Optimal control of the bidomain system (I): The monodomain approximation with the Rogers-McCulloch model

Karl Kunisch[†] *and Marcus Wagner*[‡]

Abstract. For the monodomain approximation of the bidomain equations, a weak solution concept is proposed. We analyze it for the FitzHugh-Nagumo and the Rogers-McCulloch ionic models, obtaining existence and uniqueness theorems. Subsequently, we investigate optimal control problems subject to the monodomain equations. After proving the existence of global minimizers, the system of the first-order necesssary optimality conditions is rigorously characterized. For the adjoint system, we prove an existence and regularity theorem as well.

Key words. PDE constrained optimization, monodomain equations, Rogers-McCulloch model, uniqueness theorem, weak local minimizer, existence theorem, necessary optimality conditions.

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1. Introduction.

The present work opens a series of papers where we will set forth a basic framework for *optimal control of the bidomain equations* together with related uniqueness and regularity results. [01] The full bidomain system, which represents a well-established description of the electrical activity of the heart, is given by

$$\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \operatorname{div}\big(\, M_i \, \nabla \Phi_i \,\big) \; = \; I_i \quad \text{for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{1.1}$$

$$\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) + \operatorname{div}\big(\, M_e \, \nabla \Phi_e \,\big) \; = \; -I_e \quad \text{for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{1.2}$$

$$\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \; = \; 0 \quad \text{for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{1.3}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \, \nabla \Phi_i \; = \; 0 \;\text{ and }\; \mathfrak{n}^{\mathrm{T}} M_e \, \nabla \Phi_e \; = \; 0 \quad \text{for all } (x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{1.4}$$

$$\Phi_{tr}(x,0) \; = \; \Phi_0(x) \;\text{ and }\; W(x,0) \; = \; W_0(x) \quad \text{for a. a. } x \in \Omega\,, \tag{1.5}$$

together with appropriate specifications of the ionic current I_{ion} and the function G within the gating equation (1.3). [02] Within the cardiac muscle, which occupies the spatial domain $\Omega \subset \mathbb{R}^3$, the anisotropic properties of the intracellular and extracellular tissue parts will be described by conductivity tensors M_i and M_e. The variables $\Phi_i = \Phi_i(x,t)$ and $\Phi_e = \Phi_e(x,t)$ represent the intracellular and extracellular electrical potential; their difference $\Phi_{tr} = \Phi_i - \Phi_e$ is the transmembrane potential. Further, I_i and I_e model the intracellular and extracellular stimulation current, respectively. W, the so-called gating variable, is related to the ion transport through the cell membrane. On a microscopical level, the intracellular and extracellular quantities should be concentrated on disjoint subdomains Ω_i and Ω_e of Ω, whose common boundary represents the total of the cell membranes. [03] After an averaging procedure, [04] the macroscopic model (1.1) – (1.5) is obtained, where the superimposed intracellular and extracellular media occupy the same domain Ω.

The present paper is concerned with the *monodomain equations*, which arise from (1.1) – (1.5) as a special case if the conductivity tensors satisfy $M_e = \lambda \, M_i$ with a constant parameter $\lambda > 0$. Then Φ_e can be eliminated from (1.1) – (1.5), and we get the monodomain system

$$(\mathrm{M})_1 \quad \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \frac{\lambda}{1+\lambda} \operatorname{div}\big(\, M_i \, \nabla \Phi_{tr} \,\big) \; = \; \frac{1}{1+\lambda}\big(\, \lambda \, I_i - I_e \,\big) \quad \text{for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{1.6}$$

$$\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \; = \; 0 \quad \text{for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{1.7}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \, \nabla \Phi_{tr} \; = \; 0 \quad \text{for all } (x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{1.8}$$

$$\Phi_{tr}(x,0) \; = \; \Phi_0(x) \;\text{ and }\; W(x,0) \; = \; W_0(x) \quad \text{for a. a. } x \in \Omega \tag{1.9}$$

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[01] For an introduction to PDE-constrained optimal control problems, cf. [Ito/Kunisch 08] and [Tröltzsch 09].

[02] First considered in [Tung 78]. For a more detailed introduction to the model, we refer to [Sundnes/Lines/Cai/Nielsen/Mardal/Tveito 06], pp. 21 – 56, and the references therein.

[03] See [Colli Franzone/Savaré 02], pp. 49 – 52, and [Veneroni 06].

[04] Described in [Colli Franzone/Savaré 02], pp. 71 – 75.

as a considerable simplification of (1.1) – (1.5) which, nevertheless, conserves some essential features of the full bidomain model as excitability phenomena. For this reason, this system has been deserved considerable attention for itself.

We are now ready to state the optimal control problem to be investigated:

$$\text{(P)}\quad F(\Phi_{tr}, W, I_e) \,=\, \int_0^T \int_\Omega r\big(\,x, t, \Phi_{tr}(x,t), W(x,t)\,\big)\,dx\,dt + \frac{\mu}{2}\int_0^T \int_\Omega I_e(x,t)^2\,dx\,dt \longrightarrow \inf! \qquad (1.10)$$

subject to the state equations (1.6) – (1.9) in its weak formulation (see (2.3) – (2.5) below) and

$$\text{the control restriction } \big|\,I_e(x,t)\,\big| \leqslant R \text{ for a. a. } (x,t) \in \Omega \times [\,0\,,\,T\,] \qquad (1.11)$$

with $R > 0$ and a function $r(x, t, \varphi, w)$ to be specified below. A typical choice is a tracking-type integrand $r = \frac{1}{2}\big(\,\varphi - \Phi_{st}(x)\,\big)^2$ where Φ_{st} is taken, for example, from a steady-state solution (Φ_{st}, W_{st}) of (1.6) – (1.9). The control restriction reflects the obvious fact that one cannot apply arbitrary large electrical stimulations to living tissue without damaging it.

Although the bidomain system has been extensively studied under computational aspects, [05] only little work related to its optimal control is available in the literature as yet. Problem (1.10) – (1.11) was already considered in [Nagaiah/Kunisch/Plank 09] and [Nagaiah/Kunisch 11] . In these papers, the control problem has been successfully numerically accessed on the base of gradient or inexact Newton techniques, respectively, but the optimality system has been derived only formally without proof. [Ainseba/Bendahmane/Ruiz-Baier 10] study an optimal control problem on a tridomain model. Even here, the optimality conditions have been derived only formally. Another related control problem was investigated in [Brandão/Fernández-Cara/Magalhães/Rojas-Medar 08] where the authors study to a tracking-type functional, restricting themselves in (1.6) – (1.9) to the FitzHugh-Nagumo model for I_{ion} and G and replacing (1.8) by Dirichlet boundary conditions. In this particular case, the authors obtain necessary optimality conditions by means of the Dubovitskij-Milyutin formalism. In the context of defibrillation, [Muzdeka/Barbieri 05] pursued a different approach. After disregarding the nonlinearities, the authors perform a spectral approximation and solve a time-optimal control problem for the ODE, which arises for the lumped mass system resulting from the eigenmode expansion.

The structure of the paper is as follows. In the next Section 2, we study the existence and uniqueness of weak solutions to (1.5) – (1.9) for the Rogers-McCulloch and the FitzHugh-Nagumo model. [06] Even for the monodomain equations, these results may not be available in the literature. In part, they could be deduced from results on the bidomain system, but then one has to impose additional conditions on the spectral properties of the conductivity tensors. [07] In our approach, such conditions become unnecessary (Theorem 2.8.). In Section 3, we turn to the study of the related optimal control problems. After confirming the existence of global minimizers, we prove directly the existence of solutions for the adjoint equations. Then, treating (P) as a *weakly singular problem* in the sense of Ito/Kunisch, [08] we obtain in Theorem 3.7. the following set of first-order necessary optimality conditions for weak local minimizers $(\hat\Phi_{tr}, \hat W, \hat I_e)$ of (P), consisting of the variational inequality

$$\int_0^T \int_\Omega \Big(\,\mu\,\hat I_e(x,t) + \frac{1}{1+\lambda}\,P_1(x,t)\,\Big)\,\big(\,I_e(x,t) - \hat I_e(x,t)\,\big)\,dx\,dt \,\geqslant\, 0 \;\text{ for all feasible controls } I_e \qquad (1.12)$$

[05] We refer e. g. to [Colli Franzone/Deuflhard/Erdmann/Lang/Pavarino 06] , [Vigmond/Aguel/Trayanova 02] and [Weber dos Santos/Plank/Bauer/Vigmond 04] .

[06] In a subsequent publication, the linearized Aliev-Panfilov model will be considered as well, cf. [Aliev/Panfilov 96] and [Bourgault/Coudière/Pierre 09] , p. 480.

[07] Cf. [Boulakia/Fernández/Gerbeau/Zemzemi 08] , p. 8, (2.18), and [Bourgault/Coudière/Pierre 09] , p. 478 f., Theorem 32.

[08] [Ito/Kunisch 08] , p. 17 f.

and the adjoint system

$$-\frac{\partial P_1}{\partial t} - \nabla\cdot\big(M_i\,\nabla P_1 \big) + \frac{\partial I_{ion}}{\partial\varphi}(\hat\Phi_{tr},\hat W)\,P_1 \;=\; -\frac{\partial G}{\partial\varphi}(\hat\Phi_{tr},\hat W)\,P_2 - \frac{\partial r}{\partial\varphi}(\hat\Phi_{tr},\hat W) \;\text{ for a. a. } (x,t)\in\Omega_T\,; \quad (1.13)$$

$$\mathfrak{n}^{\mathrm T}\,M_i\,\nabla P_1 \;=\; 0 \;\text{ for all } (x,t)\in\partial\Omega\times[\,0\,,\,T\,]\,;\quad P_1(x,T)\;=\;0 \;\text{ for a. a. } x\in\Omega\,; \quad (1.14)$$

$$-\frac{\partial P_2}{\partial t} + \frac{\partial G}{\partial w}(\hat\Phi_{tr},\hat W)\,P_2 \;=\; -\frac{\partial I_{ion}}{\partial w}(\hat\Phi_{tr},\hat W)\,P_1 - \frac{\partial r}{\partial w}(\hat\Phi_{tr},\hat W) \;\text{ for a. a. } (x,t)\in\Omega_T\,; \quad (1.15)$$

$$P_2(x,T) \;=\; 0 \;\text{ for a. a. } x\in\Omega \quad (1.16)$$

for the multipliers P_1 and P_2 related to (1.6) and (1.7), respectively. The section is closed with the derivation of an a. e. pointwise formulated optimality condition. For the conveniency of the reader, we collect some facts about Bochner integrable mappings in an appendix (Section 4).

Notations.

We denote by $L^p(\Omega)$ the space of functions which are in the pth power integrable ($1 \leqslant p < \infty$), or are measurable and essentially bounded ($p = \infty$), and by $W^{1,p}(\Omega)$ the Sobolev space of functions $\psi\colon\ \Omega \to \mathbb{R}$ which, together with their first-order weak partial derivatives, belong to the space $L^p(\Omega,\mathbb{R})$ ($1 \leqslant p < \infty$). For spaces of Bochner integrable mappings, e. g. $L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big]$, we refer to Section 4. Ω_T is an abbreviation for $\Omega\times[\,0\,,\,T\,]$. The gradient ∇ is always taken only with respect to the spatial variables x. The abbreviation "$(\forall)\,t\in$ A" has to be read as "for almost all $t\in$ A" or "for all $t\in$ A except a Lebesgue null set", and the symbol $\mathfrak{o}$ denotes, depending on the context, the zero element or the zero function of the underlying space.

2. Weak solutions of the monodomain system.

a) The monodomain system.

Let $\Omega\subset\mathbb{R}^3$ be a bounded, open set. For the monodomain system (1.6) – (1.9) with Neumann boundary conditions, we introduce the following notion of a strong solution:

Definition 2.1. (Strong solution of the monodomain system) [09] Let $T > 0$. A pair (Φ_{tr}, W) is called a strong solution of the monodomain system $(\mathrm{M})_1$ on $[\,0\,,\,T\,]$ iff Φ_{tr} and W satisfy the equations in $(\mathrm{M})_1$ a. e. on $\Omega\times[\,0\,,\,T\,]$ as well as the initial and boundary conditions on $\partial\Omega\times[\,0\,,\,T\,]$, respectively. Moreover, the functions belong to the spaces

$$\Phi_{tr}\in W^{1,2}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]\,\cap\,L^2\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]\,; \quad (2.1)$$

$$W\in W^{1,2}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]\,\cap\,C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]\,. \quad (2.2)$$

The corresponding weak formulation of the monodomain system, on which the formulation of the optimal control problems in Section 3 will be based, reads as follows:

$$(\mathrm{M})_2 \quad \int_\Omega\Big(\frac{\partial\Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr},W)\Big)\,\psi\,dx + \int_\Omega\frac{\lambda}{1+\lambda}\,\nabla\psi^{\mathrm T}M_i\,\nabla\Phi_{tr}\,dx \;=\; \int_\Omega\frac{1}{1+\lambda}\,\big(\,\lambda\,I_i - I_e\,\big)\,\psi\,dx \quad (2.3)$$

$$\forall\,\psi\in W^{1,2}(\Omega)\;\;(\forall)\,t\in[\,0\,,\,T\,]\,;$$

[09] Slightly modified from [Bourgault/Coudière/Pierre 09], p. 469, Definition 18.

$$\int_\Omega \Big(\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \Big)\, \psi \, dx \;=\; 0 \quad \forall\, \psi \in L^2(\Omega) \;\; (\forall)\, t \in [\,0\,,\, T\,]\,; \tag{2.4}$$

$$\Phi_{tr}(x,0) \;=\; \Phi_0(x) \quad (\forall)\, x \in \Omega\,; \quad W(x,0) \;=\; W_0(x) \quad (\forall)\, x \in \Omega\,. \tag{2.5}$$

Definition 2.2. (Weak solution of the monodomain system) [10] Let $T > 0$. A pair (Φ_{tr}, W) is called a weak solution of the monodomain system $(\mathrm{M})_2$ on $[\,0\,,\,T\,]$ iff Φ_{tr} and W satisfy the equations in $(\mathrm{M})_2$ on $[\,0\,,\,T\,]$ in the distributional sense and obey the initial conditions. Moreover, the functions belong to the spaces

$$\Phi_{tr} \in C^0\big[\,[\,0\,,\,T\,]\,,\, L^2(\Omega)\,\big] \,\cap\, L^2\big[\,(\,0\,,\,T\,)\,,\, W^{1,2}(\Omega)\,\big] \,\cap\, L^p(\Omega_T) \quad \text{with } \; 2 \leqslant p \leqslant 6\,; \tag{2.6}$$

$$W \in C^0\big[\,[\,0\,,\,T\,]\,,\, L^2(\Omega)\,\big]\,. \tag{2.7}$$

Assumptions 2.3. (Basic assumptions about the data) About the data in $(\mathrm{M})_1$ and $(\mathrm{M})_2$, the following will be assumed.

1) $\Omega \subset \mathbb{R}^3$ is a bounded strongly Lipschitz domain.

2) $M_i :\; \mathrm{cl}\,(\Omega) \to \mathbb{R}^{3\times 3}$ is a symmetric, positive definite matrix function with $L^\infty(\Omega)$-coefficients, which obeys a uniform ellipticity condition with $\mu_1,\, \mu_2 > 0$:

$$0 \leqslant \mu_1 \,\|\,\xi\,\|^2 \leqslant \xi^{\mathrm{T}} M_i(x)\, \xi \leqslant \mu_2 \,\|\,\xi\,\|^2 \quad \forall\, \xi \in \mathbb{R}^3 \;\; \forall\, x \in \Omega \tag{2.8}$$

3) I_{ion} and G are affine-linear with respect to W with

$$I_{ion}(\varphi, w) = F_1(\varphi) + F_2(\varphi)\, w \quad \text{and} \quad G(\varphi, w) = G_1(\varphi) + g_2\, w \tag{2.9}$$

with continuous functions $F_1,\, F_2,\, G_1 :\; \mathbb{R} \to \mathbb{R}$ and $g_2 \in \mathbb{R}$.

4) The functions F_1, F_2 and G_1 obey the following growth conditions: For all $\varphi \in \mathbb{R}$, it holds that

$$\big|\, F_1(\varphi) \,\big| \;\leqslant\; c_1 + c_2 \,\big|\, \varphi \,\big|^{p-1}\,; \tag{2.10}$$

$$\big|\, F_2(\varphi) \,\big| \;\leqslant\; c_3 + c_4 \,\big|\, \varphi \,\big|^{p/2-1}\,; \tag{2.11}$$

$$\big|\, G_1(\varphi) \,\big| \;\leqslant\; c_5 + c_6 \,\big|\, \varphi \,\big|^{p/2} \tag{2.12}$$

with nonnegative constants $c_1,\, \dots\,,\, c_6 \geqslant 0$ and some $2 \leqslant p \leqslant 6$. Further, for all $\varphi,\, w \in \mathbb{R}$, it holds that

$$a\,|\,\varphi\,|^p - b\,\big(\,\varrho\,|\,\varphi\,|^2 + |\,w\,|^2\,\big) - c \;\leqslant\; \lambda\,\big(\,F_1(\varphi) + F_2(\varphi)\, w\,\big)\cdot \varphi + \big(\,G_1(\varphi) + g_2\, w\,\big)\cdot w \tag{2.13}$$

with constants $a > 0$, $\varrho > 0$, $b,\, c \geqslant 0$ and $2 \leqslant p \leqslant 6$ as above.

5) The initial values Φ_0, W_0 belong to the space $L^2(\Omega)$.

6) I_i and I_e belong to $L^2\big[\,(\,0\,,\,T\,)\,,\, \big(\,W^{1,2}(\Omega)\,\big)^*\,\big]$.

Let us remark that the monodomain system is solvable without the compatibility condition $\int_\Omega \big(\, I_i(x,t) + I_e(x,t)\,\big)\, dx = 0 \;(\forall)\, t \in (\,0\,,\,T\,)$, which is a mandatory assumption in the full bidomain case.

b) The models for the ionic current.

The ionic current through the cell membranes will be described with the help of a so-called gating variable W, which is coupled with the transmembrane voltage Φ_{tr} by an ODE. We will consider the following models:

[10] See [Bourgault/Coudière/Pierre 09], p. 472, Definition 26.

a) The Rogers-McCulloch model. [11]

$$I_{ion}(\varphi, w) \,=\, b \cdot \varphi\,(\varphi - a)\,(\varphi - 1) + \varphi \cdot w \,=\, b\,\varphi^3 - (a+1)\,b\,\varphi^2 + a\,b\,\varphi + \varphi\,w\,; \tag{2.14}$$

$$G(\varphi, w) \,=\, \varepsilon\,w - \varepsilon\,\kappa\,\varphi \tag{2.15}$$

with $0 < a < 1$, $b > 0$, $\kappa > 0$ and $\varepsilon > 0$. Consequently, the gating variable obeys the linear ODE

$$\frac{\partial W}{\partial t} + \varepsilon\,W \,=\, \varepsilon\,\kappa\,\Phi_{tr}\,. \tag{2.16}$$

b) The FitzHugh-Nagumo model. [12]

$$I_{ion}(\varphi, w) \,=\, \varphi\,(\varphi - a)\,(\varphi - 1) + w \,=\, \varphi^3 - (a+1)\,\varphi^2 + a\,\varphi + w\,; \tag{2.17}$$

$$G(\varphi, w) \,=\, \varepsilon\,w - \varepsilon\,\kappa\,\varphi \tag{2.18}$$

with $0 < a < 1$, $\kappa > 0$ and $\varepsilon > 0$. Consequently, the gating variable obeys the same linear ODE (2.16) as before.

Proposition 2.4. (Analytical properties of the ionic current models) [13] *The Rogers-McCulloch model and the FitzHugh-Nagumo model satisfy Assumptions 2.3., 3) and 4) with* $p = 4$.

c) Existence and uniqueness of weak solutions for the monodomain system.

Theorems 2.5. and 2.6. can be obtained by slight modifications of the arguments presented in [Bourgault/ Coudière/Pierre 09] and [Nagaiah/Kunisch/Plank 09].

