REDUCED-ORDER OPTIMAL CONTROL BASED ON APPROXIMATE INERTIAL MANIFOLDS FOR NONLINEAR DYNAMICAL SYSTEMS

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Abstract. A reduced-order method for optimal control problems in infinite dimensions based on approximate inertial manifolds is developed. Convergence of the cost, optimal controls, and optimal states of the finite dimensional, reduced-order, optimal control problems to the original optimal control problem is analyzed. Special attention is given to the particular case when the dynamics are described by the Navier–Stokes equations in dimension two.

Key words. reduced-order methods, approximate inertial manifold, nonlinear Galerkin, decomposition of state space, infinite dimensional system

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1. Introduction. In this research we consider optimal control problems governed by partial differential equations. For such problems there has recently been an increased interest in developing reduced-order control methods. This is motivated by computational needs and by systems science considerations as well. For nonlinear distributed parameter control systems, let us specifically mention model reduction based on proper orthogonal decomposition approach [1, 15, 16, 24] and the reduced-basis method [11, 12, 13, 17]. The key issue for the reduction consists in selecting basis elements which are rich in information in the sense that they capture well the essential dynamical properties of the original control system. After the basis elements are selected by these methods the standard Galerkin approach is applied to obtain a reduced-order control system. For linear control systems many alternative reduction methods were proposed and analyzed, including the Hankel-norm approximation [7] and the LQG-balanced truncation realization [2]. The LQG-balanced truncation method was introduced in the finite dimensional literature by [14] and other interpretations followed in [19, 20]. For the infinite dimensional theory we refer to [3]. For linear systems, model reduction based on specific versions of proper orthogonal decomposition and of balanced truncation coincide [22]. The selection of references here is by no means complete and we refer to the citations in the quoted papers for references.

In this paper we discuss an order-reduction method based on the concept of inertial manifolds. Let us recall that an important feature of inertial manifolds consists in the description of the small-scale dynamics as a graph of the large-scale dynamics. The inertial manifold is a finite dimensional invariant manifold that attracts all orbits exponentially. Thus it is natural to expect that controlling the inertial manifold dynamics results in the ability to control the full underlying control system. However,
there are some technical difficulties associated with this approach: (1) the existence of the inertial manifold can only be proved for a limited class of systems, (2) the construction of the inertial manifold is highly involved and technical, and (3) the analysis of the behavior of the closed-loop system has not been fully addressed.

Here we therefore consider approximate inertial manifolds which are not necessarily invariant under the dynamics but which approximate all orbits starting from a bounded set with any desired accuracy. We use nonlinear Galerkin approximations as proposed, e.g., in [5, 18], and the manifold is represented by a stationary graph determined by the residual dynamics. If one would simply truncate the residual dynamics (the flat manifold), then this approach coincides with the standard Galerkin approximation. In [10] we demonstrated the effectiveness of the approximate inertial manifold approach for the linear quadratic regulator problem.

Error estimates for the uncontrolled solution based on the approximate inertial manifold method were investigated extensively. We refer, e.g., to [5, 18], where the manifold approach for the linear quadratic regulator problem.

The outline of this paper is as follows. In section 2 we describe the general inertial manifold approach. An upper bound for the performance of the reduced-order control as well as an error estimate for the reduced-order control to the infinite dimensional control in terms of the gap estimate of the approximate inertial manifold are established. In section 3 we discuss error estimates for the two-dimensional Navier–Stokes equations in $L^2(0, T; X)$, where $X$ denotes the state space. Section 4 is devoted to the necessary optimality condition for the optimal control problem, a second order sufficient optimality condition, and the gap estimate for the approximate inertial manifold.

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2. Approximate inertial manifold and reduced-order control system. In this section we describe a general strategy for the approximation of nonlinear control systems by nonlinear Galerkin schemes. Throughout this paper, $X$ and $U$ denote separable Hilbert spaces. We consider the optimal control problem

\begin{equation}
2.1 \quad \min_{u \in L^2(0, T; U)} J(x, u) = \int_0^T \ell(x(t)) + h(u(t)) \, dt,
\end{equation}

subject to

\begin{equation}
2.2 \quad \frac{d}{dt} x(t) = A_0 x(t) + F(x(t)) + B u(t), \quad x(0) = x_0 \in X,
\end{equation}

where $T > 0$, $F : X \to X^*$ is a nonlinear mapping, $A_0$ is a linear self-adjoint negative definite operator in $X$ with dense domain denoted by $\text{dom}(A_0)$, and $B$ is a bounded linear operator from $U$ to $X$. We further set $V = \text{dom}(-A_0)^{1/2}$ endowed with $|v|_V = \sqrt{-A_0 v, v}$ as norm. We assume that (2.2) is a well-posed control system, i.e., given any $x_0 \in X$ and $u \in L^2(0, T; U)$, there exists a unique weak solution $x = x(t; x_0, u) \in C(0, T; X) \cap L^2(0, T; V)$ to (2.2) which depends continuously on $(x_0, u) \in X \times L^2(0, T; U)$. For example, we can formulate the control problem (2.1)–(2.2) in a
Gelfand triple formulation or as semilinear control systems; see, e.g., [9, 21]. Further, \( \ell : X \to \mathbb{R} \) and \( h : U \to \mathbb{R} \) are \( C^1 \) mappings, \( \ell \) is supposed to be uniformly Lipschitz continuous on bounded sets of \( X \), and \( h \) is such that the associated substitution operator from \( L^2(0,T;U) \) to \( \mathbb{R} \) is well-defined.

We assume that \(-A_0\) has eigenpairs \((\lambda_i, \phi_i)\) in ascending order and that \( \{\phi_i\}_{i=1}^\infty \) forms an orthonormal basis of \( X \). Let \( P_1 \) be the orthogonal projection of \( X \) onto

\[ X_1 = \text{span} \{ \phi_i : 1 \leq i \leq N \}. \]

Further we set \( X_2 = (I - P_1)X \). Expressing \( x \) as

\[ x = p + q, \quad \text{with} \quad p(t) = P_1x(t) \quad \text{and} \quad q(t) = P_2x(t), \]

where \( P_2 = I - P_1 \), we have from (2.2)

\[
\begin{align*}
\frac{d}{dt}p(t) & = A_0p(t) + P_1F(p(t) + q(t)) + P_1Bu(t), \\
\frac{d}{dt}q(t) & = A_0q(t) + P_2F(p(t) + q(t)) + P_2Bu(t).
\end{align*}
\]

In the linear Galerkin approach, the higher order modes \( q \) are neglected. This results in the control system

\[
\frac{d}{dt}\hat{x}_1(t) = A_0\hat{x}_1(t) + P_1F(\hat{x}_1(t)) + P_1Bu(t).
\]

In the nonlinear Galerkin approach, \( \frac{d}{dt}q(t) \) is assumed to be negligible compared to \( A_0q(t) \) in (2.3). This suggests considering the control system

\[
\begin{align*}
\frac{d}{dt}\hat{x}_1(t) & = A_0\hat{x}_1(t) + P_1F(\hat{x}_1(t) + \hat{x}_2(t)) + P_1Bu(t), \\
0 & = A_0\hat{x}_2(t) + P_2F(\hat{x}_1(t) + \hat{x}_2(t)) + P_2Bu(t).
\end{align*}
\]

We set

\[ \hat{x} = \hat{x}_1 + \hat{x}_2. \]

and the first equation in (2.4) can be expressed as

\[
\frac{d}{dt}P_1\hat{x}(t) = A_0\hat{x}(t) + P_1F(\hat{x}(t)) + P_1Bu(t).
\]

The second equation in (2.4) is nonlinear and is replaced in the approach that follows by the linear equation

\[ A_0\hat{x}_2(t) + P_2F(\hat{x}_1(t)) = 0, \]

where the coupling in the nonlinearity and \( P_2Bu(t) \) are neglected. An alternative to this may be to consider

\[ A_0\hat{x}_2 + P_2(F(\hat{x}_1) + F'(\hat{x}_1)\hat{x}_2) = 0, \]
which should lead to better approximation properties. The reduced-order system that we consider in this paper is given by

\[
\frac{d}{dt} \hat{x}_1(t) = A_0\hat{x}_1(t) + P_1 F(\hat{x}_1(t) + \hat{x}_2(t)) + P_1 Bu(t),
\]

\[
0 = A_0\hat{x}_2(t) + P_2 F(\hat{x}_1(t)).
\]

Given \( \hat{x}_1 \in X_1 = P_1 X \), we denote the unique solution to the second equation in (2.5) by

\[
(2.6) \quad \hat{x}_2 = \Phi(\hat{x}_1), \quad \text{where} \quad \Phi(\phi) = -A_0^{-1} P_2 F(\phi).
\]

In this way we obtain the finite dimensional reduced-order control system in \( X_1 \):

\[
(2.7) \quad \min_{u \in L^2(0,T;U)} \int_0^T \ell(z(t) + \Phi(z(t))) + h(u(t)) \, dt
\]

subject to

\[
(2.8) \quad \frac{d}{dt} z(t) = A_0 z(t) + P_1 F(z(t) + \Phi(z(t))) + P_1 Bu(t), \quad z(0) = P_1 x_0;
\]

i.e., we set \( z = \hat{x}_1 \) with \( \hat{x}_1 \) as in (2.5). Here we assume the existence of an optimal control \( u \) to (2.7) as well as \( u^* \) to (2.1). To obtain error estimates for \( u - u^* \), we first establish an error estimate for the reduced-order equation. This involves an estimate of the gap

\[
\Delta(t) = |q(t) - \Phi(p(t))|,
\]

where \( p(t) \) and \( q(t) \) are defined in (2.3); see, e.g., [5] and section 3. Then, we can argue that under appropriate conditions

\[
(2.9) \quad |z(t) - p(t)|^2 \leq M \int_0^t |q(s) - \Phi(p(s))|^2 \, ds \quad \text{for} \quad t \in [0,T]
\]

for some \( M > 0 \). In fact, for \( S = \{ v \in V : |v| \leq K \} \), with \( K \geq 0 \), let us assume the properties

\[
(2.10) \quad \langle F(x) - F(y), \phi \rangle \leq c |x - y| |\phi|_V \quad \text{for all} \quad x, y, \phi \in S \subset V,
\]

and

\[
(2.11) \quad x(t) \in S \quad \text{and} \quad z(t) + \Phi(z(t)) \in S \quad \text{for a.e.} \quad t \in (0,T).
\]

