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REDUCED-ORDER OPTIMAL CONTROL BASED ON APPROXIMATE INERTIAL MANIFOLDS FOR NONLINEAR DYNAMICAL SYSTEMS*

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Abstract. A reduced-order method for optimal control problems in infinite dimensions based on approximate inertial manifolds is developed. Convergence of the cost, optimal controls, and optimal states of the finite dimensional, reduced-order, optimal control problems to the original optimal control problem is analyzed. Special attention is given to the particular case when the dynamics are described by the Navier–Stokes equations in dimension two.

Key words. reduced-order methods, approximate inertial manifold, nonlinear Galerkin, decomposition of state space, infinite dimensional system

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1. Introduction.

In this research we consider optimal control problems governed by partial differential equations. For such problems there has recently been an increased interest in developing reduced-order control methods. This is motivated by computational needs and by systems science considerations as well. For nonlinear distributed parameter control systems, let us specifically mention model reduction based on proper orthogonal decomposition approach [1, 15, 16, 24] and the reduced-basis method [11, 12, 13, 17]. The key issue for the reduction consists in selecting basis elements which are rich in information in the sense that they capture well the essential dynamical properties of the original control system. After the basis elements are selected by these methods the standard Galerkin approach is applied to obtain a reduced-order control system. For linear control systems many alternative reduction methods were proposed and analyzed, including the Hankel-norm approximation [7] and the LQG-balanced truncation realization [2]. The LQG-balanced truncation method was introduced in the finite dimensional literature by [14] and other interpretations followed in [19, 20]. For the infinite dimensional theory we refer to [3]. For linear systems, model reduction based on specific versions of proper orthogonal decomposition and of balanced truncation coincide [22]. The selection of references here is by no means complete and we refer to the citations in the quoted papers for references.

In this paper we discuss an order-reduction method based on the concept of inertial manifolds. Let us recall that an important feature of inertial manifolds consists in the description of the small-scale dynamics as a graph of the large-scale dynamics. The inertial manifold is a finite dimensional invariant manifold that attracts all orbits exponentially. Thus it is natural to expect that controlling the inertial manifold dynamics results in the ability to control the full underlying control system. However,

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there are some technical difficulties associated with this approach: (1) the existence of the inertial manifold can only be proved for a limited class of systems, (2) the construction of the inertial manifold is highly involved and technical, and (3) the analysis of the behavior of the closed-loop system has not been fully addressed.

Here we therefore consider approximate inertial manifolds which are not necessarily invariant under the dynamics but which approximate all orbits starting from a bounded set with any desired accuracy. We use nonlinear Galerkin approximations as proposed, e.g., in [5, 18], and the manifold is represented by a stationary graph determined by the residual dynamics. If one would simply truncate the residual dynamics (the flat manifold), then this approach coincides with the standard Galerkin approximation. In [10] we demonstrated the effectiveness of the approximate inertial manifold approach for the linear quadratic regulator problem.

Error estimates for the uncontrolled solution based on the approximate inertial manifold method were investigated extensively. We refer, e.g., to [5, 18], where the estimates are obtained for $t \geq t^* > 0$ for some sufficiently large t^*. In this paper we establish error estimates for control systems based on the reduced-order systems governed by approximate inertial manifolds. Such estimates are obtained by the necessary optimality condition for the optimal control problem, a second order sufficient optimality condition, and the gap estimate for the approximate inertial manifold.

The outline of this paper is as follows. In section 2 we describe the general inertial manifold approach. An upper bound for the performance of the reduced-order control as well as an error estimate for the reduced-order control to the infinite dimensional control in terms of the gap estimate of the approximate inertial manifold are established. In section 3 we discuss error estimates for the two-dimensional Navier–Stokes equations in $L^2(0,T;X)$, where X denotes the state space. Section 4 is devoted to the necessary optimality condition, a second order sufficient optimality condition as well as a first concrete estimate for the finite dimensional optimal controls to the infinite dimensional one. In section 5 a sharper estimate of the convergence rate of the reduced-order control is established on the basis of a sensitivity analysis for the necessary optimality condition.

2. Approximate inertial manifold and reduced-order control system.

In this section we describe a general strategy for the approximation of nonlinear control systems by nonlinear Galerkin schemes. Throughout this paper, X and U denote separable Hilbert spaces. We consider the optimal control problem

$$\min_{u\in L^2(0,T;U)} \quad J(x,u) = \int_0^T \ell(x(t)) + h(u(t))\,dt, \tag{2.1}$$

subject to

$$\frac{d}{dt}x(t) = A_0x(t) + F(x(t)) + Bu(t), \quad x(0) = x_0 \in X, \tag{2.2}$$

where $T > 0$, $F : X \to X^*$ is a nonlinear mapping, A_0 is a linear self-adjoint negative definite operator in X with dense domain denoted by $dom(A_0)$, and B is a bounded linear operator from U to X. We further set $V = dom(-A_0)^{\frac{1}{2}}$ endowed with $|v|_V = \sqrt{\langle -A_0v\,, v\rangle}$ as norm. We assume that (2.2) is a well-posed control system, i.e., given any $x_0 \in X$ and $u \in L^2(0,T;U)$, there exists a unique weak solution $x = x(t;x_0,u) \in C(0,T;X) \cap L^2(0,T;V)$ to (2.2) which depends continuously on $(x_0,u) \in X \times L^2(0,T;U)$. For example, we can formulate the control problem (2.1)–(2.2) in a

Gelfand triple formulation or as semilinear control systems; see, e.g., [9, 21]. Further, $\ell : X \to \mathbb{R}$ and $h : U \to \mathbb{R}$ are C^1 mappings, ℓ is supposed to be uniformly Lipschitz continuous on bounded sets of X, and h is such that the associated substitution operator from $L^2(0,T;U)$ to $\mathbb{R}$ is well-defined.

We assume that $-A_0$ has eigenpairs (λ_i, ϕ_i) in ascending order and that $\{\phi_i\}_{i=1}^\infty$ forms an orthonormal basis of X. Let P_1 be the orthogonal projection of X onto

$$X_1 = \text{ span } \{\phi_i : \ 1 \le i \le N\}.$$

Further we set $X_2 = (I - P_1)X$. Expressing x as

$$x = p + q, \quad \text{with } \ p(t) = P_1 x(t) \ \text{ and } \ q(t) = P_2 x(t),$$

where $P_2 = I - P_1$, we have from (2.2)

$$\begin{aligned} &\frac{d}{dt}p(t) = A_0 p(t) + P_1 F(p(t) + q(t)) + P_1 Bu(t), \\ &\frac{d}{dt}q(t) = A_0 q(t) + P_2 F(p(t) + q(t)) + P_2 Bu(t). \end{aligned} \tag{2.3}$$