Theorem 2.5. (Existence of weak solutions) [14] *Assume that the data within* $(\mathrm{M})_2$ *obey Assumptions 2.3., 1) – 6) with* $p = 4$. *Then for arbitrary initial values* Φ_0, $W_0 \in L^2(\Omega)$, *the monodomain system* $(\mathrm{M})_2$ *admits on* $[\,0\,,\,T\,]$ *at least one weak solution* (Φ_{tr}, W) *in the sense of Definition 2.2. Consequently, for both models from Subsection 2.b), weak solutions with* $p = 4$ *exist.*

Theorem 2.6. (A priori estimate for weak solutions) [15] *Assume that the data within* $(\mathrm{M})_2$ *obey Assumptions 2.3., 1) – 6) together with* $p = 4$. *If a pair*

$$(\Phi_{tr}, W) \in \Big(C^0\big[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\big] \,\cap\, L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\big] \,\cap\, L^p(\Omega_T) \Big) \,\times\, C^0\big[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\big] \tag{2.19}$$

forms a weak solution of the monodomain system $(\mathrm{M})_2$ *on* $[\,0,\ T\,]$ *then the following estimate holds:*

$$\begin{aligned} &\|\,\Phi_{tr}\,\|^2_{C^0[\,[0,T),\,L^2(\Omega)]} + \|\,\Phi_{tr}\,\|^2_{L^2[\,(0,T),\,W^{1,2}(\Omega)]} + \|\,\Phi_{tr}\,\|^p_{L^p(\Omega_T)} + \|\,\partial\Phi_{tr}/\partial t\,\|^q_{L^q[\,(0,T),\,(W^{1,2}(\Omega))^*]} \\ &\qquad + \|\,W\,\|^2_{C^0[\,[0,T),\,L^2(\Omega)]} + \|\,\partial W/\partial t\,\|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*]} \\ &\leqslant C \cdot \Big(1 + \|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,I_i\,\|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*]} + \|\,I_e\,\|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*]} \Big) \end{aligned} \tag{2.20}$$

where $q = 4/3$. *The constant* $C > 0$ *does not depend on* Φ_0, W_0, I_i *and* I_e.

The following uniqueness theorem, which is based on an error estimate for the weak solutions, allows to dispense with the eigenvalue conditions for the Jacobi matrix of the model functions I_{ion} and G.

11) [Rogers/McCulloch 94].

12) [FitzHugh 61], together with [Nagumo/Arimoto/Yoshizawa 62].

13) [Bourgault/Coudière/Pierre 09], pp. 479 – 481.

14) Compare with [Bourgault/Coudière/Pierre 09], p. 473, Theorem 30.

15) Compare with [Nagaiah/Kunisch/Plank 09], p. 10, Lemma 3.5.

Theorem 2.7. (Error estimates for weak solutions) *Assume that the data within* $(\mathrm{M})_2$ *obey Assumptions 2.3., 1) − 6) together with* $p = 4$*, and specify within* $(\mathrm{M})_2$ *one of the models from Subsection 2.b). If two weak solutions* $({\Phi_{tr}}', W')$, $({\Phi_{tr}}'', W'') \in \big(C_0^0\big[\,[\,0\,,T\,]\,,\, L^2(\Omega)\big] \,\cap\, L^2\big[\,(\,0\,,T\,)\,,W^{1,2}(\Omega)\big] \,\cap\, L^p(\Omega_T)\big) \times C^0\big[\,[\,0\,,T\,]\,,\, L^2(\Omega)\big]$ *of* $(\mathrm{M})_2$ *correspond with initial values* $\Phi_0' = \Phi_0'' = \Phi_0 \in L^2(\Omega)$, $W_0' = W_0'' = W_0 \in L^4(\Omega)$ *and inhomogeneities* ${I_i}'$, ${I_e}'$, ${I_i}''$ *and* ${I_e}'' \in L^\infty\big[\,(\,0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]$*, whose norms are bounded by* $R > 0$*, then the following estimates hold:*

$$\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,W^{1,2}(\Omega)\right]} + \|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{C^0\left[\,[\,0\,,T\,]\,,L^2(\Omega)\right]} \tag{2.21}$$
$$+ \|\,W' - W''\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,L^2(\Omega)\right]} + \|\,W' - W''\,\|^2_{C^0\left[\,[\,0\,,T\,]\,,L^2(\Omega)\right]} + \|\,W' - W''\,\|^2_{W^{1,2}\left[\,(\,0\,,T\,)\,,L^2(\Omega)\right]}$$
$$\leqslant C\Big(\,\|\,{I_i}' - {I_i}''\,\|^2_{L^\infty\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]} + \|\,{I_e}' - {I_e}''\,\|^2_{L^\infty\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]}\,\Big)\,.$$

$$\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|_{W^{1,4/3}\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]} \leqslant C\cdot \mathrm{Max}\Big(\,\|\,{I_i}' - {I_i}''\,\|_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]}\,, \tag{2.22}$$
$$\|\,{I_e}' - {I_e}''\,\|_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]}\,,\ \|\,{I_i}' - {I_i}''\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]}\,,\ \|\,{I_e}' - {I_e}''\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\right]}\,\Big)\,.$$

Theorem 2.8. (Uniqueness of weak solutions) *Assume that the data within* $(\mathrm{M})_2$ *obey Assumptions 2.3., 1) − 6) together with* $p = 4$*, and specify within* $(\mathrm{M})_2$ *one of the models from Subsection 2.b). Then for initial values* $\Phi_0 \in L^2_{(\Omega)}$, $W_0 \in L^4(\Omega)$ *and inhomogeneities* I_i, $I_e \in L^\infty\big[\,(\,0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]$*, the monodomain system* $(\mathrm{M})_2$ *admits a unique weak solution* (Φ_{tr}, W) *in the sense of Definition 2.2. on* $[\,0\,,T\,]$.

d) Proofs.

Throughout the following proofs, C denotes a generical positive constant, which may appropriately change from line to line. C will never depend on the data Φ_0, W_0, I_i and I_e but, possibly, on Ω and $p = 4$.

Proof of Theorem 2.5. Observe first that the reformulated bidomain system in [Bourgault/Coudière/Pierre 09] , p. 473, Lemma 28, and the monodomain system $(\mathrm{M})_2$ have the same structure. In $(\mathrm{M})_2$, however, the bilinear form $M:\ W^{1,2}(\Omega)\times W^{1,2}(\Omega) \to \mathbb{R}$ reads as

$$M(\psi_1\,,\,\psi_2) \;=\; \frac{\lambda}{1+\lambda}\int_\Omega \nabla\psi_1^{\mathrm{T}} M_i\,\nabla\psi_2\,dx\,. \tag{2.23}$$

Lemma 2.9. [16] *The bilinear form* M *is symmetric, continuous and coercive, satisfying with* β, $\gamma > 0$

$$\beta\,\|\,\psi\,\|^2_{W^{1,2}(\Omega)} \;\leqslant\; M(\psi\,,\,\psi) + \beta\,\|\,\psi\,\|^2_{L^2(\Omega)} \quad \forall\,\psi\in W^{1,2}(\Omega) \quad \text{and} \tag{2.24}$$

$$\big|\,M(\psi_1\,,\,\psi_2)\,\big| \;\leqslant\; \gamma\cdot\|\,\psi_1\,\|_{W^{1,2}(\Omega)}\cdot\|\,\psi_2\,\|_{W^{1,2}(\Omega)} \quad \forall\,\psi_1\,,\ \psi_2\in W^{1,2}(\Omega)\,. \tag{2.25}$$

Proof. As a consequence of Assumption 2.3., 2), we have

$$\frac{\lambda\,\mu_1}{1+\lambda}\int_\Omega \big|\,\nabla\psi\,\big|^2\,dx \;\leqslant\; \frac{\lambda}{1+\lambda}\int_\Omega \nabla\psi^{\mathrm{T}} M_i\,\nabla\psi\,dx \;=\; M(\psi\,,\,\psi) \quad \forall\,\psi\in W^{1,2}(\Omega) \quad \Longrightarrow \tag{2.26}$$
$$\frac{\lambda\,\mu_1}{1+\lambda}\,\|\,\psi\,\|^2_{W^{1,2}(\Omega)} \;\leqslant\; M(\psi\,,\,\psi) + \frac{\lambda\,\mu_1}{1+\lambda}\,\|\,\psi\,\|^2_{L^2(\Omega)} \quad \forall\,\psi\in W^{1,2}(\Omega)\,.$$

The uniform ellipticity of M_i implies the second inequality as well. ∎

Obviously, the form M generates a weak operator on $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$. Consequently, the existence proof from [Bourgault/Coudière/Pierre 09] , pp. 473 ff., Subsections 5.2.1. − 5.3. can be carried over to $(\mathrm{M})_2$ after replacing the bidomain bilinear form by M. ∎

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16) Compare with [Bourgault/Coudière/Pierre 09] , p. 464, Theorem 6.

Proof of Theorem 2.6. For the same reasons as in the proof of Theorem 2.5., the arguments from [NAGAIAH/KUNISCH/PLANK 09] , p. 10 f., Lemma 3.5., as well as the underlying estimates from [BOURGAULT/COUDIÈRE/PIERRE 09] , pp. 474 – 476, may be carried over to the monodomain system $(\mathrm{M})_2$. ∎

Proof of Theorem 2.7. The proof will be divided in two parts according to the underlying ionic current model.

Part A. The Rogers-McCulloch model.

• **Step A1.** *The difference of the parabolic equations.* The pairs $({\Phi_{tr}}', W')$ and $({\Phi_{tr}}'', W'')$ satisfy for almost all $t \in [\,0\,,\,T\,]$ the equations

$$\langle \frac{d}{dt}\,{\Phi_{tr}}'(t)\,,\,\psi\,\rangle + M\big(\,{\Phi_{tr}}'(t)\,,\,\psi\,\big) + \int_\Omega I_{ion}({\Phi_{tr}}'(t), W'(t))\,\psi\,dx \;=\; \langle\,\frac{1}{1+\lambda}\,\big(\,\lambda\,{I_i}'(t) - {I_e}'(t)\,\big)\,,\,\psi\,\rangle \qquad (2.27)$$
$$\forall\,\psi \in W^{1,2}(\Omega)\,;$$

$$\langle \frac{d}{dt}\,{\Phi_{tr}}''(t)\,,\,\psi\,\rangle + M\big(\,{\Phi_{tr}}''(t)\,,\,\psi\,\big) + \int_\Omega I_{ion}({\Phi_{tr}}''(t), W''(t))\,\psi\,dx \;=\; \langle\,\frac{1}{1+\lambda}\,\big(\,\lambda\,{I_i}''(t) - {I_e}''(t)\,\big)\,,\,\psi\,\rangle \qquad (2.28)$$
$$\forall\,\psi \in W^{1,2}(\Omega)\,.$$

Consequently, we obtain the equation

$$\langle \frac{d}{dt}\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,,\,\psi\,\rangle + M\big(\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,,\,\psi\,\big) + \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}'(t), W'(t)) - I_{ion}({\Phi_{tr}}''(t), W''(t))\,\Big)\,\psi\,dx$$
$$= \langle\,\frac{1}{1+\lambda}\,\Big(\,\lambda\,\big(\,{I_i}'(t) - {I_i}''(t)\,\big) - \big(\,{I_e}'(t) - {I_e}''(t)\,\big)\,\Big)\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega)\,. \qquad (2.29)$$

Since ${\Phi_{tr}}'(t)$, ${\Phi_{tr}}''(t) \in W^{1,2}(\Omega)$, we may insert $\psi = {\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)$ as a feasible test function into these equations. Using the constant $\beta > 0$ from Lemma 2.9., (2.24), we arrive at

$$\frac{1}{2}\,\frac{d}{dt}\,\|\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\|^2_{L^2(\Omega)} + M\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big) + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+ \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'')\,\Big)\,\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)\,dx$$
$$= \langle\,\frac{1}{1+\lambda}\,\Big(\,\lambda\,\big(\,{I_i}' - {I_i}''\,\big) - \big(\,{I_e}' - {I_e}''\,\big)\,\Big)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} \quad \Longrightarrow \qquad (2.30)$$

$$\frac{1}{2}\,\frac{d}{dt}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'')\,\Big)\,\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)\,dx$$
$$\leqslant\,\Big|\,\langle\,\frac{1}{1+\lambda}\,\Big(\,\lambda\,\big(\,{I_i}' - {I_i}''\,\big) - \big(\,{I_e}' - {I_e}''\,\big)\,\Big)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle\,\Big| + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}\,. \qquad (2.31)$$

The first term on the right-hand side can be estimated through

$$\Big|\,\langle\,\frac{1}{1+\lambda}\,\Big(\,\lambda\,\big(\,{I_i}'(t) - {I_i}''(t)\,\big) - \big(\,{I_e}'(t) - {I_e}''(t)\,\big)\,\Big)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle\,\Big| \;\leqslant\; \frac{1}{2\varepsilon}\cdot C\,\Big(\,\|\,{I_i}'(t) - {I_i}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}$$
$$+\,\|\,{I_e}'(t) - {I_e}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,\Big) + \frac{3\varepsilon}{4}\,\|\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\|^2_{W^{1,2}(\Omega)} \qquad (2.32)$$

with arbitrary $\varepsilon_1 > 0$. The second term will be estimated with the help of the following lemma.

Lemma 2.10. *For all φ_1, $\varphi_2 \in \mathbb{R}$, the following identity holds:*

$$\big(\,\varphi_1^3 - (a+1)\,\varphi_1^2 + a\,\varphi_1\,\big) - \big(\,\varphi_2^3 - (a+1)\,\varphi_2^2 + a\,\varphi_2\,\big)$$
$$= (\varphi_1 - \varphi_2)\cdot\big(\,\varphi_1^2 + \varphi_1\,\varphi_2 + \varphi_2^2 - (a+1)\,(\varphi_1 + \varphi_2) + a\,\big)\,. \quad ∎ \qquad (2.33)$$

Consequently, we get

$$\int_\Omega \Big(I_{ion}(\Phi_{tr}{}', W') - I_{ion}(\Phi_{tr}{}'', W'') \Big) \left(\Phi_{tr}{}' - \Phi_{tr}{}'' \right) dx \tag{2.34}$$
$$= \int_\Omega (\Phi_{tr}{}' - \Phi_{tr}{}'')\, b \left((\Phi_{tr}{}')^2 + \Phi_{tr}{}' \,\Phi_{tr}{}'' + (\Phi_{tr}{}'')^2 + a \right) (\Phi_{tr}{}' - \Phi_{tr}{}'')\, dx$$
$$- (a+1)\, b \int_\Omega (\Phi_{tr}{}' - \Phi_{tr}{}'') \left(\Phi_{tr}{}' + \Phi_{tr}{}'' \right) (\Phi_{tr}{}' - \Phi_{tr}{}'')\, dx + \int_\Omega \left(\Phi_{tr}{}' \,W' - \Phi_{tr}{}'' \,W'' \right) (\Phi_{tr}{}' - \Phi_{tr}{}'')\, dx\,.$$

Since $\Phi_{tr}{}'(x,t)^2 + \Phi_{tr}{}'(x,t)\,\Phi_{tr}{}''(x,t) + \Phi_{tr}{}''(x,t)^2 \geqslant 0$ for almost all $(x,t) \in \Omega_T$ and a, $b > 0$, the inequalities (2.31), (2.32) and (2.34) imply

$$\frac{d}{dt} \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} + 2\,\beta \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} \leqslant 2\,C \int_\Omega \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right| \cdot \left| \Phi_{tr}{}' + \Phi_{tr}{}'' \right| \cdot \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right| dx$$
$$+ \; 2 \int_\Omega \left(\Phi_{tr}{}' \,W' - \Phi_{tr}{}'' \,W'' \right) (\Phi_{tr}{}'' - \Phi_{tr}{}')\, dx \; + \; \frac{C}{\varepsilon_1} \left(\| I_i{}' - I_i{}'' \|^2_{\left(W^{1,2}(\Omega) \right)^*} + \| I_e{}' - I_e{}'' \|^2_{\left(W^{1,2}(\Omega) \right)^*} \right)$$
$$+ \frac{3\varepsilon_1}{2} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} + 2\,\beta \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)}\,. \tag{2.35}$$

Applying the generalized Cauchy's inequality with $\varepsilon_2 > 0$ to the first term on the right-hand side of (2.35), we get

$$2\,C \int_\Omega \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right| \cdot \left| \Phi_{tr}{}' + \Phi_{tr}{}'' \right| \cdot \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right| dx$$
$$\leqslant C\, \frac{\varepsilon_2}{2} \int_\Omega \left| \Phi_{tr}{}' + \Phi_{tr}{}'' \right|^2 \cdot \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right|^2 dx + \frac{C}{2\,\varepsilon_2} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} \tag{2.36}$$
$$\leqslant C\, \frac{\varepsilon_2}{2} \left(\int_\Omega \left| \Phi_{tr}{}' + \Phi_{tr}{}'' \right|^4 dx \right)^{1/2} \left(\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^4_{L^4(\Omega)} \right)^{1/2} + \frac{C}{2\,\varepsilon_2} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} \tag{2.37}$$
$$\leqslant C\, \frac{\varepsilon_2}{2} \left(\int_\Omega \left| \Phi_{tr}{}' + \Phi_{tr}{}'' \right|^4 dx \right)^{1/2} \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} + \frac{C}{2\,\varepsilon_2} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)}\,. \tag{2.38}$$

The second term on the right-hand side of (2.35) will be estimated through

$$2 \int_\Omega \left(\Phi_{tr}{}' \,W' \pm \Phi_{tr}{}'' \,W' - \Phi_{tr}{}'' \,W'' \right) (\Phi_{tr}{}'' - \Phi_{tr}{}')\, dx$$
$$= 2 \int_\Omega (\Phi_{tr}{}' - \Phi_{tr}{}'')\, W' \,(\Phi_{tr}{}' - \Phi_{tr}{}'')\, dx + 2 \int_\Omega (W' - W'')\, \Phi_{tr}{}'' \,(\Phi_{tr}{}' - \Phi_{tr}{}'')\, dx \tag{2.39}$$
$$\leqslant \varepsilon_3 \int_\Omega \left(W' \right)^2 \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right|^2 dx + \frac{1}{\varepsilon_3} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} \tag{2.40}$$
$$+ \,\varepsilon_4 \int_\Omega \left(\Phi_{tr}{}'' \right)^2 \left| \Phi_{tr}{}' - \Phi_{tr}{}'' \right|^2 dx + \frac{1}{\varepsilon_4} \,\| W' - W'' \|^2_{L^2(\Omega)}$$
$$\leqslant \varepsilon_3 \left(\int_\Omega \left(W' \right)^4 dx \right)^{1/2} \left(\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^4_{L^4_{(\Omega)}} \right)^{1/2} + \frac{1}{\varepsilon_3} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} \tag{2.41}$$
$$+ \,\varepsilon_4 \left(\int_\Omega \left(\Phi_{tr}{}'' \right)^4 dx \right)^{1/2} \left(\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^4_{L^4(\Omega)} \right)^{1/2} + \frac{1}{\varepsilon_4} \,\| W' - W'' \|^2_{L^2(\Omega)}$$
$$\leqslant C\,\varepsilon_3 \,\| W' \|^2_{L^4(\Omega)} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_3} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2(\Omega)} \tag{2.42}$$
$$+ \,C\,\varepsilon_4 \,\| \Phi_{tr}{}'' \|^2_{L^4(\Omega)} \,\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_4} \,\| W' - W'' \|^2_{L^2(\Omega)}\,,$$

applying the generalized Cauchy's inequality again with ε_3, $\varepsilon_4 > 0$. Assembling now (2.35), (2.38) and (2.42), we arrive at

$$\frac{d}{dt}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + 2\,\beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} \tag{2.43}$$
$$\leqslant C\,\frac{\varepsilon_2}{2}\,\Big(\,\|\,{\Phi_{tr}}'\,\|^4_{L^4(\Omega)} + \|\,{\Phi_{tr}}''\,\|^4_{L^4(\Omega)}\Big)^{1/2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{C}{2\,\varepsilon_2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+ \; C\,\varepsilon_3\,\|\,W'\,\|^2_{L^4(\Omega)}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_3}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+ \; C\,\varepsilon_4\,\|\,{\Phi_{tr}}''\,\|^2_{L^4(\Omega)}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_4}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}$$
$$+ \; \frac{C}{\varepsilon_1}\,\Big(\,\|\,{I_i}'(t) - {I_i}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,{I_e}'(t) - {I_e}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)$$
$$+ \; \frac{3\varepsilon_1}{2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + 2\,\beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}\,.$$

Analogously to [Nagaiah/Kunisch/Plank 09], p. 10, (33), with the aid of (2.13) we may derive

$$\frac{\varrho\,\beta}{4}\,\|\,\Phi_{tr}(t)\,\|^2_{W^{1,2}(\Omega)} + \int_\Omega a\,|\,\Phi_{tr}(t)\,|^p\,dx$$
$$\leqslant C\,\Big(\,\varrho\,\|\,\Phi_{tr}(0)\,\|^2_{L^2(\Omega)} + \|\,W(0)\,\|^2_{L^2(\Omega)} + \int_0^t c\,|\,\Omega\,|\,d\tau + C\int_0^t \|\,I_i(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,d\tau \tag{2.44}$$
$$+ C\int_0^t \|\,I_e(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,d\tau\Big) + c\,|\,\Omega\,| + \frac{\varrho\,C}{2\beta}\,\Big(\,\|\,I_i(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)\,.$$