From (2.3) and (2.8) we have

\[
(2.12) \quad \frac{d}{dt} (z(t) - p(t)) = A_0(z(t) - p(t)) + P_1 F(z(t) + \Phi(z(t))) - P_1 F(p(t) + q(t)),
\]

\[
\quad z(0) - p(0) = 0.
\]

By (2.10) we find for \( x(t) \in S \) and \( z(t) + \Phi(z(t)) \in S \) that

\[
\langle P_1 F(z(t) + \Phi(z(t))) - P_1 F(p(t) + q(t)), z(t) - p(t) \rangle
\]

\[
\leq c( |z(t) - p(t)| + |\Phi(z(t)) - \Phi(p(t))| + |q(t) - \Phi(p(t))| ) |z(t) - p(t)|_V
\]

\[
\leq c((1 + \|\Phi\|)|z(t) - p(t)| + |q(t) - \Phi(p(t))|) |z(t) - p(t)|_V,
\]
where $\|\Phi\|$ is the Lipschitz constant of $\Phi$ on $S$. Taking the inner product of (2.11) with $z(t) - p(t)$ and using (2.12) results in

$$\frac{1}{2} \frac{d}{dt} |z(t) - p(t)|^2 \leq -|z(t) - p(t)|_V^2 + c\left(1 + \|\Phi\|\right)|z(t) - p(t)|_V \left|\Phi(p(t))\right| \left|z(t) - p(t)\right|_V,$$

which implies, using $ab \leq a^2 + \frac{1}{4}b^2$, that

$$\frac{1}{2} \frac{d}{dt} |z(t) - p(t)|^2 \leq \frac{c^2}{2} \left(1 + \|\Phi\|^2\right)|z(t) - p(t)|^2 + |q(t) - \Phi(p(t))|^2.$$

Estimate (2.9) with $M = c^2 \exp\left(c^2 (1 + \|\Phi\|^2)T\right)$ follows from Gronwall’s inequality since $z(0) = p(0)$.

Once (2.9) is established, it implies the error estimate

$$|\hat{x}(t) - x(t)| = |z(t) + \Phi(z(t)) - x(t)| \leq |z(t) - p(t)| + |\Phi(z(t)) - \Phi(p(t))| + |q(t) - \Phi(p(t))| \leq (1 + \|\Phi\|)|z(t) - p(t)| + |q(t) - \Phi(p(t))|,$$

for $t \in [0, T]$.

To demonstrate the use of the a priori estimate (2.13) in the context of the optimal control problem, let us assume the existence of optimal controls $u$ and $u^*$ to (2.7)–(2.8) and (2.1)–(2.2), respectively. Moreover, let $x$ and $x^*$ in $C(0, T; X) \cap L^2(0, T; V)$ be the solutions to (2.2) with $u$ and $u^*$, respectively. Thus $x^*$ is the optimal trajectory to (2.1)–(2.2), whereas $x$ is the trajectory of the infinite dynamical system controlled by the optimal control computed on the bases of the reduced-order system. Let

$$\dot{x} = z + \Phi(z) \quad \text{and} \quad \dot{x}^* = z^* + \Phi(z^*),$$

where $z$ and $z^*$ are the solutions in $C(0, T; X) \cap L^2(0, T; V)$ to (2.8), corresponding to $u$ and $u^*$, respectively. Then, for the finite horizon control on $[0, T]$ we have

$$0 \leq J(x, u) - J(x^*, u^*)$$

$$= J(x, u) - J(\hat{x}, u) + J(\hat{x}, u) - J(\hat{x}^*, u^*) + J(\hat{x}^*, u^*) - J(x^*, u^*)$$

$$\leq J(x, u) - J(\hat{x}, u) + J(\hat{x}^*, u^*) - J(x^*, u^*),$$

where we used that $J(\hat{x}, u) - J(\hat{x}^*, u^*) \leq 0$. Thus

$$0 \leq J(x, u) - J(x^*, u^*) \leq C (|x^* - \hat{x}^*|_{L^2(0, T; X)} + |x - \hat{x}|_{L^2(0, T; X)}),$$

with $C$ the Lipschitz constant of $\ell : X \to R$ on the ball determined by the $C(0, T; X)$ norm of $x, \dot{x}, x^*$, and $\dot{x}^*$. The error estimate (2.9) now provides an upper bound for the performance of the suboptimal control $u$ based on the reduced-order control problem (2.7)–(2.8) imposed onto the original control problem (2.1)–(2.2). In fact, if

$$x(t), \dot{x}(t), x^*(t), \dot{x}^*(t) \text{ are in } S \text{ for a.e. } t \in (0, T)$$

with $\hat{x} = z + \Phi(z)$ and $\hat{x}^* = z^* + \Phi(z^*)$, then by (2.9) and (2.13)–(2.14)

$$0 \leq J(x, u) - J(x^*, u^*) \leq \hat{C} (|q - \Phi(p)|_{L^2(0, T; X)} + |q^* - \Phi(p^*)|_{L^2(0, T; X)}),$$

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where \( \dot{C} = \dot{C}(\|\Phi\|, c, C) \) and \((p, q), (p^*, q^*)\) are the orthogonal projections of \( x \) and \( x^* \) according to (2.3).

Suppose, moreover, that for some \( \sigma > 0 \) the second order sufficient optimality condition

\[
J(x(v), v) - J(x^*, u^*) \geq \sigma |v - u^*|^2_{L^2(0,T;U)}
\]

holds for all \( v \in L(0,T;U) \) in a neighborhood of \( u^* \), where \( x(v) \) denotes the solution to (2.2) with \( u \) replaced by \( v \). Then, if \( u \) is contained in this neighborhood for second order sufficient optimality, we have the estimate

\[
|u - u^*|^2_{L^2(0,T;U)} \leq \frac{c}{\sigma} (|x - \tilde{x}|_{L^2(0,T;X)} + |x^* - \tilde{x}^*|_{L^2(0,T;X)})
\]

and thus the optimal controls for the finite and the original optimal control systems are estimated in terms of the gap \( q - \Phi(p) \) and \( q^* - \Phi(p^*) \).

3. Basic estimates for two-dimensional Navier–Stokes systems. In this section we analyze the case where the dynamical system (2.2) is given by the Navier–Stokes equations in dimension two. We establish estimates for \( q \) and the gap \( \Delta = q - \Phi(p) \) following [5]. Our estimate here, however, differs from those in [5] in that we consider the transient situation on the time horizon \([0, T]\) as opposed to the case where \( t \) is sufficiently large as in [5]. We then turn to analyzing properties of the mapping \( \Phi \) and derive error estimates for the reduced-order system.

Let us consider the two-dimensional incompressible Navier–Stokes equations:

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v + \nabla \rho &= \nu \Delta v + Bu & \text{for} \ (t, x) \in (0, T) \times \Omega, \\
\nabla \cdot v &= 0 & \text{in} \ (0, T) \times \Omega, \\
v(x, 0) &= v_0(x) \quad \text{in} \ \Omega, \quad v(t, x) = 0 & \text{for} \ (t, x) \in (0, T) \times \Gamma,
\end{align*}
\]

where \( \Omega \) is a bounded open domain in \( \mathbb{R}^2 \) with sufficiently smooth boundary \( \Gamma \), \( v = v(t, x) \in \mathbb{R}^2 \) is the velocity field, \( \rho = \rho(t, x) \in \mathbb{R} \) is the pressure, and \( \nu > 0 \) is the normalized viscosity. For convenience, in order to exclude special cases in the estimate we also assume that \( \nu \leq 1 \). Let

\[
V = \{ \phi \in H^1_0(\Omega)^2 : \nabla \cdot \phi = 0 \},
\]

and let

\[
X = \{ \phi \in L^2(\Omega) : \nabla \cdot \phi = 0, \ n \cdot \phi = 0 \text{ on } \partial \Omega \}
\]

denote the closure of \{ \phi \in \mathcal{D}(\Omega) : \nabla \cdot \phi = 0 \} in \( H^1_0(\Omega)^2 \) and \( L^2(\Omega)^2 \), respectively. Let \( P \) stand for the orthogonal projection of \( L^2(\Omega)^2 \) onto the closed subspace \( X \), and let \( A = P\Delta \) in \( X \) denote the Stokes operator defined by

\[
(A\phi, \psi) = -\langle \nabla \phi, \nabla \psi \rangle_{L^2} \quad \text{for} \ \phi, \ \psi \in V.
\]