In the linear Galerkin approach, the higher order modes q are neglected. This results in the control system

$$\frac{d}{dt}\hat{x}_1(t) = A_0 \hat{x}_1(t) + P_1 F(\hat{x}_1(t)) + P_1 Bu(t).$$

In the nonlinear Galerkin approach, $\frac{d}{dt}\, q(t)$ is assumed to be negligible compared to $A_0\, q(t)$ in (2.3). This suggests considering the control system

$$\frac{d}{dt}\hat{x}_1(t) = A_0 \hat{x}_1(t) + P_1 F(\hat{x}_1(t) + \hat{x}_2(t)) + P_1 Bu(t), \tag{2.4}$$

$$0 = A_0 \hat{x}_2(t) + P_2 F(\hat{x}_1(t) + \hat{x}_2(t)) + P_2 Bu(t).$$

We set

$$\hat{x} = \hat{x}_1 + \hat{x}_2.$$

and the first equation in (2.4) can be expressed as

$$\frac{d}{dt}P_1 \hat{x}(t) = A_0 \hat{x}(t) + P_1 F(\hat{x}(t)) + P_1 Bu(t).$$

The second equation in (2.4) is nonlinear and is replaced in the approach that follows by the linear equation

$$A_0 \hat{x}_2(t) + P_2 F(\hat{x}_1) = 0,$$

where the coupling in the nonlinearity and $P_2 Bu(t)$ are neglected. An alternative to this may be to consider

$$A_0 \hat{x}_2 + P_2 (F(\hat{x_1}) + F'(\hat{x_1})\hat{x}_2) = 0,$$

which should lead to better approximation properties. The reduced-order system that we consider in this paper is given by

$$\begin{aligned}\frac{d}{dt}\hat{x}_1(t) &= A_0\hat{x}_1(t) + P_1F(\hat{x}_1(t)+\hat{x}_2(t)) + P_1Bu(t),\\ 0 &= A_0\hat{x}_2(t) + P_2F(\hat{x}_1(t)).\end{aligned} \tag{2.5}$$

Given $\hat{x}_1 \in X_1 = P_1X$, we denote the unique solution to the second equation in (2.5) by

$$\hat{x}_2 = \Phi(\hat{x}_1), \quad \text{where } \ \Phi(\phi) = -A_0^{-1}P_2F(\phi). \tag{2.6}$$

In this way we obtain the finite dimensional reduced-order control system in X_1:

$$\min_{u\in L^2(0,T;U)} \quad \int_0^T \ell\big(z(t)+\Phi(z(t))\big) + h(u(t))\,dt \tag{2.7}$$

subject to

$$\frac{d}{dt}z(t) = A_0z(t) + P_1F(z(t)+\Phi(z(t))) + P_1Bu(t), \quad z(0) = P_1x_0; \tag{2.8}$$

i.e., we set $z = \hat{x}_1$ with $\hat{x}_1$ as in (2.5). Here we assume the existence of an optimal control u to (2.7) as well as u^* to (2.1). To obtain error estimates for $u - u^*$, we first establish an error estimate for the reduced-order equation. This involves an estimate of the gap

$$\Delta(t) = |q(t) - \Phi(p(t))|,$$

where $p(t)$ and $q(t)$ are defined in (2.3); see, e.g., [5] and section 3. Then, we can argue that under appropriate conditions

$$|z(t)-p(t)|^2 \le M \int_0^t |q(s)-\Phi(p(s))|^2\,ds \text{ for } t\in[0,T] \tag{2.9}$$

for some $M > 0$. In fact, for $S = \{v \in V : |v| \le K\}$, with $K \ge 0$, let us assume the properties

$$\langle F(x)-F(y),\phi\rangle \le c\,|x-y||\phi|_V \quad \text{for all} \quad x,y,\phi\in S\subset V, \tag{2.10}$$

and

$$x(t)\in S \text{ and } z(t)+\Phi(z(t))\in S \text{ for a.e. } t\in(0,T).$$

From (2.3) and (2.8) we have

$$\begin{aligned}&\tfrac{d}{dt}(z(t)-p(t)) = A_0(z(t)-p(t)) + P_1\,F(z(t)+\Phi(z(t))) - P_1\,F(p(t)+q(t)),\\ &\quad z(0)-p(0) = 0.\end{aligned} \tag{2.11}$$

By (2.10) we find for $x(t)\in S$ and $z(t)+\Phi(z(t))\in S$ that

$$\begin{aligned}&\langle P_1\,F\big(z(t)+\Phi(z(t))\big) - P_1\,F\big(p(t)+q(t)\big), z(t)-p(t)\rangle\\ &\quad \le c\big(|z(t)-p(t)| + |\Phi(z(t))-\Phi(p(t))| + |q(t)-\Phi(p(t))|\,\big)|z(t)-p(t)|_V\\ &\qquad \le c\big((1+\|\Phi\|)|z(t)-p(t)| + |q(t)-\Phi(p(t))|\big)|z(t)-p(t)|_V,\end{aligned} \tag{2.12}$$

where $\|\Phi\|$ is the Lipschitz constant of Φ on S. Taking the inner product of (2.11) with $z(t)-p(t)$ and using (2.12) results in

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|z(t)-p(t)|^2 &\le -|z(t)-p(t)|_V^2\\ &\quad + c\big((1+\|\Phi\|)\,|z(t)-p(t)| + |q(t)-\Phi(p(t))|\,\big)\,|z(t)-p(t)|_V,\end{aligned}$$

which implies, using $ab \le a^2 + \frac{1}{4}b^2$, that

$$\frac{1}{2}\frac{d}{dt}|z(t)-p(t)|^2 \le \frac{c^2}{2}\big((1+\|\Phi\|)^2|z(t)-p(t)|^2 + |q(t)-\Phi(p(t))|^2\big).$$

Estimate (2.9) with $M = c^2 exp\big(c^2(1+\|\Phi\|)^2T\big)$ follows from Gronwall's inequality since $z(0)=p(0)$.

Once (2.9) is established, it implies the error estimate

$$\begin{aligned}|\hat{x}(t)-x(t)| &= |z(t)+\Phi(z(t))-x(t)|\\ &\le |z(t)-p(t)| + |\Phi(z(t))-\Phi(p(t))| + |q(t)-\Phi(p(t))|\\ &\le (1+\|\Phi\|)\,|z(t)-p(t)| + |q(t)-\Phi(p(t))|,\end{aligned} \tag{2.13}$$

for $t\in[0,T]$.