Due to (2.44), the L^4-norms of ${\Phi_{tr}}'(t)$, ${\Phi_{tr}}''(t)$ are bounded through

$$\int_\Omega {\Phi_{tr}}'(t)^4\,dx \;\leqslant\; C\,\Big(\,1 + \varrho\,\|\,{\Phi_{tr}}'(0)\,\|^2_{L^2(\Omega)} + \|\,W'(0)\,\|^2_{L^2(\Omega)} + C\int_0^T \big(\,\|\,{I_i}'(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}$$
$$+ \|\,{I_e}'(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\big)\,d\tau\Big) \;+\; c\,|\,\Omega\,| + \frac{\varrho\,C}{2\beta}\,\Big(\,\|\,{I_i}'(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,{I_e}'(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big) \tag{2.45}$$
$$\leqslant C\,\Big(\,1 + \varrho\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,{I_i}'\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,{I_e}'\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}$$
$$+ \|\,{I_i}'\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,{I_e}'\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\Big)\,; \tag{2.46}$$

$$\int_\Omega {\Phi_{tr}}''(t)^4\,dx \;\leqslant\; C\,\Big(\,1 + \varrho\,\|\,{\Phi_{tr}}''(0)\,\|^2_{L^2(\Omega)} + \|\,W''(0)\,\|^2_{L^2(\Omega)} + C\int_0^T \big(\,\|\,{I_i}''(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}$$
$$+ \|\,{I_e}''(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\big)\,d\tau\Big) \;+\; c\,|\,\Omega\,| + \frac{\varrho\,C}{2\beta}\,\Big(\,\|\,{I_i}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,{I_e}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big) \tag{2.47}$$
$$\leqslant C\,\Big(\,1 + \varrho\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,{I_i}''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,{I_e}''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}$$
$$+ \|\,{I_i}''\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,{I_e}''\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\Big)\,. \tag{2.48}$$

Here the assumed L^∞-regularity of the excitations ${I_i}'$, ${I_e}'$, ${I_i}''$ and ${I_e}''$ has been used. As the unique (weak or strong) solution of the initial value problem (1.7), (1.9), W' admits the representation [17)]

$$W'(x,t) \;=\; W_0(x)\,e^{-\varepsilon t} + \varepsilon\,\kappa\,e^{-\varepsilon t}\int_0^t {\Phi_{tr}}'(x,\tau)\,e^{\varepsilon\tau}\,d\tau\,, \tag{2.49}$$

consequently, it belongs to the space $C^1\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$. Together with Theorem 2.6., the L^4-norm of $W'(t)$ can be estimated through

17) [Warga 72], p. 192, Theorem II.4.6.

$$\int_\Omega W'(t)^4\,dx \leqslant C\,\|\,W_0\,\|^4_{L^4(\Omega)} + C\,\varepsilon\,\kappa\,\|\,\Phi_{tr}{}'\,\|^4_{L^4(\Omega_T)} \tag{2.50}$$

$$\leqslant C\,\|\,W_0\,\|^4_{L^4(\Omega)} + C\,\varepsilon\,\kappa\,\Big(1 + \|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} \tag{2.51}$$
$$+ \|\,I_i{}'\,\|^2_{L^2\left[\,(0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_e{}'\,\|^2_{L^2\left[\,(0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\Big)\,.$$

Inserting now (2.46), (2.48) and (2.51) into (2.43), we may fix the numbers ε_1, ε_2, ε_3, $\varepsilon_4 > 0$ in such a way that the terms with $\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{W^{1,2}(\Omega)}$ will be annihilated. We arrive at

$$\frac{d}{dt}\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} \leqslant \Big(\frac{C}{2\,\varepsilon_2} + \frac{1}{\varepsilon_3} + 2\,\beta\Big)\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} \tag{2.52}$$
$$+ \frac{1}{\varepsilon_4}\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon_1}\,\Big(\,\|\,I_i{}'(t) - I_i{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e{}'(t) - I_e{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)\,.$$

• **Step A2.** *The difference of the gating equations.* The weak solutions $(\Phi_{tr}{}', W')$ and $(\Phi_{tr}{}'', W'')$ satisfy for almost all $t \in [\,0\,,\,T\,]$

$$\langle\,\frac{d}{dt}\,W'(t)\,,\,\psi\,\rangle = -\int_\Omega \big(\,\varepsilon\,W'(t) - \varepsilon\,\kappa\,\Phi_{tr}{}'(t)\,\big)\,\psi\,dx \quad \forall\,\psi \in L^2(\Omega)\,; \tag{2.53}$$

$$\langle\,\frac{d}{dt}\,W''(t)\,,\,\psi\,\rangle = -\int_\Omega \big(\,\varepsilon\,W''(t) - \varepsilon\,\kappa\,\Phi_{tr}{}''(t)\,\big)\,\psi\,dx \quad \forall\,\psi \in L^2(\Omega) \quad \Longrightarrow \tag{2.54}$$

$$\langle\,\frac{d}{dt}\,W'(t) - W''(t)\,,\,\psi\,\rangle = -\,\varepsilon\,\int_\Omega \big(\,W'(t) - W''(t)\,\big)\,\psi\,dx \tag{2.55}$$
$$+\,\varepsilon\,\kappa\,\int_\Omega \big(\,\Phi_{tr}{}'(t) - \Phi_{tr}{}''(t)\,\big)\,\psi\,dx \quad \forall\,\psi \in L^2(\Omega)\,.$$

Here the test function $\psi = W'(t) - W''(t)$ is feasible; consequently, we get

$$\langle\,\frac{d}{dt}\,\big(\,W'(t) - W''(t)\,\big)\,,\,W'(t) - W''(t)\,\rangle \tag{2.56}$$
$$= -\,\varepsilon\,\int_\Omega \big(\,W' - W''\,\big)^2\,dx + \varepsilon\,\kappa\,\int_\Omega \big(\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\big)\,\big(\,W' - W''\,\big)\,dx \quad \Longrightarrow$$

$$\frac{d}{dt}\,\Big(\,\frac{1}{2}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \leqslant \varepsilon\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\int_\Omega \big|\,\Phi_{tr}{}'(t) - \Phi_{tr}{}''(t)\,\big|\cdot\big|\,W'(t) - W''(t)\,\big|\,dx \quad \Longrightarrow \tag{2.57}$$

$$\frac{d}{dt}\,\Big(\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \leqslant \Big(2\,\varepsilon + \varepsilon\,\kappa\Big)\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)}\,. \tag{2.58}$$

• **Step A3.** *The estimates for the differences* $\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{C^0\left[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\right]}$, $\|\,W' - W''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}$ *and* $\|\,W' - W''\,\|^2_{C^0\left[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\right]}$. After equalization of the constants on the right-hand sides, the inequalities (2.52) and (2.58) yield together

$$\frac{d}{dt}\,\Big(\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \leqslant C\cdot\Big(\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \tag{2.59}$$
$$+ \frac{C}{\varepsilon_1}\,\Big(\,\|\,I_i{}'(t) - I_i{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e{}'(t) - I_e{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)\,,$$

and Gronwall's inequality finally implies that

$$\|\,\Phi_{tr}{}'(t) - \Phi_{tr}{}''(t)\,\|^2_{L^2(\Omega)} + \|\,W'(t) - W''(t)\,\|^2_{L^2(\Omega)} \leqslant e^{C\,t}\,\Big(\,\|\,\Phi_{tr}{}'(0) - \Phi_{tr}{}''(0)\,\|^2_{L^2(\Omega)} \tag{2.60}$$
$$+ \|\,W'(0) - W''(0)\,\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon_1}\int_0^t \big(\,\|\,I_i{}'(\tau) - I_i{}''(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e{}'(\tau) - I_e{}''(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,\big)\,d\tau\Big)$$

$$\leqslant e^{C\,T}\,\frac{C}{\varepsilon_1}\,\Big(\,\|\,I_i{}' - I_i{}''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_e{}' - I_e{}''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\Big)\,. \tag{2.61}$$

From the last inequality, we get the following estimates:

$$\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{C^0\big[[0,T],L^2(\Omega)\big]} \leqslant e^{CT}\,\frac{C}{\varepsilon_1}\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big); \tag{2.62}$$

$$\|W' - W''\|^2_{C^0\big[[0,T],L^2(\Omega)\big]} \leqslant e^{CT}\,\frac{C}{\varepsilon_1}\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big); \tag{2.63}$$

$$\|W' - W''\|^2_{L^2\big[(0,T),L^2(\Omega)\big]} \leqslant T\,e^{CT}\,\frac{C}{\varepsilon_1}\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big). \tag{2.64}$$

• **Step A4.** *The estimate for the difference* $\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2\big[(0,T),W^{1,2}(\Omega)\big]}$. In (2.43), the numbers ε_1, ... , $\varepsilon_4 > 0$ may be alternatively chosen in such a way that

$$\frac{d}{dt}\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} + \beta\,\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{W^{1,2}(\Omega)} \leqslant \Big(\frac{C}{2\,\varepsilon_2} + \frac{1}{\varepsilon_3} + 2\,\beta\Big)\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon_4}\|W' - W''\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon_1}\Big(\|I_i{}'(t) - I_i{}''(t)\|^2_{\big(W^{1,2}(\Omega)\big)^*} + \|I_e{}'(t) - I_e{}''(t)\|^2_{\big(W^{1,2}(\Omega)\big)^*}\Big). \tag{2.65}$$

This implies the following modification of (2.59):

$$\frac{d}{dt}\Big(\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} + \|W' - W''\|^2_{L^2(\Omega)}\Big) + \beta\,\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{W^{1,2}(\Omega)} \leqslant C\cdot\Big(\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} + \|W' - W''\|^2_{L^2(\Omega)}\Big) + \frac{C}{\varepsilon_1}\Big(\|I_i{}'(t) - I_i{}''(t)\|^2_{\big(W^{1,2}(\Omega)\big)^*} + \|I_e{}'(t) - I_e{}''(t)\|^2_{\big(W^{1,2}(\Omega)\big)^*}\Big). \tag{2.66}$$

Together with (2.61), we arrive at:

$$\|\Phi_{tr}{}'(t) - \Phi_{tr}{}''(t)\|^2_{W^{1,2}(\Omega)} \leqslant C\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big) \implies \tag{2.67}$$

$$\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2\big[(0,T),W^{1,2}(\Omega)\big]} \leqslant C\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big). \tag{2.68}$$

• **Step A5.** *The estimate for the difference* $\|W' - W''\|^2_{W^{1,2}\big[(0,T),L^2(\Omega)\big]}$. Into equation (2.55), we insert the test function $\psi = (\partial W'(t)/\partial t) - (\partial W''(t)/\partial t)$, which obviously belongs to $L^2(\Omega_T)$ and is therefore admissible. Then we get with the generalized Cauchy's inequality

$$\langle\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}, \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\rangle = \|\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\|^2_{L^2(\Omega)} \leqslant \varepsilon\,\frac{\varepsilon_5}{2}\|\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\,\varepsilon_5}\|W' - W''\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\,\frac{\varepsilon_6}{2}\|\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\|^2_{L^2(\Omega)} + \frac{\varepsilon\,\kappa}{2\,\varepsilon_6}\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} \tag{2.69}$$

for arbitrary ε_5, $\varepsilon_6 > 0$. Fixing the numbers ε_5 and ε_6 in such a way that $\varepsilon\,\varepsilon_5 + \varepsilon\,\kappa\,\varepsilon_6 = 1$, we find together with (2.62) and (2.63):

$$\|\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\|^2_{L^2(\Omega)} \leqslant \frac{\varepsilon}{\varepsilon_5}\|W' - W''\|^2_{L^2(\Omega)} + \frac{\varepsilon\,\kappa}{\varepsilon_6}\|\Phi_{tr}{}' - \Phi_{tr}{}''\|^2_{L^2(\Omega)} \leqslant C\Big(\|I_i{}' - I_i{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]} + \|I_e{}' - I_e{}''\|^2_{L^2\big[(0,T),\big(W^{1,2}(\Omega)\big)^*\big]}\Big) \implies \tag{2.70}$$

$$\| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|^2_{L^2\left[\,(0,T),\,L^2(\Omega)\right]} \leqslant C\,T \Big(\| I_i' - I_i'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} + \| I_e' - I_e'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} \Big), \tag{2.71}$$

and with (2.64), we get finally:

$$\| W' - W'' \|^2_{W^{1,2}\left[\,(0,T),\,L^2(\Omega)\right]} \tag{2.72}$$

$$\leqslant C \Big(\| I_i' - I_i'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} + \| I_e' - I_e'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} \Big).$$

• **Step A6.** *The estimate for the difference* $\| {\Phi_{tr}}' - {\Phi_{tr}}'' \|_{W^{1,4/3}\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]}$. Let $q = 4/3$. Exploiting the definition of the dual norm, we see that

$$\int_0^T \| \frac{\partial {\Phi_{tr}}'(t)}{\partial t} - \frac{\partial {\Phi_{tr}}''(t)}{\partial t} \|^q_{\left(W^{1,2}(\Omega)\right)^*} \, dt = \int_0^T \sup_{\|\psi\|_{W^{1,2}(\Omega)}=1} \big| \langle \frac{\partial {\Phi_{tr}}'}{\partial t} - \frac{\partial {\Phi_{tr}}''}{\partial t}\,,\,\psi \rangle \big|^q \, dt \tag{2.73}$$

$$\leqslant C \cdot \int_0^T \Big(\sup_{\dots} \big| \langle \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big)\,,\,\psi \rangle \big|^q \tag{2.74}$$

$$+ \sup_{\dots} \big| M\big({\Phi_{tr}}' - {\Phi_{tr}}''\,,\,\psi \big) \big|^q + \sup_{\dots} \big| \int_\Omega \Big(I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'')\,\psi \, dx \,\big|^q \Big) \, dt \quad \Longrightarrow$$

$$\Big(\int_0^T \| \frac{\partial {\Phi_{tr}}'}{\partial t} - \frac{\partial {\Phi_{tr}}''}{\partial t} \|^q_{\left(W^{1,2}(\Omega)\right)^*} \, dt \Big)^{1/q} \tag{2.75}$$

$$\leqslant C \cdot \Big(\int_0^T \sup_{\dots} \big| \langle \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big)\,,\,\psi \rangle \big|^2 \, dt \Big)^{1/2}$$

$$+ \, C \cdot \Big(\int_0^T \sup_{\dots} \big| M\big({\Phi_{tr}}' - {\Phi_{tr}}''\,,\,\psi \big) \big|^2 \, dt \Big)^{1/2}$$

$$+ C \cdot \Big(\int_0^T \big(\sup_{\dots} \| I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'') \|_{L^q(\Omega)} \cdot \| \psi \|_{L^p(\Omega)} \big)^q \, dt \Big)^{1/q}.$$

We estimate the three terms on the right-hand side of (2.75) separately. For the first term, we get

$$\big| \langle \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big)\,,\,\psi \rangle \big|^2$$

$$\leqslant \| \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big) \|^2_{\left(W^{1,2}(\Omega)\right)^*} \| \psi \|^2_{W^{1,2}(\Omega)} \quad \Longrightarrow \tag{2.76}$$

$$\sup_{\dots} \big| \langle \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big)\,,\,\psi \rangle \big|^2$$

$$\leqslant \| \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big) \|^2_{\left(W^{1,2}(\Omega)\right)^*} \tag{2.77}$$

$$\leqslant C \Big(\| I_i'(t) - I_i''(t) \|^2_{\left(W^{1,2}(\Omega)\right)^*} + \| I_e'(t) - I_e''(t) \|^2_{\left(W^{1,2}(\Omega)\right)^*} \Big) \quad \Longrightarrow \tag{2.78}$$

$$\Big(\int_0^T \sup_{\dots} \big| \langle \frac{1}{1+\lambda} \Big(\lambda \big(I_i'(t) - I_i''(t) \big) - \big(I_e'(t) - I_e''(t) \big) \Big)\,,\,\psi \rangle \big|^2 \, dt \Big)^{1/2}$$

$$\leqslant C \Big(\| I_i' - I_i'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} + \| I_e' - I_e'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} \Big). \tag{2.79}$$

For the second term, we obtain from Lemma 2.9. and (2.68):

$$\big| M\big({\Phi_{tr}}' - {\Phi_{tr}}''\,,\,\psi \big) \big|^2 \leqslant \gamma^2 \, \| {\Phi_{tr}}' - {\Phi_{tr}}'' \|^2_{W^{1,2}(\Omega)} \cdot \| \psi \|^2_{W^{1,2}(\Omega)} \quad \Longrightarrow \tag{2.80}$$

$$\Big(\int_0^T \sup_{\dots} \big| M\big({\Phi_{tr}}' - {\Phi_{tr}}''\,,\,\psi \big) \big|^2 \, dt \Big)^{1/2} \leqslant C \, \| {\Phi_{tr}}' - {\Phi_{tr}}'' \|_{L^2\left[\,(0,T),\,W^{1,2}(\Omega)\right]} \tag{2.81}$$

$$\leqslant C \Big(\| I_i' - I_i'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} + \| I_e' - I_e'' \|^2_{L^2\left[\,(0,T),\,\left(W^{1,2}(\Omega)\right)^*\right]} \Big). \tag{2.82}$$

In order to estimate the third term, we write, relying on Lemma 2.10.,

$$\| I_{ion}(\Phi_{tr}', W') - I_{ion}(\Phi_{tr}'', W'') \|_{L^q(\Omega)}$$

$$\leqslant \| \, b \, \big((\Phi_{tr}')^2 + \Phi_{tr}' \, \Phi_{tr}'' + (\Phi_{tr}'')^2 - (a+1) \, (\Phi_{tr}' + \Phi_{tr}'') + a \big) \, \big(\Phi_{tr}' - \Phi_{tr}'' \big) \, \|_{L^q(\Omega)} \tag{2.83}$$

$$+ \| \, (\Phi_{tr}' - \Phi_{tr}'') \, W' \, \|_{L^q(\Omega)} + \| \, (W' - W'') \, \Phi_{tr}'' \, \|_{L^q(\Omega)} \; = \; J_1 + J_2 + J_3 \, . \tag{2.84}$$

For J_1, we obtain

$$J_1 \; = \; \Big(\int_\Omega b^q \, \big((\Phi_{tr}')^2 + \Phi_{tr}' \, \Phi_{tr}'' + (\Phi_{tr}'')^2 - (a+1) \, (\Phi_{tr}' + \Phi_{tr}'') + a \big)^q \, \big(\Phi_{tr}' - \Phi_{tr}'' \big)^q \, dx \Big)^{1/q} \tag{2.85}$$

$$\leqslant \Big(\Big(\int_\Omega b^{3q/2} \, \big((\Phi_{tr}')^2 + \Phi_{tr}' \, \Phi_{tr}'' + (\Phi_{tr}'')^2 - (a+1) \, (\Phi_{tr}' + \Phi_{tr}'') + a \big)^{3q/2} \, dx \Big)^{2/3} \, \Big(\int_\Omega \big(\Phi_{tr}' - \Phi_{tr}'' \big)^{3q} \Big)^{1/3} \Big)^{1/q} \tag{2.86}$$

$$\leqslant C \Big(\big(1 + \| \, \Phi_{tr}'(t) \, \|^4_{L^4(\Omega)} + \| \, \Phi_{tr}''(t) \, \|^4_{L^4(\Omega)} \big)^{2/3} \cdot \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{4/3}_{L^4(\Omega)} \Big)^{4/3} \tag{2.87}$$

$$\leqslant C \, \big(1 + \| \, \Phi_{tr}'(t) \, \|^4_{L^4(\Omega)} + \| \, \Phi_{tr}''(t) \, \|^4_{L^4(\Omega)} \big)^{8/9} \cdot \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{16/9}_{W^{1,2}(\Omega)} \, . \tag{2.88}$$

Further, with (2.51) we get

$$J_2 \; = \; \Big(\int_\Omega \big(\Phi_{tr}' - \Phi_{tr}'' \big)^q \, (W')^q \, dx \Big)^{1/q} \; \leqslant \; \Big(\Big(\int_\Omega \big(\Phi_{tr}' - \Phi_{tr}'' \big)^2 \, dx \Big)^{2/3} \, \Big(\int_\Omega (W')^4 \, dx \Big)^{1/3} \Big)^{1/q} \tag{2.89}$$

$$= \Big(\| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{4/3}_{L^2(\Omega)} \cdot \| \, W' \, \|^{4/3}_{L^4(\Omega)} \Big)^{3/4} \; = \; \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|_{L^2(\Omega)} \cdot \| \, W' \, \|_{L^4(\Omega)} \tag{2.90}$$

$$\leqslant C \, \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|_{L^2(\Omega)} \, . \tag{2.91}$$

