align*}
\frac{\partial z_v}{\partial t} &= \Delta z_v - cz_v - a(v^3 - \bar{v}^3) + b(v^2 - \bar{v}^2) - d(vw - \bar{v}\bar{w}) + Bu, \\
\frac{\partial z_w}{\partial t} &= \gamma z_v - \delta z_w,
\end{align*}
\]

which, using that

\[
\begin{align*}
vw - \bar{v}\bar{w} &= (z_v + \bar{v})(z_w + \bar{w}) - \bar{v}\bar{w} = z_v z_w + \bar{v} z_w + \bar{w} z_v, \\
v^3 - \bar{v}^3 &= (y_v + \bar{v})^3 - \bar{v}^3 = z_v^3 + 3\bar{v}z_v^2 + 3\bar{v}^2z_v, \\
v^2 - \bar{v}^2 &= (z_v + \bar{v})^2 - \bar{v}^2 = z_v^2 + 2\bar{v}z_v,
\end{align*}
\]

leads to the time-varying control system

\[
\dot{\tilde{z}}(t) = A^{RM}(t) \tilde{z}(t) + F^{RM}(t, \tilde{z}) + Bu(t), \quad \tilde{z}(0) = \tilde{z}_0,
\]
where the operators $A^{RM}$ and $F^{RM}$ are given as
\[
A^{RM}(t)\vec{z} = \begin{pmatrix}
\Delta - (3a\bar{v}^2 - 2\bar{v}b + c + d\bar{w}) & -d\bar{v} \\
\gamma & -\delta
\end{pmatrix}
\begin{pmatrix}
z_v \\
z_w
\end{pmatrix},
\]
\[
F^{RM}(t, \vec{z}) = \begin{pmatrix}
-a\bar{v}^3 - (-b + 3a\bar{v})z_v^2 - d\bar{v}z_w \\
0
\end{pmatrix},
B u = \begin{pmatrix}
Bu \\
0
\end{pmatrix},
\]
(1.7)

Analogously, for the FitzHugh-Nagumo model we obtain
\[
A^{FN}(t)\vec{z} = \begin{pmatrix}
\Delta - (3a\bar{v}^2 - 2\bar{v}b + c + d\bar{w}) & -d\bar{v} \\
\gamma & -\delta
\end{pmatrix}
\begin{pmatrix}
z_v \\
z_w
\end{pmatrix},
\]
\[
F^{FN}(t, \vec{z}) = \begin{pmatrix}
-a\bar{v}^3 - (-b + 3a\bar{v})z_v^2 \\
0
\end{pmatrix},
B u = \begin{pmatrix}
Bu \\
0
\end{pmatrix}.
\]
(1.8)

The feedback stabilization approach to (1.5) will mainly consist in two nested subproblems. In the first one, similar to the approach taken in [2, 3, 12], we focus on the linearized system, arising from (1.5), which is given by
\[
\dot{\vec{z}}(t) = A(t)\vec{z}(t) + Bu(t), \quad \vec{z}(0) = \vec{z}_0.
\]
(1.9)

In the second, the inner, subproblem, we decouple the PDE part of the system, i.e. we consider the (1, 1) block of (1.9), for which we study a stabilization problem together with an associated differential Riccati equation. In this way, we can compensate for the lack of null controllability of the coupled linear system (1.9), see [4, Section 2.2] and [8, Theorem 1.2 and Remark 1.3].

The feedback law we will be based on an infinite-horizon optimal control problem associated with (1.9), for which we shall use the cost functional
\[
J(u) = \frac{1}{2} \left( \int_0^\infty |M\vec{z}|^2 + |R\dot{\vec{z}}| \right) dt,
\]
(1.10)

where the particular structure of the pair $(M, R)$ will be specified subsequently.

The structure of the paper is as follows. In Section 2 we investigate stabilization to zero for a system of the form
\[
\frac{\partial z}{\partial t} + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + Bu = 0,
\]
(1.11a)
\[
\frac{\partial z}{\partial \nu} = 0, \quad z(0) = z_0.
\]
(1.11b)

which can be seen to contain the linearizations of the Rogers-McCulloch and the FitzHugh-Nagumo nonlinearities as special cases. These results will provide the stabilization of the decoupled (1, 1) block of (1.9) described above. In Section 3 it will be shown that under suitable assumptions on the system parameters, the obtained feedback formula is shown to stabilize the linearized PDE-ODE system, resulting from the Rogers-McCulloch and the FitzHugh-Nagumo nonlinearities. In Section 4, we show the local exponential stabilization of the full nonlinear system. The theoretical results are illustrated by means of different numerical examples in Section 5.

**Notation.** We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_a := (a, +\infty)$ for all $a \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}_0$, a bounded domain with a smooth boundary $\Gamma = \partial \Omega$. 3
Given a function \( v: (t, x_1, x_2, \ldots, x_n) \mapsto v(t, x_1, x_2, \ldots, x_n) \in \mathbb{R} \), defined in an open subset of \( \mathbb{R} \times \Omega \), its partial time derivative \( \frac{\partial v}{\partial t} \) will be denoted by \( \partial_t v \), and its normal derivative \( \frac{\partial v}{\partial n} \) at the boundary will be denoted \( \partial_n v |_{\Gamma} \).

We use the standard notation for Bochner spaces \( L^p(I, X) \) where \( I \subseteq \mathbb{R} \), and \( X \) is a Banach space. The Lebesgue spaces \( L^p(\Omega)^m \) will be denoted by simply \( L^p \) whenever there is no ambiguity concerning the superscript \( m \in \mathbb{N}_0 \).

Given an open interval \( I \subseteq \mathbb{R} \), and Banach spaces \( X \) and \( Y \), then we write \( W(I, X, Y) := \{ f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y) \} \), where the derivative \( \partial_t f \) is taken in the sense of distributions. This space is endowed with the natural norm \( |f|_{W(I, X, Y)} \). The inclusion \( X \hookrightarrow Y \) will be denoted by \( \mathcal{L}(X, Y) \). In case \( X = Y \) we write \( \mathcal{L}(X) := \mathcal{L}(X, X) \) instead. If the inclusion \( X \subseteq Y \) is continuous, we write \( X \hookrightarrow Y \); we write \( X \overset{d}{\rightarrow} Y \), respectively \( X \overset{c}{\rightarrow} Y \), if the inclusion is also dense, respectively compact. The kernel and range \( \text{Ker} A := \{ x \in Z \mid Ax = 0 \} \) and \( \text{Ran} A := \{ Ax \in X \} \), respectively.

Given a function \( v \), take cost functionals different from those in [2, 12], with respect to the state. In Lemma 2.6 we give a property of global solutions of (1.11) which will allow us to take advantage of the results obtained in [2, 12] for the systems of the form (1.11). We can take advantage of the results obtained in [2, 12] for optimal control theoretic tools here we follow a functional analytic approach. Finally over in exploiting the relation between null controllability of (1.11) and observability conditions. Here we deal with homogeneous Neumann boundary conditions. Moreover in exploiting the relation between null controllability of (1.11) and observability of its adjoint we also follow a different procedure. While the one in [2, 12] is based on optimal control theoretic tools here we follow a functional analytic approach. Finally in Lemma 2.6 we give a property of global solutions of (1.11) which will allow us to take cost functionals different from those in [2, 12], with respect to the state.

2. Stabilization for parabolic equations with homogeneous Neumann boundary conditions. The section is devoted to the stabilization to zero for systems of the form (1.11). We can take advantage of the results obtained in [2, 12] for Oseen–Burgers and Oseen–Stokes equations under homogeneous Dirichlet boundary conditions. Here we deal with homogeneous Neumann boundary conditions. Moreover in exploiting the relation between null controllability of (1.11) and observability of its adjoint we also follow a different procedure. While the one in [2, 12] is based on optimal control theoretic tools here we follow a functional analytic approach. Finally in Lemma 2.6 we give a property of global solutions of (1.11) which will allow us to take cost functionals different from those in [2, 12], with respect to the state.

2.1. Some regularity results. We start by deriving some regularity results for

\[
\begin{align*}
\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + f &= 0, \quad (2.1a)\\
\partial_n z |_{\Gamma} &= 0, \quad z(0) = z_0. \quad (2.1b)
\end{align*}
\]

in the form which will be required further below.

For simplicity we denote \( H = L^2 \), and \( V = H^1(\Omega) \). We consider \( H \) as the pivot space and define the operator \( A: V \rightarrow V' \) by \( (Au, v)_V := (f, v)_H \). We have that, \( V \overset{d}{\rightarrow} H \overset{d}{\rightarrow} V' \), and if \( (f, v) \in H \times V \) we have \( (f, v)|_{V'} := (f, v)_H \).

By the Lax-Milgram lemma (cf. [23, Section II.2.1, Theorem 2.1]), \( A: V \rightarrow V' \) is a bijective isometry. The domain \( D(A) \) of \( A \) is defined as \( D(A) := \{ z \in H \mid (\Delta + 1)z \in H \} \). Let \( \text{Lem} \text{ma} \text{ } 2.1 \text{. We have that} \text{ } D(A) = \{ z \in H \mid (\Delta + 1)z \in H \text{ and } \partial_n z |_{\Gamma} = 0 \} = \{ z \in H^2(\Omega) \mid \partial_n z |_{\Gamma} = 0 \} \) and the norms \( z \mapsto |z|_{H^2(\Omega)} \) and \( z \mapsto |Az|_H = |(\Delta + 1)z|_H \) are equivalent in \( D(A) \).

Proof. We can derive the above identities by following the arguments in [23, Section II.2.2, Example 2.5]. In order to check the equivalence of the norms, let us fix \( z \in D(A) \). Clearly \( |(\Delta + 1)z|_H \leq C \| z \|_{H^2(\Omega)} \). From [21, Chapter 5, Proposition 7.2] we also have \( |z|_{H^2(\Omega)}^2 \leq C \| (\Delta z)_{\text{div}} \|_H^2 + |\nabla z|_L^2 \). This implies \( |z|_{H^2(\Omega)}^2 \leq C \| (\Delta + 1)z\|_H^2 \). For \( m \in \mathbb{N}_0 \), in order to simplify the writing we denote
\[ W^{j,m} := L^\infty(J, L^\infty(\Omega, \mathbb{R}^m)) = L^\infty(J \times \Omega, \mathbb{R}^m), \]
\[ W^m := L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^m), \]

where \( J \subseteq (0, +\infty) \) is an open interval. In the case \( m = 1 \), we will omit the superscript ‘\( m \)’.

The notation for the interval \( I = (s_0, s_1) \) with \( 0 \leq s_0 < s_1 \) is fixed throughout the paper, and its length is denoted by \( |I| \). We also fix \( \rho \) and \( \sigma \), which may depend on time and space, and a constant \( C_W \geq 0 \), satisfying
\[
|\rho|_{W^2} + |\sigma|_{W^2} \leq C_W. \tag{2.3}
\]

**Lemma 2.2.** Given \( f \in L^2(I, V') \) and \( z_0 \in H \), there is a weak (variational) solution \( z \in W(I, V, V') \) for (2.1). Moreover \( z \) is unique and depends continuously on the data:
\[
|z|^2_{W(I, V, V')} \leq C_{|I|, C_W} \left( |z(0)|^2_H + |f|^2_{L^2(I, V')} \right).
\]

**Proof.** While this result can be found in the literature we will provide a proof since the explicit estimates will be used later on.

Weak solutions for system (2.1) are understood in the variational sense. We restrict ourselves to the derivation of some a priori (like) estimates. In fact those estimates will also hold for Galerkin approximations of the system, for example using a basis of eigenfunctions of the operator \( A = \Delta - 1 \), thus the estimates can be used to precisely derive the existence of weak solutions. For more details on the procedure we refer to [15, Chapter 1, Section 6], [22, Chapter 1, Section 3], and [24, Chapter 3, Sections 1.3, 1.4, and 3.2].

By standard arguments, multiplying (2.1a) by \( 2z \), formally we find that
\[
\frac{d}{dt} |z|^2_H + 2|z|^2_V \leq 2|\rho|_{L^\infty} |z|^2_H + 2|\sigma \cdot \nabla z|_H |z|_H + 2|f|_{V'} |z|_V,
\]
and since \( |\sigma \cdot \nabla z|^2_H \leq 3|\sigma|^2_{L^\infty} |\nabla z|^2_{L^2} \), we find
\[
\frac{d}{dt} |z|^2_H + |z|^2_V \leq 2|\rho|_{L^\infty} |z|^2_H + 6|\sigma|^2_{L^\infty} |z|^2_H + 2|f|^2_{V'} . \tag{2.4}
\]

By the Gronwall inequality it follows that for all \( s \in I \),
\[
|z(s)|^2_H \leq e^{(2|\rho|_{W^1} + 6|\sigma|^2_{W^1, H})(s-s_0)} \left( |z(0)|^2_H + 2|f|^2_{L^2(I, V')} \right) \tag{2.5}
\]
and, integrating (2.4),
\[
|z(s)|^2_H + |z|^2_{L^2((s_0,s), V')} \leq |z(s_0)|^2_H + (2|\rho|_{W^1} + 6|\sigma|^2_{W^1, H}) |z|^2_{L^2(I, H)} + 2|f|^2_{L^2(I, V')} . \tag{2.6}
\]

From (2.1a) and \( H \hookrightarrow V' \), with \( |.|_H \leq |.|_H \), we also have
\[
|\partial_t z|^2_{L^2(I, V')} \leq |z|^2_{L^2(I, V') + |\rho z + \sigma \cdot \nabla z|_{L^2(I, H)} + |f|^2_{L^2(I, V')} \]
from which, using (2.5) and (2.6) we can conclude that
\[
|z|^2_{W(I, V, V')} \leq C_{s_1-s_0, |\rho|_{W^1}, |\sigma|^2_{W^1, H}} \left( |z(0)|^2_H + |f|^2_{L^2(I, V')} \right).
\]
Finally the uniqueness of $z$, follows from the fact that if $\tilde{z}$ is another weak solution, then $\delta z = z - \tilde{z}$, solves (2.1) with $\delta z(s_0) = 0$ and $f = 0$. From (2.5) it will follow that $|\delta z(s)|_H = 0$ for all $s \in I$. □

**Lemma 2.3.** Given $f \in L^2(I, H)$ and $z_0 \in V$, there is a strong solution $z \in W(I, D(A), H)$ for system (2.1), which depends continuously on the data:

$$|z|^2_{W(I, D(A), H)} \leq C_{||I||, C_{W}} \left( |z(s_0)|_V^2 + |f|^2_{L^2(I, H)} \right).$$

**Proof.** Multiplying (2.1a) by $2(-\Delta + 1)z$, formally we find that

$$\frac{d}{dt} |z|^2_V + 2 |z|^2_{D(A)} \leq 2 |\rho|_{L^\infty} |z|_H |z|_{D(A)} + 2 |\sigma \cdot \nabla z|_H |z|_{D(A)} + 2 |f|_H |z|_{D(A)},$$

which implies

$$\frac{d}{dt} |z|^2_V + |z|^2_{D(A)} \leq 3 |\rho|_{L^\infty} |z|^2_H + 9 |\sigma|_{W^{1,\infty}}^2 |z|^2_V + 3 |f|^2_H.$$

Thus, for all $s \in I$,

$$|z(s)|_H^2 \leq e^{9|\sigma|^2_{W^{1,\infty}}(s-s_0)} \left( |z(s_0)|_V^2 + 3(s-s_0) |\rho|_{W^1}^2 + 3 |f|^2_{L^2(I, H)} \right);$$

$$|z(s)|_V^2 + |z|^2_{L^2((s_0, s), W^{1,2})} \leq |z(s_0)|_V^2 + 3 |\rho|_{W^1}^2 |z|^2_{L^2((s_0, s), H)} + 9 |\sigma|_{W^{1,\infty}}^2 |z|^2_{L^2((s_0, s), V)} + 3 |f|^2_{L^2(I, H)}.$$

From (2.1a) we also have

$$|\partial_t z|^2_{L^2(I, H)} \leq |z|^2_{L^2(I, D(A))} + |\rho z + \sigma \cdot \nabla z|_{L^2(I, H)} + |f|_{L^2(I, H)}$$

and we can conclude that

$$|z|^2_{W(I, D(A), H)} \leq C_{|s_1-s_0|, |\rho|_{W^{1,\infty}}, |\sigma|_{W^{1,\infty}}} \left( |z(s_0)|_V^2 + |f|^2_{L^2(I, H)} \right),$$

which ends the proof. □

The next lemma shows a certain smoothing property of system (2.1).