To demonstrate the use of the a priori estimate (2.13) in the context of the optimal control problem, let us assume the existence of optimal controls u and u^* to (2.7)–(2.8) and (2.1)–(2.2), respectively. Moreover, let x and x^* in $C(0,T;X)\cap L^2(0,T;V)$ be the solutions to (2.2) with u and u^*, respectively. Thus x^* is the optimal trajectory to (2.1)–(2.2), whereas x is the trajectory of the infinite dynamical system controlled by the optimal control computed on the bases of the reduced-order system. Let

$$\hat{x} = z + \Phi(z) \ \text{ and } \hat{x}^* = z^* + \Phi(z^*),$$

where z and z^* are the solutions in $C(0,T;X)\cap L^2(0,T;V)$ to (2.8), corresponding to u and u^*, respectively. Then, for the finite horizon control on $[0,T]$ we have

$$\begin{aligned}0 &\le J(x,u) - J(x^*,u^*)\\ &= J(x,u) - J(\hat{x},u) + J(\hat{x},u) - J(\hat{x}^*,u^*) + J(\hat{x}^*,u^*) - J(x^*,u^*)\\ &\le J(x,u) - J(\hat{x},u) + J(\hat{x}^*,u^*) - J(x^*,u^*),\end{aligned}$$

where we used that $J(\hat{x},u) - J(\hat{x}^*,u^*) \le 0$. Thus

$$0 \le J(x,u) - J(x^*,u^*) \le C\,(|x^*-\hat{x}^*|_{L^2(0,T;X)} + |x-\hat{x}|_{L^2(0,T;X)}), \tag{2.14}$$

with C the Lipschitz constant of $\ell : X \to R$ on the ball determined by the $C(0,T;X)$ norm of $x, \hat{x}, x^*$, and $\hat{x}^*$. The error estimate (2.9) now provides an upper bound for the performance of the suboptimal control u based on the reduced-order control problem (2.7)–(2.8) imposed onto the original control problem (2.1)–(2.2). In fact, if

$$x(t),\ \hat{x}(t),\ x^*(t),\ \hat{x}^*(t) \ \text{ are in } S \text{ for a.e. } t\in(0,T)$$

with $\hat{x} = z + \Phi(z)$ and $\hat{x}^* = z^* + \Phi(z^*)$, then by (2.9) and (2.13)–(2.14)

$$0 \le J(x,u) - J(x^*,u^*) \le \hat{C}(|q-\Phi(p)|_{L^2(0,T;X)} + |q^*-\Phi(p^*)|_{L^2(0,T;X)}), \tag{2.15}$$

where $\hat{C} = \hat{C}(\|\Phi\|, c, C)$ and (p, q), (p^*, q^*) are the orthogonal projections of x and x^* according to (2.3).

Suppose, moreover, that for some $\sigma > 0$ the second order sufficient optimality condition

$$J(x(v), v) - J(x^*, u^*) \geq \sigma\, |v - u^*|^2_{L^2(0,T;U)} \tag{2.16}$$

holds for all $v \in L(0, T; U)$ in a neighborhood of u^*, where $x(v)$ denotes the solution to (2.2) with u replaced by v. Then, if u is contained in this neighborhood for second order sufficient optimality, we have the estimate

$$\begin{aligned} |u - u^*|^2_{L^2(0,T;U)} &\leq \tfrac{C}{\sigma}\left(|x - \hat{x}|_{L^2(0,T;X)} + |x^* - \hat{x}^*|_{L^2(0,T;X)}\right) \\ &\leq \tfrac{\hat{C}}{\sigma}(|q - \Phi(p)|_{L^2(0,T;X)} + |q^* - \Phi(p^*)|_{L^2(0,T;X)}), \end{aligned} \tag{2.17}$$

and thus the optimal controls for the finite and the original optimal control systems are estimated in terms of the gap $q - \Phi(p)$ and $q^* - \Phi(p^*)$.

3. Basic estimates for two-dimensional Navier–Stokes systems.

In this section we analyze the case where the dynamical system (2.2) is given by the Navier–Stokes equations in dimension two. We establish estimates for q and the gap $\Delta = q - \Phi(p)$ following [5]. Our estimate here, however, differs from those in [5] in that we consider the transient situation on the time horizon $[0, T]$, as opposed to the case where t is sufficiently large as in [5]. We then turn to analyzing properties of the mapping Φ and derive error estimates for the reduced-order system.

Let us consider the two-dimensional incompressible Navier–Stokes equations:

$$\begin{aligned} &v_t + (v \cdot \nabla)v + \nabla\rho = \nu\, \Delta v + Bu && \text{for } (t, x) \in (0, T] \times \Omega, \\ &\nabla \cdot v = 0 && \text{in } (0, T] \times \Omega, \\ &v(x, 0) = v_0(x) \text{ in } \Omega, \quad v(t, x) = 0 && \text{for } (t, x) \in (0, T] \times \Gamma, \end{aligned} \tag{3.1}$$

where Ω is a bounded open domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ, $v = v(t, x) \in \mathbb{R}^2$ is the velocity field, $\rho = \rho(t, x) \in \mathbb{R}$ is the pressure, and $\nu > 0$ is the normalized viscosity. For convenience, in order to exclude special cases in the estimate we also assume that $\nu \leq 1$. Let

$$V = \{\phi \in H^1_0(\Omega)^2 : \nabla \cdot \phi = 0\},$$

and let

$$X = \{\phi \in L^2(\Omega) : \nabla \cdot \phi = 0, \;\; n \cdot \phi = 0 \text{ on } \partial\Omega\}$$

denote the closure of $\{\phi \in \mathcal{D}(\Omega) : \nabla \cdot \phi = 0\}$ in $H^1_0(\Omega)^2$ and $L^2(\Omega)^2$, respectively. Let P stand for the orthogonal projection of $L^2(\Omega)^2$ onto the closed subspace X, and let $A = P\Delta$ in X denote the Stokes operator defined by

$$(A\phi, \psi) = -(\nabla\phi, \nabla\psi)_{L^2} \text{ for } \phi,\ \psi \in V.$$

It is a closed self-adjoint operator with $dom(A) = H^2(\Omega)^2 \cap V$ and $dom((-A)^{\frac{1}{2}}) = V$. The operator $(A)^{-1}$ is compact from X to $dom(A)$ and hence the eigenvalue-eigenvector pairs of $-A$ define a complete orthonormal basis of X.