Finally, J_3 will be estimated together with (2.48) through

$$J_3 \; = \; \Big(\int_\Omega \big(W' - W'' \big)^q \, (\Phi_{tr}'')^q \, dx \Big)^{1/q} \; \leqslant \; \Big(\Big(\int_\Omega \big(W' - W'' \big)^2 \, dx \Big)^{2/3} \, \Big(\int_\Omega (\Phi_{tr}'')^4 \, dx \Big)^{1/3} \Big)^{1/q} \tag{2.92}$$

$$= \Big(\| \, W' - W'' \, \|^{4/3}_{L^2(\Omega)} \cdot \| \, \Phi_{tr}'' \, \|^{4/3}_{L^4(\Omega)} \Big)^{3/4} \; = \; \| \, W' - W'' \, \|_{L^2(\Omega)} \cdot \| \, \Phi_{tr}'' \, \|_{L^4(\Omega)} \tag{2.93}$$

$$\leqslant C \, \| \, W' - W'' \, \|_{L^2(\Omega)} \, . \tag{2.94}$$

Together, the estimates (2.88), (2.91) and (2.94) imply that

$$\Big(\int_0^T \big(\sup_{\dots} \| \, I_{ion}(\Phi_{tr}', W') - I_{ion}(\Phi_{tr}'', W'') \, \|_{L^{4/3}(\Omega)} \cdot \| \, \psi \, \|_{L^4(\Omega)} \big)^{4/3} \, dt \Big)^{3/4} \tag{2.95}$$

$$\leqslant C \cdot \Big(\int_0^T \Big(\big(1 + \| \, \Phi_{tr}'(t) \, \|^4_{L^4(\Omega)} + \| \, \Phi_{tr}''(t) \, \|^4_{L^4(\Omega)} \big)^{8/9} \cdot \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{16/9}_{W^{1,2}(\Omega)} + \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|_{L^2(\Omega)} + \| \, W' - W'' \, \|_{L^2(\Omega)} \Big)^{4/3} \, dt \Big)^{3/4} \tag{2.96}$$

$$\leqslant C \cdot \Big(\int_0^T \big(1 + \| \, \Phi_{tr}'(t) \, \|^4_{L^4(\Omega)} + \| \, \Phi_{tr}''(t) \, \|^4_{L^4(\Omega)} \big)^{32/27} \cdot \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{64/27}_{W^{1,2}(\Omega)} + \int_0^T \| \, \Phi_{tr}' - \Phi_{tr}'' \, \|^{4/3}_{W^{1,2}(\Omega)} \, dt + \int_0^T \| \, W' - W'' \, \|^{4/3}_{L^2(\Omega)} \, dt \Big)^{3/4} . \tag{2.97}$$

With (2.46), (2.48), (2.61) and (2.67), we find

$$\Big(\int_0^T \big(\sup_{\cdots} \| I_{ion}(\Phi_{tr}', W') - I_{ion}(\Phi_{tr}'', W'') \|_{L^{4/3}(\Omega)} \cdot \| \psi \|_{L^4(\Omega)} \big)^{4/3} dt \Big)^{3/4}$$

$$\leqslant C \cdot \Big(\int_0^T \Big(\| I_i' - I_i'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} + \| I_e' - I_e'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big)^{64/27} dt$$

$$+ 2 \int_0^T \Big(\| I_i' - I_i'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} + \| I_e' - I_e'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big)^{4/3} dt \Big)^{3/4} \tag{2.98}$$

$$\leqslant C \cdot \mathrm{Max} \Big(\| I_i' - I_i'' \|^{16/9}_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_e' - I_e'' \|^{16/9}_{L^2[(0,T),(W^{1,2}(\Omega))^*]},$$

$$\| I_i' - I_i'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_e' - I_e'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big). \tag{2.99}$$

Summing up, we get from (2.79), (2.82) and (2.99)

$$\| \frac{\partial \Phi_{tr}'}{\partial t} - \frac{\partial \Phi_{tr}''}{\partial t} \|_{L^{4/3}[(0,T),(W^{1,2}(\Omega))^*]} \leqslant C \cdot \mathrm{Max} \Big(\| I_i' - I_i'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \tag{2.100}$$

$$\| I_e' - I_e'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_i' - I_i'' \|^{16/9}_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_e' - I_e'' \|^{16/9}_{L^2[(0,T),(W^{1,2}(\Omega))^*]},$$

$$\| I_i' - I_i'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_e' - I_e'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big)$$

and, considering (2.62),

$$\| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,4/3}[(0,T),(W^{1,2}(\Omega))^*]} \leqslant C \cdot \mathrm{Max} \Big(\| I_i' - I_i'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \tag{2.101}$$

$$\| I_e' - I_e'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_i' - I_i'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]}, \| I_e' - I_e'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big).$$

• **Step A7.** *Conclusion of the proof of Part A.* Since $L^\infty[(0,T),(W^{1,2}(\Omega))^*]$ is continuously embedded into $L^2[(0,T),(W^{1,2}(\Omega))^*]$, we have

$$\| I_e' - I_e'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \leqslant C \| I_e' - I_e'' \|_{L^\infty[(0,T),(W^{1,2}(\Omega))^*]}, \tag{2.102}$$

and the proof of Part A is complete.

Part B. The FitzHugh-Nagumo model.

• **Step B1.** *The difference of the parabolic equations.* The only differences between the Rogers-McCulloch and the FitzHugh-Nagumo model are the replacement of the nonlinear coupling term φw by w and the setting $b = 1$ within the ionic current. Consequently, proceeding as in Step A1, the first change applies to (2.34) and (2.35):

$$\int_\Omega \Big(I_{ion}(\Phi_{tr}', W') - I_{ion}(\Phi_{tr}'', W'') \Big) (\Phi_{tr}' - \Phi_{tr}'') \, dx \tag{2.103}$$

$$= \int_\Omega (\Phi_{tr}' - \Phi_{tr}'') \big((\Phi_{tr}')^2 + \Phi_{tr}' \Phi_{tr}'' + (\Phi_{tr}'')^2 + a \big) (\Phi_{tr}' - \Phi_{tr}'') \, dx$$

$$- (a+1) \int_\Omega (\Phi_{tr}' - \Phi_{tr}'') (\Phi_{tr}' + \Phi_{tr}'') (\Phi_{tr}' - \Phi_{tr}'') \, dx + \int_\Omega (W' - W'') (\Phi_{tr}' - \Phi_{tr}'') \, dx.$$

Now (2.31), (2.32) and (2.103) imply

$$\frac{d}{dt} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} + 2\beta \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{1,2}(\Omega)} \leqslant 2C \int_\Omega | \Phi_{tr}' - \Phi_{tr}'' | \cdot | \Phi_{tr}' + \Phi_{tr}'' | \cdot | \Phi_{tr}' - \Phi_{tr}'' | \, dx$$

$$+ 2 \int_\Omega (W' - W'') (\Phi_{tr}'' - \Phi_{tr}') \, dx + \frac{C}{\varepsilon_1} \Big(\| I_i' - I_i'' \|^2_{(W^{1,2}(\Omega))^*} + \| I_e' - I_e'' \|^2_{(W^{1,2}(\Omega))^*} \Big)$$

$$+ \frac{3\varepsilon_1}{2} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{1,2}(\Omega)} + 2\beta \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}. \tag{2.104}$$

The first term on the right-hand side of (2.104) will be estimated through (2.38) again; for the second term, we get

$$2\int_\Omega \big(W' - W'' \big)\,(\Phi_{tr}'' - \Phi_{tr}')\,dx \;\leqslant\; 2\,\varepsilon_7\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon_7}\,\|\,W' - W''\,\|^2_{L^2(\Omega)} \tag{2.105}$$

after application of the generalized Cauchy inequality with $\varepsilon_7 > 0$. (2.38), (2.104) and (2.105) yield together

$$\begin{aligned}
&\frac{d}{dt}\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)} + 2\,\beta\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{W^{1,2}(\Omega)} \\
&\quad\leqslant C\,\frac{\varepsilon_2}{2}\,\Big(\,\|\,\Phi_{tr}'\,\|^4_{L^4(\Omega)} + \|\,\Phi_{tr}''\,\|^4_{L^4(\Omega)}\Big)^{1/2}\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{W^{1,2}(\Omega)} + \frac{C}{2\,\varepsilon_2}\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)} \\
&\qquad + 2\,\varepsilon_7\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon_7}\,\|\,W' - W''\,\|^2_{L^2(\Omega)} \\
&\qquad\quad + \frac{C}{\varepsilon_1}\,\Big(\,\|\,I_i'(t) - I_i''(t)\,\|^2_{\big(W^{1,2}(\Omega)\big)^*} + \|\,I_e'(t) - I_e''(t)\,\|^2_{\big(W^{1,2}(\Omega)\big)^*}\Big) \\
&\qquad\qquad + \frac{3\varepsilon_1}{2}\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{W^{1,2}(\Omega)} + 2\,\beta\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)}\,,
\end{aligned} \tag{2.106}$$

which replaces (2.43). Considering (2.46) and (2.48), the numbers ε_1, ε_2 and $\varepsilon_7 > 0$ may be fixed in such a way that the terms with $\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{W^{1,2}(\Omega)}$ on both sides of (2.106) will vanish. Up to a change in the constants, we arrive at (2.52) again.

• **Step B2.** *The difference of the gating equations.* Since the gating equation is the same as in the Rogers-McCulloch model, Step A2 can be carried over without changes.

• **Step B3.** *The estimates for the differences* $\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{C^0\big[\,[\,0\,,T\,]\,,\,L^2(\Omega)\big]}$, $\|\,W' - W''\,\|^2_{L^2\big[\,(\,0\,,T\,)\,,\,L^2(\Omega)\big]}$ *and* $\|\,W' - W''\,\|^2_{C^0\big[\,[\,0\,,T\,]\,,\,L^2(\Omega)\big]}$. Again, Step A3 can be carried over without alterations, and the estimates (2.62), (2.63) and (2.64) will be obtained.

• **Step B4.** *The estimate for the difference* $\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2\big[\,(\,0\,,T\,)\,,\,W^{1,2}(\Omega)\big]}$. In (2.106), the numbers ε_1, ε_2 and $\varepsilon_7 > 0$ may be alternatively chosen in order to obtain

$$\begin{aligned}
\frac{d}{dt}\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)} + \beta\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{W^{1,2}(\Omega)} \;\leqslant\; C\,\Big(\,\|\,\Phi_{tr}' - \Phi_{tr}''\,\|^2_{L^2(\Omega)}& \\
+ \|\,W' - W''\,\|^2_{L^2(\Omega)} + \|\,I_i'(t) - I_i''(t)\,\|^2_{\big(W^{1,2}(\Omega)\big)^*} + \|\,I_e'(t) - I_e''(t)\,\|^2_{\big(W^{1,2}(\Omega)\big)^*}\Big)&\,.
\end{aligned} \tag{2.107}$$

Then the remaining part of Step A4 can be carried over, and we get (2.68) again.

• **Step B5.** *The estimate for the difference* $\|\,W' - W''\,\|^2_{W^{1,2}\big[\,(\,0\,,T\,)\,,\,L^2(\Omega)\big]}$. Step A5 may be carried over without alterations, and we obtain (2.72).

• **Step B6.** *The estimate for the difference* $\|\,\Phi_{tr}' - \Phi_{tr}''\,\|_{W^{1,4/3}\big[\,(\,0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\big]}$. The calculations from Step A6 may be repeated until (2.82). The estimation of the norm difference in (2.83) ff. simplifies to

$$\|\,I_{ion}(\Phi_{tr}', W') - I_{ion}(\Phi_{tr}'', W'')\,\|_{L^q(\Omega)} \tag{2.108}$$

$$\leqslant \|\,\Big((\Phi_{tr}')^2 + \Phi_{tr}'\,\Phi_{tr}'' + (\Phi_{tr}'')^2 - (a+1)\,(\Phi_{tr}' + \Phi_{tr}'') + a\Big)\,\Big(\,\Phi_{tr}' - \Phi_{tr}''\Big)\,\|_{L^q(\Omega)} + \|\,W' - W''\,\|_{L^q(\Omega)}$$

$$= \Big(\int_\Omega \Big((\Phi_{tr}')^2 + \Phi_{tr}'\,\Phi_{tr}'' + (\Phi_{tr}'')^2 - (a+1)\,(\Phi_{tr}' + \Phi_{tr}'') + a\Big)^q\,\Big(\,\Phi_{tr}' - \Phi_{tr}''\Big)^q\,dx\Big)^{1/q} + \|\,W' - W''\,\|_{L^q(\Omega)} \tag{2.109}$$

$$\leqslant \Big(\Big(\int_\Omega \big((\Phi_{tr}{}')^2 + \Phi_{tr}{}' \, \Phi_{tr}{}'' + (\Phi_{tr}{}'')^2 - (a+1)\,(\Phi_{tr}{}' + \Phi_{tr}{}'') + a \big)^{3q/2} \, dx \Big)^{2/3} \Big(\int_\Omega \big(\Phi_{tr}{}' - \Phi_{tr}{}'' \big)^{3q} \Big)^{1/3} \Big)^{1/q} + \| W' - W'' \|_{L^q(\Omega)} \tag{2.110}$$

$$\leqslant C \Big(\big(1 + \| \Phi_{tr}{}'(t) \|^4_{L^4(\Omega)} + \| \Phi_{tr}{}''(t) \|^4_{L^4(\Omega)} \big)^{2/3} \cdot \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^{4/3}_{L^4(\Omega)} \Big)^{4/3} + \| W' - W'' \|_{L^q(\Omega)} \tag{2.111}$$

$$\leqslant C \big(1 + \| \Phi_{tr}{}'(t) \|^4_{L^4(\Omega)} + \| \Phi_{tr}{}''(t) \|^4_{L^4(\Omega)} \big)^{8/9} \cdot \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^{16/9}_{W^{1,2}(\Omega)} + \| W' - W'' \|_{L^q(\Omega)} \, . \tag{2.112}$$

Consequently, (2.96) and (2.97) are replaced by

$$\Big(\int_0^T \big(\sup_{\dots} \| I_{ion}(\Phi_{tr}{}', W') - I_{ion}(\Phi_{tr}{}'', W'') \|_{L^{4/3}(\Omega)} \cdot \| \psi \|_{L^4(\Omega)} \big)^{4/3} \, dt \Big)^{3/4} \leqslant C \cdot \Big(\int_0^T \Big(\big(1 + \| \Phi_{tr}{}'(t) \|^4_{L^4(\Omega)} + \| \Phi_{tr}{}''(t) \|^4_{L^4(\Omega)} \big)^{8/9} \cdot \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^{16/9}_{W^{1,2}(\Omega)} \, + \, \| W' - W'' \|_{L^2(\Omega)} \Big)^{4/3} \, dt \Big)^{3/4} \tag{2.113}$$

$$\leqslant C \Big(\int_0^T \big(1 + \| \Phi_{tr}{}'(t) \|^4_{L^4(\Omega)} + \| \Phi_{tr}{}''(t) \|^4_{L^4(\Omega)} \big)^{32/27} \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^{64/27}_{W^{1,2}(\Omega)} + \int_0^T \| W' - W'' \|^{4/3}_{L^2(\Omega)} \, dt \Big)^{3/4},$$

and we arrive again at (2.99) (up to a change of the constant C). The further conclusions of Step A6 remain unchanged, and we obtain (2.101).

• **Step B7.** *Conclusion of the proof of Part B.* By application of (2.102), the proof of Part B will be completed. ■

Proof of Theorem 2.8. In order to confirm uniqueness, apply Theorem 2.7. to $I_i{}' = I_i{}'' = I_i$ and $I_e{}' = I_e{}'' = I_e$. ■

3. Optimal control problems for the monodomain system.

a) Statement of the control problems.

The monodomain system $(\mathrm{M})_2$ will be controlled by means of the excitation variables. In practical situations, the application of an excitation to the intracellular part of the tissue is impossible; consequently, we have $I_i = \mathfrak{o}$, and the single control variable is I_e. The fact that one cannot apply arbitrary large electrical stimulations to living tissue without damaging it gives rise to a uniform bound

$$|\, I_e(x,t) \,| \leqslant R \quad (\forall)\,(x,t) \in \Omega_T \tag{3.1}$$

for the control variable. The objective will not be specified closer here, but we consider a regularization term with respect to I_e, which will be considered as an element of the space $L^\infty\big[\,(0\,,\,T\,)\,,\,\big(L^2(\Omega)\big)^*\,\big]$. Then the control problems $(\mathrm{P})_i$, $i = 1\,,2$, may be formulated as follows:

$$(\mathrm{P})_i \quad F(\Phi_{tr}, W, I_e) \, = \, \int_0^T \int_\Omega r\big(\, x, t, \Phi_{tr}(x,t), W(x,t) \,\big) \, dx\, dt + \frac{\mu}{2} \int_0^T \int_\Omega I_e(x,t)^2 \, dx\, dt \longrightarrow \inf! \, ; \tag{3.2}$$

$$E_1(\Phi_{tr}, W, I_e) \, = \, \mathfrak{o} \iff \int_\Omega \Big(\frac{\partial \Phi_{tr}(t)}{\partial t} + I_{ion}(\Phi_{tr}(t), W(t)) + \frac{1}{1+\lambda} \, I_e(t) \Big) \, \psi \, dx + \frac{\lambda}{1+\lambda} \int_\Omega \nabla \psi^{\mathrm{T}} M_i \, \nabla \Phi_{tr}(t) \, dx \, = \, 0 \quad \forall \, \psi \in W^{1,2}(\Omega) \ \ (\forall)\, t \in [\, 0\,,\, T\,] \, ; \tag{3.3}$$

$$E_2(\Phi_{tr}, W) \, = \, \mathfrak{o} \iff \int_\Omega \Big(\frac{\partial W(t)}{\partial t} + G(\Phi_{tr}(t), W(t)) \Big) \, \psi \, dx \, = \, 0 \quad \forall \, \psi \in L^2(\Omega) \ \ (\forall)\, t \in [\, 0\,,\, T\,] \, ; \tag{3.4}$$

$$E_3(\Phi_{tr}) \, = \, \mathfrak{o} \iff \Phi_{tr}(x,0) - \Phi_0(x) \, = \, 0 \quad (\forall)\, x \in \Omega \, ; \tag{3.5}$$

$$E_4(W) \, = \, \mathfrak{o} \iff W(x,0) - W_0(x) \, = \, 0 \quad (\forall)\, x \in \Omega \, ; \tag{3.6}$$

$$I_e \in \mathcal{C} \, = \, \big\{ \, Z \in L^\infty\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\big] \ \big| \ |\, Z(x,t) \,| \leqslant R \ \ (\forall)\,(x,t) \in \Omega_T \,\big\} \, . \tag{3.7}$$

About the data, we take Assumptions 2.3., 1) – 6). The numbers $T > 0$, $\lambda > 0$, $\mu > 0$ and $R > 0$ are fixed. The functions I_{ion} and G will be specified according to the models from Subsection 2.b) as the Rogers-McCulloch model in $(\mathrm{P})_1$ and the FitzHugh-Nagumo model in $(\mathrm{P})_2$. Since the monodomain equations have been included in their weak formulation, the domains of definition and the ranges of the operators will be identified as follows:

$$F:\ \mathrm{X}_1 \times \mathrm{X}_2 \times \mathrm{X}_3 \to \mathbb{R} \quad \text{with} \tag{3.8}$$
$$\mathrm{X}_1 = L^2\big[\,(0,T),\, W^{1,2}(\Omega)\,\big]\,; \quad \mathrm{X}_2 = L^2\big[\,(0,T),\, L^2(\Omega)\,\big]\,; \quad \mathrm{X}_3 = L^\infty\big[\,(0,T),\, L^2(\Omega)\,\big]\,.$$

Note that $\mathcal{C}$ forms a closed, convex subset of X_3 (see Proposition 3.1. below). In view of Theorems 2.6. and 2.7., we may further specify the subspaces

$$\widetilde{\mathrm{X}}_1 = \mathrm{X}_1 \cap W^{1,4/3}\big[\,(0,T),\, \big(W^{1,2}(\Omega)\big)^*\,\big] \cap C^0\big[\,[0,T],\, L^2(\Omega)\,\big]\,; \tag{3.9}$$
$$\widetilde{\mathrm{X}}_2 = \mathrm{X}_2 \cap W^{1,2}\big[\,(0,T),\, \big(L^2(\Omega)\big)^*\,\big] \cap C^0\big[\,[0,T],\, L^2(\Omega)\,\big]\,, \tag{3.10}$$

which contain all polynomials and, consequently, lie dense in X_1 and X_2, and the target spaces $\mathrm{Z}_1, \ldots, \mathrm{Z}_4$ for the operators $E_1, \ldots, E_4$:

$$E_1:\ \widetilde{\mathrm{X}}_1 \times \widetilde{\mathrm{X}}_2 \times \mathrm{X}_3 \to \mathrm{Z}_1 = L^{4/3}\big[\,(0,T),\, \big(W^{1,2}(\Omega)\big)^*\,\big]\,; \tag{3.11}$$
$$E_2:\ \widetilde{\mathrm{X}}_1 \times \widetilde{\mathrm{X}}_2 \to Z_2 = L^2\big[\,(0,T),\, \big(L^2(\Omega)\big)^*\,\big]\,; \tag{3.12}$$
$$E_3:\ \widetilde{\mathrm{X}}_1 \to Z_3 = L^2(\Omega)\,; \tag{3.13}$$
$$E_4:\ \widetilde{\mathrm{X}}_2 \to Z_4 = L^2(\Omega)\,. \tag{3.14}$$

b) Structure of the feasible domain and existence of global minimizers.