**Lemma 2.4.** Given $f \in L^2(I, H)$ and $z_0 \in H$, let $z$ be the weak solution for system (2.1). Then $y(t) := (t-s_0)z(t)$ is in $W(I, D(A), H)$ and satisfies the estimates

$$|y|^2_{W(I, D(A), H)} \leq C_{||I||, C_{W}} \left( (s_1-s_0)^2 |f|^2_{L^2(I, H)} + |z|^2_{L^2(I, H)} \right)$$

$$\leq C_{||I||, C_{W}} \left( |z(s_0)|_H^2 + |f|^2_{L^2(I, H)} \right),$$

**Proof.** Notice that $y(t) = (t-s_0)z(t)$ solves (2.1a) with $g = g(t) = (t-s_0)f(t) + z(t)$ in place of $f$, and $y(s_0) = 0$. Hence by Lemma 2.3,

$$|y|^2_{W(I, D(A), H)} \leq C_{||I||, C_{W}} |g|^2_{L^2(I, H)},$$

and using (2.5),

$$|z|^2_{L^2(I, H)} \leq C_{||I||, C_{W}} (s_1 - s_0) \left( |z(s_0)|_H^2 + |f|^2_{L^2(I, V)} \right).$$
which implies that \( |y|_{W(I, D(A), H)}^2 \leq \overline{C}_{[l, C_w]} \left( |z(s_0)|_{H}^2 + |f|_{L^2(I, H)}^2 \right) \). \( \Box \)

**Definition 2.5.** For \( f \in L^2_{L^2}(\mathbb{R}_{s_0} \times \Omega) \) and \( y_0 \in H \) the function \( z \) defined in \( \mathbb{R}_{s_0} \times \Omega \) by the property that \( z_{|[s_0, \tau)} \) coincides with the weak solution of (2.1) in \( (s_0, \tau) \), for all \( \tau > s_0 \) is well defined. It is called the global weak solution of (2.1) in \( \mathbb{R}_{s_0} \times \Omega \).

We have the following property for the solutions of (2.1) on the infinite time interval \( \mathbb{R}_{s_0} = (s_0, +\infty) \), \( s_0 \geq 0 \).

**Lemma 2.6.** For \( f \in L^2(\mathbb{R}_{s_0}, V') \) and \( z_0 \in H \), let \( z \) be the global weak solution of (2.1) in \( \mathbb{R}_{s_0} \), with \( z(s_0) = z_0 \). If \( z \in L^2(\mathbb{R}_{s_0}, H) \), then \( z \in W(\mathbb{R}_{s_0}, V, V') \), and

\[
|z|_{W(\mathbb{R}_{s_0}, V, V')} \leq \overline{C}_{[C_w]} \left( |z(s_0)|_{H}^2 + |f|_{L^2(\mathbb{R}_{s_0}, V')}^2 + |z|_{L^2(\mathbb{R}_{s_0}, H)}^2 \right) .
\]

(2.10)

**Proof.** Integrating (2.4) over \((s_0, \tau)\), we find

\[
|z(\tau)|_{H}^2 + |z|_{L^2((s_0, \tau), V')}^2 \leq |z(s_0)|_{H}^2 + \overline{C}_{[C_w]} |z|_{L^2((s_0, \tau), H)}^2 + 2 |f|_{L^2((s_0, \tau), V')}^2 ,
\]

which leads us to

\[
|z|_{L^2(\mathbb{R}_{s_0}, V')}^2 \leq |z(s_0)|_{H}^2 + \overline{C}_{[C_w]} |z|_{L^2(\mathbb{R}_{s_0}, H)}^2 + 2 |f|_{L^2(\mathbb{R}_{s_0}, V')}^2 .
\]

(2.11)

Finally, from (2.1a) is follows also that

\[
|\partial_t z|_{L^2(\mathbb{R}_{s_0}, V')} \leq \overline{C}_{[C_w]} \left( |z(s_0)|_{H}^2 + |z|_{L^2(\mathbb{R}_{s_0}, V')}^2 + |f|_{L^2(\mathbb{R}_{s_0}, V')}^2 \right) ,
\]

which, together with (2.11), gives us (2.10). \( \Box \)

**2.2. Null controllability.** Here we recall the relation between null controllability of system (2.1) and a suitable observability inequality for the adjoint system.

Consider, in the bounded cylinder \( I \times \Omega \), the controlled system

\[
\begin{alignat}{2}
\partial_t z + (\Delta + 1) z + \rho z + \sigma \cdot \nabla z + Bu &= 0, \\
\partial_t z |_{\Gamma} &= 0, \quad z(s_0) = z_0,
\end{alignat}
\]

(2.12a)

(2.12b)

where \( u \in L^2(I, H) \) and \( B \in \mathcal{L}(H) \), with adjoint denoted by \( B^\ast \). Let us also consider in \( I \times \Omega \) the adjoint system

\[
\begin{alignat}{2}
- \partial_t q + (\Delta + 1) q + \rho q - \nabla \cdot (q \sigma) &= 0, \\
(q \sigma \cdot \nu)|_{\Gamma} &= 0, \quad q(s_1) = q_1 \in H,
\end{alignat}
\]

(2.13a)

(2.13b)

and let \( z(q, u)(t) := z(t) \) and \( q(q_1)(t) := q(t) \) denote the solutions of (2.12) and (2.13), for given data \((z_0, u)\) and \( q_1 \), respectively.

Weak solutions \( q \in W(I, V, V') \) for system (2.13) are understood again in the variational sense as in [7]. In [7, Section 2] weak solutions are asked to be in \( L^2(I, V) \cap C([s_0, s_1], H) \), but from \( \rho \in L^2(I, H) \) and \( \sigma q \in L^2(I, H^\ast) \) we can obtain that the variational solution is indeed in the space \( W(I, V, V') \).

Let \( z(\cdot) = z(z_0, u)(\cdot) \) and \( q(\cdot) = q(q_1)(\cdot) \) solve (2.12) and (2.13), respectively.

**Definition 2.7.** (i) We say that (2.12) is null controllable in \( I \) if there exists a family \( \{u(z_0) \mid z_0 \in H\} \subset L^2(I, H) \) such that \( z(z_0, u(z_0))(s_1) = 0 \), for \( z_0 \in H \).
(ii) We say that (2.13) is observable in $I$ if there exists a constant $C_2 > 0$ such that for all $q_1 \in H$ we have that the corresponding weak solution $q$ satisfies the inequality

$$|q(q_1)(s_0)|_H \leq C_2 |B^* q(q_1)|_{L^2(I, H)}.$$  

(2.14)

The constant $C_2$ in (2.14) depends, in general, on $\Omega$, $\omega$, $I$, $B$, and on the coefficients $\rho$ and $\sigma$.

**Lemma 2.8.** System (2.13) is observable in $I$ if, and only if, system (2.12) is null controllable in $I$ and the family of controls $\{u(z_0) \mid z_0 \in H\}$ is a bounded linear function of $z_0$:

$$|u(z_0)|_{L^2(I, H)} \leq C_2 |z_0|_H,$$  

where $C_2$ is as in (2.14).

**Proof.** From [24, Chapter 3, Section 1.4, Lemma 1.2], we can write

$$\frac{d}{dt}(z, q)_H = \frac{1}{2} \left( |z + q|^2 - |z|^2 - |q|^2 \right) = \langle \partial_t (z + q), z + q \rangle_{V^\prime, V} - \langle \partial_t z, z \rangle_{V^\prime, V} - \langle \partial_t q, q \rangle_{V^\prime, V},$$

for a.e. $t \in I$, and therefore

$$\frac{d}{dt}(z, q)_H = \langle \partial_t z, q \rangle_{V^\prime, V} + \langle z, \partial_t q \rangle_{V, V^\prime} = (-Bu, q)_H,$$  

(2.15)

$$\langle z(s_1), q_1 \rangle_H - (z_0, q(s_0))_H = -\int_{s_0}^{s_1} (u(s), B^* q(s))_H \, ds.$$  

(2.16)

Thus if there is a family $u = u(z_0) \in L^2(I, H)$ with $|u(z_0)|_{L^2(I, H)} \leq C_2 |z_0|_H$ such that $z(z_0, u)(s_1) = 0$, then we find

$$\langle z_0, q(q_1)(s_0) \rangle_H \leq C_2 |z_0|_H |B^* q(q_1)|_{L^2(I, H)},$$

for all $z_0 \in H$, that is, $|q(q_1)(s_0)|_H \leq C_2 |B^* q(q_1)|_{L^2(I, H)}$.

On the other hand, if there is $C_2 > 0$ such that $|q(q_1)(s_0)|_H \leq C_2 |B^* q(q_1)|_{L^2(I, H)}$, then null controllability in $I$ of (2.12) can be proven by the following arguments (see, e.g., [5, Chapter 2]). We note that the literature typically considers the case of autonomous systems but this does not change the proof (cf. [1, Section 2]). Let us define the mappings

$$\mathcal{F}: L^2(I, H) \to H \quad u \mapsto z(0, u)(s_1) \quad \text{and} \quad \mathcal{G}: H \to H \quad z_0 \mapsto z(z_0, 0)(s_1).$$

From (2.16), we have

$$\langle \mathcal{F}u, q_1 \rangle_H = -(u, B^* q(q_1))_{L^2(I, H)},$$

$$\langle \mathcal{G}z_0, q_1 \rangle_H = (z_0, q(q_1)(s_0))_H,$$

which show that the adjoints of $\mathcal{F}$ and $\mathcal{G}$ are given, respectively, by

$$\mathcal{F}^*: H \to L^2(I, H) \quad q_1 \mapsto -B^* q(q_1) \quad \text{and} \quad \mathcal{G}^*: H \to H \quad q_1 \mapsto q(q_1)(s_0).$$

Now we can write the observability inequality (2.14) as

$$|\mathcal{G}^* q_1|_H \leq C_2 |\mathcal{F}^* q_1|_{L^2(I, H)},$$  

(2.17)
and from $z(0, u(z_0))(s_1) = F u(z_0) + G z_0$ we can conclude that null controllability of system (2.12) holds if, and only if,

$$\text{Ran} \mathcal{G} \subseteq \text{Ran} \mathcal{F}$$

(2.18)

and by Lemma 2.48 in [5, Section 2.3.2] (cf. Theorem 2.2 in [25, Chapter 2]), we have that (2.18) is equivalent to (2.17).

It remains to prove that the family $\{u(z_0) \mid z_0 \in H\}$ can be chosen as a linear and continuous mapping of $z_0$, with $|u(z_0)|_{L^2(I, H)} \leq C_2 |z_0|_{H}$. This fact follows also from Lemma 2.48 in [5, Section 2.3.2], which states the existence of a mapping $\mathcal{F}^* \in \mathcal{L}(H \to L^2(I, H))$ such that $\mathcal{G} = \mathcal{F} \mathcal{F}^*$ and $|\mathcal{F}^*|_{\mathcal{L}(H \to L^2(I, H))} \leq C_2$ with $C_2$ as in (2.17). The family $\{u(z_0) \mid z_0 \in H\}$ is constructed by setting $u \in \mathcal{L}(H \to L^2(I, H))$:

$$u(z_0) := -\mathcal{F}^* z_0.$$  

Notice that $z_0, u(z_0))(s_1) = z(0, 0)(s_1) + z(0, u(z_0))(s_1) = G z_0 - \mathcal{F} \mathcal{F}^* z_0 = 0$, and thus this choice of control (also) provides the desired null controllability, and we have the announced inequality $|u(z_0)|_{L^2(I, H)} \leq C_2 |z_0|_{H}$. $\Box$

**Controls supported in a subset.** From now on, we will deal with controls supported in any given open subset $\omega \subseteq \Omega$. From [7] we know that in the case we take $B = 1_\omega \in \mathcal{L}(H)$ with

$$1_\omega u(x) := \begin{cases} u(x), & \text{if } x \in \omega, \\ 0, & \text{if } x \in \Omega \setminus \overline{\omega}. \end{cases}$$

we have that (2.13) is observable and (2.12) is null controllable. More precisely we know (cf. [7, Theorem 2]) that the following theorem holds true

**Theorem 2.9.** Let $B = 1_\omega$ and let $I = (s_0, s_1)$ be arbitrary, then, there exists a family $\{u(z_0) \mid z_0 \in H\} \subseteq L^2(I, H)$ such that the solutions $z_0, u(z_0)$ to (2.12) satisfy $z(0, u(z_0))(s_1) = 0$ and, for a constant $C = C(\omega, \Omega)$, we have that

$$|u(z_0)|_{L^2(I, H)} \leq e^{\tilde{C} \Theta} |z_0|_{H} \quad \text{with} \quad \Theta = \Theta(s_1 - s_0, |\rho|_{W^{1, \infty}}, |\sigma|_{W^{1, \infty}})$$

given by

$$\Theta(\theta_1, \theta_2, \theta_3) = 1 + \theta_2^2 + \theta_3^2 + \frac{1}{\theta_1} + \theta_1 (\theta_2 + \theta_3^2).$$

(2.19)

Notice that Theorem 2.9 and Lemma 2.8 imply that (2.14) holds with $C_2 = e^{\tilde{C} \Theta}$ and $B = 1_\omega$. Proceeding as in [2, Section A.2] we can conclude that (2.14) also holds with $C_2 = C_\chi e^{\tilde{C}_\chi \Theta} \leq e^{\tilde{D} \Theta}$ and $B^* q := \chi \omega q = 1_{\omega} \chi \omega q$, where $\tilde{D} = \log(C_\chi) + \tilde{C}_\chi$ and $\chi \in C^\infty(\overline{\Omega})$ is any given smooth function with $\emptyset \neq \omega \cap \text{supp} \chi$. Here $\tilde{D} = \tilde{D}(\chi, \omega, \Omega) > 0$ depends only on $(\chi, \omega, \Omega)$.