Let $B \in \mathcal{L}(U, X)$, and define $A_0 = \nu A$, and $F : X \to X^*$

$$-F(\phi) = b(\phi, \phi) = P(\phi \cdot \nabla\phi).$$

Then (3.1) can equivalently be expressed in abstract form as (2.2). For results on the Navier–Stokes equations that we utilize here, we refer to [23]. Note that we chose to endow V with $\sqrt{\langle -A_0 v\,, v\rangle}$ as norm in section 2, whereas $\sqrt{\langle -Av\,, v\rangle}$ is the norm for V in this and the following sections in order to specify the dependency of our estimates on $\nu > 0$ as in [5]. We use the following properties [5, 23] of the bilinear form $b(\phi, \psi) = P(\phi \cdot \nabla\psi)$: There exist constants c_i such that for all ϕ, ψ, w in *dom* A

$$|b(\phi,\psi)|_{L^2} \le c_1 \begin{cases} |\phi|^{\frac12}|\phi|_V^{\frac12}, |\psi|_V^{\frac12}|A\psi|^{\frac12}, \\ |\phi|^{\frac12}|A\phi|^{\frac12}|\psi|_V, \end{cases}$$

$$|(b(\phi,\psi), w)| \le c_2\, |\phi|^{\frac12}|\phi|_V^{\frac12}|\psi|_V|w|^{\frac12}|w|_V^{\frac12},$$

$$|\phi|_{L^\infty(\Omega)} \le c_3\, |\phi|_V \left(1 + \log\left(\frac{|A\phi|^2}{\lambda_1|\phi|_V^2}\right)\right)^{\frac12}, \tag{3.2}$$

$$|b(\phi,\psi)|_{L^2} \le c_4 \begin{cases} |\phi|_V|\psi|_V \left(1 + \log\left(\frac{|A\phi|^2}{\lambda_1|\phi|_V^2}\right)\right)^{\frac12}, \\ |\phi|\,|A\psi| \left(1 + \log\left(\frac{|A^{\frac32}\psi|^2}{\lambda_1|A\psi|^2}\right)\right)^{\frac12}, \end{cases}$$

and satisfies

$$(b(\phi,\psi),\psi) = 0 \quad \text{for } \phi, \psi \in V. \tag{3.3}$$

It follows from [23, proof of Theorem III.3.10], that for $x_0 \in V$ and $u \in L^2(0, T; U)$ there exists a unique solution to (2.2),

$$x(t) \in C(0, T; V) \cap L^2(0, T; dom\,(A)),$$

and

$$|x|_{C(0,T;V)\cap L^2(0,T;dom\,(A))} \le C(|x_0|_V, |u|_{L^2(0,T;U)}),$$

for a constant C depending continuously on its arguments. We let

$$M_0 = \sup_{t\in[0,T]} |x(t)| \text{ and } M_1 = \sup_{t\in[0,T]} |x(t)|_V.$$

Throughout the remainder of this paper we assume that the cut off for the subspace X_1 is such that

$$\lambda_{N+1} \ge \left(\frac{4c_2M_1}{\nu}\right)^2. \tag{3.4}$$

Note that

$$|A\phi|^2 \le \lambda_N\, |\phi|_V^2 \text{ for } \phi \in X_1, \text{ and } |\phi|_V^2 \ge \lambda_{N+1}|\phi|^2 \text{ for } \phi \in X_2. \tag{3.5}$$

Taking the inner product of (2.2) with $q(t) \in X_2$ we obtain

$$\frac12 \frac{d}{dt}|q|^2 + \nu\, |q|_V^2 = -(b(p+q, p+q), q) + (P_2Bu, q) \text{ for a.e. } t \in (0, T).$$

From (3.3)

$$(b(p+q,p+q),q) = (b(p,p),q) + (b(q,p),q).$$

Together with (3.2)–(3.5) it follows that

$$|(b(p+q,p+q),q)| \le |(b(p,p),q)| + |(b(q,p),q)| \le c_4 L^{\frac{1}{2}}\,|p|_V^2|q| + c_2\,|q||q|_V|p|_V,$$

at a.e. $t \in (0,T)$, where

$$L = 1 + \log\frac{\lambda_{N+1}}{\lambda_1}. \tag{3.6}$$

Referring to (3.5) once again we obtain

$$\begin{aligned}\frac{d}{dt}|q|^2 + (2\nu - 2c_2\lambda_{N+1}^{-\frac{1}{2}}M_1|)\,|q|_V^2 &\le 2|P_2Bu||q| + 2c_4M_1L^{\frac{1}{2}}\,|p|_V|q|\\ &\le \tfrac{4}{\nu\lambda_{N+1}}(c_4^2LM_1^2|p|_V^2 + |P_2Bu|^2) + \tfrac{\nu}{2}|q|_V^2.\end{aligned}$$

By (3.4) the above inequality implies that

$$\frac{d}{dt}|q|^2 + \frac{3\nu}{2}\,|q|_V^2 \le \frac{\nu}{2}|q|_V^2 + \frac{4}{\nu\lambda_{N+1}}(|P_2Bu|^2 + c_4^2M_1^2L|p|_V^2)$$

and

$$\frac{d}{dt}|q|^2 + \nu\lambda_{N+1}\,|q|^2 \le \frac{4}{\nu\lambda_{N+1}}(|P_2Bu|^2 + c_4^2M_1^2L|x|_V^2).$$

Multiplying this inequality with $e^{\nu\lambda_{N+1}t}$ and integrating over $(0,t)$ implies that

$$|q(t)|^2 \le e^{-\nu\lambda_{N+1}t}|q(0)|^2 + \frac{4}{\nu\lambda_{N+1}}\int_0^t e^{\nu\lambda_{N+1}(s-t)}(|P_2Bu(s)|^2 + c_4^2M_1^4L)\,ds, \tag{3.7}$$

for every $t \in [0,T]$. Consequently,

$$\int_0^T |q|^2\,dt \le \frac{1}{\nu\lambda_{N+1}}|q(0)|^2 + \frac{4}{\nu^2\lambda_{N+1}^2}\int_0^T(|P_2Bu|^2 + c_4^2M_1^4L)\,dt$$

and this implies the existence of a constant $k_1 = k_1(M_1, c_4, T)$ such that

$$\int_0^T |q|^2\,dt \le \frac{1}{\nu^2\lambda_{N+1}^2}(|q(0)|_V^2 + 4|P_2Bu|^2_{L^2(0,T;X)} + k_1\,L). \tag{3.8}$$

Taking the inner product of (2.2) with Aq we find that

$$\frac{1}{2}\frac{d}{dt}|q|_V^2 + \nu\,|Aq|^2 = -(b(p+q,p+q),Aq) + (P_2Bu,Aq).$$

From (3.2)–(3.5)

$$\begin{aligned}&|(b(p+q,p+q),Aq)| + (P_2Bu,Aq)\\ &\le c_4L^{\frac{1}{2}}|p|_V|Aq|(|p|_V + |q|_V) + c_1\,|q|^{\frac{1}{2}}|Aq|^{\frac{3}{2}}(|p|_V + |q|_V) + (P_2Bu,Aq).\end{aligned}$$