It is a closed self-adjoint operator with \( \text{dom}(A) = H^2(\Omega)^2 \cap V \) and \( \text{dom}((-A)^{\frac{1}{2}}) = V \). The operator \( (A)^{-1} \) is compact from \( X \) to \( \text{dom}(A) \) and hence the eigenvalue-eigenvector pairs of \(-A\) define a complete orthonormal basis of \( X \).
Let $B \in \mathcal{L}(U, X)$, and define $A_0 = \nu A$, and $F : X \to X^*$

$$-F(\phi) = b(\phi, \phi) = P(\phi \cdot \nabla \phi).$$

Then (3.1) can equivalently be expressed in abstract form as (2.2). For results on the Navier–Stokes equations that we utilize here, we refer to [23]. Note that we chose to endow $V$ with $\sqrt{(-A_0\nu, v)}$ as norm in section 2, whereas $\sqrt{(-Av, v)}$ is the norm for $V$ in this and the following sections in order to specify the dependency of our estimates on $\nu > 0$ as in [5]. We use the following properties [5, 23] of the bilinear form $b(\phi, \psi) = P(\phi \cdot \nabla \psi)$: There exist constants $c_i$ such that for all $\phi, \psi, w$ in $\text{dom } A$

$$|b(\phi, \psi)|_{L^2} \leq c_1 \left\{ \begin{array}{l} |\phi|^2|\phi|_V^2, |\psi|^2|A\psi|^2, \\
|\phi|^2|A\phi|^2|\psi|_V, \end{array} \right.$$  

and satisfies

$$|b(\phi, \psi), w| \leq c_2 |\phi|^2|\phi|_V^2|\psi|_V^2|w|^2, \quad |b(\phi, \psi)|_{L^2} \leq c_4 \left\{ \begin{array}{l} |\phi|_V|\psi|_V \left( 1 + \log \left( \frac{|A\phi|^2}{\lambda_1|\phi|_V} \right) \right)^{\frac{1}{2}}, \\
|\phi|_V|A\psi| \left( 1 + \log \left( \frac{|A\phi|^2}{\lambda_1|\phi|_V} \right) \right)^{\frac{1}{2}}, \end{array} \right.$$  

and satisfies

$$(3.2) \quad (b(\phi, \psi), \psi) = 0 \quad \text{for } \phi, \psi \in V.$$  

It follows from [23, proof of Theorem III.3.10], that for $x_0 \in V$ and $u \in L^2(0, T; U)$ there exists a unique solution to (2.2),

$$x(t) \in C(0, T; V) \cap L^2(0, T; \text{dom } (A)),$$

and

$$|x|_{C(0, T; V) \cap L^2(0, T; \text{dom } (A))} \leq C(|x_0|_V, |u|_{L^2(0, T; U)}),$$  

for a constant $C$ depending continuously on its arguments. We let

$$M_0 = \sup_{t \in [0, T]} |x(t)|_V \text{ and } M_1 = \sup_{t \in [0, T]} |x(t)|_V.$$  

Throughout the remainder of this paper we assume that the cut off for the subspace $X_1$ is such that

$$(3.4) \quad \lambda_{N+1} \geq \left( \frac{4c_2 M_1}{\nu} \right)^2.$$  

Note that

$$(3.5) \quad |A\phi|^2 \leq \lambda_N |\phi|^2_V \quad \text{for } \phi \in X_1, \text{ and } |\phi|^2_V \geq \lambda_{N+1} |\phi|^2 \quad \text{for } \phi \in X_2.$$  

Taking the inner product of (2.2) with $q(t) \in X_2$ we obtain

$$\frac{1}{2} \frac{d}{dt} |q|^2 + \nu |q|^2_V = -(b(p + q, p + q), q) + (P_2 Bu, q) \quad \text{for a.e. } t \in (0, T).$$
From (3.3)
\[ (b(p+q,p+q), q) = (b(p,p), q) + (b(q,p), q). \]
Together with (3.2)–(3.5) it follows that
\[ |(b(p+q,p+q), q)| \leq |(b(p,p), q)| + |(b(q,p), q)| \leq c_4 L^{\frac{1}{2}} |p|_V^2 |q| + c_2 |q| |q|_V |p|_V, \]
at a.e. \( t \in (0,T) \), where
\[ L = 1 + \log \frac{\lambda_{N+1}}{\lambda_1}. \]
Referring to (3.5) once again we obtain
\[
\frac{d}{dt} |q|^2 + (2\nu - 2c_2 \lambda_{N+1}^{-1} M_1) |q|_V^2 \leq 2 |P_2 Bu| |q| + 2c_4 M_1 L^{\frac{1}{2}} |p|_V |q|
\leq \frac{4}{\nu \lambda_{N+1}} (c_4^2 LM_1^2 |p|_V^2 + |P_2 Bu|^2) + \frac{\nu}{2} |q|_V^2.
\]
By (3.4) the above inequality implies that
\[
\frac{d}{dt} |q|^2 + \frac{3\nu}{2} |q|_V^2 \leq \nu |q|_V^2 + \frac{4}{\nu \lambda_{N+1}} (|P_2 Bu|^2 + c_4^2 M_1^2 L |x|_V^2)
\]
and
\[
\frac{d}{dt} |q|^2 + \nu \lambda_{N+1} |q|^2 \leq \frac{4}{\nu \lambda_{N+1}} (|P_2 Bu|^2 + c_4^2 M_1^2 L |x|_V^2).
\]
Multiplying this inequality with \( e^{\nu \lambda_{N+1} t} \) and integrating over \((0,t)\) implies that
\[ |q(t)|^2 \leq e^{-\nu \lambda_{N+1} t} |q(0)|^2 + \frac{4}{\nu \lambda_{N+1}} \int_0^t e^{\nu \lambda_{N+1} (s-t)} (|P_2 Bu(s)|^2 + c_4^2 M_1^4 L) \, ds, \]
for every \( t \in [0,T] \). Consequently,
\[
\int_0^T |q|^2 \, dt \leq \frac{1}{\nu \lambda_{N+1}} |q(0)|^2 + \frac{4}{\nu^2 \lambda_{N+1}^2} \int_0^T (|P_2 Bu|^2 + c_4^2 M_1^4 L) \, dt
\]
and this implies the existence of a constant \( k_1 = k_1(M_1,c_4,T) \) such that
\[ \int_0^T |q|^2 \, dt \leq \frac{1}{\nu^2 \lambda_{N+1}^2} (|q(0)|^2 + 4|P_2 Bu|_{L^2(0,T;X)}^2 + k_1). \]
Taking the inner product of (2.2) with \( Aq \) we find that
\[ \frac{1}{2} \frac{d}{dt} |q|_V^2 + \nu |Aq|^2 = -(b(p+q,p+q), Aq) + (P_2 Bu, Aq). \]
From (3.2)–(3.5)
\[
|(b(p+q,p+q), Aq)| + (P_2 Bu, Aq)
\leq c_4 L^{\frac{1}{2}} |p|_V |Aq| + (|p|_V + |q|_V) + c_1 |q|^{\frac{1}{2}} |Aq|^{\frac{1}{2}} (|p|_V + |q|_V) + (P_2 Bu, Aq).
\]
Expressing $|Aq|^2 (|p_V| + |q_V|)$ as $\nu^4 |Aq|^2 \frac{1}{\nu^4} (|p_V| + |q_V|)$ and analogously for the remaining terms which contain $A(q)$, we have

$$|(b(p + q, p + q), Aq)| + (P_2 Bu, Aq) \leq \frac{\nu}{2} |Aq|^2 + \frac{1}{\nu} |P_2 Bu| + \frac{\tilde{c}_1 M_1^2 L |x|_V^2}{\nu} + \frac{\tilde{c}_1 M_2^2 |x|_V^4}{\nu^3},$$

where $\tilde{c}_1, \tilde{c}_3$ are independent of $\nu, L, x, u$. Hence

$$\frac{d}{dt} |q|_V^2 + \nu |A q|^2 \leq \frac{2}{\nu} |P_2 Bu|^2 + \frac{2\tilde{c}_1 M_1^2 L |x|_V^2}{\nu} + \frac{2\tilde{c}_1 M_2^2 |x|_V^4}{\nu^3},$$

and

$$|q(t)|_V^2 \leq e^{-\nu \lambda_{N+1} t} |q(0)|_V^2 + \int_0^t e^{\nu \lambda_{N+1} (s-t)} \left( \frac{2}{\nu} |P_2 Bu(s)| + \frac{2\tilde{c}_1 M_1^2 L |x|_V^2}{\nu} + \frac{2\tilde{c}_1 M_2^2 |x|_V^4}{\nu^3} \right).$$

Thus there exists a constant $k_2 = k_2(M_1)$ such that

$$\int_0^T |q(t)|_V^2 \, dt \leq \frac{1}{\nu \lambda_{N+1}} \left( |q(0)|_V^2 + \frac{2}{\nu} |P_2 Bu|^2_{L^2(0,T;X)} + \frac{k_2 L}{\nu} + \frac{k_2}{\nu^3} \right).$$

For this estimate the assumption $\lambda_{N+1} \geq (\frac{4\tilde{c}_2 M_1^2}{\nu})^2$ is not needed. We now summarize these results.

**Theorem 3.1.** Assume that $x_0 \in V, u \in L^2(0,T;U)$, and that $|x|_{C([0,T];V)} \leq M_1$ and $\lambda_{N+1} \geq (\frac{4\tilde{c}_2 M_1}{\nu})^2$. Then there exist constants $k_1 = k_1(M_1)$ and $k_2 = k_2(M_1)$ such that

$$\int_0^T |q(t)|_V^2 \, dt \leq \delta^2 (|q(0)|_V^2 + 4|P_2 Bu|^2_{L^2(0,T;X)} + k_1 L)$$

and

$$\int_0^T |q(t)|_V^2 \, dt \leq \delta \left( |q(0)|_V^2 + \frac{2}{\nu} |P_2 Bu|^2_{L^2(0,T;X)} + \frac{k_2 L}{\nu} + \frac{k_2}{\nu^3} \right),$$

where $\delta = \frac{1}{\nu \lambda_{N+1}}$ and $L = 1 + \log \frac{\lambda_{N+1}}{M_1^2}$.

If, moreover, $u \in L^\infty(0,T;U)$ and $x_0 \in \text{dom}(A_0)$, then for constants $k_3 = k_3(M_1)$ and $k_4 = k_4(M_1)$

$$|q(t)|^2 \leq \delta^2 (|A_0 q(0)|^2 + 4|P_2 Bu|^2_{L^\infty(0,T;X)} + k_3 L),$$

and

$$|q(t)|_V^2 \leq \delta \left( |A_0 q(0)|^2 + \frac{2}{\nu} |P_2 Bu|^2_{L^\infty(0,T;X)} + \frac{k_4 L}{\nu} + \frac{k_4}{\nu^3} \right),$$

for all $t \in [0,T]$.