**Corollary 2.10.** Theorem 2.9 holds in the more general case $B = 1_\omega \chi \omega$, with $\tilde{D}$ in the place of $\tilde{C}$.

**Remark 2.11.** We point out that the observability constants $C_2 = e^{\tilde{C} \Theta}$ and $C_2 = e^{\tilde{D} \Theta}$, in Theorem 2.9 and Corollary 2.10, do depend on the triple $(I, \rho, \sigma)$, but that dependence is in terms of the triple $(|I|, |\rho|_{W^{1, \infty}}, |\sigma|_{W^{1, \infty}})$ only. This particular dependence on $I$ is of crucial importance in this section. This dependence holds for the control operators $B = 1_\omega \chi \omega$, but we do not know what happens for a general $B.$
2.3. Stabilization to zero by finite dimensional controls. Here we analyze the case when stabilization can be achieved by finite dimensional control action. Earlier related results are contained in [11, 12]. Let $\mathcal{C} = \{ \Psi_i \in H \mid i \in \{1, 2, \ldots, M\} \}$ and denote by $P_M$ the orthogonal projection in $H$ onto $\mathcal{S}_C := \text{span}\mathcal{C}$. Henceforth we also fix a positive constant $\lambda > 0$ and an open subset $\omega \subseteq \Omega$.

Let us consider, in $\mathbb{R}_{s_0} \times \Omega$, the system:

$$
\begin{align*}
\partial_t z + (-\Delta + 1) z + \rho z + \sigma \cdot \nabla z + 1_\omega \chi P_M 1_\omega u &= 0, \\
\partial_t z|_\Gamma &= 0, \quad z(s_0) = z_0. 
\end{align*}
$$

(2.20a) (2.20b)

**Definition 2.12.** We say that (2.20) is exponentially stabilizable to zero, with rate $\frac{\lambda}{2}$, if there are a constant $C > 0$ and a family $\{u = u(z_0) \mid z_0 \in H\} \subseteq L^2(\mathbb{R}_{s_0}, H)$ such that the corresponding global solution $z(t) = z(z_0, u(z_0))(t)$ satisfies

$$
|z(t)|^2_H \leq C e^{-\lambda(t-s_0)} |z_0|^2_H, \quad \text{for all } t \geq s_0.
$$

(2.21)

The stabilizing control $\zeta$ takes its values in the finite dimensional space span $\{1_\omega \chi \Psi_i \in H \mid i \in \{1, 2, \ldots, M\}\}$, for all $t \in \mathbb{R}_{s_0}$. Henceforth we use the control operator

$$
B_M = 1_\omega \chi P_M 1_\omega.
$$

(2.22)

Further $\theta$ and $\widehat{D}$ are the constants of Theorem 2.9 and Corollary 2.10.

Let us consider the function $\Phi: (0, +\infty) \to (0, +\infty)$ defined by

$$
\Phi: (0, +\infty) \to (0, +\infty), \quad \Phi(\tau) := 2e^{(2|\rho - \frac{\lambda}{2}|_W + 6|\sigma|^2_W)\tau} e^{2\widehat{D} \theta (|\rho - \frac{\lambda}{2}|_W + |\sigma|^2_W)},
$$

which we can extend to a function $\Phi: [0, +\infty) \to (0, +\infty]$ setting

$$
\Phi_e(\tau) := \begin{cases}
\Phi(\tau), & \text{if } \tau \in (0, +\infty), \\
\lim_{\tau \to 0} \Phi(\tau), & \text{if } \tau \in \{0, +\infty\}.
\end{cases}
$$

The minimum and minimizer of $\Phi_e$ are denoted by $\Upsilon$ and $T_*$, respectively. From

$$
\frac{d\Phi}{d\tau} \bigg|_{\tau=t} = \left(2|\rho - \frac{\lambda}{2}|_W + 6|\sigma|^2_W + 2\widehat{D} \left(-t^{-2} + |\rho - \frac{\lambda}{2}|_W + |\sigma|^2_W \right)\right) \Phi(t)
$$

we can conclude that $T_* > 0$ can be defined by

$$
T_*^2 = 2\widehat{D} \left|\rho - \frac{\lambda}{2}\right|_W + 6|\sigma|^2_W + 2\widehat{D} \left(|\rho - \frac{\lambda}{2}|_W + |\sigma|^2_W \right).
$$

Further $T_* = +\infty$ if, and only if, both $|\rho - \frac{\lambda}{2}|_W$ and $|\sigma|^2_W$ vanish.

The following result gives us a sufficient condition on the family $\mathcal{C}$ for the existence of a stabilizing control.

**Theorem 2.13.** Let us be given $\chi \in C^\infty(\overline{\Omega})$ satisfying $\emptyset \neq \omega \cap \text{supp } \chi$. If

$$
T_* \in \mathbb{R}_0 \quad \text{and} \quad |1_\omega \chi (1 - P_M)1_\omega|_{\mathcal{L}(H, V')} \leq \Upsilon^{-1},
$$

(2.23)
then system (2.20) is stabilizable to zero with rate $\frac{1}{2}$. Moreover, we can set the stabilizing control function $u = u(z_0)$ such that

$$|z(t)|^2_H \leq \left( \Theta_0 + \Upsilon |B_M|^2_{\mathcal{L}(H,V')} \right) e^{-\lambda(t-s_0)} |z_0|^2_H, \quad \text{for } t \geq s_0,$$

$$|e^{\frac{1}{2}t} u(z_0)|^2_{L^2(\mathbb{R},H)} \leq \frac{1}{1-e^{(\lambda-\lambda')T_*}} e^{2\lambda' \Theta} |z_0|^2_H, \quad \text{for } \lambda < \lambda',$$

with $\Theta_0 := e^{(2|\rho-\frac{1}{2}|W+6|\sigma|_W^2)T_*}$ and $\Theta_* := \Theta(T_*, |\rho-\lambda|_W, |\sigma|_W)$.

If $T_* = +\infty$, then setting $u = u(z_0) = 0$ the solution $z$ of system (2.20) satisfies

$$|z(t)|^2_H \leq e^{-\lambda(t-s_0)} |z_0|^2_H, \quad \text{for } t \geq s_0.$$ 

Proof. We consider separately the two cases $T_* \in \mathbb{R}_0$ and $T_* = +\infty$.

(a) The case $T_* \in \mathbb{R}_0$. Let $I_0 := (s_0, s_0 + T_*)$ and let $z$ solve

$$\partial_z + (-\Delta + 1)z + (\rho - \frac{\lambda}{2})z + \sigma \cdot \nabla z + 1 \omega \chi u = 0, \quad (2.24a)$$

$$\partial_z |z| = 0, \quad z(s_0) = z_0, \quad (2.24b)$$

in $I_0 \times \Omega$, with $u = u(z_0)$ given by Corollary 2.10, sending the solution of (2.24) to zero at time $t = s_0 + T_*$ (notice that Corollary 2.10 holds true with $\rho - \frac{1}{2}$ in the place of $\rho$), and let $z_M$ be the solution of (2.24) with $u = P_M 1_\omega u(z_0)$ in the place of $u(z_0)$. Then, the difference $d := z - z_M$ satisfies (2.24) with $d(s_0) = 0$ and $u = (1 - P_M) 1_\omega u(z_0)$.

The analogues to (2.5) for $z_M$ and $d$ read: for all $s \in I_0$,

$$|z_M(s)|^2_H \leq \Theta_0 \left( |z_0|^2_H + 2 |B_M u(z_0)|^2_{L^2(I_0, V')} \right)$$

$$|d(s)|^2_H \leq \Theta_0 2 |1_\omega \chi (1 - P_M) 1_\omega u(z_0)|^2_{L^2(I_0, V')}.$$ 

From Corollary 2.10 it follows that

$$|z_M(s)|^2_H \leq (\Theta_0 + \Upsilon |B_M|^2_{\mathcal{L}(H,V')} |z_0|^2_H; \quad (2.25)$$

$$|d(s)|^2_H \leq \Theta_0 |1_\omega \chi (1 - P_M) 1_\omega u(z_0)|^2_{\mathcal{L}(H,V')} |z_0|^2_H. \quad (2.26)$$

Then, from (2.23) we obtain

$$|z_M(s_0 + T_*)|^2_H = |d(s_0 + T_*)|^2_H \leq |z(s_0 + T_*)_0|^2_H. \quad (2.27)$$

Repeating the argument in the time intervals $I_i := (s_0 + iT_*, s_0 + (i + 1)T_*)$ with initial state $z_i^0 := z(s_0 + iT_*) = z_M(s_0 + iT_*)$ in (2.24b), finding $u^1_i = P_M 1_\omega u(z_i^0) \in L^2(I_i, H)$, leads to the analogues to (2.25), (2.26), and (2.27): for all $s \in I_i$,

$$|z_M(s)|^2_H \leq (\Theta_0 + \Upsilon |B_M|^2_{\mathcal{L}(H,V')} |z_i^0|^2_H)$$

$$|d(s)|^2_H \leq \Theta_0 |1_\omega \chi (1 - P_M) 1_\omega u(z_i^0)|^2_{\mathcal{L}(H,V')} |z_i^0|^2_H;$$

$$|z_i(s_0 + (i + 1)T_*)|^2_H \leq |z_i^0|^2_H.$$ 

Concatenating these controls we can see that the corresponding solution $z_M$ will remain bounded: $|z_M|^2_{L^\infty(\mathbb{R},H)} \leq (\Theta_0 + \Upsilon |B_M|^2_{\mathcal{L}(H,V')}) |z_0|^2_H$.

Next, we notice that $\hat{z}(t) := e^{-\frac{1}{2}(t-s_0)} z_M(t)$ solves (2.20) in $\mathbb{R}_0 \times \Omega$, with the concatenated control $u = \hat{u}$ defined by $\hat{u}|_{I_i} := e^{-\frac{1}{2}(t-s_0)} u(z_i^0)$. Moreover, we have the estimates

$$|\hat{z}(t)|^2_H \leq e^{-\lambda(t-s_0)} |z_M(t)|^2_H \leq (\Theta_0 + \Upsilon |B_M|^2_{\mathcal{L}(H,V')}) e^{-\lambda(t-s_0)} |z_0|^2_H$$
and, using Corollary 2.10 and \( |z_i|^2_H \leq |z_0|^2_H \), for all \( i \in \mathbb{N}_0 \),

\[
|z_i|_H^2 \leq \lim_{i \to +\infty} \sum_{j=0}^{i} \int_{I_j} e^{\lambda_j - \lambda_j(s-s_0)} |u(z_0)(s)|_H^2 ds
\]

\[
\leq \lim_{i \to +\infty} \sum_{j=0}^{i} e^{\lambda_j - \lambda_j T_j} \int_{I_j} |u(z_0)(s)|_H^2 ds
\]

\[
\leq e^{2\hat{D}\Theta} \lim_{i \to +\infty} \sum_{j=0}^{i} e^{(\lambda_j - \lambda_j T_j)} \leq e^{2\hat{D}\Theta} \leq |z_0|^2_H
\]

which ends the proof in the case \( T_\ast \in \mathbb{R}_0 \).

(b) The case \( T_\ast = +\infty \). In this case the solution of system (2.24) remains bounded with zero control \( u = 0 \). Indeed the analogue to (2.5) reads \( |z(s)|_H^2 \leq e^0 |z(s_0)|_H^2 \), for all \( s \geq s_0 \).

Now we give 2 examples of families \( \mathcal{C} \) which satisfy (2.23). For simplicity we suppose that \( \omega := \Pi_{j=1}^{n}(l_j, 1, l_j, 2) \subset \Omega \) is an open nonempty rectangle.

**Example 2.14. Eigenfunctions of the Laplacian operator.** Here we choose \( 0 \neq \chi \in C^\infty(\overline{\Omega}) \) such that \( \text{supp} \chi \subseteq \overline{\omega} \) and we let \( \{ \Psi_{R,i} \mid i \in \mathbb{N}_0 \} \) be a complete system of eigenfunctions of the negative Laplacian in \( \omega \) with homogeneous Dirichlet boundary conditions, which are ordered according to the increasing sequence of the (repeated) eigenvalues: \( 0 < \lambda_1 \leq \lambda_{i+1}, \lim_{i \to \infty} = \infty \). We define

\[
\Psi_i(x) := \begin{cases} 
\Psi_{R,i}(x), & \text{if } x \in \omega \\
0, & \text{if } x \in \Omega \setminus \overline{\omega},
\end{cases}
\]

and set \( \mathcal{C} = \{ \Psi_i \mid i \in \{1, 2, \ldots, M\} \} \). Let \( P_M : H \to \mathcal{S}_C \) be the orthogonal projection in \( H \) onto \( \mathcal{S}_C = \text{span} \mathcal{C} \). We observe that \( 1_\omega (1-P_M)\chi 1_\omega \nu \) and \( (1-P_M^R)\chi (\nu_\omega)_\omega \) coincide in \( \omega \). Here \( P_M^R : L^2(\omega) \to \mathcal{S}_{\mathcal{C}^R} \) is the orthogonal projection in \( L^2(\omega) \) onto \( \mathcal{S}_{\mathcal{C}^R} := \text{span} \{ \Psi_{R,i} \mid i \in \{1, 2, \ldots, M\} \} \}. \) Thus we obtain

\[
(1_\omega (1-P_M)1_\omega \nu, \nu)_\omega = (z|_\omega , (1-P_M^R)\chi_\omega)_\omega \leq |z|_H (1-P_M^R)_{(L^2(\omega), L^2(\omega))} |\chi_\omega|_{H^1(\omega)}
\]

and, since by assumption \( \chi|_{\partial \omega} = 0 \), we arrive at

\[
|1_\omega (1-P_M)1_\omega |_{L^2(H_V)} \leq |1-P_M^R|_{(L^2(\omega), L^2(\omega))} |\chi_\omega|_{L^2(\omega)} \leq 2 |\chi|_{C^1(\overline{\Omega})} (\lambda_M + 1)^{-\frac{1}{2}}.
\]

Consequently condition (2.23) is satisfied provided that \( \lambda_M + 1 \geq (2 |\chi|_{C^1(\overline{\Omega})} \cdot \mathcal{Y})^2 \).

Furthermore, from the asymptotic behavior \( \lambda_M \geq C_0 M^{\frac{2}{p}} \) (cf. [14, Corollary 1]) we can also arrive at the sufficient condition \( M \geq C_0^{-\frac{2}{p}} (2 |\chi|_{C^1(\overline{\Omega})} \cdot \mathcal{Y})^n \), which gives us an upper bound on the number \( M \) of controls which are needed to stabilize the system.