Expressing $|Aq|^{\frac{3}{2}}(|p|_V+|q|_V)$ as $\nu^{\frac{3}{4}}|Aq|^{\frac{3}{2}}\frac{1}{\nu^{\frac{3}{4}}}(|p|_V+|q|_V)$ and analogously for the remaining terms which contain $A(q)$, we have

$$\begin{aligned}&|(b(p+q,p+q),Aq)|+(P_2Bu,Aq)\\ &\qquad\le \frac{\nu}{2}|Aq|^2+\frac{1}{\nu}|P_2Bu|+\frac{\tilde{c}_3M_1^2L\,|x|_V^2}{\nu}+\frac{\tilde{c}_1M_0^2|x|_V^4}{\nu^3},\end{aligned}$$

where $\tilde{c}_1,\tilde{c}_3$ are independent of ν,L,x, and u. Hence

$$\frac{d}{dt}|q|_V^2+\nu\,|A\,q|^2\le\frac{2}{\nu}|P_2Bu|^2+\frac{2\tilde{c}_3M_1^2L\,|x|_V^2}{\nu}+\frac{2\tilde{c}_1M_0^2|x|_V^4}{\nu^3},$$

and

$$|q(t)|_V^2\le e^{-\nu\lambda_{N+1}t}|q_0|_V^2+\int_0^t e^{\nu\lambda_{N+1}(s-t)}\left(\frac{2}{\nu}|P_2Bu(s)|+\frac{2\tilde{c}_3M_1^4L}{\nu}+\frac{2\tilde{c}_1M_0^2M_1^4}{\nu^3}\right).$$

Thus there exists a constant $k_2=k_2(M_1)$ such that

$$\int_0^T|q(t)|_V^2\,dt\le\frac{1}{\nu\lambda_{N+1}}\left(|q(0)|_V^2+\frac{2}{\nu}|P_2Bu|_{L^2}^2+\frac{k_2L}{\nu}+\frac{k_2}{\nu^3}\right).\tag{3.9}$$

For this estimate the assumption $\lambda_{N+1}\ge(\frac{4c_2M_1}{\nu})^2$ is not needed. We now summarize these results.

THEOREM 3.1. *Assume that* $x_0\in V$, $u\in L^2(0,T;U)$, *and that* $|x|_{C(0,T;V)}\le M_1$ *and* $\lambda_{N+1}\ge(\frac{4c_2M_1}{\nu})^2$. *Then there exist constants* $k_1=k_1(M_1)$ *and* $k_2=k_2(M_1)$ *such that*

$$\int_0^T|q(t)|^2\,dt\le\delta^2(|q(0)|_V^2+4|P_2Bu|_{L^2(0,T;X)}^2+k_1L)$$

and

$$\int_0^T|q(t)|_V^2\,dt\le\,\delta\left(|q(0)|_V^2+\frac{2}{\nu}|P_2Bu|_{L^2(0,T;X)}^2+\frac{k_2L}{\nu}+\frac{k_2}{\nu^3}\right),$$

where $\delta=\frac{1}{\nu\lambda_{N+1}}$ *and* $L=1+\log\frac{\lambda_{N+1}}{\lambda1}$.

If, moreover, $u\in L^\infty(0,T;U)$ *and* $x_0\in dom(A_0)$, *then for constants* $k_3=k_3(M_1)$ *and* $k_4=k_4(M_1)$

$$|q(t)|^2\le\delta^2(|A_0q(0)|^2+4|P_2Bu|_{L^\infty(0,T;X)}^2+k_3L),$$

and

$$|q(t)|_V^2\le\,\delta\left(|A_0q(0)|^2+\frac{2}{\nu}|P_2Bu|_{L^\infty(0,T;X)}^2+\frac{k_4L}{\nu}+\frac{k_4}{\nu^3}\right),$$

for all $t\in[0,T]$.

The second part of the theorem follows from (3.7) and the equation above (3.9). For the optimal solution u to (2.1)–(2.2), the assumption that $u\in L^\infty(0,T;U)$ is not restrictive as can be seen from the associated optimality system which will be given below.

3.1. Gap estimate. In this subsection we prove an estimate for the gap

$$\Delta(t) = q(t) - \Phi(p(t)) \text{ for } t \in [0,T].$$

Throughout we assume that $x(t) \in C(0,T;V) \cap L^2(0,T;dom\,(A_0))$, and we again set $M_1 = \sup_{t\in[0,T]} |x(t)|_V$. By (2.3) and (2.6) we have

$$-A_0\Delta + b(p,q) + b(q,p) + b(q,q) + \frac{d}{dt}\,q - P_2Bu = 0 \text{ for a.e. } t \in (0,T). \tag{3.10}$$

Using (3.2), (3.3), and (3.6) we find that

$$\begin{aligned} &|b(p,q) + b(q,p) + b(q,q)| \\ &\le (c_4L^{\frac{1}{2}} + c_1)|p|_V|q|_V + \tfrac{c_1}{\sqrt{\lambda_{N+1}}}|q|_V|Aq| \quad \text{ for a.e. } t \in (0,T). \end{aligned} \tag{3.11}$$

Taking the inner product of (2.2) with $\frac{d}{dt}q$,

$$\frac{\nu}{2}\frac{d}{dt}|q|_V^2 + |\frac{d}{dt}q|^2 = -\left(b(p+q,p+q)\,,\frac{d}{dt}q\right) + \left(P_2Bu,\frac{d}{dt}q\right),$$

and therefore

$$\begin{aligned} &\tfrac{\nu}{2}\tfrac{d}{dt}|q|_V^2 + |\tfrac{d}{dt}q|^2 \\ &\le |b(p,q)+b(q,p)+b(q,q)|^2 + |P_2(b(p,p)+Bu)|^2 + \tfrac{1}{2}|\tfrac{d}{dt}q|^2. \end{aligned}$$

This implies that

$$\begin{aligned} &\nu|q(T)|_V^2 + \textstyle\int_0^T |\tfrac{d}{dt}q|^2\,dt \le \nu|q(0)|_V^2 \\ &+2\textstyle\int_0^T |b(p,q)+b(q,p)+b(q,q)|^2\,dt + 2\int_0^T |P_2(b(p,p)+Bu)|^2\,dt \\ &\le \nu|q(0)|_V^2 + 4\textstyle\int_0^T \left((c_4L^{\frac{1}{2}}+c_1)^2M_1^2|q|_V^2 + \tfrac{c_1^2}{\lambda_{N+1}}M_1^2|Aq|^2\right)dt \\ &+2\textstyle\int_0^T |P_2(b(p,p)+Bu)|^2\,dt. \end{aligned} \tag{3.12}$$

From (3.10)–(3.12) there exists a constant $\hat{k}$ such that

$$\begin{aligned} \textstyle\int_0^T |A_0\Delta|^2\,dt \le \hat{k}\big[\textstyle\int_0^T[(c_4L^{\frac{1}{2}}+c_1)^2M_1^2|q|_V^2 + \tfrac{c_1^2}{\lambda_{N+1}}M_1^2|A\,q|^2]\,dt \\ +\nu|q(0)|_V^2 + \textstyle\int_0^T |P_2(b(p,p))|^2\,dt + \int_0^T |P_2Bu|^2\,dt\big]. \end{aligned}$$