The second part of the theorem follows from (3.7) and the equation above (3.9). For the optimal solution $u$ to (2.1)–(2.2), the assumption that $u \in L^\infty(0,T;U)$ is not restrictive as can be seen from the associated optimality system which will be given below.
3.1. Gap estimate. In this subsection we prove an estimate for the gap
\[ \Delta(t) = q(t) - \Phi(p(t)) \text{ for } t \in [0, T]. \]
Throughout we assume that \( x(t) \in C([0, T); V) \cap L^2(0, T; \text{dom}(A_0)) \), and we again set
\[ M_1 = \sup_{t \in [0, T]} |x(t)|_V. \]
By (2.3) and (2.6) we have
\[ \frac{1}{2} \int_0^T \|q\|^2 V + \int_0^T \left( \frac{d}{dt} \|q\|^2 V - P_2 Bu, \frac{d}{dt} q \right) \leq 0 \text{ for a.e. } t \in (0, T). \]
Using (3.2), (3.3), and (3.6) we find that
\[ \|b(p, q) + b(q, p) + b(q, q)\| \leq (c_4 L^\frac{1}{2} + c_1) \|p\|_V \|q\|_V + \frac{c_1}{\sqrt{\lambda_{N+1}}} \|q\|_V \|Aq\| \text{ for a.e. } t \in (0, T). \]
Taking the inner product of (2.2) with \( \frac{d}{dt} q \),
\[ \frac{\nu}{2} \frac{d}{dt} \|q\|^2 V + \int_0^T \left( b(p, q) + b(q, p) + b(q, q) \right) dt + \left( P_2 Bu, \frac{d}{dt} q \right) \]
and therefore
\[ \frac{\nu}{2} \frac{d}{dt} \|q\|^2 V + \int_0^T \frac{d}{dt} \|q\|^2 V \]
\[ \leq \|b(p, q) + b(q, p) + b(q, q)\|^2 + \|P_2 b(p, p + Bu)\|^2 + \frac{1}{2} \|\frac{d}{dt} q\|^2. \]
This implies that
\[ \int_0^T \|q(t)\|_V^2 dt + \int_0^T \left( b(p, q) + b(q, p) + b(q, q) \right) dt + \int_0^T \|P_2(b(p, p + Bu)\|^2 dt \]
\[ \leq \nu |q(0)|_V^2 + 4 \int_0^T \left( (c_4 L^\frac{1}{2} + c_1) \|p\|_V \|q\|_V + \frac{c_1}{\sqrt{\lambda_{N+1}}} \|q\|_V \|Aq\| \right) dt \]
\[ + 2 \int_0^T \|P_2(b(p, p + Bu)\|^2 dt. \]
From (3.10)–(3.12) there exists a constant \( \tilde{k} \) such that
\[ \int_0^T \|A_0 \Delta\|^2 dt \leq \tilde{k} \left[ \int_0^T \left( (c_4 L^\frac{1}{2} + c_1) \|p\|^2_2 + \frac{c_1}{\sqrt{\lambda_{N+1}}} \|q\|^2_2 \right) dt \right. \]
\[ + \int_0^T \|P_2(b(p, p)\|^2 dt + \int_0^T \|P_2 Bu\|^2 dt \left. \right]. \]
Hence using (3.5) and \( L \geq 1 \) there exists a constant \( k_5 \) independent of \( \nu \) such that
\[ \int_0^T \|A_0 \Delta\|^2 dt \leq k_5 \frac{L}{\lambda_{N+1}} \|Aq\|^2_{L^2(0, T; X)} \]
\[ + \int_0^T \|P_2(b(p, p)\|^2 dt + \int_0^T \|P_2 Bu\|^2 dt. \]
We consider the term \( P_2 b(p, p) \). Recall that \( b(p, p) = P(p \cdot \nabla)p \). Since \( p \in X_1 \) and hence, in particular, \( p \in \text{dom}(A) \), then we have
\[ g := (p \cdot \nabla)p \in H^1_0(\Omega). \]
Then $P_g$ satisfies
\[ P_g = g - \nabla \tilde{p} \in X, \]
where $\tilde{p} \in H^1(\Omega)/\mathbb{R}$ is given by
\[
\begin{cases}
\Delta \tilde{p} = \nabla \cdot g, \\
\frac{\partial \tilde{p}}{\partial n} = 0.
\end{cases}
\]
Note that $\tilde{p} \in H^2(\Omega)$ and $|\nabla \tilde{p}|_{H^1} \leq c |g|_{H^1}$. Hence
\[
|P_g|_{H^1} = |g - \nabla \tilde{p}|_{H^1} \leq c |g|_{H^1}
\]
and $P_g \in H^1(\Omega) \cap X$, with $P_g \cdot n = 0$. (But $P_g$ is not in $V$ since $\tau \cdot P_g \neq 0$, in general.)

Recall from \cite{6} that $\text{dom}((-A)\alpha) = \text{dom}((-\Delta)\alpha) \cap X$ for $\alpha \in (0,1)$, where $\Delta$ denotes the vector Laplacian. We further refer to \cite[Chapter 8]{4}, for characterizations and properties of the domains of $(-\Delta)\alpha$. It follows from \cite{4} that for $\epsilon > 0$ there exists $C_\epsilon$ such that
\[
|P_g|_{\text{dom}((-A)^{\frac{1}{2}-\epsilon})} \leq C_\epsilon |P_g|_{H^1(\Omega)}
\]
and hence we have
\[
|P_2 b(p,p)|_X = |P_2 P(p \cdot \nabla) p|_X \leq \frac{1}{\lambda_{N+1}^{2-\epsilon}} |(-A)^{\frac{1}{2}-\epsilon} P((p \cdot \nabla) p)| \leq \frac{C_\epsilon}{\lambda_{N+1}^{2-\epsilon}} |(p \cdot \nabla) p|_{H^1}.
\]
Here and below $C_\epsilon$ is a constant independent of $N$ and $p$. By (3.2) we have
\[
|(p \cdot \nabla) p|_{H^1} \leq |(p \cdot \nabla) p|^2 + |\nabla (p \cdot \nabla) p|^2 \leq |(p \cdot \nabla) p|^2 + 2(|p \cdot \nabla) p|^2 + |p \cdot \nabla) p|^2 + |p \cdot \nabla) p|^2 \leq C |p|^2 |Ap|^2 L.
\]
Combining these estimates, we find
\[
|P_2 b(p,p)|_X \leq C_\epsilon \frac{L^{1/2}}{\lambda_{N+1}^{2-\epsilon}} |p|_V |Ap| \leq C_\epsilon \frac{L^{1/2}}{\lambda_{N+1}^{2-\epsilon}} M_1 |Ap|.
\]
This estimate can now be used for the last term in (3.13).

For Neumann boundary conditions $V = \{ \phi \in H^1(\Omega)^2 : \nabla \cdot \phi = 0 \}$. Since $p \in H^2(\Omega) \subset L^\infty(\Omega)$ $g = (p \cdot \nabla) p \in H^1(\Omega)$ and $P_g = g - \nabla \tilde{p}$, where $\tilde{p} \in H^1(\Omega)/\mathbb{R}$ with
\[
\begin{cases}
\Delta \tilde{p} = \nabla \cdot g, \\
\frac{\partial \tilde{p}}{\partial n} = n \cdot g.
\end{cases}
\]
Hence $\tilde{p} \in H^2(\Omega)$, $P_g \in H^1(\Omega)$, and $P_g \in V$. In this case, as well as for periodic boundary conditions, $\lambda_{N+1}^{2-\epsilon}$ and the above estimates can be replaced by $\lambda_{N+1}^{2}$.

We summarize these estimates in the following theorem.

\textbf{Theorem 3.2.} \textit{If $x_0 \in \text{dom} (A_0)$, $P_2 B u \in L^2(0,T;V)$, and $\epsilon > 0$, then there exists a constant $k_\tau$ such that}
\[
\int_0^T |A_0(q - \Phi(p))|^2 dt \leq k_\tau \frac{L}{(\lambda_{N+1})^\alpha},
\]
\textit{where $\alpha = 1/2 - \epsilon$ in the case of Dirichlet boundary conditions and $\alpha = 1$ for Neumann or periodic boundary conditions. Here $k_\tau$ depends on $\epsilon$ and also on $k_5, |Aq|_{L^2(0,T;X)}, |Aq(0)|_{X^*}, |P_2 B u|_{L^2(0,T;V)}, \nu$.}
3.2. Lipschitz continuity of $\Phi$. We derive Lipschitz estimates for
\[
e = \Phi(p_1) - \Phi(p_2),
\]
in $X$ and $V$, where $p_1 \in X_1$ and $p_2 \in X_1$ satisfy $|p_1|_V, |p_2|_V \leq M_1$. From (2.6) and (3.2), (3.5) we have
\[
\nu(Ae, e) = -(b(p_1, p_1 - p_2) + b(p_1 - p_2, p_2), e) \leq c_4(|p_1|_V + |p_2|_V)|p_1 - p_2|L^{\frac{1}{2}}|e|_V
\]
and hence
\[
|\Phi(p_1) - \Phi(p_2)|_X \leq ks |p_1 - p_2|_X, \quad \text{where} \quad ks = \frac{2c_4 L^{\frac{1}{2}} M_1}{\nu \sqrt{\lambda_{N+1}}}. \tag{3.14}
\]
Note that $ks = k_s(N) \to 0$ for $N \to \infty$. We also find
\[
\nu|e|^2_V \leq -(b(p_1, p_1 - p_2) + b(p_1 - p_2, p_2), e) \\
\leq c_4(|p_1|_V + |p_2|_V)|p_1 - p_2|_V L^{\frac{1}{2}}|e| \leq \frac{2}{\sqrt{\lambda_{N+1}}} c_4 M_1 L^{\frac{1}{2}} |p_1 - p_2|_V |e|_V,
\]
and hence
\[
|\Phi(p_1) - \Phi(p_2)|_V \leq ks |p_1 - p_2|_V. \tag{3.15}
\]

**Lemma 3.1.** Let $|p_1|_V \leq M_1, |p_2|_V \leq M_1$ and set $ks = \frac{2c_4 L^{\frac{1}{2}} M_1}{\nu \sqrt{\lambda_{N+1}}}$. Then
\[
|\Phi(p_1) - \Phi(p_2)|_V \leq ks |p_1 - p_2|_V, \quad \text{and} \quad |\Phi(p_1) - \Phi(p_2)|_X \leq ks |p_1 - p_2|_X.
\]

3.3. Error analysis for reduced-order system. We derive error estimates for $z - p$, where $z$ and $p$ are the components of the reduced systems given by (2.8) and the first equation in (2.3). The following a priori estimates will be needed.