**Example 2.15. Piecewise constant controls.**

Here we consider a uniform partition of \( \omega \) where each interval \( (l_j, 1, l_j, 2) \) is divided into \( p \) intervals: \( l_j, k = (l_j, 1 + k_j \frac{1}{p_j}, l_j, 1 + (k_j + 1) \frac{1}{p_j}) \), with \( k_j \in \{0, 1, \ldots, p_j - 1\} \) and \( \bar{l}_j := l_j, 2 - l_j, 1 \). In this way, our rectangle is divided into \( M = \Pi_{j=1}^{n} p_j \) sub-rectangles

\[
\{ R_k \mid i \in \{1, 2, \ldots, M\} \} = \{ \Pi_{j=1}^{n} I_{l_j, k_j} \mid k_j \in \{0, 1, \ldots, p_j - 1\} \}.
\]
Let us set \( C = \left\{ \Psi_i = \frac{1}{|1_R_i|} 1_{R_i} \mid i \in \{1, 2, \ldots, M\} \right\} \in H \), and \( \chi = 1 \). For given \( v \in V \) and \( z \in H \) we find that

\[
(1_\omega (1 - P_M) 1_\omega z, v)_V = (z, 1_\omega (1 - P_M) 1_\omega v)_H \\
= (z |_{1_R_i}, v |_{1_R_i} - \sum_{i=1}^M (v, \Psi_i)_H \Psi_i |_{1_R_i})_{L^2(\omega)} = \sum_{i=1}^M (z |_{1_R_i}, \varphi_i)_{L^2(\omega)},
\]

where \( \varphi_i := v |_{1_R_i} - (v |_{1_R_i}, \Psi_i |_{1_R_i})_{L^2(\omega)} \Psi_i |_{1_R_i} = v |_{1_R_i} - \frac{1}{|1_R_i|} (v |_{1_R_i}, 1)_{L^2(\omega)} \) has zero average in \( R_i \). This allows to conclude that \( |\nabla \varphi|^2_{L^2(\omega)} \geq \beta_i |\varphi|^2_{L^2(\omega)} \) where \( \beta_i \) is the smallest positive eigenvalue of the Laplace–Neumann problem in the rectangle \( R_i \):

\[
-\Delta \phi = \beta_i \phi \quad \text{in } R_i, \\
\partial_n \phi = 0 \quad \text{on } \partial R_i.
\]

Since \( \beta_i = \pi_j^2 \mu_M \) where \( \mu_M := \min \{ \frac{\pi_j^2}{1} \mid j \in \{1, 2, \ldots, M\} \} \), we find for \( z \in H \) and \( v \in V \) with \( |z|_H = 1 \), \( |v|_V = 1 \) the estimates

\[
(1_\omega (1 - P_M) 1_\omega z, v)_V \leq \sum_{i=1}^M (\mu_M \pi_j^2)^{-\frac{1}{2}} |z |_{1_R_i} |_L^2(\omega) \left| \nabla v |_{1_R_i} \right|_{L^2(\omega)} \leq (\mu_M \pi_j^2)^{-\frac{1}{2}}.
\]

Since \( \mu_M \rightarrow \infty \) as the meshsize tends to 0, we conclude that condition (2.23) is satisfied provided that \( \mu_M \geq \frac{T}{T} \). Furthermore, in the case we take \( p_j = p \in \mathbb{N}_0 \), we arrive at the sufficient condition \( M \geq \frac{T}{T} \) with \( T := \max \{ T_j \mid j \in \{1, 2, \ldots, M\} \} \), which gives us an upper bound on the number \( M \) of controls we need to stabilize the system: \( M \geq \left( \frac{T}{T} \right)^n \). For the treatment in dimension 1 we refer to [11, Section IV.A].

2.4. Feedback stabilizing rule and Riccati equation. From Theorem 2.13 we know that system (2.20) is stabilizable. Here we show that the control can be taken in feedback form, i.e.

\[
u = K(t)z = B_M \Pi(t)z,
\]

with \( B_M \) given in (2.22). To specify the structure of the feedback operator \( K \) a suitably defined optimal control problem together with the dynamical programming principle will be used. It will turn out that \( \Pi \) satisfies a differential Riccati equation.

We shall require the spaces \( X_{z_0} := W(\mathbb{R}_{z_0}, V, V') \times L^2(\mathbb{R}_{z_0}, H) \), and

\[
X_{z_0}^\lambda := \{ (z, u) \in X_{z_0} \mid e^{\frac{\lambda}{2}}(z, u) \in X_{z_0} \},
\]

and the cost functionals

\[
J_{z_0}^{(\lambda)}(z, u) := \frac{1}{2} \left( |e^{\frac{\lambda}{2}} z |_{L^2(\mathbb{R}_{z_0}, H)}^2 + |e^{\frac{\lambda}{2}} R^\frac{1}{2} u |_{L^2(\mathbb{R}_{z_0}, H)}^2 \right),
\]

where \( \lambda \geq 0 \) and we set \( J_{z_0} := J_{z_0}^{(0)} \). For each \( z_0 \in H \) and \( s_0 \geq 0 \), we consider

\[
\text{Minimize } J_{z_0}^{(\lambda)}(z, u), \text{ over } (z, u) \in X_{z_0}^\lambda \text{ satisfying (2.20)}.
\]

Notice that \( (z, u) \) solves (2.20) if, and only if, \( (y, v) = e^{\frac{\lambda}{2}(\cdot - s_0)}(z, u) \) solves

\[
\begin{align}
\partial_t y + (-\Delta + 1)y + (\rho - \frac{\lambda}{2})y + \sigma \cdot \nabla y + B_M v &= 0, \\
\partial_t y |_{\Gamma} &= 0, \\
y(s_0) &= z_0.
\end{align}
\]
Consequently \((\bar{z}, \bar{u})\) is a minimizer for (2.28) if, and only if, 
\((\bar{y}, \bar{v}) = \mathcal{e}^\xi(\cdot; -s_0)(\bar{z}, \bar{u})\)
is a minimizer for:

\[
\text{Minimize } J_{s_0}(y, v), \text{ over } (y, u) \in X_{s_0} \text{ satisfying (2.29).} \tag{2.30, s_0}
\]

From now on we focus on problem (2.30). First of all, notice that from Theorem 2.13 both problems (2.28) and (2.30) are well-defined with \((\mathcal{M}, \mathcal{R}) = (1, 1)\) (for example, taking \((\lambda, 2\lambda)\) for \((\hat{\lambda}, \lambda)\)). Subsequently, from Lemma 2.6 it follows that they are also well-defined for the choice \((\mathcal{M}, \mathcal{R}) = (-(\Delta + 1)^2, 1)\).

Let us denote

\[\hat{X} := \{(z, u) \in X_{s_0} \mid (z, u) \text{ satisfy (2.29)}\}\]

and observe that, from Theorem 2.13 the mapping \(A_1 \in L(\hat{X}, H), A_1(y, v) := y(s_0)\) is surjective. Moreover, for given \((z_0, c) \in H \times \mathbb{R}_0\) it follows that the set \(S = \{(y, v) \in A_1^{-1}(\{z_0\}) \mid J(y, v) \leq c\}\) is bounded in \(\hat{X}\), if \((\mathcal{M}, \mathcal{R}) = (1, 1)\) or \((\mathcal{M}, \mathcal{R}) = (-(\Delta + 1)^2, 1)\). Hence, from [19, Lemma A.14 and Remark A.15], we know that Problem (2.30, s_0) has a unique minimizer which we denote by \((y^*_{s_0}, v^*_{s_0}) = (y^*_{s_0}, v^*_{s_0})(z_0)\).

Furthermore, the mapping \(z_0 \mapsto (y^*_{s_0}, v^*_{s_0})(z_0)\) is linear. From Theorem 2.13 and from the fact that \((y, v) \mapsto J_{s_0}(y, v)\) is quadratic we can conclude that there exists an operator \(\Pi_{s_0} \in L(H)\) such that

\[J_{s_0}(y^*_{s_0}, v^*_{s_0}) = \frac{1}{2} (\Pi_{s_0} z_0, z_0)_{H}, \text{ with } |\Pi_{s_0}|_{L(H)} \leq \overline{C}_{[C_W, \lambda, \frac{1}{4}]}\]

with \(\overline{C}_{[C_W, \lambda, \frac{1}{4}]}\) independent of \(s_0\), and where \(C_W\) is as in (2.3).

Motivated by the dynamical programming principle we define

\[
\mathcal{X}_I := W(I, V, V') \times L^2(I, H)
\]

\[
\mathcal{I}(y, v) := \frac{1}{2} \left( |\mathcal{M} y|_{L^2(I, H)}^2 + |\mathcal{R} y|_{L^2(I, H)}^2 + (\Pi_{s_1} y(s_1), y(s_1))_{H} \right). \tag{2.31, s_0}
\]

For arbitrary \(z_0 \in H\), we consider the finite horizon problem:

\[
\text{Minimize } \mathcal{I}(y, v), \text{ over } (y, v) \in \mathcal{X}_I, \text{ satisfying (2.29).} \tag{2.32, s_0, s_1}
\]

Proceeding as above we can prove that Problem 2.32 has a unique minimizer we denote \((y^*_{I}, v^*_{I}) = (y^*_{I}, v^*_{I})(z_0)\), with \(z_0 \mapsto (y^*_{I}, v^*_{I})(z_0)\) linear.

The next Lemma is the dynamical programming principle for problem (2.30, s_0). Since the result is standard we omit the proof (cf. Lemma 3.10 in [2].)

**Lemma 2.16.** The minimizers of Problems (2.30, s_0) and (2.32, s_0, s_1) have the following properties:

\[
(y^*_{s_0}, v^*_{s_0})(z_0)|_I = (y^*_{I}, v^*_{I})(z_0) \text{ and } (y^*_{s_0}, v^*_{s_0})(z_0)|_{G_{s_1}} = (y^*_{s_1}, v^*_{s_1})(y^*_{s_0}(s_1)).
\]

Next we describe how the optimal control \(Bv^*_{s_0}\) can be expressed in feedback form. For this purpose we define

\[
\tilde{X} := \left\{(y, v) \in \mathcal{X}_I \mid (y, v) \text{ satisfies (2.29), with } y(s_0) = y_0 \text{ for some } y_0 \in H \right\}.
\]
We observe that
\[ F: \tilde{X} \rightarrow \mathcal{Y} := H \times L^2(I, V'), \]
\[ (y, v) \mapsto (y(s_0) - z_0, \partial_t y + Ay + (\rho - \frac{A}{2})y + \sigma \cdot \nabla y + B_M v) \]
is a differentiable mapping and \( dF|_{(y^*, v^*)} : (z, u) \mapsto F(z, u) + (z_0, 0) \) is surjective. By the Karush–Kuhn–Tucker Theorem (e.g., see [2, Theorem A.1]) there is a Lagrange multiplier \((\mu^I, q^I) \in H \times L^2([s_0, s_1), V)\) such that
\[ dI|_{(y^*, v^*)} + (\mu^I, q^I) \circ dF|_{(y^*, v^*)} = 0. \]
That is, for all \((z, \xi) \in \tilde{X}, \) we have
\[ 0 = (\Pi_s y^I(s_1), z(s_1))_H + (\mu^I, z(s_0))_H + \int_{s_0}^{s_1} \langle M^* M y^I_s, z \rangle_H(t) \, dt \]
\[ + \int_{s_0}^{s_1} \langle \partial_t z + (-\Delta + 1)z + (\rho - \frac{A}{2})z + \sigma \cdot \nabla z, q^I \rangle_{V', V}(t) \, dt, \quad (2.33) \]
\[ 0 = \int_{s_0}^{s_1} \langle R v^I_\xi, \xi \rangle_H(t) \, dt + \int_{s_0}^{s_1} \langle B_M \xi, q^I \rangle_{V', V}(t) \, dt. \quad (2.34) \]
Relation (2.33) implies that \( q = q^I \) solves
\[ - \partial_t q + (-\Delta + 1)q + \rho q - \nabla \cdot (q \sigma) + M^* M y^I_s = 0, \quad (2.35a) \]
\[ (q \sigma + \nabla q) \cdot v^I = 0, \quad q(s_1) = -\Pi_s y^I(s_1). \quad (2.35b) \]
On the other hand (2.34) implies that \( R v^I_\xi = -B_M^* q^I. \) Using Lemma 2.16, we find
\[ v^I_{s_0}(s_1) = -R^{-1} B_M^* q^I(s_1) = R^{-1} B_M^* \Pi_s y^I_{s_0}(s_1). \] That is, the optimal control \( \zeta = B_M v^I_{s_0} \) is given in feedback form
\[ \zeta(s) = B_M K(s)y^I_{s_0}(s), \text{ with } K(s) := R^{-1} B_M^* \Pi_s, \quad s > s_0 \quad (2.36) \]
In particular, we observe that \( K(s) \) does not depend on the past \( t < s. \)

Let us now consider the closed-loop system
\[ \partial_t y + (-\Delta + 1)y + (\rho - \frac{A}{2})y + \sigma \cdot \nabla y + B_M K y = 0, \quad (2.37a) \]
\[ \partial_t y|_\Gamma = 0, \quad y(s_0) = z_0. \quad (2.37b) \]

**Theorem 2.17.** Let \( \chi \) and \( P_M \) satisfy the conditions in Theorem 2.13, let \( (\mathcal{M}, \mathcal{R}) = (1, 1) \) or \( (\mathcal{M}, \mathcal{R}) = ((-\Delta + 1)^2, 1), \) and let \( z_0 \in H. \) Then the solution \( y \) for (2.37) is defined for all \( t \geq s_0, \) and it satisfies
\[ |y|^2_{W([s_0, +\infty), V')} \leq \overline{C}_{[\gamma \cdot W, \lambda, \frac{1}{2}]} |z_0|^2_H, \quad \text{and} \quad (2.38a) \]
\[ |y|^2_{L^2([s_0, +\infty), V')} + \sup_{\tau \geq s_0} |y|^2_{L^2((\tau, \tau+1), D(\mathcal{A}))} \leq \overline{C}_{[\gamma \cdot W, \lambda, \frac{1}{2}]} |z_0|^2_V, \quad (2.38b) \]
if in addition \( z_0 \in V. \)

**Proof.** We know that \( y^I_{s_0}(s) \) solves (2.37). From (2.31) we have the uniform boundedness of \( |K(s)|_{C(H)} \), in \( s \geq 0. \) Thus, proceeding as in the proof of Lemma 2.2 we can arrive to the estimate (cf. (2.5))
\[ |y(s)|^2_H \leq e^{\overline{C}_{[\gamma \cdot W, \lambda, \frac{1}{2}]} |s-s_0|} |y(s_0)|^2_H, \]
globally and satisfies, for all unique solution of \( y_{s_0}^* \) with \( A \) |
\( \nabla (z_0) \) \( H \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \) \( |z_0|^2 \) (cf. Lemma 2.6 and (2.31)). On the other hand, from Lemma 2.4, (2.31), and (2.38a), we can derive that
\[
\left| y_{s_0}^* \right|^2_{W((\tau, \tau+1), H)} \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \left( \left| y_{s_0}^*(\tau) \right|^2_H + \left| B_M K_y s_0 \right|^2_{L^2((\tau, \tau+1), H)} \right) \\
\leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \left( \left| y_{s_0}^*(\tau) \right|^2_H + \left| y_{s_0}^*(s_0) \right|^2_H \right) \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \left| z_0 \right|^2_H , \text{ for all } t \geq s_0.
\]

Finally, from \( W((\tau, \tau+1), D(A), H) \hookrightarrow C(\tau, \tau+1), V \) uniformly with respect to \( \tau \geq 0 \), we obtain the inequality \( |y|^2_{C((s_0, +\infty), V)} \leq C \sup_{\tau \geq s_0} |y|^2_{L^2((\tau, \tau+1), D(A))} \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \left| z_0 \right|^2_V \).