Proposition 3.1. *The set $\mathcal{C}$ of the admissible controls according to* (3.7) *forms a closed, convex, weak*-sequentially compact subset of $L^\infty\big[\,(0,T),\, L^2(\Omega)\,\big]$.*

Proof. The convexity of $\mathcal{C}$ is obvious. In order to prove closedness, consider a sequence $\{\,Z^N\,\} \subset L^\infty\big[\,(0,T),\, L^2(\Omega)\,\big]$, which converges in norm to a limit element $\hat{Z}$. Then there exists a subsequence, which converges a. e. pointwise on Ω_T to $\hat{Z}$, and the limit element obeys the a. e. pointwise restriction as well. Then the weak*-sequential compactness follows from [Rolewicz 76], p. 301, Theorem VI.6.6., together with p. 152, Theorem IV.4.11. ∎

Proposition 3.2. *Under the assumptions from Subsection 3.a), the feasible domains $\mathcal{B}_i$ of the problems* $(\mathrm{P})_i$*, $i = 1, 2$, are nonempty and closed with respect to the following topology in $\mathrm{X}_1 \times \mathrm{X}_2 \times \mathrm{X}_3$: weak convergence with respect to the first and second component, weak*-convergence with respect to the third component.*

Proof. The existence of feasible solutions for $(\mathrm{P})_i$ follows from Theorem 2.5. Assume that a sequence $\{\,(\Phi_{tr}{}^N, W^N, I_e{}^N)\,\} \subset \mathcal{B}_i$ with $\Phi_{tr}{}^N \rightharpoonup \hat{\Phi}_{tr}$, $W^N \rightharpoonup \hat{W}$ and $I_e{}^N \overset{*}{\rightharpoonup} \hat{I}_e$ is given. Then, by Proposition 3.1., $\hat{I}_e$ belongs to $\mathcal{C}$, and from the a-priori estimate (Theorem 2.6.) it follows that, together with $\|\,I_e{}^N\,\|_{\mathrm{X}_3}$, the norms $\|\,\Phi_{tr}{}^N\,\|_{L^4(\Omega_T)}$, $\|\,\Phi_{tr}{}^N\,\|_{\tilde{\mathrm{X}}_1}$ and $\|\,W^N\,\|_{\tilde{\mathrm{X}}_2}$ are uniformly bounded. In order to confirm that $(\hat{\Phi}_{tr}, \hat{W})$ solves $(\mathrm{M})_2$ with right-hand side $\hat{I}_e$, we may repeat now the arguments from [Bourgault/Coudière/Pierre 09], pp. 476 – 478, Subsection 5.4.3. As in the proof of Theorem 2.5., all conclusions may be carried over to $(\mathrm{M})_2$. ∎

Theorem 3.3. (Existence of global minimizers) *We take over the assumptions from Subsection 3.a). Assume that the integrand $r(x,t,\varphi,w):\ \Omega \times [\,0,T\,] \times \mathbb{R} \times \mathbb{R}$ is bounded from below, measurable with respect* *to x and t and continuous and convex with respect to φ and w. Then the problems* $(\mathrm{P})_i$, $i = 1, 2$, *admit global minimizers.*

Proof. Together with r, the objective F is bounded from below, and the problem $(\mathrm{P})_i$ admits a minimizing sequence $\{ ({\Phi_{tr}}^N, W^N, {I_e}^N) \}, \mathcal{B}_i$. Since $\| {I_e}^N \|_{\mathrm{X}_3}$ is uniformly bounded, Theorem 2.6. implies the boundedness of $\| {\Phi_{tr}}^N \|_{\mathrm{X}_1}$ and $\| W^N \|_{\mathrm{X}_2}$ as well, and we may pass to a subsequence $\{ ({\Phi_{tr}}^{N'}, W^{N'}, {I_e}^{N'}) \}$ with ${\Phi_{tr}}^{N'} m \rightharpoonup \hat{\Phi}_{tr}$, $W^{N'} \rightharpoonup \hat{W}$ and ${I_e}^{N'} \overset{*}{\rightharpoonup} \hat{I}_e$. By Proposition 3.2., $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ is feasible. The lower semicontinuity of the objective may be confirmed as in [Dacorogna 08], p. 96, Theorem 3.23., and p. 97, Remark 3.25.(ii). Consequently, denoting the minimal value of $(\mathrm{P})_i$ by m_i, we have

$$m_i = \lim_{N' \to \infty} F({\Phi_{tr}}^{N'}, W^{N'}, {I_e}^{N'}) \geqslant \liminf_{N' \to \infty} F({\Phi_{tr}}^{N'}, W^{N'}, {I_e}^{N'}) \geqslant F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \geqslant m_i \quad (3.15)$$

and $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ is a global minimizer of $(\mathrm{P})_i$. ■

c) The system of adjoint equations.

For the optimal control problems $(\mathrm{P})_i$, $i = 1, 2$, we introduce the formal Lagrange function

$$\mathcal{L}(\Phi_{tr}, W, I_e, P_1, P_2, P_3, P_4) = F(\Phi_{tr}, W, I_e) + \langle P_1 \, , \, E_1(\Phi_{tr}, W, I_e) \rangle \quad (3.16)$$
$$+ \langle P_2 \, , \, E_2(\Phi_{tr}, W) \rangle + \langle P_3 \, , \, E_3(\Phi_{tr}) \rangle + \langle P_4 \, , \, E_4(W) \rangle$$

with multipliers $P_1 \in L^4\big[(0, T), W^{1,2}(\Omega) \big]$, $P_2 \in L^2\big[(0, T), L^2(\Omega) \big]$, and P_3, $P_4 \in \big(L^2(\Omega) \big)^*$. Differentiating $\mathcal{L}$ at the point $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ in a formal way with respect to the variables Φ_{tr} and W, we find the adjoint equations

$$D_\Phi F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) + \langle P_1 \, , \, D_\Phi \, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \rangle + \langle P_2 \, , \, D_\Phi \, E_2(\hat{\Phi}_{tr}, \hat{W}) \rangle + \langle P_3 \, , \, D_\Phi \, E_3(\hat{\Phi}_{tr}) \rangle = 0 \quad (3.17)$$

$$\Longleftrightarrow \int_0^T \int_\Omega \Big(-\frac{\partial P_1}{\partial t} + \frac{\partial I_{ion}}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) \, P_1 \Big) \, \psi \, dx \, dt + \frac{\lambda}{1+\lambda} \int_0^T \int_\Omega \nabla \psi^{\mathrm{T}} \, M_i \, \nabla P_1 \, dx \, dt \quad (3.18)$$
$$= - \int_0^T \int_\Omega \Big(\frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) + \frac{\partial G}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) \, P_2 \Big) \, \psi \, dx \, dt - \int_\Omega P_3 \, \psi(x, 0) \, dx$$
$$\forall \, \psi \in L^2\big[(0, T), W^{1,2}(\Omega) \big], \; P_1(x, T) \equiv 0 \, ;$$

$$D_W \, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) + \langle P_1 \, , \, D_W \, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \rangle + \langle P_2 \, , \, D_W \, E_2(\hat{\Phi}_{tr}, \hat{W}) \rangle + \langle P_4 \, , \, D_W \, E_4(\hat{W}) \rangle = 0 \quad (3.19)$$

$$\Longleftrightarrow \int_0^T \int_\Omega \Big(-\frac{\partial P_2}{\partial t} + \frac{\partial G}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \, P_2 \Big) \, \psi \, dx \, dt \quad (3.20)$$
$$= - \int_0^T \int_\Omega \Big(\frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) + \frac{\partial I_{ion}}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \, P_1 \Big) \, \psi \, dx \, dt - \int_\Omega P_4 \, \psi(x, 0) \, dx$$
$$\forall \, \psi \in L^2\big[(0, T), L^2(\Omega) \big], \; P_2(x, T) \equiv 0 \, .$$

In Theorem 3.7. below, we will prove directly that this system is part of the necessary optimality conditions for $(\mathrm{P})_i$. The adjoint system consists of a parabolic PDE in its weak formulation, which is coupled with a linear ODE. Assuming that P_3 and P_4 may be set to zero, the corresponding strong formulation of the adjoint system reads as follows:

$$-\frac{\partial P_1}{\partial t} - \nabla \cdot \big(M_i \, \nabla P_1 \big) + \frac{\partial I_{ion}}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) \, P_1 = -\frac{\partial G}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) \, P_2 - \frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}) \;\; (\forall) \, (x, t) \in \Omega_T \, ; \quad (3.21)$$

$$\mathfrak{n}^{\mathrm{T}} \, M_i \, \nabla P_1 = 0 \;\; \forall \, (x, t) \in \partial\Omega \times [0, T] \, ; \quad P_1(x, T) = 0 \;\; (\forall) \, x \in \Omega \, ; \quad (3.22)$$

$$- \frac{\partial P_2}{\partial t} + \frac{\partial G}{\partial w}(\hat{\Phi}_{tr}, \hat{W})\, P_2 \;=\; -\frac{\partial I_{ion}}{\partial w}(\hat{\Phi}_{tr}, \hat{W})\, P_1 - \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \;\; (\forall)\, (x,t) \in \Omega_T\,; \tag{3.23}$$

$$P_2(x,T) \;=\; 0 \;\; (\forall)\, x \in \Omega\,. \tag{3.24}$$

In order to prove the existence of a solution for (3.21) − (3.24), we adopt the following stronger set of analytical assumptions, which corresponds to the assumptions of Theorem 3.10. below.

Assumptions 3.4. (Stronger assumptions about the data) [18] About the data in $(\mathrm{M})_1$ and $(\mathrm{M})_2$, the following will be assumed.

$1)'$ $\Omega \subset \mathbb{R}^3$ is a bounded domain with $C^{2,\varepsilon}$-boundary, $0 < \varepsilon \leqslant 1$.

$2)'$ $M_i:\ \mathrm{cl}\,(\Omega) \to \mathbb{R}^{3\times3}$ is a symmetric, positive definite matrix function with $W^{1,\infty}(\Omega)$-coefficients, satisfying a uniform ellipticity condition:

$$0 \leqslant \mu_1 \,\|\, \xi \,\|^2 \leqslant \xi^{\mathrm{T}} M_i(x)\, \xi \leqslant \mu_2 \,\|\, \xi \,\|^2 \quad \forall\, \xi \in \mathbb{R}^3 \;\; \forall\, x \in \Omega \tag{3.25}$$

with $\mu_1,\, \mu_2 > 0$. The boundary values of M_i belong even to $C^{1,\varepsilon}(\partial\Omega)$, $0 < \varepsilon \leqslant 1$.

$3)'$ and $4)'$ are identical with Assumptions 2.3., 3) and 4).

$5)'$ The initial values belong to the following spaces: $\Phi_0 \in W^{2,2}(\Omega)$, $W_0 \in L^\infty(\Omega)$.

$6)'$ I_i and I_e belong to the space $L^\infty\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$.

Theorem 3.5. (Existence of solutions for the adjoint system) *We study the optimal control problems* $(\mathrm{P})_i$, $i = 1, 2$, *under Assumptions 3.4., $1)' - 6)'$ with* $p = 4$. *If* $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ *is a feasible solution of* $(\mathrm{P})_i$ *with*

$$\frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W})\,,\;\; \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \in L^r(\Omega_T) \tag{3.26}$$

where $4 < r < 6$ *then the adjoint system* (3.21) − (3.24) *admits a unique (weak or strong) solution* (P_1, P_2) *with*

$$P_1 \,\in\, L^r\big[\,(\,0\,,\,T\,)\,,\,W^{2,r}(\Omega)\,\big] \,\cap\, W^{1,r}\big[\,(\,0\,,\,T\,)\,,\,L^r(\Omega)\,\big]\,; \tag{3.27}$$

$$P_2 \,\in\, C^1\big[\,(\,0\,,\,T\,)\,,\,L^r(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^r(\Omega)\,\big]\,. \tag{3.28}$$

The proof will be based on a parabolic maximal regularity theorem (Theorem 3.10. below) and a fixed point argument.

d) Necessary optimality conditions.

We search for weak local minimizers according to the following definition:

Definition 3.6. (Weak local minimizer) *A triple* $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$, *which is feasible in* $(\mathrm{P})_i$, $i = 1, 2$, *is called a weak local minimizer of* $(\mathrm{P})_i$ *iff there exists a number* $\varepsilon > 0$ *such that for all admissible* (Φ_{tr}, W, I_e) *the conditions*

$$\|\, \Phi_{tr} - \hat{\Phi}_{tr} \,\|_{\mathrm{X}_1} \;\leqslant\; \varepsilon\,,\;\; \|\, W - \hat{W} \,\|_{\mathrm{X}_2} \;\leqslant\; \varepsilon\,,\;\; \|\, I_e - \hat{I}_e \,\|_{\mathrm{X}_3} \;\leqslant\; \varepsilon \tag{3.29}$$

imply the relation $F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \leqslant F(\Phi_{tr}, W, I_e)$.

The existence of at least one weak local minimizer for $(\mathrm{P})_i$ is confirmed by Theorem 3.3. Treating $(\mathrm{P})_1$ and $(\mathrm{P})_2$ as "weakly singular problems", we may follow the approach outlined in [Ito/Kunisch 08] , p. 17 f. and

[18] Note that Assumption 3.4., $k)'$ from the present definition implies Assumption 2.3., $k)$ above, $1 \leqslant k \leqslant 6$.

pp. 129 ff., [19] and prove the necessary optimality conditions without recourse to the regularity conditions of Kurcyusz-Zowe or Ioffe-Tichomirow. [20] Instead, the existence of a solution for the adjoint system will be ensured by Theorem 3.5., the assumptions of which, consequently, must be carried over.

Theorem 3.7. (First-order necessary optimality conditions for the control problems $(\mathrm{P})_i$ **)** *We study the problems* $(\mathrm{P})_i$, $i = 1, 2$, *under Assumptions 3.4, 1)′ – 6)′ with* $p = 4$. *If* $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \in L^2\big[\,(0\,,T)\,,W^{1,2}(\Omega)\big] \times L^2\big[\,(0\,,T)\,,L^2(\Omega)\big] \times L^\infty\big[\,(0\,,T)\,,L^2(\Omega)\big]$ *is a weak local minimizer of* $(\mathrm{P})_i$ *with*

$$\frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W})\,,\ \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \in L^r(\Omega_T) \tag{3.30}$$

where $4 < r < 6$ *then there exist multipliers* $P_1 \in L^r\big[\,(0\,,T)\,,W^{2,r}(\Omega)\big] \cap W^{1,r}\big[\,(0\,,T)\,,L^r(\Omega)\big]$ *and* $P_2 \in C^1\big[\,(0\,,T)\,,L^r(\Omega)\big] \cap C^0\big[\,[0\,,T]\,,L^r(\Omega)\big]$, *satisfying together with* $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ *the optimality condition*

$$\int_0^T\!\!\int_\Omega \Big(\mu\,\hat{I}_e(x,t) + \frac{1}{1+\lambda}\,P_1(x,t)\Big)\,\big(\,I_e(x,t) - \hat{I}_e(x,t)\,\big)\,dx\,dt \,\geqslant\, 0 \quad \forall\, I_e \in \mathcal{C} \tag{3.31}$$

as well as the adjoint equations (3.21) – (3.24). *The functions* P_1 *and* P_2 *solve the adjoint system in the weak as well as in the strong sense.*

Corollary 3.8. (Pointwise formulation of the optimality condition) *The optimality condition* (3.31) *from Theorem 3.7. implies the following Pontryagin minimum condition, which holds a. e. pointwise:*

$$\Big(\,\mu\,(1+\lambda)\,\hat{I}_e(x,t) + P_1(x,t)\Big)\,\hat{I}_e(x,t) \,=\, \underset{-R\,\leqslant\,\eta\,\leqslant\,R}{\mathrm{Min}}\ \Big(\,\mu\,(1+\lambda)\,\hat{I}_e(x,t) + P_1(x,t)\Big)\,\eta \quad (\forall)\,(x,t) \in \Omega_T\,. \tag{3.32}$$

Consequently, we have

$$\hat{I}_e(x,t) \,=\, \begin{cases} -R & \big|\; Q(x,t) > R\,; \\ -Q(x,t) & \big|\; Q(x,t) \,\in\, [-R\,,\,R]\,; \\ R & \big|\; Q(x,t) < -R \end{cases} \quad \text{where } Q(x,t) = \frac{1}{\mu\,(1+\lambda)}\,P_1(x,t)\,. \tag{3.33}$$

We may conclude that an optimal control, which never becomes active, admits the same regularity as the adjoint variable P_1.

Corollary 3.9. (Higher regularity of weak local minimizers) *Under the assumptions of Theorem 3.7., consider a weak local minimizer* $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ *of* $(\mathrm{P})_i$, $i = 1, 2$, *with* $\operatorname{ess\,sup}_{(x,t)\,\in\,\Omega_T}\,|\,\hat{I}_e(x,t)\,| < R$. *Then* $\hat{I}_e$ *belongs to the space* $W^{1,r}\big[\,(0\,,T)\,,L^r(\Omega)\big] \cap L^r\big[\,(0\,,T)\,,W^{2,r}(\Omega)\big]$ *with* $4 < r < 6$.

In a subsequent publication, we will prove that this corollary implies an improvement of the regularity of $\hat{\Phi}_{tr}$ and $\hat{W}$ as well.

e) Proofs.

We start with the following parabolic maximal regularity theorem:

Theorem 3.10. (Maximal regularity for the adjoint parabolic equation) [21] *Consider the parabolic initial-boundary value problem*

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[19] Although the problems $(\mathrm{P})_i$, $i = 1, 2$, do not fit all assumptions of [Ito/Kunisch 08], p. 18, Theorem 1.17., the proof scheme of the theorem can be carried over.

[20] Cf. [Ioffe/Tichomirow 79], p. 74, Assumption c), and [Ito/Kunisch 08], p. 5, Definition 1.5.

[21] [Weidemaier 02], p. 50, Theorem 3.2.

$$\frac{\partial P}{\partial t}(x,t) - \sum_{i,j} \nabla \cdot \big(a_{i,j}(x,t)\, \nabla P(x,t) \big) + a_0(x,t)\, P(x,t) \;=\; f(x,t) \;\; (\forall)\, (x,t) \in \Omega \,\times\, (\,0\,,\,T\,)\,; \tag{3.34}$$

$$\sum_{i,j} \mathfrak{n}_i(x)\, a_{i,j}(x,t)\, \frac{\partial P}{\partial x_j}(x,t) \;=\; g(x,t) \;\; \forall\, (x,t) \in \partial\Omega \,\times\, (\,0\,,\,T\,)\,; \quad P(x,0) \;=\; P_0(x) \;\; (\forall)\, x \in \Omega \tag{3.35}$$

with the following assumptions about the data:

a) $\Omega \subset \mathbb{R}^n$ *is a bounded domain of the class* $C^{2+\varepsilon}$, $\varepsilon > 0$.

b) The matrix function $a_{ij} : \mathrm{cl}\,(\Omega) \,\times\, [\,0\,,\,T\,] \to \mathbb{R}^{n\times n}$ *is symmetric and satisfies a uniform ellipticity condition. The entries* a_{ij} *belong to* $C^0\big[\,[\,0\,,\,T\,]\,,\,C^{\alpha_1}\big(\,\mathrm{cl}\,(\Omega)\,\big)\,\big]$ *and admit boundary values in*

$$B\big[\,[\,0\,,\,T\,]\,,\,C^{1+\varepsilon}(\partial\Omega)\,\big] \,\cap\, C^{\alpha_2}\big[\,[\,0\,,\,T\,]\,,\,C^0(\partial\Omega)\,\big] \tag{3.36}$$

where $\varepsilon > 0$, $\alpha_1 > 1 - 1/p$ *and* $\alpha_2 > 0.5\,(1 - 1/p)$. *Further, it holds that* $a_0 \in C^0\big[\,[\,0\,,\,T\,]\,,\,L^r(\Omega)\,\big]$ *with* $r > n/2$.

c) The right-hand side f *belongs to* $L^q\big[\,(\,0\,,\,T\,)\,,\,L^p(\Omega)\,\big]$ *with* $3 < p \leqslant q < (\,+\infty\,)$.

d) P_0 *belongs to the Besov space* $\mathrm{Y}_1 \;=\; B^{2\,(1-1/q)}_{p,q}(\Omega)$, *and* g *to* $\mathrm{Y}_2 \;=\; L^q\big[\,(\,0\,,\,T\,)\,,\,W^{1-1/p,p}(\partial\Omega)\,\big] \,\cap\, F^{0.5\,(1-1/p)}_{q,p}\big[\,(\,0\,,\,T\,)\,,\,L^p(\partial\Omega)\,\big]$ *where the second space is a Triebel-Lizorkin space. The coefficients satisfy* $3 < p \leqslant q < (\,+\infty\,)$.

e) For all $x \in \Omega$, *it holds that* $\sum_{i,\,j} \mathfrak{n}_i(x)\, a_{i,j}(x,0)\, \dfrac{\partial P_0}{\partial x_j} \;=\; g(x,0) \quad \forall\, x \in \partial\Omega\,.$

Then the problem (3.34) $-$ (3.35) *admits a unique weak solution*

$$P \,\in\, L^q\big[\,(\,0\,,\,T\,)\,,\,W^{2,p}(\Omega)\,\big] \,\cap\, W^{1,q}\big[\,(\,0\,,\,T\,)\,,\,L^p(\Omega)\,\big]\,. \tag{3.37}$$

Moreover, this solution satisfies the estimate

$$\|\,P\,\|_{L^q\left[\,(\,0\,,\,T\,)\,,\,W^{2,p}(\Omega)\right]} + \|\,P\,\|_{W^{1,q}\left[\,(\,0\,,\,T\,)\,,\,L^p(\Omega)\right]} \,\leqslant\, C\,\Big(\,\|\,f\,\|_{L^q\left[\,(\,0\,,\,T\,)\,,\,L^p(\Omega)\right]} + \|\,P_0\,\|_{\mathrm{Y}_1} \,+\, \|\,g\,\|_{\mathrm{Y}_2}\Big) \tag{3.38}$$

with a constant $C > 0$ *depending on* p, q *and* T *only.*

Proof of Theorem 3.5. The proof will be divided in two parts according to the underlying ionic current model.