Hence using (3.5) and $L \ge 1$ there exists a constant k_5 independent of ν such that

$$\begin{aligned} &\textstyle\int_0^T |A_0\Delta|^2\,dt \le k_5\tfrac{L}{\lambda_{N+1}}[|Aq|^2_{L^2(0,T;X)} \\ &\qquad +\nu|Aq(0)|^2 + |P_2Bu|^2_{L^2(0,T;V)}] + k_5|P_2b(p,p)|^2_{L^2(0,T;X)}. \end{aligned} \tag{3.13}$$

We consider the term $P_2b(p,p)$. Recall that $b(p,p) = P\,(p\cdot\nabla)p$. Since $p \in X_1$ and hence, in particular, $p \in dom(A)$, then we have

$$g := (p\cdot\nabla)p \,\in\, H_0^1(\Omega).$$

Then Pg satisfies

$$P\,g = g - \nabla\tilde{p} \in X,$$

where $\tilde{p} \in H^1(\Omega)/\mathbb{R}$ is given by

$$\begin{cases} \Delta\tilde{p} = \nabla\cdot\, g, \\ \frac{\partial\tilde{p}}{\partial n} = 0. \end{cases}$$

Note that $\tilde{p} \in H^2(\Omega)$ and $|\nabla\tilde{p}|_{H^1} \le c\,|g|_{H^1}$. Hence

$$|Pg|_{H^1} = |g - \nabla\tilde{p}|_{H^1} \le c\,|g|_{H^1}$$

and $Pg \in H^1(\Omega)\cap X$, with $Pg\cdot n = 0$. (But Pg is not in V since $\tau\cdot Pg \neq 0$, in general.) Recall from [6] that $dom((-A)^\alpha) = dom((-\Delta)^\alpha)\cap X$ for $\alpha \in (0,1)$, where Δ denotes the vector Laplacian. We further refer to [4, Chapter 8], for characterizations and properties of the domains of $(-\Delta)^\alpha$. It follows from [4] that for $\epsilon > 0$ there exists C_ϵ such that

$$|Pg|_{dom((-A)^{\frac{1}{4}-\epsilon})} \le C_\epsilon |Pg|_{H^1(\Omega)}$$

and hence we have

$$|P_2 b(p,p)|_X = |P_2 P(p\cdot\nabla)p|_X \le \frac{1}{\lambda_{N+1}^{\frac{1}{4}-\epsilon}}|(-A)^{\frac{1}{4}-\epsilon}P((p\cdot\nabla)p)| \le \frac{cC_\epsilon}{\lambda_{N+1}^{\frac{1}{4}-\epsilon}}|(p\cdot\nabla)p|_{H^1}.$$

Here and below C_ϵ is a constant independent of N and p. By (3.2) we have

$$\begin{aligned} |(p\cdot\nabla)p|_{H^1}^2 &\le |(p\cdot\nabla)p|^2 + |\nabla(p\cdot\nabla)p|^2 \le |(p\cdot\nabla)p|^2 + 2\big(|(p\cdot\nabla)p_{x_1}|^2 \\ &+|(p\cdot\nabla)p_{x_2}|^2 + |(p_{x_1}\cdot\nabla)p|^2 + |(p_{x_2}\cdot\nabla)p|^2\big) \le C|p|_V^2|Ap|^2 L. \end{aligned}$$

Combining these estimates, we find

$$|P_2\,b(p,p)|_X \le \frac{C_\epsilon\,L^{1/2}}{\lambda_{N+1}^{\frac{1}{4}-\epsilon}}|p|_V|Ap| \le \frac{C_\epsilon\,L^{1/2}}{\lambda_{N+1}^{\frac{1}{4}-\epsilon}}M_1|Ap|.$$

This estimate can now be used for the last term in (3.13).

For Neumann boundary conditions $V = \{\phi \in H^1(\Omega)^2 : \nabla\cdot\phi = 0\}$. Since $p \in H^2(\Omega) \subset L^\infty(\Omega)$ $g = (p\cdot\nabla)p \in H^1(\Omega)$ and $Pg = g - \nabla\tilde{p}$, where $\tilde{p} \in H^1(\Omega)/\mathbb{R}$ with

$$\begin{cases} \Delta\tilde{p} = \nabla\cdot\, g, \\ \frac{\partial\tilde{p}}{\partial n} = n\cdot g. \end{cases}$$

Hence $\tilde{p} \in H^2(\Omega)$, $Pg \in H^1(\Omega)$, and $Pg \in V$. In this case, as well as for periodic boundary conditions, $\lambda_{N+1}^{\frac{1}{4}-\epsilon}$ and the above estimates can be replaced by $\lambda_{N+1}^{\frac{1}{2}}$.

We summarize these estimates in the following theorem.

THEOREM 3.2. *If* $x_0 \in dom\,(A_0), P_2Bu \in L^2(0,T;V)$, *and* $\epsilon > 0$, *then there exists a constant* k_7 *such that*

$$\int_0^T |A_0(q-\Phi(p))|^2\,dt \le k_7\,\frac{L}{(\lambda_{N+1})^\alpha},$$

where $\alpha = 1/2-\epsilon$ *in the case of Dirichlet boundary conditions and* $\alpha = 1$ *for Neumann or periodic boundary conditions. Here* k_7 *depends on* ϵ *and also on* $k_5, |Aq|_{L^2(0,T;X)}$, $|Aq(0)|_X, |P_2Bu|_{L^2(0,T;V)}, \nu$.

3.2. Lipschitz continuity of Φ. We derive Lipschitz estimates for

$$e = \Phi(p_1) - \Phi(p_2),$$

in X and V, where $p_1 \in X_1$ and $p_2 \in X_1$ satisfy $|p_1|_V,\ |p_2|_V \le M_1$. From (2.6) and (3.2), (3.5) we have

$$\nu(Ae, e) = -\big(b(p_1, p_1 - p_2) + b(p_1 - p_2, p_2), e\big) \le c_4(|p_1|_V + |p_2|_V)|p_1 - p_2|L^{\frac{1}{2}}|e|_V$$

and hence

$$|\Phi(p_1) - \Phi(p_2)|_X \le k_8\, |p_1 - p_2|_X, \qquad \text{where } k_8 = \frac{2c_4 L^{\frac{1}{2}} M_1}{\nu\sqrt{\lambda_{N+1}}}. \tag{3.14}$$

Note that $k_8 = k_8(N) \to 0$ for $N \to \infty$. We also find

$$\begin{aligned}\nu|e|_V^2 &\le -\big(b(p_1, p_1 - p_2) + b(p_1 - p_2, p_2), e\big)\\ &\le c_4\,(|p_1|_V + |p_2|_V|)\,|p_1 - p_2|_V L^{\frac{1}{2}}|e| \le \tfrac{2}{\sqrt{\lambda_{N+1}}} c_4 M_1 L^{\frac{1}{2}}\ |p_1 - p_2|_V|e|_V,\end{aligned}$$

and hence

$$|\Phi(p_1) - \Phi(p_2)|_V \le k_8\, |p_1 - p_2|_V. \tag{3.15}$$

Lemma 3.1. *Let $|p_1|_V \le M_1, |p_2|_V \le M_1$ and set $k_8 = \frac{2c_4 L^{\frac{1}{2}} M_1}{\nu\,\sqrt{\lambda_{N+1}}}$. Then*

$$|\Phi(p_1) - \Phi(p_2)|_V \le k_8|p_1 - p_2|_V, \text{ and } |\Phi(p_1) - \Phi(p_2)|_X \le k_8|p_1 - p_2|_X.$$

3.3. Error analysis for reduced-order system. We derive error estimates for $z - p$, where z and p are the components of the reduced systems given by (2.8) and the first equation in (2.3). The following a priori estimates will be needed.