The next Lemma can be derived following the arguments in [2, Remark 3.11(b) and proof of Lemma 3.8] and in [12, Section 3.4].

**Lemma 2.18.** The function \( \Pi: s \mapsto \Pi(s) := \Pi_s, s \geq 0 \), belongs to
\[
\mathcal{P} := \left\{ P \in L^\infty(\mathbb{R}, \mathcal{L}(H)) \left| \begin{array}{l}
P(t) \text{ is self-adjoint positive definite for all } t \geq 0, \\
\text{the family } \{P(t) \mid t \geq 0\} \text{ is continuous in the weak operator topology}
\end{array} \right. \right\}
\]
and satisfies the differential Riccati equation
\[
\hat{\Pi} + \Pi A + A^* \Pi - \Pi B_M R^{-1} B_M^* \Pi + \lambda \Pi + \lambda^* \mathcal{M} = 0,
\]
with \( \lambda y := (\Delta - 1)y - (\rho - \frac{\lambda}{2})y - \sigma \cdot \nabla y \). Moreover, \( \Pi \) is the unique solution of (2.39) in the class \( \mathcal{P} \).

Recall that \( y \) solves (2.37) if, and only if, \( z = e^{-(-s_0)^\frac{1}{2}} y(t) \) solves
\[
\partial_t z + (-\Delta + 1)z + \rho z + \sigma \cdot \nabla z + B_M R^{-1} B_M^* \Pi z = 0,
\]
\[
\partial_t z|_{t=0} = 0, \quad z|_{t=s_0} = z_0.
\]

Therefore we can conclude the next result.

**Corollary 2.19.** Under the assumptions of Theorem 2.17 let \( \Pi \in \mathcal{P} \) be the unique solution of (2.39). Then for any \( z_0 \in H \), the solution \( z \) of (2.40) is defined globally and satisfies, for all \( t \geq s_0, \)
\[
e^{(t-s_0)\lambda} |z(t)|^2_H + \int_{s_0}^t e^{(r-s_0)\lambda} (|z(r)|^2_V + |\partial_t z(r)|^2_V) \, dr \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} \left| z_0 \right|^2_H , \tag{2.41a}
\]
and
\[
|z(t)|^2_V + |z|^2_{L^2((t, t+1), D(A))} \leq C_{[C_{\infty}, \lambda, \frac{1}{2}]} e^{-(t-s_0)\lambda} \left| z_0 \right|^2_V , \tag{2.41b}
\]
if \( z_0 \in V \).

**Remark 2.20.** We have shown that Theorem 2.17 holds for the two choices \( (\mathcal{M}, \mathcal{R}) = (1, 1) \) and \( (\mathcal{M}, \mathcal{R}) = ((-\Delta + 1)^{\frac{1}{2}}, 1) \), for \( B_M = 1 \omega P_M 1 \omega \). It may be of interest to investigate alternative triples \( (\mathcal{M}, \mathcal{R}, B_M) \) for which Theorem 2.17 holds. One such example is the following: suppose \( \rho \) is constant and \( \sigma = 0 \). Then we can restrict ourselves to the subspaces \( H_{av} \subset H \) and \( V_{av} \subset V \) containing the functions with zero mean in \( \Omega \), and in that case we can take \( \mathcal{M} = (-\Delta)^{\frac{1}{2}} \), since the norms \( |\cdot|_V \) and \( |\nabla \cdot|_{L^2} \) are equivalent in \( V_{av} \).
3. Stabilization of the coupled system. Here we address the stabilization of the coupled linear system (1.9) where $A$ is either $A^M$ or $A^N$.

3.1. Conditional stabilization of the coupled system. Let $\Pi(t) = \Pi_0(t)$ be the solution of (2.39) with $\lambda = 2\alpha > 0$ and let $U_v(t, s)$ denote the evolution operator generated by $A_v(t) - BR^{-1}B_M^*\Pi_0(t)$. Then, from (2.41a), we have that

$$\|U_v\| = \sup_{t \geq s_0 \geq 0} \|e^{(t-s_0)}U_v(t, s_0)\|_{L(H)} \leq C_{\Pi_0, \lambda, \frac{1}{2}}$$

(3.1)

Recall the parameters $d, \gamma, \delta$, and the reference trajectory $\tilde{z} := \left(\frac{v}{\bar{w}}\right)$ in systems (1.7) and (1.8). To deal with these systems simultaneously we set the model indicator

$$t_m := \begin{cases} \bar{v} & \text{for system (1.7)}, \\ 1 & \text{for system (1.8)}, \end{cases} \quad \text{and } ||t|| := |t_m|^L([\mathbb{R} \times H]).$$

Though it will play no role hereafter, notice that systems (1.7) and (1.8) take the form (2.40) with $\sigma = 0$, therefore we have that $C_W = \overline{C}_{\Pi_0} = \overline{C}_{\Pi_0, \Pi_0}$, see (2.3).

**Theorem 3.1.** Let us be given $0 < \varepsilon < \min\{\alpha, \delta\}$. If

$$\|U_v\| ||t|| \gamma d < (\alpha - \varepsilon)(\delta - \varepsilon),$$

(3.2)

then for the evolution operator $U(t, s_0)$ generated by $A(t) - B_M R^{-1} B_M^* \Pi_0(t)$ it holds that $|e^{(t-s_0)}U(t, s_0)|_{L(H \times H)} \leq C_1$ for all $t \geq s_0 \geq 0$. For a constant $C_1$ depending on the bound $C_W$ and on the parameters in (3.2).

**Proof.** Notice that $e^{(t-s_0)U(t, s_0)}$ is the evolution operator generated by $A(t) - B_M R^{-1} B_M^* \Pi_0(t) + \varepsilon$. Therefore we want to show that, for all $\tilde{z}_{e, 0} = \left(\frac{z_{e, v}}{z_{e, w}}\right)$, the global solution $\tilde{z}_e(t) = \left(\frac{z_{e, v}}{z_{e, w}}\right)$ of the system

$$\dot{\tilde{z}}_e(t) = A(t)\tilde{z}_e(t) + \varepsilon \tilde{z}_e(t) - B_M R^{-1} B_M^* \Pi_0(t)\tilde{z}_e(t), \quad \tilde{z}_e(s_0) = \tilde{z}_{e, 0},$$

is bounded. Using Duhamel (variation of constants) formula we integrate the equation

$$\dot{z}_{e, w}(t) = e^{(\varepsilon - \delta)(t-s_0)}z_{e, w}(s_0) + \gamma \int_{s_0}^{t} e^{(\varepsilon - \delta)(t-s)}z_{e, v}(s) \, ds,$$

(3.3a)

$$\dot{z}_{e, v}(t) = L_z z_{e, v} - d t_m z_{e, w}(t), \quad z_{e, v}(s_0) = z_{e, v, 0},$$

(3.3b)

where for simplicity we have denoted $L_z := A_v(t) - B_M R^{-1} B_M^* \Pi_0(t) + \varepsilon$. Notice that for the evolution operator $U_v,e(t, s) = e^{(t-s)}U_v(t, s)$ generated by $L_z$ it holds

$$|U_v,e(t, s)|_{L(H)} \leq \|U_v\| \|e^{(t-s)}\|_{L(H \times H)}$$

Therefore, with $t \geq s_0$, we arrive to

$$|z_{e, v}(t)|_H$$

$$\leq \|U_v\| \|e^{(t-s_0)}z_{e, v, 0}\|_H + \|U_v\| \|s\|d \int_{s_0}^{t} e^{(t-s_0)} \|z_{e, w}(s)\|_H \, ds$$

(3.4)

$$\leq \|U_v\| \|e^{(t-s_0)}z_{e, w}\|_H + \|U_v\| \|s\|d \int_{s_0}^{t} e^{(t-s_0)} \|e^{(\varepsilon - \delta)(s-s_0)}z_{e, w, 0}\|_H \, ds$$

$$+ \|U_v\| \|s\| \gamma d \int_{s_0}^{t} e^{(t-s_0)} \int_{s}^{t} e^{(\varepsilon - \delta)(s-\tau)} \|z_{e, v}(\tau)\|_H \, d\tau \, ds.$$
That is,

\[ |z_{v}(t)|_{H} \leq \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} |z_{v,0}|_{H} + \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} \int_{s_{0}}^{t} e^{(\alpha - \varepsilon)s} \left| z_{v,0} \right|_{H} ds \]

+ \frac{1}{\alpha - \varepsilon} \int_{s_{0}}^{t} e^{(\alpha - \varepsilon)s} \left| z_{v,0} \right|_{H} ds,

where \( s_{0} \) is the initial time.

For the last single integral we have

\[ \int_{\tau}^{+\infty} \left( e^{(\alpha - \varepsilon)(t - \tau)} - e^{(\alpha - \varepsilon)(t - \tau)} \right) dt = \frac{1}{\delta - \varepsilon} \frac{1}{(\delta - \varepsilon)(\alpha - \varepsilon)} \]

which leads us to

\[ |z_{v}|_{L_{1}(\mathbb{R}, H)} \leq \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} |z_{v,0}|_{H} + \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} \int_{s_{0}}^{t} e^{(\alpha - \varepsilon)s} \left| z_{v,0} \right|_{H} ds \]

+ \frac{1}{\alpha - \varepsilon} \int_{s_{0}}^{t} e^{(\alpha - \varepsilon)s} \left| z_{v,0} \right|_{H} ds,

and from (3.2) it follows that, with \( \xi := \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} \|d_{\varepsilon}\|_{\alpha - \varepsilon} < 1 \),

\[ |z_{v}|_{L_{1}(\mathbb{R}, H)} \leq \frac{1}{1 - \xi} \left( \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} |z_{v,0}|_{H} + \|U_{v}^{\alpha}\|_{\alpha - \varepsilon} \int_{s_{0}}^{t} e^{(\alpha - \varepsilon)s} \left| z_{v,0} \right|_{H} ds \right). \]

Therefore, from (3.3a), it follows that \( |z_{v}|_{L_{\infty}(\mathbb{R}, H)} \leq C_{1}(|z_{v,0}|_{H} + |z_{w,0}|_{H}) \), and consequently from (3.4), it follows that \( |z_{w}|_{L_{\infty}(\mathbb{R}, H)} \leq C_{2}(|z_{v,0}|_{H} + |z_{w,0}|_{H}) \), for suitable constants \( C_{1} \) and \( C_{2} \).

**Corollary 3.2.** If \( 0 < \varepsilon < \min\{\alpha, \delta\}, \bar{z}_{0} \in H \times H \) and (3.2) holds true, then the solution of the system \( \dot{z} = A(t)\bar{z} - B_{M}R^{-1}B_{M}^{\dagger}A_{\alpha}(t)\bar{z}, \bar{z}(s_{0}) = \bar{z}_{0} \) satisfies \( |\bar{z}(t)|_{H \times H} \leq C_{1}\varepsilon^{-(t-s_{0})} |\bar{z}_{0}|_{H \times H} \), for all \( t \geq s_{0} \geq 0 \).

**3.2. Remarks on the conditional result. Lack of null controllability of the coupled system.** Let us consider the system

\[ \dot{z} = Zz, \ z(0) \in \mathbb{R}^{2}, \ \text{with} \ Z = \begin{bmatrix} -\varepsilon & d \\ \gamma & -(\delta - \varepsilon) \end{bmatrix}, \ t \geq 0. \]
This system is stable if, and only if, the eigenvalues of $Z$ have a nonpositive real part. Since those eigenvalues are the solutions of $(\alpha - \varepsilon + \lambda)(\delta - \varepsilon + \lambda) - \gamma d = 0$, then the stability holds if, and only if, the real part of each of the two values $-(\alpha + \delta) \pm \sqrt{(\alpha + \delta)^2 + 4\gamma d - 4(\alpha - \varepsilon)(\delta - \varepsilon)}$ is nonpositive, which is equivalent to the inequality $\gamma d - (\alpha - \varepsilon)(\delta - \varepsilon) \leq 0$.

We see that, for given $\gamma, d, \delta, \alpha$, we can choose $\alpha$ big enough such that this condition $\gamma d - (\alpha - \varepsilon)(\delta - \varepsilon) < 0$ holds true. This condition may look like (3.2) where $\alpha$ is also at our disposal. However the transient bound $\|U^c\|$ does depend on $\alpha$. From [12, Theorem 3.4 and Figure 10(a)] we can also guess that this dependence could be like $C_1 e^{C_2 \delta^2}$ with suitable constants $C_1$ and $C_2$, for big $\alpha$. See also [9].

In other words we expect to have $\inf_{\alpha > 0} \frac{\|U^c\|}{\alpha} > 0$. In that case (3.2) is truly conditional on the parameters $\gamma, d, \delta, \varepsilon$. In particular, the necessity of the condition (3.2), in Theorem 3.1, would mean that the parameters $\gamma, d, \delta$ in systems (1.7) and (1.8) cannot be taken arbitrarily. However, we do not know whether (3.2) is necessary, we have only proven its sufficiency.

We would also like to remark that though null controllability holds for the uncoupled linearized system, it does not hold for the coupled one (cf. [4, Section 2.2]).

### 4. Local stabilization of the nonlinear system.

Here we show that the feedback rule $-B_M R^{-1} B_M^* \Pi_\alpha(t)$ constructed to stabilize exponentially the linear system (1.9) to zero, with rate $\alpha = \frac{1}{2}$, also stabilizes the nonlinear system (1.5) to zero, with the same rate, provided $\tilde{z}_0$ is small enough.