**Lemma 3.2.** Let $u \in L^2(0, T; U)$ and $x_0 \in V$, and let $x$ denote the solution to (2.2). Then for all $N$ sufficiently large, the solution $\hat{x}$ to (2.5) exists. Moreover, there exists a constant $M_1$, such that $|x(t)|_V \leq M_1$ and $|\hat{x}(t)|_V \leq M_1$ for all $t \in [0, T]$, and this constant is uniform with respect to $(u, x_0)$ in bounded sets of $L^2(0, T; U) \times V$ and all $N$ sufficiently large.

An outline for the proof is given in the appendix.

From (2.8) we have
\[
\frac{d}{dt}(z(t) - p(t)) - \nu A(z(t) - p(t)) \\
+ P_1 b(\hat{x}(t), \hat{x}(t) - x(t)) + P_1 b(\hat{x}(t) - x(t), x(t)) = 0, \quad z(0) - p(0) = 0,
\]
where
\[
x(t) = p(t) + q(t), \quad \hat{x} = z(t) + \Phi(z(t)).
\]

Taking the inner product with $z(t) - p(t),$
\[
\frac{1}{2} \frac{d}{dt}|z(t) - p(t)|^2 + \nu|z(t) - p(t)|_V^2 \\
\leq |(b(\hat{x}, \hat{x} - x), z - p)| + |(b(\hat{x} - x, x), z - p)| \\
\leq |(b(\hat{x}, \Phi(z) - q), z - p)| + |(b(z - p, x), z - p)| + |(b(\Phi(z) - q, x), z - p)|,
\]

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where we suppressed the dependence on $t$ on the right-hand side and used

$$
(b(\hat{x}(t), \hat{x}(t) - x(t)), z(t) - p(t)) = (b(\hat{x}(t), \Phi(z(t)) - q(t)), z(t) - p(t)).
$$

From (3.2) and the fact that $(b(\hat{x}, \dot{x} - x), z - p) = -(b(\hat{x}, \dot{x} - x), z - p)$, we find that

$$
\frac{1}{2} \frac{d}{dt}|z(t) - p(t)|^2 + \nu|z(t) - p(t)|^2_V
\leq |\dot{z}|_L^\infty|z(t) - p(t)|_V (|z(t) - p(t)|_X + |q(t) - \Phi(p(t))|_X)
+ M_1 (c_2 |z(t) - p(t)|_X + c_3 L^{\frac{1}{2}} (k_8 |z(t) - p(t)|_X + |q(t) - \Phi(p(t))|_X)) (z(t) - p(t))_V.
$$

Concerning the term $|\dot{z}(t)|_L^\infty$, note that for $z \in V$

$$
|A_0 \Phi(z)| \leq c_4 L^{\frac{1}{2}} |z|^2_V,
$$

and hence

$$
\nu |A \Phi(z)| \leq c_4 L^{\frac{1}{2}} |z|^2_V.
$$

Moreover, for every $\epsilon > 0$ there exists $c > 0$ such that

$$
|\phi|_{L^\infty} \leq c |A^{\frac{1}{2} + \epsilon} \phi| \text{ for all } \phi \in \text{dom}(A).
$$

Consequently,

$$
|\Phi(z(t))|_{L^\infty} \leq c \frac{\lambda_N^{\frac{1}{2} + \epsilon} L^{\frac{1}{2}}}{\nu} |z(t)|^2_V \quad \text{and} \quad |z(t)|_{L^\infty} \leq c_3 L^{\frac{1}{2}} |z(t)|_V,
$$

where (3.2) was used for the second estimate. Since $L = L(N) \sim \log \lambda_{N+1}$, we can assume without loss of generality that

$$
|\dot{z}(t)|_{L^\infty} = |z(t) + \Phi(z(t))|_{L^\infty} \leq c L^{\frac{1}{2}} M_1^2 \quad \text{for } t \in [0, T],
$$

where $c = \max(c_3, \frac{c_4}{\nu})$. Without loss of generality we also assume that $M_1 \geq 1$ and, as before, that $\nu \leq 1$. Combining these estimates we find that

$$
\frac{1}{2} \frac{d}{dt}|z(t) - p(t)|^2 + \nu|z(t) - p(t)|^2_V
\leq M_1^2 (c_2 + 2k_8 c L^{\frac{1}{2}}) |z(t) - p(t)|_X |z(t) - p(t)|_V + 2M_1^2 c L^{\frac{1}{2}} |q(t) - \Phi(p(t))|_V
- \Phi(p(t)) |z(t) - p(t)|_V
$$

and thus

$$
\frac{d}{dt}|z(t) - p(t)|^2 + \nu|z(t) - p(t)|^2_V
\leq 2 \left( \frac{c_2 M_1^2 + 2k_8 M_1^2 c L^{\frac{1}{2}}}{\nu} \right)^2 |z(t) - p(t)|^2 + \frac{8c^2 L M_1^4}{\nu} |q(t) - \Phi(p(t))|^2.
$$

Gronwall’s lemma now implies the following theorem.

**Theorem 3.3.** Let $z$ and $p$ denote the solutions to (2.8) and the first equation in (2.2), with $x_0 \in V$. Then for all $N$ sufficiently large

$$
|z(t) - p(t)|^2 + \nu \int_0^t |z(s) - p(s)|^2_V ds \leq \frac{8c^2 L M_1^4}{\nu} \int_0^t e^{(t-s)} |q(s) - \Phi(p(s))|^2 ds,
$$

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where \( \bar{c} = \max(c_3, \frac{c_4}{\nu}) \) and \( c = \frac{(c_2 M^2 + 2 k_M M^2 \bar{c} L^2)}{\nu} \), and \( t \in [0, T] \).

Considering the definitions of \( k_N \) and \( \bar{c} \), we deduce that \( c \) behaves like \( \frac{L^2}{\lambda N + 1} \), where \( L(N) = 1 + \log \frac{\lambda N + 1}{\lambda_1} \), for \( N \to \infty \) and, in particular, is bounded with respect to \( N \) for \( N \to \infty \).

At the end of section 2 we addressed the question of the performance of the finite dimensional controller applied to the infinite dimensional optimal control problem, measured in terms of the difference of the control-costs. For the two-dimensional Navier–Stokes equations we obtain, using the results of this section, the following performance indicator in terms of the cost.

**Corollary 3.1.** Let \( x_0 \in V, J \) as in section 2, and let \( u^* \) denote an optimal solution to (2.1)–(2.2) and \( u \) an optimal solution to (2.7)–(2.8), with dynamics given by the Navier–Stokes equations. Then there exists a constant \( k_9 = k_9(C, \nu, c, M_1, c, k_7, k_8) \) independent of \( N \), such that

\[
0 \leq J(x, u) - J(x^*, u^*) \leq k_9 \frac{L}{(\lambda N + 1)^{1+\frac{2}{p}}},
\]

where \( x \) denotes the solution to the infinite dimensional system (2.2) controlled by \( u \).

**Proof.** Let \( x^* = x(u^*) \) and \( x = x(u) \) denote the solutions to (2.2) with controls \( u \) and \( u^* \), respectively. Then by Lemma 3.2 there exists \( M_1 \) such that \( x^*, x, \hat{x}(u^*) \), and \( \hat{x}(u) \) are all bounded in \( C([0, T]; V) \) by \( M_1 \). By (2.14)

\[
0 \leq J(x, u) - J(x^*, u^*) \leq C(|x^* - \hat{x}^*|_{L^2(0, T; X)} + |x - \hat{x}|_{L^2(0, T; X)}).
\]

From (2.13) and Lemma 3.1 we have with \( \hat{C} = C(1 + k_8) \) that

\[
J(x, u) - J(x^*, u^*) \leq \hat{C}([p(u_N) - z(u_N)]_{L^2(0, T; X)} + |q(u_N) - \Phi(z(u_N))|_{L^2(0, T; X)} + |p(u^*) - z(u^*)|_{L^2(0, T; X)} + |q(u^*) - \Phi(z(u^*))|_{L^2(0, T; X)}).
\]

In the following estimates \( k \) denotes a generic constant depending continuously on the indicated arguments but independent of \( N \). From Theorem 3.3, (3.5), and Theorem 3.2, we have

\[
J(x, u) - J(x^*, u^*) \leq k(\hat{C}, \nu, c, M_1, c, k_7, k_8) \frac{L}{(\lambda N + 1)^{1+\frac{2}{p}}}.
\]

In section 4 these estimates will be carried further to obtain estimates on \( u_N - u^* \), where \( u_N \) is the solution to (2.7)–(2.8), as \( N \to \infty \).

**4. Optimality condition.** In this section we discuss a direct approach to obtain necessary and sufficient optimality conditions for (2.1)–(2.2), with the dynamical system given by (3.1). As above, let \( V = dom((-A)^{\frac{1}{2}}) \) and \(-A\phi, \phi\) = \( ||\phi||_{V}^{2} \) for \( \phi \in V \). Let \( (x^*, u^*) \) be an optimal solution to (2.1)–(2.2) with

\[
J(x, u) = \frac{1}{2} \int_{0}^{T} (Q(x(t) - \bar{x}(t)), x(t) - \bar{x}(t)) dt + \frac{\beta}{2} \int_{0}^{T} |u(t)|_{V}^{2} dt.
\]
where $Q$ is symmetric and positive definite on $X$, $\bar{x} \in L^2(0, T; X)$, and $\beta > 0$. We shall use the following properties:

- Given $(x_0, u) \in X \times L^2(0, T; U)$, there exists a unique weak solution $x$ to \eqref{eq:2.2} in $W = L^2(0, T; V) \times H^1(0, T; V^*)$ satisfying $x(0) = x_0$ and
  \[
  \left\langle \frac{d}{dt} x(t), \psi \right\rangle = \langle A_0 x(t) + F(x(t)) + B u(t), \psi \rangle \text{ for all } \psi \in V.
  \]

- $F : V \to V^*$ is continuously differentiable and there exists a $c > 0$ such that
  \[
  \langle F(y) - F(x), y - x \rangle \leq c |y - x| |y - x|_V \quad \forall x, y \in V,
  \]
  \[
  |F'(x)y, y| \leq c |y|_V |y|_V \quad \forall x, y \in V,
  \]
  \[
  |F(y) - F(x) - F'(x)(y - x)|_V \leq c |y - x| |y - x|_V
  \]
  for all $x, y \in V$.