Part A. The Rogers-McCulloch model. Here the adjoint equations (3.21) and (3.23) read as follows:

$$-\,\frac{\partial P_1}{\partial t} - \nabla\cdot\big(\,M_i\,\nabla P_1\,\big) + \Big(\,3\,b\,(\,\hat{\Phi}_{tr})^2 - 2\,(a+1)\,b\,\hat{\Phi}_{tr} + a\,b + \hat{W}\,\Big)\,P_1 \;=\; \varepsilon\,\kappa\,P_2 - \frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W})\,; \tag{3.39}$$

$$-\,\frac{\partial P_2}{\partial t} + \varepsilon\,P_2 \;=\; -\hat{\Phi}_{tr}\,P_1 - \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W})\,. \tag{3.40}$$

• **Step A1.** *Since* $\hat{I}_e \in L^\infty\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$, *we observe that* $\hat{\Phi}_{tr} \in C^0\big[\,[\,0\,,\,T\,]\,,\,W^{1,2}(\Omega)\,\big]$. From the proof of Theorem 2.6., we take the estimate (2.44), which reads after the insertion of $I_i = \mathfrak{o}$ as follows:

$$\frac{\varrho\,\beta}{4}\,\|\,\hat{\Phi}_{tr}(t)\,\|^2_{W^{1,2}(\Omega)} \,\leqslant\, C\,\Big(\,\varrho\,\|\,\Phi_{tr}(0)\,\|^2_{L^2(\Omega)} + \|\,W(0)\,\|^2_{L^2(\Omega)} + \int_0^t c\,|\,\Omega\,|\,d\tau \tag{3.41}$$

$$+\,C\int_0^t \|\,\hat{I}_e(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,d\tau\,\Big) + c\,|\,\Omega\,| + \frac{\varrho\,C}{2\beta}\,\|\,\hat{I}_e(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}$$

$$\leqslant\, C\,\Big(\,\lambda\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + c\,T\,|\,\Omega\,| + C\,\|\,\hat{I}_e\,\|^2_{L^2(\Omega_T)}\,d\tau\,\Big) + c\,|\,\Omega\,| + \frac{\varrho\,C}{2\beta}\,\|\,\hat{I}_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\right]} \tag{3.42}$$

for arbitrary $t \in [\,0\,,\,T\,]$.

• **Step A2.** *For any* $\widetilde{P}_1 \in L^r\big[\,(\,0\,,\,T\,)\,,\,L^{6r/(6-r)}(\Omega)\,\big]$*, the terminal problem for the adjoint ODE admits a unique (weak or strong) solution* $P_2 \in C^1\big[\,(\,0\,,\,T\,)\,,\,L^r(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^r(\Omega)\,\big]$. It is obvious that the problem

$$-\,\frac{\partial P_2}{\partial t} + \varepsilon\, P_2 \;=\; -\hat{\Phi}_{tr}\,\widetilde{P}_1 - \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \quad (\forall)\,(x,t) \in \Omega_T\,, \;\; P_2(x,T) \equiv 0 \tag{3.43}$$

admits the unique (weak or strong) solution

$$P_2(x,t) \;=\; -\int_t^T \Big(\,\hat{\Phi}_{tr}\widetilde{P}_1 + \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W})\,\Big)\, e^{\varepsilon\,(t-\tau)}\, d\tau\,, \tag{3.44}$$

which is continuous in time on $[\,0\,,\,T\,]$ and even differentiable in time on $(\,0\,,\,T\,)$. In order to confirm the integrability with respect to x, we estimate

$$\int_\Omega \big(\,\hat{\Phi}_{tr}(t)\,\widetilde{P}_1(t)\,\big)^r\,dx \;\leqslant\; \Big(\int_\Omega |\,\hat{\Phi}_{tr}(t)\,|^6\,dx\Big)^{r/6}\,\Big(\int_\Omega |\,\widetilde{P}_1(t)\,|^{6r/(6-r)}\,dx\Big)^{(6-r)/6} \tag{3.45}$$

where the right-hand side is finite due to the continuous imbedding $\hat{\Phi}_{tr}(t) \in W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$. Consequently, P_2 belongs to the space $C^1\big[\,(\,0\,,\,T\,)\,,\,L^r(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^r(\Omega)\,\big]$.

• **Step A3.** *For any* $\widetilde{P}_2 \in L^r(\Omega_T)$*, the terminal-boundary value problem for the parabolic adjoint equation admits a unique (weak or strong) solution* $P_1 \in L^r\big[\,(\,0\,,\,T\,)\,,\,W^{2,r}(\Omega)\,\big] \cap W^{1,r}\big[\,(\,0\,,\,T\,)\,,\,L^r(\Omega)\,\big]$. In order to confirm this claim, we must check whether the assumptions of Theorem 3.10. are satisfied. Note first that the assertions of the theorem will not be afflicted by the fact that time runs backwards from T. In consequence of Assumptions 3.4., Assumptions a) and b) of Theorem 4.4. about Ω and $a_{ij} = (M_i)_{ij}$ hold true. By Step A1, $\hat{\Phi}_{tr}$ belongs particularly to $C^0\big[\,[\,0\,,\,T\,]\,,\,L^4(\Omega)\,\big]$, which implies $\big(\,\hat{\Phi}_{tr}\,\big)^2 \in C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$. Since $\hat{W} \in C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ as well, we find that

$$a_0 \;=\; \Big(\,3\,b\,(\,\hat{\Phi}_{tr}\,)^2 - 2\,(a+1)\,b\,\hat{\Phi}_{tr} + a\,b + \hat{W}\,\Big) \;\in\; C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]\,, \tag{3.46}$$

and Assumption b) is complete. With $\widetilde{P}_2 \in L^r(\Omega_T)$, $4 < r < 6$, Assumption c) is obviously satisfied. Since $P_1(x,T)$ and $g(x,t)$ are the zero functions, Assumption d) holds with $p = q = r$. Finally, Assumption e) holds true since M_i does not depend on t and $P_1(x,T) \equiv 0$.

• **Step A4.** *For two functions* P_1'*,* $P_1'' \in L^r\big[\,(\,0\,,\,T\,)\,,\,L^{6r/(6-r)}(\Omega)\,\big]$*, the corresponding solutions of the terminal problem for the adjoint ODE satisfy*

$$\|\,P_2'(t) - P_2''(t)\,\|^r_{L^r(\Omega)} \;\leqslant\; C \cdot \int_t^T \|\,P_1'(\tau) - P_1''(\tau)\,\|^r_{L^{6r/(6-r)}(\Omega)}\,d\tau\,. \tag{3.47}$$

The solutions of the ODE have been calculated in Step A2. Consequently, applying Jensen's integral inequality and Hölder's inequality, we find

$$P_2'(x,t) - P_2''(x,t) \;=\; \int_t^T \hat{\Phi}_{tr}\,\big(\,P_1'' - P_1'\,\big)\,e^{\varepsilon\,(t-\tau)}\,d\tau \;\;\Longrightarrow\;\; |\,P_2' - P_2''\,| \;\leqslant\; \int_t^T |\,\hat{\Phi}_{tr}\,| \cdot |\,P_1' - P_1''\,|\,d\tau \;\;\Longrightarrow \tag{3.48}$$

$$|\,P_2' - P_2''\,|^r \;\leqslant\; (T-t)^r\,\Big(\frac{1}{T-t}\int_t^T |\,\hat{\Phi}_{tr}\,| \cdot |\,P_1' - P_1''\,|\,d\tau\,\Big)^r \;\leqslant\; T^{r-1}\int_t^T |\,\hat{\Phi}_{tr}\,|^r \cdot |\,P_1' - P_1''\,|^r\,d\tau \;\;\Longrightarrow \tag{3.49}$$

$$\int_\Omega |\,P_2' - P_2''\,|^r\,dx \;\leqslant\; C\int_t^T\int_\Omega |\,\hat{\Phi}_{tr}\,|^r \cdot |\,P_1' - P_1''\,|^r\,dx\,d\tau \;\leqslant\; C\int_t^T \Big(\int_\Omega |\,\hat{\Phi}_{tr}\,|^6\,dx\Big)^{r/6} \tag{3.50}$$

$$\cdot\,\Big(\int_\Omega |\,P_1' - P_1''\,|^{6r/(6-r)}\,dx\Big)^{(6-r)/6}\,d\tau \;=\; C\int_t^T \|\,\hat{\Phi}_{tr}(\tau)\,\|^r_{L^6(\Omega)} \cdot \|\,P_1'(\tau) - P_1''(\tau)\,\|^r_{L^{6r/(6-r)}(\Omega)}\,d\tau\,. \tag{3.51}$$

From Step A1, we know that $\hat{\Phi}_{tr} \in C^0\big[\,[\,0\,,\,T\,]\,,\,W^{1,2}(\Omega)\,\big] \hookrightarrow C^0\big[\,[\,0\,,\,T\,]\,,\,L^6(\Omega)\,\big]$. Consequently, the norms $\|\,\hat{\Phi}_{tr}(\tau)\,\|^r_{L^6(\Omega)}$ are uniformly bounded with respect to τ, and we arrive at the claimed inequality.

• **Step A5.** *For two functions P_2', $P_2'' \in L^r(\Omega_T)$, the corresponding solutions of the terminal-boundary value problem for the parabolic adjoint equation satisfy*

$$\|\,P_1'(t) - P_1''(t)\,\|^r_{L^{6r/(6-r)}(\Omega)} \;\leqslant\; C\cdot\int_t^T \|\,P_2'(\tau) - P_2''(\tau)\,\|^r_{L^r(\Omega)}\,d\tau\,. \tag{3.52}$$

In their strong form, the parabolic equations read as

$$-\,\frac{\partial P_1'}{\partial t} - \nabla\cdot\big(\,M_i\,\nabla P_1'\,\big) + \Big(\,3\,b\,(\,\hat{\Phi}_{tr}\,)^2 - 2\,(a+1)\,b\,\hat{\Phi}_{tr} + a\,b + \hat{W}\,\Big)\,P_1' \;=\; \varepsilon\,\kappa\,P_2' - \frac{\partial r}{\partial\varphi}(\hat{\Phi}_{tr},\hat{W}) \tag{3.53}$$

$$-\,\frac{\partial P_1''}{\partial t} - \nabla\cdot\big(\,M_i\,\nabla P_1''\,\big) + \Big(\,3\,b\,(\,\hat{\Phi}_{tr}\,)^2 - 2\,(a+1)\,b\,\hat{\Phi}_{tr} + a\,b + \hat{W}\,\Big)\,P_1'' \;=\; \varepsilon\,\kappa\,P_2'' - \frac{\partial r}{\partial\varphi}(\hat{\Phi}_{tr},\hat{W})\,. \tag{3.54}$$

Subtraction yields an analogous problem for the difference

$$-\,\frac{\partial}{\partial t}\big(\,P_1' - P_1''\,\big) - \nabla\cdot\big(\,M_i\,\nabla\big(\,P_1' - P_1''\,\big)\,\big) \tag{3.55}$$

$$+\Big(\,3\,b\,(\,\hat{\Phi}_{tr}\,)^2 - 2\,(a+1)\,b\,\hat{\Phi}_{tr} + a\,b + \hat{W}\,\Big)\,\big(\,P_1' - P_1''\,\big) \;=\; \varepsilon\,\kappa\,\big(\,P_2' - P_2''\,\big)\,;$$

$$\mathfrak{n}^{\mathrm{T}}\,M_i\,\nabla\big(\,P_1' - P_1''\,\big) \;=\; 0\,; \quad \big(\,P_1' - P_1''\,\big)(x,T) \equiv 0\,, \tag{3.56}$$

and from Theorem 3.10. we get the a-priori estimate

$$C\,\varepsilon\,\kappa\,\Big(\int_t^T \|\,P_2' - P_2''\,\|^r_{L^r(\Omega)}\,d\tau\Big)^{1/r} \;=\; C\,\varepsilon\,\kappa\,\|\,P_2' - P_2''\,\|_{L^r\big[\,(\,t\,,\,T\,)\,,\,L^r(\Omega)\,\big]} \tag{3.57}$$

$$\geqslant\; C\,\Big(\,\|\,P_1' - P_1''\,\|_{L^r\big[\,(\,t\,,\,T\,)\,,\,W^{2,r}(\Omega)\,\big]} + \|\,P_1' - P_1''\,\|_{W^{1,r}\big[\,(\,t\,,\,T\,)\,,\,L^r(\Omega)\,\big]}\,\Big) \tag{3.58}$$

$$\geqslant\; C\,\Big(\,\|\,P_1' - P_1''\,\|_{L^r\big[\,(\,t\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]} + \|\,P_1' - P_1''\,\|_{W^{1,r}\big[\,(\,t\,,\,T\,)\,,\,L^2(\Omega)\,\big]}\,\Big) \tag{3.59}$$

$$\geqslant\; \|\,P_1' - P_1''\,\|_{C^0(\,\Omega\,\times\,[\,t\,,\,T\,]\,)} \;=\; \underset{(x,t)\,\in\,\Omega\,\times\,[\,t\,,\,T\,]}{\mathrm{Max}}\,\big|\,P_1'(x,t) - P_1''(x,t)\,\big|\,, \tag{3.60}$$

cf. [Veneroni 09] , p. 864 f. We find that

$$\|\,P_1'(t) - P_1''(t)\,\|^{6r/(6-r)}_{L^{6r/(6-r)}(\Omega)} = \int_\Omega \big|\,P_1'(t) - P_1''(t)\,\big|^{6r/(6-r)}\,dx \;\leqslant\; |\,\Omega\,|\cdot\Big(\,\underset{\cdots}{\mathrm{Max}}\,\big|\,P_1'(x,t) - P_1''(x,t)\,\big|\,\Big)^{6r/(6-r)} \tag{3.61}$$

$$\leqslant\; C\,\Big(\int_t^T \|\,P_2' - P_2''\,\|^r_{L^r(\Omega)}\,d\tau\Big)^{6/(6-r)} \;\Longrightarrow\; \|\,P_1'(t) - P_1''(t)\,\|^r_{L^{6r/(6-r)}(\Omega)} \;\leqslant\; C\int_t^T \|\,P_2' - P_2''\,\|^r_{L^r(\Omega)}\,d\tau \tag{3.62}$$

• **Step A6.** *Application of Banach's fixed point theorem.* We consider the operator

$$\mathcal{I}\colon\;\Big(\,L^r\big[\,(\,0\,,\,T\,)\,,\,L^{6r/(6-r)}(\Omega)\,\big]\times L^r(\Omega_T)\,\Big) \to \Big(\,L^r\big[\,(\,0\,,\,T\,),\,L^{6r/(6-r)}(\Omega)\,\big]\times L^r(\Omega_T)\,\Big)\,, \tag{3.63}$$

which assigns to a given pair (P_1, P_2) the new pair $(\mathcal{I}P_1, \mathcal{I}P_2)$ arising from the solution $\mathcal{I}P_2$ of the adjoint ODE after insertion of P_1 and the solution of the adjoint parabolic problem after insertion of $\mathcal{I}P_2$. In order to prove the contractivity of this operator, we start with two pairs (P_1', P_2'), $(P_1'', P_2'') \in L^r\big[\,(\,0\,,\,T\,)\,,\, L^{6r/(6-r)}(\Omega)\,\big]\times L^r(\Omega_T)$. By (3.47) and (3.52), it holds that

$$\|\,\mathcal{I}P_1'(t) - \mathcal{I}P_1''(t)\,\|^r_{L^{6r/(6-r)}(\Omega)} \;\leqslant\; C\int_t^T \|\,\mathcal{I}P_2'(\tau) - \mathcal{I}P_2''(\tau)\,\|^r_{L^r(\Omega)}\,d\tau \tag{3.64}$$

$$\leqslant\; C\int_t^T\int_\tau^T \|\,P_1'(\vartheta) - P_1''(\vartheta)\,\|^r_{L^{6r/(6-r)}(\Omega)}\,d\vartheta\,d\tau \;\leqslant\; C\,T\cdot\int_t^T \|\,P_1'(\vartheta) - P_1''(\vartheta)\,\|^r_{L^{6r/(6-r)}(\Omega)}\,d\vartheta\,. \tag{3.65}$$

Defining the functions

$$f(t) = \| \mathcal{I}P_1'(t) - \mathcal{I}P_1''(t) \|^r_{L^{6r/(6-r)}(\Omega)} \;\text{ and }\; \widetilde{f}(t) = \| P_1'(t) - P_1''(t) \|^r_{L^{6r/(6-r)}(\Omega)} \,, \tag{3.66}$$

this inequality reads as

$$0 \leqslant f(t) \leqslant C \int_t^T \widetilde{f}(\vartheta)\, d\vartheta \implies \int_0^T e^{\lambda_1 t} f(t)\, dt \leqslant C \cdot \int_0^T e^{\lambda_1 t} \Big(\int_t^T \widetilde{f}(\vartheta)\, d\vartheta \Big)\, dt \tag{3.67}$$

$$= C \Big[\frac{1}{\lambda_1} e^{\lambda_1 t} \cdot \int_t^T \widetilde{f}(\vartheta)\, d\vartheta \Big]_0^T + C \int_0^T \frac{1}{\lambda_1} e^{\lambda_1 t} \widetilde{f}(t)\, dt \tag{3.68}$$

$$= \frac{C}{\lambda_1} \Big(\int_0^T e^{\lambda_1 t} \widetilde{f}(t)\, dt - \int_0^T \widetilde{f}(\vartheta)\, d\vartheta \Big) \leqslant \frac{C}{\lambda_1} \int_0^T e^{\lambda_1 t} \widetilde{f}(t)\, dt \tag{3.69}$$

since the second member within the brackets is positive. Consequently, the operator $\mathcal{I}$ is with respect to its first component contractive on the space $L^r\big[\,(0\,,T\,)\,,\, L^{6r/(6-r)}(\Omega)\big]$ if this space is equipped with the equivalent norm

$$\| P_1 \| = \Big(\int_0^T e^{\lambda_1 t} \| P_1(t) \|^r_{L^{6r/(6-r)}(\Omega)}\, dt \Big)^{1/r} \tag{3.70}$$

with sufficiently large $\lambda_1 > C$. Analogously, we may estimate

$$\| \mathcal{I}P_2'(t) - \mathcal{I}P_2''(t) \|^r_{L^r(\Omega)} \leqslant C \cdot \int_t^T \| P_1'(\tau) - P_1''(\tau) \|^r_{L^{6r/(6-r)}(\Omega)}\, d\tau \tag{3.71}$$

$$\leqslant C \cdot \int_t^T \int_\tau^T \| P_2'(\vartheta) - P_2''(\vartheta) \|^r_{L^r(\Omega)}\, d\vartheta\, d\tau \leqslant C\,T \cdot \int_t^T \| P_2'(\vartheta) - P_2''(\vartheta) \|^r_{L^r(\Omega)}\, d\vartheta \,. \tag{3.72}$$

With the abbreviations

$$h(t) = \| \mathcal{I}P_2'(t) - \mathcal{I}P_2''(t) \|^r_{L^r(\Omega)} \;\text{ and }\; \widetilde{h}(t) = \| P_2'(t) - P_2''(t) \|^r_{L^r(\Omega)} \,, \tag{3.73}$$

the last inequality reads as

$$0 \leqslant h(t) \leqslant C \int_t^T \widetilde{h}(\vartheta)\, d\vartheta \implies \int_0^T e^{\lambda_2 t} h(t)\, dt \leqslant C \cdot \int_0^T e^{\lambda_2 t} \Big(\int_t^T \widetilde{h}(\vartheta)\, d\vartheta \Big)\, dt \tag{3.74}$$

$$= C \Big[\frac{1}{\lambda_2} e^{\lambda_2 t} \cdot \int_t^T \widetilde{h}(\vartheta)\, d\vartheta \Big]_0^T + C \int_0^T \frac{1}{\lambda_2} e^{\lambda_2 t} \widetilde{h}(t)\, dt \tag{3.75}$$

$$= \frac{C}{\lambda_2} \Big(\int_0^T e^{\lambda_2 t} \widetilde{h}(t)\, dt - \int_0^T \widetilde{h}(\vartheta)\, d\vartheta \Big) \leqslant \frac{C}{\lambda_2} \int_0^T e^{\lambda_2 t} \widetilde{h}(t)\, dt \tag{3.76}$$

since the second member within the brackets is positive again. This implies the contractivity of the operator $\mathcal{I}$ with respect to its second component on the space $L^r(\Omega_T)$ if this space is equipped with the equivalent norm

$$\| P_2 \| = \Big(\int_0^T e^{\lambda_2 t} \| P_2(t) \|^r_{L^r(\Omega)}\, dt \Big)^{1/r} \tag{3.77}$$

with sufficiently large $\lambda_2 > C$. Summing up, Banach's fixed point theorem yields the existence and uniqueness of a (weak or strong) solution for the adjoint system, which admits the improved regularity guaranteed by Theorem 3.10. and Step A2 above.