Again in order to deal with the FitzHugh–Nagumo and Rogers–McCulloch models simultaneously we define another model indicator:

$$J_m := \begin{cases} 1 & \text{for system (1.7),} \\ 0 & \text{for system (1.8).} \end{cases}$$

System (1.5), under the feedback control becomes the closed loop system

$$\dot{\tilde{z}} = A_{\Pi_\alpha} \tilde{z} + F(\tilde{z}), \quad \tilde{z}(0) = \tilde{z}_0,$$

with the operator $A_{\Pi_\alpha} := \left( \begin{array}{cc} \Delta - (3a\tilde{w}^2 - 2b\tilde{w} + c) & -d \bar{z}_m \\ \gamma & -d \end{array} \right) - B_M R^{-1} B_M^* \Pi_\alpha$

and the nonlinear function $F(\tilde{z}) = \left( \begin{array}{c} -a\tilde{z}_0^3 - (b + 3a\tilde{w})\tilde{z}_0^2 - d\bar{z}_m \tilde{z}_w \\ 0 \end{array} \right)$.

#### 4.1. Local stabilization for strong regularity.

To derive the result for the nonlinear system we will need more regularity for the solutions. Thus we will ask more regularity for the initial conditions. Here we consider initial conditions in $V \times H$, instead of in $H \times H$ as in Corollary 3.2.

**Theorem 4.1.** If $0 < \varepsilon < \min\{\alpha, \delta\}$ and (3.2) holds true, then there is $\varepsilon > 0$ with the following property: if $\|\tilde{z}_0\|_{V \times H} \leq \varepsilon$, then there exists a solution for the system (4.1), in $\mathbb{R}_0 \times \Omega$, which belongs to $L^2_{\text{loc}}(\mathbb{R}_0, D(A) \times H) \cap C([0, +\infty), V \times H)$, is unique, and satisfies

$$|\tilde{z}(t)|_{V \times H} \leq C e^{-\varepsilon(t-s_0)}|\tilde{z}_0|_{V \times H}, \quad \text{for all } t \geq 0,$$

for a suitable constant $C$ independent of $(\varepsilon, \tilde{z}_0)$.

To prove Theorem 4.1 we will use a fixed point argument, following the procedure in [2, Section 4]. We start with a more regular version of Corollary 3.2.
Corollary 4.2. If $0 < \varepsilon < \min \{ \alpha, \delta \}$, $\bar{z}_0 \in V \times H$, and (3.2) holds true, then for a suitable constant $\bar{C}_2$, independent of $\bar{z}_0$, the solution of the system
\[
\dot{\bar{z}} = A_{\Pi_\alpha} \bar{z}, \quad \bar{z}(s_0) = \bar{z}_0
\]
satisfies
\[
\sup_{t \geq s_0} \| e^{\varepsilon(t-s_0)} \bar{z}(t) \|_{W((t, t+1), D(A), H) \times H^1((t, t+1), H)} \leq \bar{C}_2 \| \bar{z}_0 \|_{V \times H}.
\]

Proof. As in the proof of Theorem 3.1 we denote \( \begin{pmatrix} z_{\varepsilon,v} \\ z_{\varepsilon,w} \end{pmatrix} : = e^{\varepsilon(t-s_0)} \bar{z}(t) \).

For $t = s_0$ we have, from Lemma 2.3,
\[
\| z_{\varepsilon,v} \|_{W((s_0, s_0+1), D(A), H)} \leq \bar{C}[C_W] \left( \| z_{\varepsilon,v}(s_0) \|_V^2 + \| f \|_{L^2((s_0, s_0+1), H)}^2 \right)
\]
with $f = \tilde{f} := B_M R^{-1} B_M' \Pi_{\alpha} z_{\varepsilon,v} - \varepsilon z_{\varepsilon,v} + d_m z_{\varepsilon,w}$ and $\bar{C}[C_W] = \bar{C}[|V|_W, |w|_W]$. Thus, from Corollary 3.2.
\[
|z_{\varepsilon,v}|_{W((s_0, s_0+1), D(A), H)} \leq C_1 \left( |z_{\varepsilon,v}(s_0)|_V^2 + |\bar{z}_0|_{H \times H}^2 \right) \leq C_2 |\bar{z}_0|_{V \times H}^2. \quad (4.3)
\]

On the other hand, from Lemma 2.4 it follows that for all $t \geq s_0$
\[
|z_{\varepsilon,v}(t+1)|_V^2 \leq \bar{C}[C_W] \left( |z_{\varepsilon,v}(t)|_H^2 + \| f \|_{L^2((t, t+1), H)}^2 \right)
\]
and, again by using Corollary 3.2,
\[
|z_{\varepsilon,v}(t+1)|_V^2 \leq C_3 |\bar{z}_0|_{H \times H}^2. \quad (4.4)
\]

Finally, (4.3), (4.4), Lemma 2.3, and Corollary 3.2, give us
\[
|z_{\varepsilon,v}|_{W((t, t+1), D(A), H)} \leq \bar{C}[C_W] \left( |z_{\varepsilon,v}(t)|_V^2 + \| f \|_{L^2((t, t+1), H)}^2 \right)
\]
\[
\leq C_4 \left( |\bar{z}_0|_{V \times H}^2 + |\bar{z}_0|_{H \times H}^2 \right) \leq 2C_4 |\bar{z}_0|_{V \times H}^2, \quad \text{for all } t \geq s_0.
\]

Further from $\dot{z}_{\varepsilon,w} = -\delta z_{\varepsilon,w} + \gamma z_{\varepsilon,v} + \varepsilon z_{\varepsilon,w}$ and Corollary 3.2 we also have
\[
|z_{\varepsilon,w}|_{H^1((t, t+1), H)} \leq C_5 |\bar{z}_0|_{H \times H}^2.
\]

Notice that $C_4$ and $C_5$ can be taken independent of $t$. The proof is complete. \qed

Inspired from Corollary 4.2, taking $s_0 = 0$, we define the Banach space
\[
Z^\varepsilon := \left\{ \bar{z} \in L^2_{\text{loc}}(\mathbb{R}_0, H \times H) \ \bigg| \ |\bar{z}|_{Z^\varepsilon} < \infty \right\}
\]
edowed with the norm $|\bar{z}|_{Z^\varepsilon} := \sup_{t \geq 0} |e^{\varepsilon t} \bar{z}|_{W((t, t+1), D(A) \times H, H \times H)}$. We also set
\[
Z^\varepsilon_{\text{loc}} := \left\{ \bar{z} \in L^2_{\text{loc}}(\mathbb{R}_0, H \times H) \ \bigg| \ |e^{\varepsilon t} \bar{z}|_{W((t, t+1), D(A) \times H, H \times H)} < \infty, \ \text{for all } t \geq 0 \right\}.
\]

For a given constant $\rho > 0$ and $\bar{z}_0 \in V \times H$ we define the subset
\[
Z^\varepsilon_\rho := \left\{ \bar{z} \in Z^\varepsilon \mid |\bar{z}|_{Z^\varepsilon} \leq \rho |\bar{z}_0|_{V \times H} \right\},
\]
and the mapping $\Psi : Z^\varepsilon_\rho \to Z^\varepsilon_{\text{loc}}, \bar{z} \mapsto \bar{z}$, taking a given vector $\bar{z}$ to the solution $\bar{z}$ of
\[
\dot{\bar{z}} = A_{\Pi_\alpha} \bar{z} + F(\bar{z}), \quad \bar{z}(0) = \bar{z}_0. \quad (4.5)
\]
Lemma 4.3. Under the hypotheses of Theorem 4.1, there exists \( g > 0 \) such that the following property holds: for any \( \gamma \in (0, 1) \) one can find a constant \( \epsilon = \epsilon_\gamma > 0 \) such that, for any \( \tilde{z}_0 \) satisfying and \( |\tilde{z}_0|_{V \times H} \leq \epsilon \), the mapping \( \Psi \) takes the set \( Z^c_\epsilon \) into itself and satisfies the inequality
\[
|\Psi(\tilde{z}_1) - \Psi(\tilde{z}_2)|_{Z} \leq g|\tilde{z}_1 - \tilde{z}_2|_{Z} \quad \text{for all} \quad \tilde{z}_1, \tilde{z}_2 \in Z^c_\epsilon.
\]

Proof. We divide the proof into 3 main steps:

\( \circ \) Step 1: a preliminary estimate. Consider the system
\[
\dot{z} = \mathcal{A}_{\text{H_*}} z + f, \quad \tilde{z}(0) = \tilde{z}_0,
\]
where \( f \in L^2_{\text{loc}}(\mathbb{R}_0, H) \). If \( \tilde{z} \) is the solution of system (4.7) with \( f = 0 \), by Corollary 4.2,
\[
\sup_{r \geq 0} |e^{r \cdot \tilde{z}}(\cdot)|_{W((r, r+1), D(A) \times H \times H)} \leq \tilde{C}_2 |\tilde{z}(0)|^2_{V \times H}.
\]
We are going to derive a version of this estimate for suitable nonzero \( f = \left( \begin{array}{c} f_1 \\ 0 \end{array} \right) \).

We denote by \( S^f_{s,t} \tilde{z}_0 \) the solution \( \tilde{z} \) of (4.7). In the case \( f = 0 \), the operator \( S^0_{s,t} \) is linear; by the Duhamel formula we can write
\[
\tilde{z}(t) = S^f_{s,t} \tilde{z}_0 = S^0_{s,t} \tilde{z}_0 + \int_0^t S^0_{t,s} f(s) \, ds
\]
where \( S^f_{s,t} w \) denotes the solution of the system (4.7), with the initial time moved to \( t = s \), and the initial condition \( S^f_{s,s} w = w \). Further, from Corollary 3.2, it follows in particular that \( |e^{(t-s) \cdot S^0_{s,t}} w|_{H \times H} \leq \tilde{C}_1^2 |w|_{H \times H} \); then
\[
|\tilde{z}(t)|^2_{H \times H} \leq 2 \left| S^0_{s,t} \tilde{z}_0 \right|^2_{H \times H} + 2 \left| \int_0^t S^0_{t,s} f(s) \, ds \right|^2_{H \times H}
\]
\[
\leq 2\tilde{C}_1^2 e^{-2\epsilon t} \left( \left| \tilde{z}_0 \right|^2_{H \times H} + \left( \int_0^t e^{\epsilon s} |f(s)|_{H \times H} \, ds \right)^2 \right).
\]

Now we can find, denoting by \( |t| \in \mathbb{N} \) the integer satisfying \( |t| \leq t < |t| + 1 \),
\[
\int_0^t e^{\epsilon s} |f(s)|_{H \times H} \, ds \leq \sum_{k=0}^{|t|} \int_k^{k+1} e^{-\epsilon s} e^{2\epsilon s} |f(s)|_{H \times H} \, ds
\]
\[
\leq \sum_{k=0}^{|t|} \left( \int_k^{k+1} e^{-2\epsilon s} \, ds \right)^{\frac{1}{2}} \left( \int_k^{k+1} e^{4\epsilon s} |f(s)|^2_{H \times H} \, ds \right)^{\frac{1}{2}}
\]
\[
\leq \sup_{j \in \mathbb{N}} \left( \int_j^{j+1} e^{4\epsilon s} |f(s)|^2_{H \times H} \, ds \right)^{\frac{1}{2}} \sum_{k=0}^{|t|} \left( \int_k^{k+1} e^{-2\epsilon s} \, ds \right)^{\frac{1}{2}}.
\]

For the sum of the series, a direct computation gives us
\[
\sum_{k=0}^{|t|} \left( \int_k^{k+1} e^{-2\epsilon s} \, ds \right)^{\frac{1}{2}} \leq \sum_{k=0}^\infty \left( \int_k^{k+1} e^{-2\epsilon s} \, ds \right)^{\frac{1}{2}} = \left( \frac{1-e^{-2\epsilon}}{2\epsilon} \right)^{\frac{1}{2}} \sum_{k=0}^\infty e^{-\epsilon k} = \left( \frac{1-e^{-2\epsilon}}{2\epsilon} \right)^{\frac{1}{2}} \sum_{k=0}^\infty e^{-\frac{1}{2} \epsilon k} = \left( \frac{1-e^{-2\epsilon}}{2\epsilon} \right)^{\frac{1}{2}} \sum_{k=0}^\infty e^{-\frac{1}{2} \epsilon k} = \left( \frac{1-e^{-2\epsilon}}{2\epsilon} \right)^{\frac{1}{2}}
\]
So, 

\[
\int_0^t e^{\epsilon s} |f(s)|_{H \times H} \, ds \leq C_1 \|z_0\|_{V}^2 + \sup_{j \in \mathbb{N}} \left( \int_0^{j+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right)^{\frac{1}{2}}
\]

and, using (4.10),

\[
e^{2\epsilon t} |\tilde{z}(t)|_{H \times H}^2 \leq C_1 \left( |z_0|_{H \times H}^2 + \sup_{j \in \mathbb{N}} \int_0^{j+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right)
\]

for all \( t \geq 0 \).

Next, denoting \( \left( \begin{array}{c} z_v \\ z_w \end{array} \right) := \tilde{z} \) and using Lemma 2.4, we can obtain for all \( r \geq 0 \)

\[
|z_v(r+1)|_V^2 \leq C_2 \left( |z_v(r)|_V^2 + |\tilde{z}(r)|_{H \times H}^2 + |f_1|_{L^2((r, r+1), H)}^2 \right)
\]

with \( \tilde{f} := -B_M R^{-1} B_M^* \Pi_\alpha z_v + d_t z_w + f_1 \), which implies, using (2.31),

\[
|z_v(r+1)|_V^2 \leq C_3 e^{-2\epsilon r} \left( |z_v(0)|_V^2 + \sup_{0 \leq k \leq [r]} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right)
\]

(4.12)

since we have \( |f_1|_{L^2((r, r+1), H)}^2 = |f_1|_{L^2((r, r+1), H \times H)}^2 \leq e^{-2\epsilon r} \int_r^{r+1} e^{2\epsilon s} |f(s)|_{H \times H}^2 \, ds \leq e^{-2\epsilon r} \int_{[r]}^{[r]+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \leq 2e^{-2\epsilon r} \sup_{0 \leq k \leq [r]} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \).