The second item follows from \eqref{eq:3.2}.

### 4.1. Necessary optimality condition

Since the bilinear form $a$ on $V \times V$ defined by
\[
t \rightarrow a(t, \phi, \psi) = -\langle A_0 \phi, \psi \rangle - \langle \phi, F'(x^*(t)) \psi \rangle
\]
satisfies
\[
a(t, \phi, \phi) \geq \nu |\phi|^2_V - c|x^*(t)|_V |\phi|_V |\phi|_V \geq \nu \frac{|\phi|^2_V}{2} - \frac{c^2 |x^*(t)|^2_V}{2\nu} |\phi|^2,
\]
with $t \rightarrow |x^*(t)|^2_V \in L^1(0, T)$, there exists a unique solution $\chi^* \in H^1(0, T; V^*) \cap L^2(0, T; V)$ to the adjoint equation
\[
- \frac{d}{dt} \chi^*(t) = A_0 \chi^*(t) + F'(x^*(t)) \chi^*(t) + Q(x^*(t) - \bar{x}(t)), \quad \chi^*(T) = 0.
\]
Recall that for $x \in W$, we have $x \in C(0, T; X)$, and for $x$, $\chi \in W$ it follows that
\[
\left\langle \frac{d}{dt} x(t), \chi(t) \right\rangle_X = \left\langle \frac{d}{dt} x(t), \chi(t) \right\rangle + \left\langle x(t), \frac{d}{dt} \chi(t) \right\rangle
\]
for a.e. $t \in (0, T)$. Denoting by $x \in W$ the weak solution corresponding to $u \in L^2(0, T; X)$, and using $a b \leq \frac{1}{4} u'^2 + b^2$ we have
\[
\frac{1}{2} \frac{d}{dt} |x(t) - x^*(t)|^2 + \nu |x - x^*|_V^2 = \langle F(x) - F(x^*), x - x^* \rangle + \langle B(u - u^*), x - x^* \rangle
\]
\[
\leq c |x^*|_V |x - x^*|_V |x - x^*|_V + \|B\|_{L(U, V^*)} |x - x^*|_V |u - u^*| \leq \nu \frac{|x - x^*|^2}{V} + \frac{c^2}{\nu} |x^*|^2_V |x - x^*|^2 + \frac{1}{\nu} \|B\|^2_{L(U, V^*)} |u - u^*|^2.
\]

By Gronwall’s inequality
\[
|x(t) - x^*(t)|^2 \leq \frac{2}{\nu^2} \|B\|^2_{L(U, V^*)} \exp \left( \frac{2c^2}{\nu} \int_0^t |x^*(s)|^2_V ds \right) \int_0^t |u(s) - u^*(s)|^2 ds
\]
and
\begin{equation}
\int_0^T |x(t) - x^*(t)|^2 dt \leq M \int_0^T |u(t) - u^*(t)|^2 dt
\end{equation}
with
\begin{equation*}
M = \frac{2}{\nu^2} \|B\|^2_{L^2(U,V')} \left( 1 + \frac{2}{\nu} \epsilon^2 \left( \int_0^T |x^*(t)|^2 dt \right) e^{\frac{\nu}{2} \epsilon^2 \int_0^T |x^*(t)|^2 dt} \right).
\end{equation*}

Note that
\begin{equation}
J(u) - J(u^*) = \beta(u - u^*, u^*)_{L^2(0,T;U)} + (Q(x^* - \bar{x}), x - x^*)_{L^2(0,T;X)}
\end{equation}
and
\begin{equation*}
+ \frac{1}{2} (\beta|u - u^*|^2_{L^2(0,T;U)} + (Q(x - x^*), (x - x^*))_{L^2(0,T;X)}),
\end{equation*}

where
\begin{equation*}
H(x - x^*, x - x^*) = F(x) - F(x^*) - F'(x^*)(x - x^*).
\end{equation*}

Then, from (4.5)
\begin{equation}
J(u) - J(u^*) = (\beta u^* + B^* \chi^*, u - u^*)_{L^2(0,T;U)} + \int_0^T (H(x - x^*, x - x^*), \chi^*) dt
\end{equation}
and
\begin{equation}
+ \frac{1}{2} (\beta|u - u^*|^2_{L^2(0,T;U)} + (Q(x - x^*), (x - x^*))_{L^2(0,T;X)}),
\end{equation}

where
\begin{equation}
\left| \int_0^T (H(x - x^*, x - x^*), \chi^*) dt \right|
\leq c \left( \int_0^T |x - x^*|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |\chi^*|^2 dt \right)^{\frac{1}{2}}.
\end{equation}

Let \( u = u^* + \tau v, \ v \in L^2(0,T;U) \). Then
\begin{equation}
0 \leq J(u^* + \tau v) - J(u^*) = (\beta u^* + B^* \chi^*, \tau v)_{L^2(0,T;U)} + o(\tau).
\end{equation}

Dividing by \( \tau \) and letting \( \tau \) tend to \( 0^+ \) implies that \( (\beta u^* + B^* \chi^*, v) \leq 0 \) for all \( v \in L^2(0,T;X) \). Hence the necessary optimality condition is given by
\begin{equation}
\beta u^* + B^* \chi^* = 0 \quad \text{in} \ L^2(0,T;U),
\end{equation}

and the optimality system consists of the primal equation (2.2), the adjoint equation (4.2), and the optimality condition (4.8).
4.2. Second order sufficient optimality. Suppose that there exists a constant \( \sigma > 0 \) such that
\[
\frac{1}{2} \left( |u - u^*|_{L^2(0,T;U)}^2 + (Q(x - x^*), (x - x^*))_{L^2(0,T;X)} \right) + \int_0^T (H(x - x^*, x - x^*), \chi^*) \, dt \geq \sigma |u - u^*|_{L^2(0,T;U)}^2 \\
\]
for all \( u \) above.

By (4.10) all \( u \) above.

Turning to (4.9), we first note that
\[
-\frac{1}{2} \frac{d}{dt} |\chi^*(t)|^2 \leq -\nu |\chi^*(t)|^2_V + c |x^*(t)|_V |\chi^*(t)|_V (Q(x^*(t) - \bar{x}(t)), \chi^*(t)),
\]
and hence
\[
-\frac{1}{2} \frac{d}{dt} |\chi^*(t)|^2 + \frac{\nu}{2} |\chi^*(t)|^2_V \leq \frac{c^2}{\nu} |x^*(t)|_V^2 |\chi^*(t)|^2_V + \frac{1}{\nu} |Q(x^*(t) - \bar{x}(t))|_{L^2}.
\]
By Gronwall’s inequality applied backwards with respect to time and terminal condition \( \chi^*(T) = 0 \), we find that
\[
\int_0^T |\chi^*(t)|^2 \leq 2 \int_0^T |Q(x^* - \bar{x})|_{L^2} \exp \left( \int_0^T \frac{2c^2}{\nu} |x^*(s)|_V^2 \right).
\]
Hence there exists a constant \( M = M(c, \frac{1}{\nu}) \) such that
\[
\int_0^T |\chi^*|_V^2 \leq M^2 \int_0^T |Q(x^* - \bar{x})|_{L^2}^2 \, dt.
\]
By (4.7) and (4.10)
\[
\frac{1}{2} \left( |u - u^*|_{L^2(U)}^2 + (Q(x - x^*), (x - x^*))_{L^2(X)} \right) + \int_0^T (H(x - x^*, x - x^*), \chi^*) \, dt \geq \frac{1}{2} \left( |u - u^*|_{L^2(U)}^2 + (Q(x - x^*), (x - x^*))_{L^2(X)} \right) - cM|x - \bar{x}|_{C(X)} |x - x^*|_{L^2(V)} |Q(\bar{x} - x^*)|_{L^2(V^*)},
\]
where \( L^2(U) \) stands for \( L^2(0,T;U) \). Let \( B(r) \) denote the ball with center 0 and radius \( r \) in \( L^2(0,T;U) \). Let \( r > 0 \) and \( x_0 \in V \). Then by Lemma 3.2 estimate (4.9) holds for all \( u \in B \), provided that
\[
|Q(\bar{x} - x^*)|_{L^2(V^*)} \text{ is sufficiently small.}
\]
These results can now be utilized to argue convergence of the reduced optimal controls \( u \) to an optimal control \( u^* \) of (2.1)–(2.2).

Theorem 4.1. Let \( x_0 \in V \), \( J \) as in (4.1), and let \( u^* \) denote an optimal solution to (2.1)–(2.2) and \( u_N \) an optimal solution to (2.7)–(2.8), with dynamics given by the Navier–Stokes equations. Then, if \( |Q(\bar{x} - x^*)|_{L^2(V^*)} \) is sufficiently small, there exists
Theorem 3.3, (3.5), and Theorem 3.2 in the same manner as that of Corollary 3.1.  
(4.9) is satisfied, then from (2.16), (2.14), (2.13), and Lemma 3.1 we have with \( \hat{C} \) 
where \( \hat{C} = C^*(1 + k_8) \) that
\[
|u_N - u^*|^2_{L^2(0,T;U)} \leq \hat{C} \left( |p(u_N) - z(u_N)|_{L^2(X)} + |q(u_N) - \Phi(z(u_N))|_{L^2(X)} + |q(u^*) - \Phi(z(u^*))|_{L^2(X)} \right) 
\]
where again \( L^2(X) \) is short for \( L^2(0,T;X) \). The proof can now be completed by 
Theorem 3.3, (3.5), and Theorem 3.2 in the same manner as that of Corollary 3.1.  