Part B. The FitzHugh-Nagumo model. In this case, (3.21) and (3.23) read as follows:

$$-\frac{\partial P_1}{\partial t} - \nabla\cdot\big(M_i\,\nabla P_1\big) + \Big(3\,(\hat\Phi_{tr})^2 - 2\,(a+1)\,\hat\Phi_{tr} + a\Big)\,P_1 \;=\; \varepsilon\,\kappa\,P_2 - \frac{\partial r}{\partial\varphi}(\hat\Phi_{tr},\hat W)\,; \tag{3.78}$$

$$-\frac{\partial P_2}{\partial t} + \varepsilon\,P_2 \;=\; -P_1 - \frac{\partial r}{\partial w}(\hat\Phi_{tr},\hat W)\,. \tag{3.79}$$

The proof runs as in Part A with minor alterations. In (3.44) and (3.52) – (3.54), a_0 must be replaced by

$$a_0 \;=\; \Big(3\,(\hat\Phi_{tr})^2 - 2\,(a+1)\,\hat\Phi_{tr} + a\Big)\,, \tag{3.80}$$

which belongs to $C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ as well. In the derivations in Steps A2 and A4, $\hat\Phi_{tr}$ is to be replaced by the constant 1. ∎

Proof of Theorem 3.7. The proof of the necessary optimality conditions is based on the error estimate (Theorem 2.7.) and the existence theorem for the adjoint system (Theorem 3.5.).

• **Step 1.** Assume that $(\hat\Phi_{tr},\hat W,\hat I_e)$ is a weak local minimizer of $(\mathrm{P})_1$. If $I_e\in\mathcal{C}$ is an arbitrary feasible control with $\|\,I_e-\hat I_e\,\|_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\leqslant\varepsilon$ then, by Proposition 3.1., all controls

$$I_e(s) \;=\; \hat I_e + s\,(I_e-\hat I_e)\,,\;\; 0\leqslant s\leqslant 1\,, \tag{3.81}$$

belong to $\mathcal{C}$ as well. By Theorem 2.5., for every $I_e(s)\in L^\infty\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$, there exists at least one corresponding weak solution $(\Phi_{tr}(s),W(s))\in \mathrm{X}_1\times\mathrm{X}_2$ for the monodomain problem on $[\,0\,,\,T\,]$. Consequently, the triples $(\Phi_{tr}(s),W(s),I_e(s))$ are feasible in $(\mathrm{P})_i$ for all $0\leqslant s\leqslant 1$. On the other hand, from Theorem 2.7. it follows that every feasible triple within the neighborhood $\mathrm{K}(\hat\Phi_{tr},C\,\varepsilon)\times\mathrm{K}(\hat W,C\,\varepsilon)\times\mathrm{K}(\hat I_e,\varepsilon)$ can be generated in this way.

• **Step 2. Lemma 3.11.** *For all* $I_e\in\mathcal{C}$, $\|\,I_e-\hat I_e\,\|_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\leqslant\varepsilon$ *implies that*

$$\lim_{s\to 0+0}\frac{1}{s}\,\|\,\Phi_{tr}(s)-\hat\Phi_{tr}\,\|^2_{\mathrm{X}_1}\;=\;0\,;\qquad \lim_{s\to 0+0}\frac{1}{s}\,\|\,\Phi_{tr}(s)-\hat\Phi_{tr}\,\|^2_{\widetilde{\mathrm{X}}_1}\;=\;0\,; \tag{3.82}$$

$$\lim_{s\to 0+0}\frac{1}{s}\,\|\,W(s)-\hat W\,\|^2_{\mathrm{X}_2}\;=\;0\;\;\text{and}\;\;\lim_{s\to 0+0}\frac{1}{s}\,\|\,W(s)-\hat W\,\|^2_{\widetilde{\mathrm{X}}_2}\;=\;0\,. \tag{3.83}$$

Proof. From Theorem 2.7., (2.21), we derive

$$\begin{aligned}\|\,\Phi_{tr}(s)-\hat\Phi_{tr}\,\|^2_{\mathrm{X}_1} \;&=\; \|\,\Phi_{tr}(s)-\hat\Phi_{tr}\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\right]}\;\leqslant\; C\cdot\|\,I_e(s)-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\\ &\leqslant\; C\cdot\|\,I_e(s)-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\;=\;C\,s^2\,\|\,I_e-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\quad\Longrightarrow\end{aligned} \tag{3.84}$$

$$\lim_{s\to 0+0}\frac{1}{s}\,\|\,\Phi_{tr}(s)-\hat\Phi_{tr}\,\|^2_{\mathrm{X}_1}\;\leqslant\;\lim_{s\to 0+0}C\,s\,\|\,I_e-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\;=\;0 \tag{3.85}$$

as well as

$$\begin{aligned}\|\,W(s)-\hat W\,\|^2_{\mathrm{X}_2}\;&=\;\|\,W(s)-\hat W\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\;\leqslant\;C\cdot\|\,I_e(s)-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\\ &\leqslant\;C\cdot\|\,I_e(s)-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\;=\;C\,s^2\,\|\,I_e-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\quad\Longrightarrow\end{aligned} \tag{3.86}$$

$$\lim_{s\to 0+0}\frac{1}{s}\,\|\,W(s)-\hat W\,\|^2_{\mathrm{X}_2}\;\leqslant\;\lim_{s\to 0+0}C\,s\,\|\,I_e-\hat I_e\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}\;=\;0\,. \tag{3.87}$$

The relation with $\| W(s) - \hat{W} \|^2_{\mathrm{X}_2}$ can be confirmed analogously. Finally, (2.22) implies that

$$\| \Phi_{tr}(s) - \hat{\Phi}_{tr} \|^2_{\widetilde{\mathrm{X}}_1} = \| \Phi_{tr}(s) - \hat{\Phi}_{tr} \|^2_{W^{1,4/3}[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \tag{3.88}$$

$$\leqslant C^2 \cdot \mathrm{Max}\Big(\| I_e(s) - \hat{I}_e \|^2_{L^\infty[\,(0,T),\,(W^{1,2}(\Omega))^*\,]}\,,\ \| I_e(s) - \hat{I}_e \|^4_{L^\infty[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big) \tag{3.89}$$

$$\leqslant C \cdot \mathrm{Max}\Big(s^2 \| I_e - \hat{I}_e \|^2_{L^\infty[\,(0,T),\,L^2(\Omega)\,]}\,,\ s^4 \| I_e - \hat{I}_e \|^4_{L^\infty[\,(0,T),\,L^2(\Omega)\,]} \Big) \quad \Longrightarrow \tag{3.90}$$

$$\lim_{s\to 0+0} \frac{1}{s} \| \Phi_{tr}(s) - \hat{\Phi}_{tr} \|^2_{\mathrm{X}_1} \tag{3.91}$$

$$\leqslant \lim_{s\to 0+0} C \cdot \mathrm{Max}\Big(s \| I_e - \hat{I}_e \|^2_{L^\infty[\,(0,T),\,L^2(\Omega)\,]}\,,\ s^3 \| I_e - \hat{I}_e \|^4_{L^\infty[\,(0,T),\,L^2(\Omega)\,]} \Big) = 0\,. \quad \blacksquare$$

• **Step 3.** By Theorem 3.5., there exist solutions $P_1 \in \big(L^{4/3}[\,(0,T),\,(W^{1,2}(\Omega))^*\,]\big)^* = L^4[\,(0,T),\,W^{1,2}(\Omega)\,]$ and $P_2 \in \big(L^2[\,(0,T),\,(L^2(\Omega))^*\,]\big)^* = L^2(\Omega_T)$ for the adjoint system in relation to $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$. Using these functions, we get the following estimates:

Lemma 3.12. *With the notations of Subsection 3.a), it holds that*

$$\lim_{s\to 0+0} \frac{1}{s} \big\langle P_1\,,\, D_\Phi\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}) + D_W\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(W(s) - \hat{W}) \big\rangle \tag{3.92}$$

$$+ \big\langle P_1\,,\, D_{I_e}\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(I_e - \hat{I}_e) \big\rangle = 0\,;$$

$$\lim_{s\to 0+0} \frac{1}{s} \big\langle P_2\,,\, D_\Phi\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}) + D_W\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(W(s) - \hat{W}) \big\rangle = 0\,. \tag{3.93}$$

Proof. The proof relies on the principal theorem of the calculus in its Bochner integral version, [22] which becomes applicable due to our assumptions about the differentiability of r. We start with the feasibility of $(\Phi_{tr}(s), W(s), I_e(s))$ and $(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$:

$$\mathfrak{o} = E_1(\Phi_{tr}(s), W(s), I_e(s)) - E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) = \int_0^1 D_{(\Phi,W,I_e)}\, E_1\big(\hat{\Phi}_{tr} + \tau\,(\Phi_{tr}(s) - \hat{\Phi}_{tr})\,,$$

$$\hat{W} + \tau\,(W(s) - \hat{W})\,,\ \hat{I}_e + \tau\,(I_e(s) - \hat{I}_e) \big)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}, W(s) - \hat{W}, I_e(s) - \hat{I}_e)\, d\tau \quad \Longrightarrow \tag{3.94}$$

$$0 = \Big\langle P_1\,,\, \int_0^1 \Big(D_{(\Phi,W,I_e)}\, E_1\big(\hat{\Phi}_{tr} + \tau\,(\Phi_{tr}(s) - \hat{\Phi}_{tr})\,,\ \hat{W} + \tau\,(W(s) - \hat{W})\,,\ \hat{I}_e + \tau\,(I_e(s) - \hat{I}_e) \big)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr},$$

$$W(s) - \hat{W}, I_e(s) - \hat{I}_e) - D_{(\Phi,W,I_e)} E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}, W(s) - \hat{W}, I_e(s) - \hat{I}_e) \Big)\, d\tau \Big\rangle$$

$$+ \big\langle P_1\,,\, D_{(\Phi,W,I_e)} E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}, W(s) - \hat{W}, I_e(s) - \hat{I}_e) \big\rangle \tag{3.95}$$

$$= \Big\langle P_1\,,\, \int_0^1 \Big(D_\Phi\, E_1(\hat{\Phi}_{tr} + \tau\,\ldots\,, \hat{W} + \tau\,\ldots\,, \hat{I}_e + \tau\,\ldots\,)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr}) - D_\Phi\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(\Phi_{tr}(s) - \hat{\Phi}_{tr})$$

$$+ D_W\, E_1(\hat{\Phi}_{tr} + \tau\,\ldots\,, \hat{W} + \tau\,\ldots\,, \hat{I}_e + \tau\,\ldots\,)\,(W(s) - \hat{W}) - D_W\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(W(s) - \hat{W}) \tag{3.96}$$

$$+ D_{I_e} E_1(\hat{\Phi}_{tr} + \tau\,\ldots\,, \hat{W} + \tau\,\ldots\,, \hat{I}_e + \tau\,\ldots\,)\,(I_e(s) - \hat{I}_e) - D_{I_e}\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,(I_e(s) - \hat{I}_e) \Big)\, d\tau \Big\rangle$$

$$+ \big\langle P_1\,,\, D_\Phi E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)(\Phi_{tr}(s) - \hat{\Phi}_{tr}) + D_W E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)(W(s) - \hat{W}) + D_{I_e} E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)(I_e(s) - \hat{I}_e) \big\rangle\,.$$

Observe now that [23]

$$\Big| \Big\langle P_1\,,\, \int_0^1 (\,\ldots\,)\, d\tau \Big\rangle \Big| \leqslant \| P_1 \|_{\mathrm{Z}_1^*} \cdot \Big\| \int_0^1 (\,\ldots\,)\, d\tau \Big\|_{\mathrm{Z}_1} \leqslant \| P_1 \|_{\mathrm{Z}_1^*} \cdot \int_0^1 \| \ldots \|_{\mathrm{Z}_1}\, d\tau\,. \tag{3.97}$$

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[22] [Berger 77], p. 68, (2.1.11).

[23] For the last inequality, cf. [Yosida 95], p. 133, Corollary 1.

Consequently, for the first term within the last equation, we have

$$\lim_{s\to 0+0} \frac{1}{s} \left| \left\langle P_1 \, , \int_0^1 (\dots)\, d\tau \right\rangle \right| \leqslant \lim_{s\to 0+0} \| P_1 \|_{\mathrm{Z}_1^*} \Big(\int_0^1 | D_\Phi \, E_1(\hat\Phi_{tr} + \tau \dots , \hat W + \tau \dots , \hat I_e + \tau \dots) \tag{3.98}$$

$$- D_\Phi \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \|_{\mathcal{L}(\widetilde{\mathrm{X}}_1,\mathrm{Z}_1)} \frac{1}{s} \| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} \, d\tau$$

$$+ \int_0^1 \big\| D_W \, E_1(\hat\Phi_{tr} + \tau \dots , \hat W + \tau \dots , \hat I_e + \tau \dots) - D_W \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \big\|_{\mathcal{L}(\widetilde{\mathrm{X}}_2,Z_1)} \frac{1}{s} \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} \, d\tau$$

$$+ \int_0^1 \big\| D_{I_e} E_1(\hat\Phi_{tr} + \tau \dots , \hat W + \tau \dots , \hat I_e + \tau \dots) - D_{I_e} \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \big\|_{\mathcal{L}(\mathrm{X}_3,\mathrm{Z}_1)} \frac{1}{s} \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \, d\tau \Big)$$

$$\leqslant \lim_{s\to 0+0} \| P_1 \|_{\mathrm{Z}_1^*} \Big(\int_0^1 L_1 \, \tau \Big(\| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} + \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} + \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \Big) \frac{1}{s} \| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} \, d\tau$$

$$+ \int_0^1 L_2 \, \tau \Big(\| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} + \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} + \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \Big) \frac{1}{s} \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} \, d\tau \tag{3.99}$$

$$+ \int_0^1 L_3 \, \tau \Big(\| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} + \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} + \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \Big) \frac{1}{s} \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \, d\tau \Big)$$

with Lipschitz constants L_1, L_2, L_3, whose existence is ensured by the twice continuous Fréchet differentiability of E_1 with respect to Φ, W and I_e. With Lemma 3.11., we may further estimate:

$$\lim_{s\to 0+0} \frac{1}{s} \left| \left\langle P_1 \, , \int_0^1 (\dots)\, d\tau \right\rangle \right| \tag{3.100}$$

$$\leqslant \lim_{s\to 0+0} \| P_1 \|_{\mathrm{Z}_1^*} \frac{1}{2} \big(L_1 + L_2 + L_3 \big) \frac{1}{s} \Big(\| \Phi_{tr}(s) - \hat\Phi_{tr} \|_{\widetilde{\mathrm{X}}_1} + \| W(s) - \hat W \|_{\widetilde{\mathrm{X}}_2} + \| I_e(s) - \hat I_e \|_{\mathrm{X}_3} \Big)^2$$

$$\leqslant \lim_{s\to 0+0} \| P_1 \|_{\mathrm{Z}_1^*} \, C \Big(\frac{1}{s} \| \Phi_{tr}(s) - \hat\Phi_{tr} \|^2_{\widetilde{\mathrm{X}}_1} + \frac{1}{s} \| W(s) - \hat W \|^2_{\widetilde{\mathrm{X}}_2} + \frac{1}{s} \| I_e(s) - \hat I_e \|^2_{\mathrm{X}_3} \Big) \; = \; 0 \, , \tag{3.101}$$

which implies that

$$\lim_{s\to 0+0} \frac{1}{s} \big\langle P_1 \, , \, D_\Phi \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \, (\Phi_{tr}(s) - \hat\Phi_{tr}) + D_W \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \, (W(s) - \hat W) \tag{3.102}$$

$$+ D_{I_e} \, E_1(\hat\Phi_{tr}, \hat W, \hat I_e) \, (I_e(s) - \hat I_e) \big\rangle \; = \; 0 \, .$$

Analogously, the following equation holds:

$$\mathfrak{o} \; = \; E_2(\Phi_{tr}(s), W(s)) - E_2(\hat\Phi_{tr}, \hat W) \tag{3.103}$$

$$= \int_0^1 D_{(\Phi,W)} \, E_2 \big(\hat\Phi_{tr} + \tau \, (\Phi_{tr}(s) - \hat\Phi_{tr}) \, , \, \hat W + \tau \, (W(s) - \hat W) \big) \, (\Phi_{tr}(s) - \hat\Phi_{tr}, W(s) - \hat W) \, d\tau \quad \Longrightarrow$$

$$0 \; = \; \Big\langle P_2 \, , \int_0^1 \Big(D_{(\Phi,W)} \, E_2 \big(\hat\Phi_{tr} + \tau \, (\Phi_{tr}(s) - \hat\Phi_{tr}) \, , \, \hat W + \tau \, (W(s) - \hat W) \big) \, (\Phi_{tr}(s) - \hat\Phi_{tr}, W(s) - \hat W) \tag{3.104}$$

$$- \; D_{(\Phi,W)} \, E_2(\hat\Phi_{tr}, \hat W) \, (\Phi_{tr}(s) - \hat\Phi_{tr}, W(s) - \hat W) \Big) \, d\tau \Big\rangle + \Big\langle P_2 \, , \, D_{(\Phi,W)} \, E_2(\hat\Phi_{tr}, \hat W) \, (\Phi_{tr}(s) - \hat\Phi_{tr}, W(s) - \hat W) \Big\rangle$$

$$= \Big\langle P_2 \, , \int_0^1 \Big(D_\Phi \, E_2(\hat\Phi_{tr} + \tau \dots , \hat W + \tau \dots) \, (\Phi_{tr}(s) - \hat\Phi_{tr}) - D_\Phi \, E_2(\hat\Phi_{tr}, \hat W) \, (\Phi_{tr}(s) - \hat\Phi_{tr}) \tag{3.105}$$

$$+ \; D_W \, E_2(\hat\Phi_{tr} + \tau \dots , \hat W + \tau \dots) \, (W(s) - \hat W) - D_W \, E_2(\hat\Phi_{tr}, \hat W) \, (W(s) - \hat W) \Big) \, d\tau \Big\rangle$$

$$+ \; \Big\langle P_2 \, , \, D_\Phi \, E_2(\hat\Phi_{tr}, \hat W) \, (\Phi_{tr}(s) - \hat\Phi_{tr}) + D_W \, E_2(\hat\Phi_{tr}, \hat W) \, (W(s) - \hat W) \Big\rangle \, .$$

For the first term, we find

$$\lim_{s\to 0+0} \frac{1}{s}\, \big|\, \big\langle\, P_2\,,\, \int_0^1 (\,...\,)\, d\tau\, \big| \tag{3.106}$$

$$\leqslant \lim_{s\to 0+0} \|\, P_2\,\|_{\mathrm{Z}_2^*} \Big(\int_0^1 \big\|\, D_\Phi\, E_2(\hat{\Phi}_{tr} + \tau\, ...\,, \hat{W} + \tau\, ...\,) - D_\Phi\, E_2(\hat{\Phi}_{tr}, \hat{W})\,\big\|_{\mathcal{L}(\widetilde{\mathrm{X}}_1, \mathrm{Z}_2)}\, \frac{1}{s}\, \|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1}\, d\tau$$

$$+ \int_0^1 \big\|\, D_W\, E_2(\hat{\Phi}_{tr} + \tau\, ...\,, \hat{W} + \tau\, ...\,) - D_W\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\,\big\|_{\mathcal{L}(\widetilde{\mathrm{X}}_2, Z_2)}\, \frac{1}{s}\, \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2}\, d\tau \Big)$$

$$\leqslant \lim_{s\to 0+0} \|\, P_2\,\|_{\mathrm{Z}_2^*} \Big(\int_0^1 L_4\, \tau\, \Big(\|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1} + \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2} \Big)\, \frac{1}{s}\, \|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1}\, d\tau \tag{3.107}$$

$$+ \int_0^1 L_5\, \tau\, \Big(\|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1} + \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2} \Big)\, \frac{1}{s}\, \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2}\, d\tau \Big)$$

with Lipschitz constants L_4, L_5, whose existence follows from the twice continuous Fréchet differentiability of E_2 with respect to Φ and W. Thus the estimate (3.107) may be continued as follows:

$$\lim_{s\to 0+0} \frac{1}{s}\, \big|\, \big\langle\, P_2\,, \int_0^1 (\,...\,)\, d\tau\, \big\rangle\, \big| \leqslant \lim_{s\to 0+0} \|\, P_2\,\|_{\mathrm{Z}_2^*}\, \frac{1}{2} \big(L_4 + L_5\, \big)\, \frac{1}{s}\, \Big(\|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1} + \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2} \Big)^2 \tag{3.108}$$

$$\leqslant \lim_{s\to 0+0} \|\, P_2\,\|_{\mathrm{Z}_2^*}\, C\, \Big(\frac{1}{s}\, \|\, \Phi_{tr}(s) - \hat{\Phi}_{tr}\,\|_{\widetilde{\mathrm{X}}_1}^2 + \frac{1}{s}\, \|\, W(s) - \hat{W}\,\|_{\widetilde{\mathrm{X}}_2}^2 \Big) \;=\; 0\,. \tag{3.109}$$