Lemma 3.2. *Let $u \in L^2(0, T; U)$ and $x_0 \in V$, and let x denote the solution to* (2.2)*. Then for all N sufficiently large, the solution $\hat{x}$ to* (2.5) *exists. Moreover, there exists a constant M_1, such that $|x(t)|_V \le M_1$ and $|\hat{x}(t)|_V \le M_1$ for all $t \in [0, T]$, and this constant is uniform with respect to (u, x_0) in bounded sets of $L^2(0, T; U) \times V$ and all N sufficiently large.*

An outline for the proof is given in the appendix.

From (2.8) we have

$$\begin{aligned}&\tfrac{d}{dt}(z(t) - p(t)) - \nu\, A(z(t) - p(t))\\ &\quad + P_1 b(\hat{x}(t), \hat{x}(t) - x(t)) + P_1 b(\hat{x}(t) - x(t), x(t)) = 0, \quad z(0) - p(0) = 0,\end{aligned}$$

where

$$x(t) = p(t) + q(t), \quad \hat{x} = z(t) + \Phi(z(t)).$$

Taking the inner product with $z(t) - p(t)$,

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}|z(t) - p(t)|^2 + \nu|z(t) - p(t)|_V^2\\ &\le |(b(\hat{x}, \hat{x} - x), z - p)| + |(b(\hat{x} - x, x), z - p)|\\ &\le |(b(\hat{x}, \Phi(z) - q), z - p)| + |(b(z - p, x), z - p)| + |(b(\Phi(z) - q, x), z - p)|,\end{aligned}$$

where we suppressed the dependence on t on the right-hand side and used

$$(b(\hat{x}(t),\hat{x}(t)-x(t)),z(t)-p(t))=(b(\hat{x}(t),\Phi(z(t))-q(t)),z(t)-p(t)).$$

From (3.2) and the fact that $\big(b(\hat{x},\hat{x}-x),z-p\big)=-\big(b(\hat{x},\hat{x}-x),z-p\big)$, we find that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}|z(t)-p(t)|^2+\nu|z(t)-p(t)|_V^2\\&\quad\le|\hat{x}|_{L^\infty}|z(t)-p(t)|_V\big(|z(t)-p(t)|_X+|q(t)-\Phi(p(t))|_X\big)\\&\quad+M_1\big(c_2|z(t)-p(t)|_X+c_3L^{\frac{1}{2}}(k_8|z(t)-p(t)|_X+|q(t)-\Phi(p(t))|_X)\big)|z(t)-p(t)|_V.\end{aligned}$$

Concerning the term $|\hat{x}(t)|_{L^\infty}$, note that for $z\in V$

$$|A_0\Phi(z)|\le c_4L^{\frac{1}{2}}|z|_V^2,$$

and hence

$$\nu|A\Phi(z)|\le c_4\,L^{\frac{1}{2}}|z|_V^2.$$

Moreover, for every $\epsilon>0$ there exists $c>0$ such that

$$|\phi|_{L^\infty}\le c|A^{\frac{1}{2}+\epsilon}\phi|\text{ for all }\phi\in dom(A).$$

Consequently,

$$|\Phi(z(t))|_{L^\infty}\le c\frac{c_4}{\nu}\lambda_{N+1}^{-\frac{1}{2}+\epsilon}L^{\frac{1}{2}}|z(t)|_V^2\quad\text{ and }\quad|z(t)|_{L^\infty}\le c_3L^{\frac{1}{2}}|z(t)|_V,$$

where (3.2) was used for the second estimate. Since $L=L(N)\sim\log\lambda_{N+1}$, we can assume without loss of generality that

$$|\hat{x}(t)|_{L^\infty}=|z(t)+\Phi(z(t))|_{L^\infty}\le\bar{c}L^{\frac{1}{2}}M_1^2\quad\text{ for }\ t\in[0,T],$$

where $\bar{c}=\max(c_3,\frac{c_4}{\nu})$. Without loss of generality we also assume that $M_1\ge1$ and, as before, that $\nu\le1$. Combining these estimates we find that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}|z(t)-p(t)|^2+\nu|z(t)-p(t)|_V^2\\&\quad\le M_1^2(c_2+2k_8\bar{c}L^{\frac{1}{2}})\,|z(t)-p(t)|\,|z(t)-p(t)|_V+2M_1^2\bar{c}L^{\frac{1}{2}}\,|q(t)\\&\qquad-\Phi(p(t))|\,|z(t)-p(t)|_V\end{aligned}$$

and thus

$$\begin{aligned}&\tfrac{d}{dt}|z(t)-p(t)|^2+\nu\,|z(t)-p(t)|_V^2\\&\quad\le2\frac{(c_2M_1^2+2k_8M_1^2\bar{c}L^{\frac{1}{2}})^2}{\nu}|z(t)-p(t)|^2+\frac{8\bar{c}^2LM_1^4}{\nu}|q(t)-\Phi(p(t)|^2.\end{aligned}$$

Gronwall's lemma now implies the following theorem.

THEOREM 3.3. *Let z and p denote the solutions to* (2.8) *and the first equation in* (2.2)*, with $x_0\in V$. Then for all N sufficiently large*

$$|z(t)-p(t)|^2+\nu\int_0^t|z(s)-p(s)|_V^2\,dt\le\frac{8\bar{c}^2LM_1^4}{\nu}\int_0^te^{c(t-s)}\,|q(s)-\Phi(p(s))|^2\,ds,$$

where $\bar{c} = \max(c_3, \frac{c_4}{\nu})$ *and* $c = 2\frac{(c_2 M_1^2 + 2k_8 M_1^2 \bar{c} L^{\frac{1}{2}})^2}{\nu}$, *and* $t \in [0, T]$.

Considering the definitions of k_8 and $\bar{c}$, we deduce that c behaves like $\frac{L^2}{\lambda_{N+1}}$, where $L(N) = 1 + \log \frac{\lambda_{N+1}}{\lambda_1}$, for $N \to \infty$ and, in particular, is bounded with respect to N for $N \to \infty$.

At the end of section 2 we addressed the question of the performance of the finite dimensional controller applied to the infinite dimensional optimal control problem, measured in term of the difference of the control-costs. For the two-dimensional Navier–Stokes equations we obtain, using the results of this section, the following performance indicator in terms of the cost.