5. Convergence rate analysis. In the previous section convergence and a first 
rate of convergence estimate were obtained on the basis of the second order sufficient 
optimality condition. In this section a rate estimate for the solutions of the reduced 
problem to the original problem is derived by means of the optimality systems. For 
the sake of simplicity we assume throughout this section that \( J \) is of the form given 
in (4.1) with \( Q = I \) and that 
\[
\text{range } (B) \subset X_1. 
\]
Further let \( (x^*, u^*) \) denote a solution to (2.1)–(2.2). Then, \( (x^*, u^*) \) with \( x^*(t) = p(t) + q(t) \) satisfies 
\[
\begin{align*}
\frac{d}{dt} p(t) &= A_0 p(t) + P_1 F(p(t) + \Phi(p(t)) + \Delta(t)) + Bu^*(t), \\
\Delta(t) &= q(t) - \Phi(p(t)) \in X_2.
\end{align*}
\]

Let 
\[
\tilde{F}(z) = P_1 F(z + \Phi(z)) \text{ for } z \in X_1
\]
and recall that \( \Phi : X_1 \to X_2 \) is \( C_1 \) regular.
The first order necessary optimality condition for (2.7), (2.8) is given by
\[
\begin{align*}
\frac{d}{dt} z(t) &= A_0 z(t) + \tilde{F}(z(t)) + Bu(t), \\
-\beta u(t) + B^* \xi(t) &= 0,
\end{align*}
\]
where \( z(0) = P_1 x_0, \xi(T) = 0, \) and \( \dot{x}(t) = z(t) + \Phi(z(t)). \) Observe that \( \beta u(t) + B^* P_\xi \dot{\xi}(t) = 0, \) and

\[
\begin{align*}
F'(z) h &= P_1 F'(\hat{x})(h + \Phi'(z) \hat{h}) = F'_{11}(\hat{x})h + F'_{12}(\hat{x})\Phi'(z)h, \\
\tilde{F}'(z)^* h &= F'_{11}(\hat{x})^* h + \Phi'(z)^* F'_{12}(\hat{x})^* h,
\end{align*}
\]  

(5.4)

for \( h \in X_1. \) Here \( F_{ij} = P_i F P_j. \) The adjoint equation split into the \( X_1 \) and \( X_2 \) components and the necessary optimality condition for (2.1)–(2.2) are given by

\[
\begin{align*}
-\frac{d}{dt} \eta &= A_0 \eta + F'_{11}(x^*)^* \eta + F'_{21}(x^*)^* \zeta + p(t) - P_1 \bar{x}, \\
-\frac{d}{dt} \zeta &= A_0 \zeta + F'_{12}(x^*)^* \eta + F'_{22}(x^*)^* \zeta + q(t) - P_2 \bar{x}, \\
\beta u^*(t) + B^* \eta(t) &= 0,
\end{align*}
\]  

(5.5)

where \( \eta(T) = 0, \zeta(T) = 0. \) Note that for \( \dot{x} = z + \Phi(z) \in X_1 + X_2 \) we have

\[
\begin{align*}
\Phi'(z) h_1 &= -A_0^{-1} F'_{21}(z) h_1, \quad h_1 \in X_1, \\
\Phi'(z)^* h_2 &= -F'_{21}(\hat{x})^* A_0^{-1} h_2, \quad h_2 \in X_2.
\end{align*}
\]  

(5.6)

From the second equation of (5.5)

\[
\zeta(t) = -A_0^{-1} (F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x}) - E(t)
\]

with

\[
E(t) = A_0^{-1} \left( \frac{d}{dt} \zeta + F'_{22}(x^*)^* \zeta \right).
\]

(5.7)

This quantity \( E \) will appear in the final estimate. So let us discuss the effect of the terms which enter into the second equation in (5.5), which is the equation for the fast modes in the adjoint system. The input to the system is \( F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x} \) which is small if the effect of the \( X_2 \)-component of the primal equation and the target state \( \bar{x} \), as well as the coupling of the nonlinearity from \( X_1 \) to \( X_2 \) are small. This results in smallness of the term \( \int_0^T \frac{d}{dt} \zeta + F'_{22}(x^*)^* \zeta \bigg|_X dt \) is small. Further \( A_0 \) acts on the fast components \( \zeta \) of the adjoint state and thus

\[
|E(t)|_X \leq \frac{1}{\lambda_{N+1}} \left| \frac{d}{dt} \zeta + F'_{22}(x^*)^* \zeta \right|_X.
\]

Inserting \( E \) into the first equation of (5.5) we find that

\[
-\frac{d}{dt} \eta = A_0 \eta + F'_{11}(x^*)^* \eta - F'_{21}(x^*)^* (A_0^{-1} (F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x}) + E) + p(t) - P_1 \bar{x},
\]
and thus by the second equations of (5.4) and (5.6)

\begin{align*}
\frac{d}{dt}(\xi - \eta) &= A_0(\xi - \eta) + \tilde{F}'(z)(\xi) - F'_{11}(x^\ast)*\eta + z - p + \Phi'(z)^\ast(\Phi(z) - P_2\bar{x}) \\
&+ (F'_{21}(x^\ast)^\ast A_0^{-1})(F'_{22}(x^\ast)^\ast \eta + q - P_2\bar{x}) + F'_{21}(x^\ast)^\ast E \\
= A_0(\xi - \eta) + \tilde{F}'(z)(\xi - \eta) + (\tilde{F}'(z)^\ast - \tilde{F}'(p)^\ast)\eta + \tilde{F}'(x^\ast)^\ast \eta - F'_{11}(x^\ast)*\eta + z - p \\
&+ \Phi'(z)^\ast \Phi(z) + F'_{21}(\hat{x})^\ast A_0^{-1}(q - P_2\bar{x}) + (F'_{21}(x^\ast)^\ast - F'_{21}(\hat{x})^\ast)A_0^{-1}(q - P_2\bar{x}) \\
&+ F'_{21}(x^\ast)^\ast A_0^{-1} F_{12}(x^\ast)^\ast \eta - \Phi'(z)^\ast P_2\bar{x} + F'_{21}(x^\ast)^\ast E \\
= A_0(\xi - \eta) + \tilde{F}'(z)(\xi - \eta) + (\tilde{F}'(z)^\ast - \tilde{F}'(p)^\ast)\eta + z - p \\
&+ \Phi'(z)^\ast (\Phi(z) - \Phi(p) - \Delta) + (F'_{21}(x^\ast)^\ast - F'_{21}(\hat{x})^\ast)A_0^{-1}(q - P_2\bar{x}) + F'_{21}(x^\ast)^\ast E.
\end{align*}

Similarly we find

\begin{align*}
\frac{d}{dt}(z - p) &= A_0(z - p) + \tilde{F}'(z)(z - p) + B(u - u^\ast) \\
&+ P_1 F'(\hat{x}) - P_1 F'(x^\ast) - P_1 F'(\hat{x})(\hat{x} - x^\ast) \\
&+ F'_{12}(\hat{x})(\Phi(z) - \Phi(p) - \Phi'(\hat{x})(z - p) - \Delta).
\end{align*}

Here we used that

\begin{align*}
P_1 F'(\hat{x})(\hat{x} - x^\ast) &= F'_{11}(\hat{x})(z - p) + F'_{12}(\hat{x})(\Phi(z) - q) \\
&= F'_{11}(\hat{x})(z - p) + F'_{12}(\hat{x})(\Phi(z) - \Phi(p) - \Delta), \\
\tilde{F}'(z)(z - p) &= F'_{11}(\hat{x})(z - p) + F'_{12}(\hat{x})(\Phi'(\hat{x})(z - p)).
\end{align*}

For the following discussion let \( f : L^2(0, T; U) \times L^2(0, T; X_2) \times L^2(0, T; X_2) \rightarrow L^2(0, T; U) \) be defined by the forward integration of the control dynamics for \( p(t) \), then the backward integration of the adjoint equation for \( \xi(t) \), followed by the application of \( B^\ast \), where \( (u, (\Delta, E)) \in L^2(0, T; U) \times L^2(0, T; X_2) \times L^2(0, T; X_2) \) are given. Thus \( f \) is the composite \( u \rightarrow p(u) \rightarrow \xi(p) \rightarrow B^\ast \xi = B^\ast P_1 \xi \). Then the optimality system for (2.1)–(2.2) can be expressed as

\[ \beta u^\ast + f(u^\ast, \Delta, E) = 0, \]

and, using (5.8), (5.9) the optimality system for (2.7), (2.8) is

\[ \beta u + f(u, 0, 0) = 0. \]

Now, if \( f \) is \( C^1 \) and

\( \beta I + H, \ H = f_u(u, 0, 0) \)

is an isomorphism on \( L^2(0, T; U) \), then an argument based on the implicit function theorem implies that

\[ |u^\ast - u|_{L^2(0, T; U)} \leq k_{12} (|\Delta|_{L^2(0, T; X_2)} + |E|_{L^2(0, T; X_2)}), \]
provided that \( u = u_N \) is sufficiently close to \( u^* \). Sufficient conditions for convergence of \( u_N \) to \( u^* \) were already given in Theorem 4.1. We prove such an estimate directly. First, observe that
\[
\beta(u - u^*, u - u^*) + (B^*(\xi - \eta), u - u^*) = 0.
\]
(5.10)