For \( t \in (0, 1) \), from Lemma 2.3 and (4.11), we can also obtain

\[
\sup_{t \in [0, 1]} |z_v(t)|_V^2 \leq C_4 \left( |z_v(0)|_V^2 + |\tilde{f}|_{L^2((0, 1), H)}^2 \right)
\]

(4.13)

\[
\leq C_5 \left( |z_v(0)|_V^2 + \sup_{t \in [0, 1]} |\tilde{z}(t)|_{H \times H}^2 + |f_1|_{L^2((0, 1), H)}^2 \right)
\]

Equations (4.12) and (4.13) allow us to conclude that

\[
|z_v(t)|_V^2 \leq C_6 e^{-2\epsilon t} \left( |z_v(0)|_V^2 + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right), \text{ for all } t \geq 0.
\]
Using again Lemma 2.3 and proceeding as above, we can now derive

\[
|z_v|_{W((r, r+1), D(A), H)}^2 \leq C_{r, H} \left( |z_v(r)|_{V}^2 + |f|_{L^2((r, r+1), H)}^2 \right)
\]

\[
\leq C_7 \left( |z_v(r)|_{V}^2 + \sup_{t \in (r, r+1)} |\tilde{x}(t)|_{H}^2 + |f(t)|_{L^2((r, r+1), H)}^2 \right)
\]

\[
\leq C_8 e^{-2\epsilon r} \left( |z_0|_{V \times H} + \sup_{0 \leq k \leq [r]} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right);
\]

which implies, since \( \partial_t(e^{\epsilon \cdot z_v(\cdot)}) = e^{\epsilon \cdot \tilde{z}(\cdot)} + \epsilon \cdot \tilde{z}_v(\cdot), \)
that

\[
\sup_{r \geq 0} |e^{\epsilon \cdot z_v(\cdot)}|_{W((r, r+1), D(A), H)}^2 \leq C_9 \left( |z_0|_{V \times H} + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right).
\]

(4.14)

Finally, for \( \left( \frac{z_{\epsilon, v}}{z_{\epsilon, w}}, \frac{\tilde{x}_{\epsilon, v}}{\tilde{x}_{\epsilon, w}} \right) \) we have \( \dot{z}_{\epsilon, w} = -\delta z_{\epsilon, w} + \gamma z_{\epsilon, v} + \epsilon z_{\epsilon, w} \)
and, after integration, \( z_{\epsilon, w}(t) = e^{(\epsilon - \delta)t} z_{\epsilon, v}(0) + \gamma \int_0^t e^{(\epsilon - \delta)(t-s)} z_{\epsilon, v}(s) \, ds. \)
Therefore, using (4.14), we arrive to

\[
|z_{\epsilon, w}(t)|_{H}^2 \leq 2 |z_{\epsilon, w}(0)|_{H}^2 + 2\gamma^2 \left( \sup_{r \geq 0} |z_{\epsilon, v}(r)|_{H} \int_0^t e^{(\epsilon - \delta)(t-s)} \, ds \right)^2
\]

\[
\leq C_{10} \left( |z_0|_{V \times H} + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right).
\]

(4.15)

Then, from \( \dot{z}_{\epsilon, w} = -\delta z_{\epsilon, w} + \gamma z_{\epsilon, v} + \epsilon z_{\epsilon, w}, \) (4.14), and (4.15), we obtain

\[
|\dot{z}_{\epsilon, w}(t)|_{H}^2 \leq C_{11} \left( |z_0|_{V \times H} + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right).
\]

(4.16)

Finally (4.14), (4.15), and (4.16), imply that

\[
\sup_{r \geq 0} |e^{\epsilon \cdot \tilde{z}(\cdot)}|_{W((r, r+1), D(A) \times H, H \times H)}^2 \leq C_{12} \left( |z_0|_{V \times H} + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{4\epsilon s} |f(s)|_{H \times H}^2 \, ds \right),
\]

(4.17)

as desired.

\( \diamond \) Step 2: \( \Psi \) maps \( Z_\psi^\epsilon \) into itself, if \( |z_0|_{V \times H} \) is small. Denoting \( \left( \frac{z_v}{\tilde{z}_w} \right) := \tilde{z}_w \), we will replace \( f \) by \( F(\tilde{z}) = \left( -a \tilde{z}_v^3 - (-b + 3a) \tilde{z}_v - j_m \tilde{z}_v \frac{\partial}{\partial z_w} \right) \) in (4.17). First we derive suitable estimates for the nonlinear term. We focus on the 3D case, that is \( \Omega \subset \mathbb{R}^3 \), however, the estimates also hold for the 2D case. We recall the inequalities

\[
|u|_{L^\infty(\Omega)} \leq C |u|_{H^\frac{1}{2}(\Omega)} |u|_{H^2(\Omega)} \quad \text{and} \quad |u|_{L^6(\Omega)} \leq C |u|_{H^1(\Omega)}
\]

which are given by the Agmon inequality and the Sobolev embedding theorem (see [23, Chapter II, Section 1.4]) and [18, Chapter 2, Theorem 3.6]).
Now, we observe that
\[
|F(\tilde{z})(s)|_{H \times H}^2 \leq C_{13} \left( |\tilde{z}_v(s)|^6_{\mathcal{H}(\Omega)} + |\tilde{z}_v(s)|^4_{L^1(\Omega)} + |\tilde{z}_w(s)|^2_{L^2(\Omega)} \right)
\]
\[
\leq C_{14} \left( |\tilde{z}_v(s)|^6_{H^1(\Omega)} + |\tilde{z}_v(s)|^4_{H^1(\Omega)} + |\tilde{z}_w(s)|^2_{H^2(\Omega)} \right)
\]
which implies,
\[
\sup_{k \in \mathbb{N}} \int_{k}^{k+1} e^{4\epsilon s} |F(\tilde{z})(s)|_{H \times H}^2 \, ds \leq \sup_{k \in \mathbb{N}} \sup_{s \in [k, k+1]} C_{14} \left( |e^{\epsilon s} \tilde{z}_v(s)|^6_{V} + |e^{\epsilon s} \tilde{z}_v(s)|^4_{V} \right)
\]
\[
+ \sup_{k \in \mathbb{N}} \sup_{s \in [k, k+1]} C_{15} |e^{\epsilon s} \tilde{z}_w(s)|^2_{H} \int_{k}^{k+1} |e^{\epsilon s} \tilde{z}_v(s)|^2_{\mathcal{D}(A)} \, ds
\]
\[
\leq C_{16} \left( |\tilde{z}|^6_{Z^*} + |\tilde{z}|^4_{Z^*} \right).
\]
Thus, inequality (4.17) with \( f = F(\tilde{z}) \) gives us
\[
|\Psi(\tilde{z})|_{Z^*}^2 \leq C_{17} \left( |\tilde{z}_0|_{V \times H}^2 + |\tilde{z}|^6_{Z^*} + |\tilde{z}|^4_{Z^*} \right).
\] (4.18)
If \( \tilde{z} \in Z^*_0 \), then
\[
|\Psi(\tilde{z})|_{Z^*}^2 \leq C_{17}(1 + \varrho^3 |\tilde{z}_0|_{V \times H}^2 + \varrho^2 |\tilde{z}_0|_{V \times H}^2) |\tilde{z}_0|_{V \times H}^2
\] (4.19)
and if we set \( \varrho = 3C_{17} \) and \( \epsilon < \min \{1, \frac{1}{\varrho} \} \), then we obtain \( C_{17}(1 + \varrho^3 \epsilon^4 + \varrho^2 \epsilon^2) \leq \varrho \)
if \( |\tilde{z}_0|_{V \times H} \leq \epsilon \), which means that \( \Psi(\tilde{z}) \in Z^*_0 \).

\( \blacklozenge \) Step 3: \( \Psi \) is a contraction, if \( |\tilde{z}_0|_{V \times H} \) is smaller. It remains to prove (4.6). Let us take two functions \( \tilde{z}_1, \tilde{z}_2 \in Z^*_0 \) and let \( \Psi(\tilde{z}_1) \) and \( \Psi(\tilde{z}_2) \) be the corresponding solutions for (4.5). Set \( e = \tilde{z}_1 - \tilde{z}_2 \) and \( d^\Psi = \Psi(\tilde{z}_1) - \Psi(\tilde{z}_2) \). Then \( d^\Psi \) solves (4.7) with \( d^\Psi(0) = 0 \)
and \( f = F(\tilde{z}_1) - F(\tilde{z}_2) \). Therefore, by inequality (4.17), we have
\[
|\Psi(\tilde{z}_1) - \Psi(\tilde{z}_2)|_{Z^*}^2 \leq C_{12} \sup_{t \geq 0} \int_{k}^{k+1} e^{4\epsilon s} |F(\tilde{z}_1)(s) - F(\tilde{z}_2)(s)|_{H \times H}^2 \, ds.
\] (4.20)
Denoting
\[
\left( \begin{array}{c}
\tilde{z}_1v \\
\tilde{z}_1w
\end{array} \right) := \tilde{z}_1, \quad \left( \begin{array}{c}
\tilde{z}_2v \\
\tilde{z}_2w
\end{array} \right) := \tilde{z}_2, \quad \text{and} \quad \left( \begin{array}{c}
e_v \\
e_w
\end{array} \right) := e = \left( \begin{array}{c}
\tilde{z}_1v - \tilde{z}_2v \\
\tilde{z}_1w - \tilde{z}_2w
\end{array} \right),
\]
we find that
\[
\begin{align*}
e_v(\tilde{z}_1v + \tilde{z}_1w, \tilde{z}_2v + \tilde{z}_2w), & \quad \tilde{z}_1v - \tilde{z}_2v = e_v(\tilde{z}_1v + \tilde{z}_2v), \\
\tilde{z}_1w - \tilde{z}_2w & = e_w(\tilde{z}_1v + \tilde{z}_2v),
\end{align*}
\] (4.21)
from which we can obtain
\[
|F(\tilde{z}_1)(s) - F(\tilde{z}_2)(s)|_{H \times H}^2
\]
\[
\leq C_{18} \left( |e_v|_{L^\infty(\Omega)}^2 + |e_w|_{L^2(\Omega)}^2 \right) \left( |\tilde{z}_1v|_{L^\infty(\Omega)}^2 + |\tilde{z}_2v|_{L^\infty(\Omega)}^2 + |\tilde{z}_1w|_{L^2(\Omega)}^2 + |\tilde{z}_2w|_{L^2(\Omega)}^2 \right)
\]
\[
+ C_{18} \left( |e_v|_{H^1(\Omega)}^2 |\tilde{z}_1v|_{H^1(\Omega)} + |e_w|_{H^2(\Omega)}^2 \right) \left( |\tilde{z}_1v|_{H^1(\Omega)}^2 + |\tilde{z}_2v|_{H^2(\Omega)}^2 + 1 \right)
\]
\[
+ C_{19} \left( |e_v|_{H^1(\Omega)}^2 |\tilde{z}_1v|_{H^2(\Omega)} + |\tilde{z}_2v|_{H^1(\Omega)} \right)\left( |\tilde{z}_1v|_{H^1(\Omega)}^2 + |\tilde{z}_2v|_{H^2(\Omega)}^2 \right)
\]
and
\[ e^{4\epsilon s} |\mathcal{F}(\vec{z}_1)(s) - \mathcal{F}(\vec{z}_2)(s)|^2_{\mathcal{H} \times \mathcal{H}} \leq C_1 |e^{\epsilon s} v|_V^2 \left( |e^{\epsilon s} z_{1v}|_{D(A)}^2 + |e^{\epsilon s} z_{2v}|_{D(A)}^2 \right) \left( |e^{\epsilon s} z_{1v}|_V^2 + |e^{\epsilon s} z_{2v}|_V^2 + 1 \right) + C_1 \left( |e^{\epsilon s} v|_{D(A)}^2 |e^{\epsilon s} z_{1v}|_{L^2(\Omega)}^2 + |e^{\epsilon s} v|_{D(A)}^2 |e^{\epsilon s} z_{2v}|_{L^2(\Omega)}^2 \right) \]
\[ \leq C_1 |e|^2_{L^2(\Omega)} \left( \left( |z_{1v}|_{\mathcal{H}}^2 + |z_{2v}|_{\mathcal{H}}^2 + 1 \right) \left( |z_{1v}|_{\mathcal{H}}^2 + |z_{2v}|_{\mathcal{H}}^2 \right) \right) + C_1 \left( |z_{1}|_{\mathcal{H}}^2 |e^{\epsilon s} v|_{D(A)}^2 + |e|^2_{L^2(\Omega)} |e^{\epsilon s} z_{2v}|_{D(A)}^2 \right) \]

Therefore, from (4.20), it follows
\[ |\Psi(\vec{z}_1) - \Psi(\vec{z}_2)|^2_{\mathcal{H}_0} \leq C_20 |e|^2_{\mathcal{H}_0} \left( \left( |z_{1v}|_{\mathcal{H}}^2 + |z_{2v}|_{\mathcal{H}}^2 + 1 \right) \left( |z_{1v}|_{\mathcal{H}}^2 + |z_{2v}|_{\mathcal{H}}^2 \right) \right), \]
and since \( \vec{z}_1 \) and \( \vec{z}_2 \) are both in \( Z_{\epsilon} \), we arrive to
\[ |\Psi(\vec{z}_1) - \Psi(\vec{z}_2)|^2_{\mathcal{H}_0} \leq C_20 (2\|\vec{z}_0\|_{\mathcal{H}_0}^2 + 1) 2\|\vec{z}_0\|_{\mathcal{H}_0}^2 |\vec{z}_1 - \vec{z}_2|_{\mathcal{H}_0}^2 \] (4.22)
Choosing \( \epsilon > 0 \) as in Step 2, that is \( \epsilon < \min \left\{ 1, \frac{1}{\theta} \right\} \), we find \( 2\|\vec{z}_0\|_{\mathcal{H}_0}^2 + 1 < 3 \). So choosing \( \epsilon > 0 \) still smaller so that \( \epsilon < \min \left\{ 1, \frac{1}{\theta} \frac{2}{\sqrt{\epsilon}} \right\} \), we see that (4.6) holds, provided \( |\vec{z}_0|_{\mathcal{H}_0}^2 \leq \epsilon \).

The proof of Lemma 4.3 is complete.

**Proof of Theorem 4.1.** From Lemma 4.3 and the contraction mapping principle it follows that if \( \vec{z}_0 \in V \times H \) is sufficiently small, \( |\vec{z}_0|_{V \times H} < \epsilon \), then there exists a unique fixed point \( \vec{z} = \Psi(\vec{z}) \in Z_{\epsilon} \) for \( \Psi \). It follows from the definitions of \( \Psi \) and \( Z_{\epsilon} \) that \( \vec{z} \) solves the system (4.5), with \( \vec{z} = \vec{z} \). We can conclude that \( \vec{z} \) solves (1.5) with the feedback control \( Bu = -B_M R^{-1} B_M^* \Pi_\alpha \vec{z} \).

Further inequality (4.2) can be concluded from (4.19).