This implies that

$$\lim_{s\to 0+0} \frac{1}{s}\, \big\langle\, P_2\,,\, D_\Phi\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(s) - \hat{\Phi}_{tr}) + D_W\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (W(s) - \hat{W})\, \big\rangle \;=\; 0\,. \quad \blacksquare \tag{3.110}$$

• **Step 4.** Choose now $\varepsilon > 0$ small enough in order to ensure that the difference $F(\Phi_{tr}(s), W(s), I_e(s)) - F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)$ of the objective values is nonnegative for all triples $(\Phi_{tr}(s), W(s), I_e(s)) \in \mathrm{K}(\hat{\Phi}_{tr}, C\,\varepsilon) \times \mathrm{K}(\hat{W}, C\,\varepsilon) \times \mathrm{K}(\hat{I}_e, \varepsilon)$. As a consequence of our analytical assumptions about the integrand r, the first variation may be written as

$$0 \leqslant \delta^+ F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(1) - \hat{\Phi}_{tr}, W(1) - \hat{W}, I_e - \hat{I}_e) \;=\; \lim_{s\to 0+0} \frac{1}{s}\, \Big(F(\Phi_{tr}(s), W(s), I_e(s)) - F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e) \Big)$$

$$= \lim_{s\to 0+0} \frac{1}{s}\, \Big(D_\Phi\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(s) - \hat{\Phi}_{tr}) \tag{3.111}$$

$$+ D_W\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (W(s) - \hat{W}) + D_{I_e}\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (I_e(s) - \hat{I}_e) \Big)\,.$$

Together with Lemma 3.12., we obtain

$$0 \leqslant \lim_{s\to 0+0} \frac{1}{s}\, \Big(D_\Phi\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(s) - \hat{\Phi}_{tr}) \tag{3.112}$$

$$+ \big\langle\, P_1\,,\, D_\Phi\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(s) - \hat{\Phi}_{tr})\, \big\rangle + \big\langle\, P_2\,,\, D_\Phi\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (\Phi_{tr}(s) - \hat{\Phi}_{tr})\, \big\rangle$$

$$+\; D_W\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (W(s) - \hat{W})$$

$$+ \big\langle\, P_1\,,\, D_W\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (W(s) - \hat{W})\, \big\rangle + \big\langle\, P_2\,,\, D_W\, E_2(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (W(s) - \hat{W})\, \big\rangle$$

$$+\; D_{I_e}\, F(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (I_e(s) - \hat{I}_e) + \big\langle\, P_1\,,\, D_{I_e}\, E_1(\hat{\Phi}_{tr}, \hat{W}, \hat{I}_e)\, (I_e - \hat{I}_e)\, \big\rangle \Big)$$

where the first two parts vanish since P_1, P_2 solve the adjoint equations

$$D_\phi\, F(\Phi_{tr}, W, I_e) \,+\, \langle\, P_1\,,\, \frac{\partial E_1}{\partial \Phi_{tr}}(\Phi_{tr}, W, I_e)\, \rangle \,+\, \langle\, P_2\,,\, \frac{\partial E_2}{\partial \Phi_{tr}}(\Phi_{tr}, W)\, \rangle \;=\; 0 \quad \text{and} \tag{3.113}$$

$$D_W\, F(\Phi_{tr}, W, I_e) \,+\, \langle\, P_1\,,\, \frac{\partial E_1}{\partial W}(\Phi_{tr}, W, I_e)\, \rangle \,+\, \langle\, P_2\,,\, \frac{\partial E_2}{\partial W}(\Phi_{tr}, W)\, \rangle \;=\; 0\,. \tag{3.114}$$

Note that, by Subsection 3.c) above, these equations take the claimed form. Consequently, we arrive at

$$0 \leqslant \lim_{s\to 0+0} \frac{1}{s} \Big(D_{I_e} F(\hat\Phi_{tr}, \hat W, \hat I_e)\,(I_e(s) - \hat I_e) + \big\langle\, P_1\,,\, D_{I_e} E_1(\hat\Phi_{tr}, \hat W, \hat I_e)\,(I_e - \hat I_e)\,\big\rangle \Big) \tag{3.115}$$

$$= \int_0^T \int_\Omega \Big(\mu\, \hat I_e(x,t) + \frac{1}{1+\lambda}\, P_1(x,t) \Big) \big(I_e(x,t) - \hat I_e(x,t) \big)\, dx\, dt\,. \tag{3.116}$$

This implies the claimed optimality condition (3.31), and the proof is complete. ■

Proof of Corollary 3.8. The non-Lebesgue points of $\hat I_e$, P_1, [24]

$$\Big(\mu\,(1+\lambda)\, \hat I_e(x,t) + P_1(x,t) \Big)\, \hat I_e(x,t) \quad \text{and} \quad \Big(\mu\,(1+\lambda)\, \hat I_e(x,t) + P_1(x,t) \Big)\, \eta \tag{3.117}$$

form null sets for arbitrary $\eta \in [\,-R\,,\, R\,]$ since the set of Lebesgue points is conserved under linear combinations. Assumption 3.4., 1)′ implies that the boundary $\partial\Omega \times [\,0\,,\,T\,]$ of Ω_T forms a null set as well. [25] Denote by N the union of all these subsets, which is still a Lebesgue null set, and consider a point $(x_0, t_0) \in \operatorname{int}(\Omega_T) \setminus \mathrm{N}$ and a number $\eta_0 \in [\,-R\,,\, R\,]$. From a Vitali covering of Ω_T, [26] choose some decreasing sequence $\{\, \mathrm{E}^N \,\}$ of closed subsets of Ω_T with $\bigcap_N \mathrm{E}^N = \{\, (x_0, t_0)\,\}$. Together with $\hat I_e$, all functions

$$I^N(x,t) = \mathbb{1}_{\mathrm{E}^N}(x,t) \cdot \eta_0 + \mathbb{1}_{(\,\Omega \,\setminus\, \mathrm{E}^N\,)}(x,t) \cdot \hat I_e(x,t) \tag{3.118}$$

belong to $\mathcal{C} \subset L^\infty(\Omega_T) \subset L^\infty\big[\,(\,0\,,\,T\,)\,,\, L^2(\Omega)\,\big]$. Consequently, we may form the Lebesgue derivative

$$\lim_{N\to\infty} \frac{1}{|\,\mathrm{E}^N\,|} \int_0^T \int_\Omega \Big(\mu\,(1+\lambda)\, \hat I_e(x,t) + P_1(x,t) \Big) \big(I^N(x,t) - \hat I_e(x,t) \big)\, dx\, dt$$

$$= \lim_{N\to\infty} \frac{1}{|\,\mathrm{E}^N\,|} \int_{\mathrm{E}^N} \Big(\mu\,(1+\lambda)\, \hat I_e(x,t) + P_1(x,t) \Big) \big(\eta_0 - \hat I_e(x,t) \big)\, d(x,t) \tag{3.119}$$

$$= \Big(\mu\,(1+\lambda)\, \hat I_e(x_0,t_0) + P_1(x_0,t_0) \Big) \big(\eta_0 - \hat I_e(x_0,t_0) \big) \;\geqslant\; 0\,, \tag{3.120}$$

and this implies the claimed conditions (3.32) and (3.33). ■

Proof of Corollary 3.9. If $(\hat\Phi_{tr}, \hat W, \hat I_e)$ is a weak local minimizer of $(\mathrm{P})_i$ with $\operatorname{ess\,sup}_{(x,t)\,\in\,\Omega_T} \,|\, \hat I_e(x,t)\,| < R$ then (3.33) implies that

$$\hat I_e(x,t) \;=\; \frac{1}{\mu\,(1+\lambda)}\, P_1(x,t) \tag{3.121}$$

for almost all $(x,t) \in \Omega_T$, and $\hat I_e$ and P_1 belong to the same space $W^{1,r}\big[\,(\,0\,,\,T\,)\,,\, L^r(\Omega)\,\big] \,\cap\, L^r\big[\,(\,0\,,\,T\,)\,,$ $W^{2,r}(\Omega)\,\big]$. ■

4. Appendix: Bochner integrable mappings.

a) Survey of spaces of Bochner integrable functions.

Let X be a Banach space. Then a mapping $f\colon [\,0\,,\,T\,] \to \mathrm{X}$ is called strongly measurable iff there is a sequence $\{\, f^N \,\}$ of simple mappings $f^N\colon [\,0\,,\,T\,] \to \mathrm{X}$ of the form $f^N(t) = \sum_{k=1}^K \mathbb{1}_{\mathrm{A}_k}(t)\, x_k$ with $x_k \in \mathrm{X}$ and Lebesgue subsets $\mathrm{A}_k \subseteq [\,0\,,\,T\,]$ such that $\|\, f^N(t) - f(t)\,\|_{\mathrm{X}} \to 0$ for almost all $t \in [\,0\,,\,T\,]$. [27]

24) For the following arguments, cf. [Wagner 09] , p. 553 f., Proof of Theorem 2.3.

25) [Wagner 06] , p. 122, Lemma 9.2.

26) [Dunford/Schwartz 88] , p. 212, Definition 2.

27) Here and in Definition 4.1., we follow [Evans 98] , pp. 285 f. and 649 f.

Definition 4.1. (Bochner function spaces) *1) Let* $1 \leqslant p < \infty$*. Then the space* $L^p\big[(0, T), \mathrm{X}\big]$ *consists of all strongly measurable functions* $f : [0, T] \to \mathrm{X}$ *with*

$$\| f \|_{L^p[(0,T),\mathrm{X}]} = \Big(\int_0^T \| f(t) \|_{\mathrm{X}} \, dt \Big)^{1/p} < \infty . \tag{4.1}$$

2) The space $L^\infty\big[(0, T), \mathrm{X}\big]$ *consists of all strongly measurable functions* $f : [0, T] \to \mathrm{X}$ *with*

$$\| f \|_{L^\infty[(0,T),\mathrm{X}]} = \operatorname{ess\,sup}_{t \in [0,T]} \| f(t) \|_{\mathrm{X}} < \infty . \tag{4.2}$$

3) The space $W^{1,p}\big[(0, T), \mathrm{X}\big]$ *consists of all functions* $f \in L^p\big[(0, T), \mathrm{X}\big]$*, admitting a weak derivative* df/dt*, which belongs to* $L^p\big[(0, T), \mathrm{X}\big]$ *as well. The weak derivative is defined by the usual formula wherein the integrals are interpreted in the Bochner sense.*
4) The space $C^0\big[[0, T], \mathrm{X}\big]$ *contains all continuous mappings* $f : [0, T] \to \mathrm{X}$ *with*

$$\| f \|_{C^0[[0,T],\mathrm{X}]} = \operatorname{Max}_{t \in [0,T]} \| f(t) \|_{\mathrm{X}} < \infty . \tag{4.3}$$

b) Imbedding theorems for Bochner spaces.

Proposition 4.2. *Assume that* $\Omega \subset \mathbb{R}^m$ *is compact and* $1 \leqslant p, q \leqslant \infty$*. Then* $L^p\big[(0, T), L^q(\Omega)\big]$ *is continuously imbedded into* $L^{\mathrm{Min}(p,q)}(\Omega_T)$*.*

Proof. This is a consequence of [Elstrodt 96], p. 232, Theorem 2.10. ∎

Proposition 4.3. [28] *If* $1 \leqslant p \leqslant q < \infty$ *then* $L^p\big[\Omega, L^q(0, T)\big]$ *is continuously imbedded into* $L^q\big[(0, T), L^p(\Omega)\big]$ *.*

Proposition 4.4. *If* $1 \leqslant p < q \leqslant \infty$ *and* X *is a Banach space then* $L^q\big[(0, T), \mathrm{X}\big]$ *is continuously imbedded into* $L^p\big[(0, T), \mathrm{X}\big]$*.*

Proof. Follows from [Elstrodt 96], p. 232, Theorem 2.10., as well. ∎

Theorem 4.5. [29] *If* $1 \leqslant p \leqslant \infty$ *and* X *is a Banach space then* $W^{1,p}\big[(0, T), \mathrm{X}\big]$ *is continuously imbedded into* $C^0\big[[0, T], \mathrm{X}\big]$*.*

Theorem 4.6. (Aubin-Dubinskij lemma) [30] *Consider three normed spaces* $\mathrm{X}_0 \subseteq \mathrm{X} \subseteq \mathrm{X}_1$ *where the imbedding* $\mathrm{X}_0 \hookrightarrow \mathrm{X}$ *is compact and the imbedding* $\mathrm{X} \hookrightarrow \mathrm{X}_1$ *is continuous. If* $p, p' \in (1, \infty)$ *then the space*

$$\mathrm{Y} = \Big\{ f \in L^p\big[(0, T), \mathrm{X}_0\big] \;\Big|\; \frac{df}{dt} \in L^{p'}\big[(0, T), \mathrm{X}_1\big] \Big\} \tag{4.4}$$

is compactly imbedded into $L^q\big[(0, T), \mathrm{X}\big]$ *for arbitrary* $q \in (1, \infty)$*.*

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28) [Garling 07], p. 52, Corollary 5.4.2.

29) [Evans 98], p. 286, Theorem 2.

30) [Dubinskij 65], p. 612, Teorema 1, and p. 615, Teorema 2.

References.

1. [Ainseba/Bendamahne/Ruiz-Baier 10] Ainseba, B.; Bendamahne, M.; Ruiz-Baier, R.: *Analysis of an optimal control problem for the tridomain model in cardiac electrophysiology.* EPF Lausanne 2010; EPFL-REPORT-155519, 20 pp.
2. [Aliev/Panfilov 96] Aliev, R. R.; Panfilov, A. V.: *A simple two-variable model of cardiac excitation.* Chaos, Solitons & Fractals **7** (1996), 293 – 301
3. [Berger 77] Berger, M. S.: *Nonlinearity and Functional Analysis.* Academic Press; New York - San Francisco - London 1977
4. [Boulakia/Fernández/Gerbeau/Zemzemi 08] Boulakia, M.; Fernández, M. A.; Gerbeau, J.-F.; Zemzemi, N.: *A coupled system of PDEs and ODEs arising in electrocardiograms modeling.* Appl. Math. Res. Express (2008), http://dx.doi.org/10.1093/amrx/abn002 (electronically published)
5. [Bourgault/Coudière/Pierre 09] Bourgault, Y.; Coudière, Y.; Pierre, C.: *Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology.* Nonlinear Analysis: Real World Appl. **10** (2009), 458 – 482
6. [Brandão/Fernández-Cara/Magalhães/Rojas-Medar 08] Brandaõ, A. J. V.; Fernández-Cara, E.; Magalhães, P. M. D.; Rojas-Medar, M. A.: *Theoretical analysis and control results for the FitzHugh-Nagumo equation.* Electronic J. of Differential Equations, Vol. 2008, No. 164, 1 – 20 (electronically published)
7. [Colli Franzone/Deuflhard/Erdmann/Lang/Pavarino 06] Colli Franzone, P.; Deuflhard, P.; Erdmann, B.; Lang, J.; Pavarino, L. F.: *Adaptivity in space and time for reaction-diffusion systems in electrocardiology.* SIAM J. Sci. Comput. **28** (2006), 942 – 962
8. [Colli Franzone/Savaré 02] Colli Franzone, P.; Savaré, G.: *Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level.* In: Lorenzi, A.; Ruf, B. (Eds.): Evolution equations, semigroups and functional analysis. In memory of Brunello Terreni. Including papers from the conference held in Milano, September 27–28, 2000. Birkhäuser; Basel - Boston - Berlin 2002 (Progress in Nonlinear Differential Equations and their Applications, Vol. 50), 49 – 78
9. [Dacorogna 08] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer; New York etc. 2008, 2nd ed.
10. [Dubinskij 65] Dubinskij, Ju. A. (Дубинский, Ю. А.): *Слабая сходимость в нелинейных эллиптических и параболических уравнениях.* Мат. Сборник **67** (1965) 4, 609 – 642
11. [Elstrodt 96] Elstrodt, J.: *Maß- und Integrationstheorie.* Springer; Berlin etc. 1996
12. [Evans 98] Evans, L. C.: *Partial Differential Equations.* American Mathematical Society; Providence 1998
13. [FitzHugh 61] FitzHugh, R.: *Impulses and physiological states in theoretical models of nerve membrane.* Biophysical J. **1** (1961), 445 – 466
14. [Garling 07] Garling, D. J. H.: *Inequalities: A Journey into Linear Analysis.* Cambridge University Press; Cambridge 2007
15. [Ioffe/Tichomirow 79] Ioffe, A. D.; Tichomirow, V. M.: *Theorie der Extremalaufgaben.* VEB Deutscher Verlag der Wissenschaften; Berlin 1979
16. [Ito/Kunisch 08] Ito, K.; Kunisch, K.: *Lagrange Multiplier Approach to Variational Problems and Applications.* SIAM; Philadelphia 2008
17. [Muzdeka/Barbieri 05] Muzdeka, S.; Barbieri, E.: *Control theory inspired considerations for the mathematical model defibrillation.* In: Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference, 7416 – 7421
18. [Nagaiah/Kunisch 11] Nagaiah, C.; Kunisch, K.: *Higher order optimization and adaptive numerical solution for optimal control of monodomain equations in cardiac electrophysiology.* Appl. Num. Math. **61** (2011), 53 – 65
19. [Nagaiah/Kunisch/Plank 09] Nagaiah, C.; Kunisch, K.; Plank, G.: *Numerical solution for optimal control of the reaction-diffusion equations in cardiac electrophysiology.* To appear in: Comput. Optim. Appl.; DOI: 10.1007/s10589-009-9280-3

20. [NAGUMO/ARIMOTO/YOSHIZAWA 62] Nagumo, J.; Arimoto, S.; Yoshizawa, S.: *An active pulse transmission line simulating nerve axon.* Proceedings of the Institute of Radio Engineers **50** (1962), 2061 – 2070

21. [ROGERS/MCCULLOCH 94] Rogers, J. M.; McCulloch, A. D.: *A collocation-Galerkin finite element model of cardiac action potential propagation.* IEEE Trans. Biomed. Eng. **41** (1994), 743 – 757

22. [ROLEWICZ 76] Rolewicz, S.: *Funktionalanalysis und Steuerungstheorie.* Springer; Berlin - Heidelberg - New York 1976

23. [SUNDNES/LINES/CAI/NIELSEN/MARDAL/TVEITO 06] Sundnes, J.; Lines, G. T.; Cai, X.; Nielsen, B. F.; Mardal, K.-A.; Tveito, A.: *Computing the Electrical Activity in the Heart.* Springer; Berlin etc. 2006

24. [TRÖLTZSCH 09] Tröltzsch, F.: *Optimale Steuerung partieller Differentialgleichungen.* Vieweg + Teubner — GWV Fachverlage GmbH; Wiesbaden 2009. 2nd ed.

25. [TUNG 78] Tung, L.: *A Bi-Domain Model for Describing Ischemic Myocardial D-C Potentials.* PhD thesis. Massachusetts Institute of Technology 1978

26. [VENERONI 06] Veneroni, M.: *Reaction-diffusion systems for the microscopic cellular model of the cardiac electric field.* Math. Meth. Appl. Sci. **29** (2006), 1631 – 1661

27. [VENERONI 09] Veneroni, M.: *Reaction-diffusion systems for the macroscopic bidomain model of the cardiac electric field.* Nonlinear Analysis: Real World Applications **10** (2009), 849 – 868

28. [VIGMOND/AGUEL/TRAYANOVA 02] Vigmond, E. J.; Aguel, F.; Trayanova, N. A.: *Computational techniques for solving the bidomain equations in three dimensions.* IEEE Trans. Biomed. Eng. **49** (2002), 1260 – 1269

29. [WAGNER 06] Wagner, M.: *Mehrdimensionale Steuerungsprobleme mit quasikonvexen Integranden.* Habilitation thesis. BTU Cottbus 2006

30. [WAGNER 09] Wagner, M.: *Pontryagin's maximum principle for multidimensional control problems in image processing.* J. Optim. Theory Appl. **140** (2009), 543 – 576

31. [WARGA 72] Warga, J.: *Optimal Control of Differential and Functional Equations.* Academic Press; New York - London 1972

32. [WEBER DOS SANTOS/PLANK/BAUER/VIGMOND 04] Weber dos Santos, R.; Plank, G.; Bauer, S.; Vigmond, E. J.: *Parallel multigrid preconditioner for the cardiac bidomain model.* IEEE Trans. Biomed. Eng. **51** (2004), 1960 – 1968

33. [WEIDEMAIER 02] Weidemaier, P.: *Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm.* Electron. Res. Announc. Amer. Math. Soc. **8** (2002), 47 – 51 (electronically published)

34. [YOSIDA 95] Yosida, K.: *Functional Analysis.* Springer; Berlin etc. 1995 (reprint of the 6th ed. from 1980)

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