From (5.8), (5.9), and (5.4) we find that
\[
\frac{d}{dt} (z - p, \xi - \eta) = (F(\dot{x}) - F(x^*) - F'(\dot{x})(\dot{x} - x^*), \xi(t) - \eta(t))
\]
\[
- (\dot{F}'(z) - \dot{F}'(p)(z - p), \eta) - |z - p|^2 - (\Phi'(z)^\ast(\Phi(z) - \Phi(p) - \Delta), z - p)
\]
\[
+ ((\Phi'(z)^\ast - \Phi'(p)^\ast)(q - P_2\tilde{x}), z - p) - (F_{21}'(x^*)^T E, z - p) + (B^*(\xi - \eta), u - u^*)
\]
\[
+ (F_{12}'(x)(\Phi(z) - \Phi(p) - \Phi'(\dot{x})(z - p) - \Delta), \xi - \eta).
\]
Integrating this expression onto \([0, T]\),
\[
\int_0^T (B^*(\xi - \eta), u - u^*) \, dt
\]
\[
= \int_0^T [\frac{d}{dt} (z - p, \xi - \eta)] = \int_0^T \left[ |z - p|^2 + \Phi'(z)^\ast(\Phi(z) - \Phi(p) - \Delta, z - p)
\right.
\]
\[
+ ((\Phi'(z)^\ast - \Phi'(p)^\ast)(q - P_2\tilde{x}), z - p) \right] dt
\]
\[
- \int_0^T [(F(\dot{x}) - F(x^*) - F'(\dot{x})(\dot{x} - x^*), \xi - \eta) + ((\dot{F}'(z) - \dot{F}'(p)(z - p), \eta)
\]
\[
+ (F_{21}'(x^*)^T E, z - p) \right] dt - \int_0^T (F_{12}'(x)(\Phi(z) - \Phi(p) - \Phi'(\dot{x})(z - p) - \Delta), \xi - \eta) \, dt.
\]
Let us now assume that the assumptions of Theorem 4.1 are satisfied, and let \( U \) denote a neighborhood of \( u^* \) in \( L^2(0, T; U) \). Let \( M_1 \) be a bound for \( |\tilde{x}|_{C([0,T];V)} \) with \( u \in U \), and let \( N \) be sufficiently large such that \( u = u_N \in U \). Using the properties of \( F \) specified at the beginning of section 4, we verify from (5.9) with techniques as in the proof of Theorem 3.3 that for some constant \( k_1 \),
\[
|z(t) - p(t)|^2 + \int_0^T |z(s) - p(s)|_V^2 \, ds \leq k_1 \int_0^T (|u(s) - u^*(s)|^2 + L|\Delta(s)|^2) \, ds
\]
for all \( t \in [0, T] \) and \( N \) sufficiently large. Similarly by (5.8), if \( x^* \in L^2(0, T; dom(-A)) \), then there exists a constant \( k_2 \) such that for all \( N \) sufficiently large so that \( u \in U \),
\[
|\xi(t) - \eta(t)|^2 + \int_0^T |\xi(s) - \eta(s)|_V^2 \, ds
\]
\[
\leq k_2 \int_0^T (|u(s) - u^*(s)|_U^2 + L|\Delta(s)|_U^2 + |E(s)|^2) \, ds.
\]
Applying the last two estimates to (5.11) we find with Lemma 3.1 and (3.2) the existence of a constant \( k_3 \) such that
\[
\int_0^T (B^*(\xi - \eta), u - u^*) \, dt \leq k_3 \int_0^T (|u(s) - u^*(s)|^2
\]
\[
+ L|\Delta(s)|^2 + |E(s)|^2) \, ds,
\]
(5.14)
for $u$ sufficiently close to $u^*$. The detailed estimates are quite similar to those necessary for proving (5.12) and (5.13). Together with (5.10) we obtain the following result.

**Theorem 5.1.** Under the assumptions of Theorem 4.1, if $\beta$ is sufficiently large, then there exists a constant $k_4$ such that

$$|u_N - u^*|_{L^2(0,T;U)} \leq \frac{k_4}{\beta} (L^{\frac{1}{2}} |\Delta|_{L^2(0,T;X_2)} + |E|_{L^2(0,T;X_2)}),$$

where $\Delta$ defines the gap and $E$ is defined by (5.7).

Referring to (5.11) let us discuss qualitatively which terms enter into the assumption that $\beta$ is sufficiently large. These are clearly the first and second terms, without $\Delta$, on the right-hand side of (5.11). The third term is quadratic in $|z - p|_{L^2(0,T;V)}$ multiplied by $|q - P_2 \tilde{x}|_{L^2(0,T;V)}$, which is small for small residue problems. Similarly, the fifth term is a quadratic multiplied by $|\eta|_{L^2(0,T;V)}$, which in view of (5.5) is again small for small residue problems when $p$ is close to $P_1 \tilde{x}$. The fourth and seventh terms are of cubic nature. The remaining terms involving $\Delta$ and $E$ imply no assumption on size of $\beta$.

**6. Appendix.** Here we sketch the proof for Lemma 3.2. The estimate for $x$ is well known [23]. We turn to the estimate for $\hat{x}$. Let $f = Bu$ and $r = \Phi(z)$. Then $\hat{x} = z + r$ satisfies

$$\left(\frac{d}{dt} z, \psi_1 \right) - \nu (Az, \psi_1) + (b(z + r, z + r), \psi_1) = (f, \psi_1),$$

$$\nu (Ar, \psi_2) - (b(z, z), \psi_2) = 0$$

for all $\psi_1 \in X_1$ and $\psi_2 \in X_2$. Letting $\psi_1 = z$ and $\psi_2 = r$, we have

$$(6.1) \quad \frac{1}{2} \frac{d}{dt} |z|^2 + \nu (|z|^2 + |r|^2) + b(r, r, z) = (f, z),$$

where we used

$$(b(z + r, z + r), z) + (b(z, z), r) = (b(r, r), z).$$

Assume that $|z(t)|_V \leq M$ on $[0, T]$. The definition of $M$ will be given at the end of the proof. From (3.2)

$$(6.2) \quad |r|_V \leq c_4 L^{\frac{1}{2}} M \frac{1}{\sqrt{\lambda_{N+1}}}|z|_V, \quad |r|_X \leq \frac{1}{\lambda_{N+1}} |r|_V, \quad |r|_X \leq c_4 L^{\frac{1}{2}} M \frac{1}{\sqrt{\lambda_{N+1}}}|z|_X.$$

Since $|(b(r, r), z)| \leq c_2 |r||r|_V|z|_V$,

$$|b(r, r, z)| \leq c_2 c_4^2 L M^3 \frac{1}{\nu^2 (\lambda_{N+1})^{3/2}} |z|_V^2.$$

We select $N$ so that $c_2 c_4^2 L M^3 \frac{1}{\nu^2 (\lambda_{N+1})^{3/2}} \leq \nu \frac{1}{2}$. Then from (6.1)

$$(6.3) \quad |z(t)|^2 + \nu \int_0^t |z|^2 \, ds \leq e^{t} |z(0)|^2 + \int_0^t e^{t-s} |f(s)|^2 \, dx \leq C,$$
for $t \in [0, T]$, where

$$C = e^T|z(0)|^2 + \int_0^T e^{T-s}|f(s)|^2 \, ds.$$

Next, letting $\psi_1 = -Az$ and $\psi_2 = -Ar$,

$$\frac{1}{2} \frac{d}{dt} |z|^2_V + \nu |Az|^2 - (b(z + r, z + r), Az) = (f, Az),$$

$$\nu |Ar|^2 - (b(z, z), Ar) = 0.$$

From the second equation

$$\nu |Ar| \leq c_4 L^\frac{1}{2} |z|_V^2.$$ 

Thus,

$$|(b(z + r, r), Az)| \leq |z + r|_{L^\infty} |r|_{V} |Az| \leq \bar{c} c_4 LM^3 \frac{1}{\nu \sqrt{\lambda N + 1}} |z|_V |Az|,$$

where, as shown in section 3, we can assume that $\bar{c} = \max c_3, \frac{c_4}{\nu}$ is chosen such that

$$|z(t) + r(t)|_{L^\infty} \leq \bar{c} M^2 L^\frac{1}{2}.$$

From (3.2) and (6.2) we deduce that

$$|(b(z + r, z), Az)| \leq 2c_1 |z|_V |z|_V |Az|_{\frac{3}{2}};$$

where we assumed

$$c_4 L^\frac{1}{2} M \frac{1}{\nu \sqrt{\lambda N + 1}} \leq 1.$$

From the first equation in (6.5) we obtain

$$\frac{1}{2} \frac{d}{dt} |z|^2_V + \nu |Az|^2 \leq \left( \frac{2}{\nu} d^2 + \frac{108}{\nu^3} c_4^4 |z|^2_V |z|^2_V \right) |z|^2_V + \frac{2}{\nu} |f|^2,$$

where

$$d = c_4^2 LM^3 \frac{1}{\nu^2 \sqrt{\lambda N + 1}}.$$

We select $N$ such that $d^2 = d(N)^2 \leq \frac{1}{4}$. Then

$$\frac{1}{2} \frac{d}{dt} |z|^2_V + \nu |Az|^2 \leq \left( 1 + \frac{216}{\nu^3} c_4^4 C |z|^2_V \right) |z|^2_V + \frac{4}{\nu} |f|^2.$$

By Gronwall’s lemma we obtain

$$|z(t)|_V^2 + \nu \int_0^t |Az|^2 \, ds \leq \exp \left( \int_0^T \left( 1 + \frac{216}{\nu^3} c_4^4 C |z|^2_V \right) \, ds \right) \left( |z(0)|_V^2 + \frac{4}{\nu} \int_0^T |f|^2 \, ds \right).$$
and hence

\[(6.6)\]

\[|z(t)|^2_V + \nu \int_0^t |A\nu|^2 ds \leq \exp \left( T + \frac{216}{\nu^4} c_4^2 C^2 \right) \left( |z(0)|^2_V + \frac{4}{\nu} \int_0^T |f|^2 ds \right) =: M.\]

Note that the estimate for \(|z(t)|_X\) by \(C\) in (6.3) involves \(|z(0)|\) and \(|f|_{L^2(0,T,X)} = |B(u)|_{L^2(0,T,X)}\) but is independent of \(M\) for all \(N\) sufficiently large. Similarly, the estimate for \(|z(t)|_V\) in (6.6) involves \(C\), \(|z(0)|\), and \(|B(u)|_{L^2(0,T,X)}\), but is independent of \(M\) for all \(N\) sufficiently large. Hence, the theory of ordinary differential equations on the ball with radius \(M\) in \(V \cap X_1\) allows one to construct a solution to (2.5) with a priori bound for the \(z(t)\) component given by \(M\). The uniform bound for the \(\Phi(z(t))\) component follows immediately from (6.2).

REFERENCES