Finally, it remains to prove the uniqueness of the solution for (4.1) in the space \( Z := L^2_{\text{loc}}(\mathbb{R}_0, D(A) \times H) \cap C([0, +\infty), V \times H) \supset Z_{\epsilon} \). Let \( \vec{z}_1 \) and \( \vec{z}_2 \) be two solutions, in \( Z \), for (4.1) and denote
\[ \begin{align*}
(\vec{z}_{1v}) := \vec{z}_1, & \quad (\vec{z}_{2v}) := \vec{z}_2, \quad \text{and} \quad (e_v, e_w) := e := \left( \begin{array}{c} z_{1v} - z_{2v} \\ z_{1w} - z_{2w} \end{array} \right). 
\end{align*} \]
It turns out that \( e \) solves (4.7) with \( f = \mathcal{F}(\vec{z}_1) - \mathcal{F}(\vec{z}_2) \), that is,
\[ \dot{e} = (\overline{A} - B_M R^{-1} B_M^* \Pi_\alpha) e + \mathcal{F}(\vec{z}_1) - \mathcal{F}(\vec{z}_2) \]
with \( \overline{A} := \begin{pmatrix} \Delta - 1 + 1 - (3\alpha \bar{u}^2 - 2b\bar{v} + c + 2m d\bar{w}) & -d \triangle m \\ -d & -\delta \end{pmatrix} \). Using (2.31) and (4.21), we can obtain
\[ \begin{align*}
(\mathcal{A} e, e)_{V \times H, V \times H} & \leq -|e_v|^2_{V} + C \left( |(\bar{v}, \bar{w})|_{L^\infty(\Omega^2)} \right) |e|^2_{H \times H}, \\
(\mathcal{B}_M R^{-1} B_M^* \Pi_\alpha e, e)_{V \times H, V \times H} & \leq C \left( |(\bar{v}, \bar{w})|_{L^\infty(\Omega^2)} \right) |e_v|^2_{H}, \\
\mathcal{F}(\vec{z}_1) - \mathcal{F}(\vec{z}_2), & \mathcal{F}(\vec{z}_1) - \mathcal{F}(\vec{z}_2) \leq C_1 \left( |z_{1v}|_{L^\infty(\Omega)}^2 + |z_{2v}|_{L^\infty(\Omega)}^2 + 1 \right) |e_v|^2_{H} \\
& + |e_v z_{1w} + z_{2v} e_w e_v|_{L^1(\Omega)}, \\
|e_v z_{1w} + z_{2v} e_w e_v|_{L^1(\Omega)} & \leq |e_v|^2_{L^2(\Omega)} |z_{1w}|_{L^2(\Omega)} + |z_{2v}|_{L^\infty(\Omega)} |e_v|^2_{H}. 
\end{align*} \]
Now, from the continuity of the inclusion $H^\frac{1}{2}(\Omega) \subset L^1(\Omega)$ (cf. [6, Chapter 4, Section 4.4, Corollary 4.53]) and the fact $H^\frac{1}{2}(\Omega)$ can be seen as an interpolation space $H^\frac{1}{2}(\Omega) = [H^0(\Omega), L^2(\Omega)]_{\frac{1}{2}}$ (cf. [16, Chapter 1, Theorem 9.6 and Remark 9.1]), we can arrive to
\[
|e_v|_{L^1(\Omega)}^2 |z_{1w}|_{L^2(\Omega)} \leq C_2 |e_v|_H^2 |z_{1w}|_H^2 \leq C_3 |e_v|_H^4 |z_{1w}|_H^4 + |e_v|_V^2.
\]
Therefore, we obtain
\[
\frac{d}{dt}|e|_{H \times H}^2 = 2(\langle \mathcal{A} - B_M R^{-1} B_M^* \Pi_a \rangle e + \mathcal{F}(z_1) - \mathcal{F}(z_2), e)_{V \times H, V \times H}
\leq C_4 \left( |z_{1v}|_{D(A)}^2 + |z_{2v}|_{D(A)}^2 + |z_{1w}|_H^4 + 1 \right) |e|_{H \times H}^2.
\]
Observe that $\psi := C_3 (|z_{1v}|_{D(A)}^2 + |z_{2v}|_{D(A)}^2 + |z_{1w}|_H^4 + 1)$ is a locally integrable function, because $\tilde{z}_1$ and $\tilde{z}_2$ are in $Z$. Thus, by the Gronwall lemma we find
\[
|e(t)|_{H \times H}^2 \leq e^{\int_0^t \psi(s) ds} |e(0)|_{H \times H}^2 = 0,
\]
that is, $\tilde{z}_1 = \tilde{z}_2$. □

4.2. Local stabilization to trajectories. As a straightforward consequence of Theorem 4.1, we have our main result on stabilization to trajectories for system (1.1).

Corollary 4.4. If $0 < \varepsilon < \min\{\alpha, \delta\}$ and (3.2) hold true, then there is $\epsilon > 0$

with the following properties: if
\[
\tilde{y} = (\tilde{v}, \tilde{w}) \in W_{loc}(\mathbb{R}_0, V \times H, V' \times H) \cap L^\infty(\mathbb{R}_0, L^2(\Omega))^2
\]
is a solution for system (1.4), with $\tilde{y}_0 = (\tilde{v}_0, \tilde{w}_0) \in H \times H$, and if $y_0 = (v_0, w_0) \in H \times H$ is such that
\[
(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0) \in V \times H \quad \text{and} \quad |(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)|_{V \times H} < \epsilon,
\]
then the solution $y = (v, w)$ of the system (1.1) with the feedback control $Bu = -B_M R^{-1} B_M^* \Pi_a (v - \tilde{v})$ goes exponentially to $\tilde{y}$ with rate $\varepsilon$, that is,
\[
|y(t) - \tilde{y}(t)|_{V \times H} \leq Ce^{-\varepsilon(t-s_0)} |y_0 - \tilde{y}_0|_{V \times H}, \quad \text{for all} \ t \geq 0,
\]
for a suitable constant $C$ independent of $(\varepsilon, y_0 - \tilde{y}_0)$, and the solution $(v, w)$ is, and is unique, in the affine space $(\tilde{v}, \tilde{w}) + L^2_{loc}(\mathbb{R}_0, D(A) \times H) \cap C([0, +\infty), V \times H)$.

5. Numerical examples. We consider the following version of the monodomain equations
\[
\begin{align*}
\partial_v v &= (\varkappa \Delta - c_1) v - dw - a v^3 + b v^2 + Bu + f_1 + f_2, \quad \text{in} \ \Omega \times (0, T), \\
\partial_t w &= \gamma v - \delta w, \quad \text{in} \ \Omega \times (0, T), \\
\partial_v v |_{\Gamma} &= 0, \quad \text{on} \ \Gamma \times (0, T), \\
v(x, 0) &= v_0(x) \quad \text{and} \quad w(x, 0) = w_0(x), \quad \text{in} \ \Omega,
\end{align*}
\]
(5.1)
where $\Omega = (0, 1) \times (0, 1)$, and the parameters are chosen as $\varkappa = 1.5 \cdot 10^{-3}, a = 1.2 \cdot 10^{-3}, b = 0.1304, c = 1.5, d = 215.6, \gamma = 1.2 \cdot 10^{-4}$ and $\delta = 1.2 \cdot 10^{-3}$. For the control we take piecewise constants as described in Example 2.15. Figure 5.1 visualizes the corresponding control domains.
5.1. Termination of a reentry wave. As a test case we consider the termination of a reentry wave modeling cardiac arrhythmia. For this purpose we initialize the system by stimulating the lower boundary of the domain. As a result, a traveling wave is obtained. Placing an external stimulus \( f_1 \) within a critical time window leads to a reentry wave as shown in Figure 5.6 (top). Outside of this time window, the stimulus only results in an excitation that immediately starts to collapse (see Figure 5.6 (bottom)). With this in mind, our setup is as follows. We assume that the desired trajectory \( \vec{y}_d = (v_d, w_d) \) is obtained from a typical heart rhythm starting at \( \vec{y}_d(0) = \vec{y}_d,0 \), such that the external stimulus is applied before the critical time window is reached. After the external stimulus has collapsed the natural heart rhythm restarts and a second traveling wave is stimulated by means of \( f_2 \), see also Figure 5.7 at time \( t = 180 \).

Considering now a perturbation of the initial condition \( \vec{y}(0) = \vec{y}_d(0) + \xi \) (postpone initial time), the external stimulus is shifted into the critical time window and causes the excitation of a reentry wave. The desired effect of the feedback law then is to stabilize the perturbed system around the natural heart beat.

5.2. Discretization and the differential Riccati equation. All simulations are generated on an Intel® Xeon(R) CPU E31270 @ 3.40 GHz x 8, 16 GB RAM, Ubuntu Linux 14.04, MATLAB® Version 8.0.0.783 (R2012b) 64-bit (ghx6a64).

For the spatial discretization of (5.1) we use a finite difference scheme on a uniform 32 \( \times \) 32 grid. The resulting ODE system then reads

\[
\begin{align*}
\partial_t v_n &= A_n v_n - d 1_n w_n + I_{ion}(v_n) + B_n u + f_1 + f_2, \quad v_n(0) = v_d,n(0) + \xi_v, \\
\partial_t w_n &= \gamma 1_n v_n - \delta 1_n w_n, \quad w_n(0) = w_d,n(0) + \xi_w,
\end{align*}
\]

where the nonlinearity is evaluated pointwise such that \( I_{ion}(v_n) = -av_n^3 + bv_n^2 \). We further have \( A_n, I_n \in \mathbb{R}^{n \times n} \) and \( B_n \in \mathbb{R}^{n \times m} \), with \( n = 1024 \) and \( m = 16 \). The desired trajectory \( (v_d,n, w_d,n) \) is computed as a solution to the uncontrolled system

\[
\begin{align*}
\partial_t v_{d,n} &= A_n v_{d,n} - d 1_n w_{d,n} + I_{ion}(v_{d,n}) + f_1 + f_2, \\
\partial_t w_{d,n} &= \gamma 1_n v_{d,n} - \delta 1_n w_{d,n}.
\end{align*}
\]

The solutions of the ODE systems are always obtained by the MATLAB routine \texttt{ode45}. The feedback control law \( u(t) = -R^{-1}B_n^* \Pi_n(t)(v_n(t) - v_{d,n}(t)) \) is computed by...
solving the matrix differential Riccati equation associated with the decoupled system, i.e.,
\[ \dot{\Pi}_n + (A_n(v_{d,n}))^*\Pi_n + \Pi_n A_n(v_{d,n}) - \Pi_n B_n R^{-1} B_n^* \Pi_n + \lambda \Pi_n + M^* M = 0, \] (5.3)
where \( A_n(v_{d,n}) = A_n - \text{diag}(3a v_{d,n}^2) + \text{diag}(2b v_{d,n}) \). Following the suggested methodology in [12], we exploit the fact that the desired trajectory is approaching a stationary state (zero). Hence, we solve (5.3) backwards in time using the initialization \( \Pi_n(t_f) = \tilde{\Pi}_n \), where \( \tilde{\Pi}_n \) solves the algebraic matrix Riccati equation
\[ A_n^* \tilde{\Pi}_n + \tilde{\Pi}_n A_n - \tilde{\Pi}_n B_n R^{-1} B_n^* \tilde{\Pi}_n + \lambda \tilde{\Pi}_n + M^* M = 0. \]

The solution of the resulting initial value problem (5.3) is determined by the MATLAB routine \texttt{ode45} rather than the Crank-Nicolson inspired scheme proposed in [12]. In this way we only need to evaluate the Riccati operator rather than solving an algebraic Riccati equation in each time step. While the latter approach generally allows for bigger time steps, in our case the performance of \texttt{ode45} was better.

5.3. The linearized system. Let us consider the effect of the feedback law when applied to the linearized system, i.e.,
\[ \partial_t y_{n,v} = (A_n(x_{d,n}) - B_n R^{-1} B_n^* \Pi_n(t)) \ y_{n,v} - d 1_n y_{n,w}, \quad y_{n,v}(0) = \xi_v, \]
\[ \partial_t y_{n,w} = \gamma 1_n y_{n,v} - \delta 1_n y_{n,w}, \quad y_{n,w}(0) = \xi_w, \]
where \( y_{n,v} = v_n - v_{d,n} \) and \( y_{n,w} = w_n - w_{d,n} \). The shift \( \lambda \) for the desired exponential decay rate of the decoupled system is chosen as \( \lambda = 1 \). Figure 5.2 shows the decay of the closed loop system for \( t \in [0, 800] \) and two different choices of \( M \). We also include a comparison with the uncontrolled solution. In this context, we remark that the system is asymptotically stable when linearized in the zero state. Since the desired trajectory \((x_{d,n}, w_{d,n})\) approaches zero, this implies that the same holds true for the uncontrolled solution. As is reflected in Figure 5.2 the controlled system performs better than the uncontrolled system. We further obtain a better performance with respect to both the \( L^2(\Omega) \)-norm as well as the \( H^1(\Omega) \)-norm in the case \( M = 1 \). The characteristic “peaks” within the error plots can be explained as follows. The first excitation \( f_1 \) modeling the undesired external stimulus happens at \( t = 9.52 \). At \( t = 176.39 \) the regular heart rhythm restarts and causes a traveling wave (due to \( f_2 \)) evolving from the center of

**Fig. 5.2.** Linearized system. Comparison of \( L^2(\Omega) \) and \( H^1(\Omega) \) error for \( R = 1 \).
the domain, see again Figure 5.7 at $t = 180$. The third peak corresponds to the sudden collapse of the traveling wave at $t \approx 305$. Figure 5.3 shows the time span between the excitation and the collapse of the traveling wave, respectively. Here, we additionally include (green axis) the time interval in which at least one of the eigenvalues of the system matrix $A_n(v_{d,n})$ has a positive real part. While it is well-known that for linear time-varying systems there is no one-to-one correspondence between spectral abscissa and stability of the system, it is still worthwhile to mention that the most significant differences to the uncontrolled system appear when $A(v_{d,n})$ is unstable. This also concerns the relation between the quality of the solutions for $M = 1$ and $M = (1 - \Delta) \frac{1}{2}$.

5.4. The nonlinear system. We now focus on the full nonlinear system (5.2). Again, the results of the simulations for two different choices of $M$ are compared with the uncontrolled solutions, see Figure 5.4. Note that the uncontrolled solution now exhibits a periodic behavior and, in particular, does not decay at all. On the other hand, both feedback control laws result in a successful termination of the reentry wave. As already indicated by the results for the linearized system, the choice $M = 1$ shows a better performance than $M = (1 - \Delta) \frac{1}{2}$. We also include a comparison for the weight matrix $R = \frac{1}{4}$ rather than $R = 1$. Since this decreases the amount of the control costs within the cost functional, we expect the control to have more influence.
Indeed, Figure 5.5 underlines this expectation. Here, the results corresponding to $M = (1 - \Delta)^{\frac{1}{2}}$ are better than those obtained for $M = 1$. In Figure 5.6 and Figure 5.7 the temporal evolution of $v_{d,n}(x)$ for the uncontrolled, desired and controlled system is shown. While for $t = 13$, the difference between desired and controlled solution is clearly visible, for larger time instances the controlled solution approaches the desired solution. Finally, Figure 5.8 visualizes the action of the piecewise constant control functions. The largest magnitude can be observed after the external stimulus has been applied (see $t = 13$.) As expected, for increasing $t$, the feedback law approaches
Acknowledgement. T. Breiten and K. Kunisch gratefully acknowledge the Austrian Science Fund (FWF) for financial support under SFB F32, “Mathematical Optimization and Applications in Biomedical Sciences.”
S. Rodrigues acknowledges partial support from the Austrian Science Fund (FWF): P 26034-N25.

REFERENCES
Fig. 5.8. Control action for different time steps.

[20] J. Rogers and A. McCulloch, A collocation-Galerkin finite element model of cardiac action