COROLLARY 3.1. *Let* $x_0 \in V$, J *as in section* 2, *and let* u^* *denote an optimal solution to* (2.1)–(2.2) *and* u *an optimal solution to* (2.7)–(2.8), *with dynamics given by the Navier–Stokes equations. Then there exists a constant* $k_9 = k_9(C, \nu, \bar{c}, M_1, c, k_7, k_8)$ *independent of* N, *such that*

$$0 \le J(x, u) - J(x^*, u^*) \le k_9 \, \frac{L}{(\lambda_{N+1})^{1+\frac{\alpha}{2}}},$$

where x *denotes the solution to the infinite dimensional system* (2.2) *controlled by* u.

Proof. Let $x^* = x(u^*)$ and $x = x(u)$ denote the solutions to (2.2) with controls u and u^*, respectively. Then by Lemma 3.2 there exists M_1 such that $x^*, x, \hat{x}(u^*)$, and $\hat{x}(u)$ are all bounded in $C([0, T]; V)$ by M_1. By (2.14)

$$0 \le J(x, u) - J(x^*, u^*) \le C\big(|x^* - \hat{x}^*|_{L^2(0,T;X)} + |x - \hat{x}|_{L^2(0,T;X)}\big).$$

From (2.13) and Lemma 3.1 we have with $\hat{C} = C(1 + k_8)$ that

$$\begin{aligned} J(x, u) - J(x^*, u^*) &\le \hat{C}\big(|p(u_N) - z(u_N)|_{L^2(0,T;X)} + |q(u_N) - \Phi(z(u_N))|_{L^2(0,T;X)} \\ &\quad + |p(u^*) - z(u^*)|_{L^2(0,T;X)} + |q(u^*) - \Phi(z(u^*))|_{L^2(0,T;X)}\big). \end{aligned}$$

In the following estimates k denotes a generic constant depending continuously on the indicated arguments but independent of N. From Theorem 3.3, (3.5), and Theorem 3.2, we have

$$\begin{aligned} &J(x, u) - J(x^*, u^*) \\ &\quad \le k(\hat{C}, \nu, \bar{c}, M_1, c)\sqrt{L}\big(|q(u) - \Phi(z(u))|_{L^2(0,T;X)} + |q(u^*) - \Phi(z(u^*))|_{L^2(0,T;X)}\big) \\ &\quad \le k(\hat{C}, \nu, \bar{c}, M_1, c)\tfrac{\sqrt{L}}{\nu\lambda_{N+1}} \\ &\qquad \big(|A_0(q(u) - \Phi(z(u)))|_{L^2(0,T;X)} + |A_0(q(u^*) - \Phi(z(u^*)))|_{L^2(0,T;X)}\big) \\ &\quad \le k(\hat{C}, \nu, \bar{c}, M_1, c, k_7)\frac{L}{(\lambda_{N+1})^{1+\frac{\alpha}{2}}} \,. \qquad \square \end{aligned}$$

In section 4 these estimates will be carried further to obtain estimates on $u_N - u^*$, where u_N is the solution to (2.7)–(2.8), as $N \to \infty$.

4. Optimality condition.

In this section we discuss a direct approach to obtain necessary and sufficient optimality conditions for (2.1)–(2.2), with the dynamical system given by (3.1). As above, let $V = dom\,((-A)^{\frac{1}{2}})$ and $-\langle A\phi, \phi\rangle = |\phi|_V^2$ for $\phi \in V$. Let (x^*, u^*) be an optimal solution to (2.1)–(2.2) with

$$J(x, u) = \frac{1}{2}\int_0^T (Q(x(t) - \bar{x}(t)), x(t) - \bar{x}(t))\, dt + \frac{\beta}{2}\int_0^T |u(t)|_U^2, \tag{4.1}$$

where Q is symmetric and positive definite on X, $\bar{x} \in L^2(0,T;X)$, and $\beta > 0$. We shall use the following properties:

- Given $(x_0, u) \in X \times L^2(0,T;U)$, there exists a unique weak solution x to (2.2) in $W = L^2(0,T;V) \times H^1(0,T;V^*)$ satisfying $x(0) = x_0$ and
$$\left\langle \frac{d}{dt}x(t), \psi \right\rangle = \langle A_0 x(t) + F(x(t)) + Bu(t), \psi\rangle \text{ for all } \psi \in V.$$
- $F : V \to V^*$ is continuously differentiable and there exists a $c > 0$ such that
$$\begin{aligned}&\langle F(y) - F(x), y - x\rangle \le c\,|y - x||y - x|_V |x|_V\\ &\langle F'(x)y, y\rangle \le c\,|y||y|_V|x|_V\\ &|F(y) - F(x) - F'(x)(y - x)|_{V^*} \le c\,|y - x||y - x|_V\end{aligned}$$
for all $x, y \in V$.

The second item follows from (3.2).

4.1. Necessary optimality condition.

Since the bilinear form a on $V \times V$ defined by
$$t \to a(t, \phi, \psi) = -\langle A_0\phi, \psi\rangle - \langle \phi, F'(x^*(t))\psi\rangle$$
satisfies
$$a(t, \phi, \phi) \ge \nu|\phi|_V^2 - c|x^*(t)|_V|\phi||\phi|_V \ge \frac{\nu}{2}|\phi|_V^2 - \frac{c^2\,|x^*(t)|_V^2}{2\nu}|\phi|^2,$$
with $t \to |x^*(t)|_V^2 \in L^1(0,T)$, there exists a unique solution $\chi^* \in H^1(0,T;V^*) \cap L^2(0,T;V)$ to the adjoint equation
$$-\frac{d}{dt}\chi^*(t) = A_0\chi^*(t) + F'(x^*(t))^*\chi^*(t) + Q(x^*(t) - \bar{x}(t)), \quad \chi^*(T) = 0. \tag{4.2}$$
Recall that for $x \in W$, we have $x \in C(0,T;X)$, and for $x,\ \chi \in W$ it follows that
$$\frac{d}{dt}(x(t), \chi(t))_X = \left\langle \frac{d}{dt}x(t), \chi(t)\right\rangle + \left\langle x(t), \frac{d}{dt}\chi(t)\right\rangle$$
for a.e. $t \in (0,T)$. Denoting by $x \in W$ the weak solution corresponding to $u \in L^2(0,T;X)$, and using $a\,b \le \frac{1}{4}a^2 + b^2$ we have
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x(t) - x^*(t)|^2 + \nu|x - x^*|_V^2 &= \langle F(x) - F(x^*), x - x^*\rangle + \langle B(u - u^*), x - x^*\rangle\\ &\le c\,|x^*|_V|x - x^*|\,|x - x^*|_V + \|B\|_{\mathcal{L}(U,V^*)}|x - x^*|_V|u - u^*|\\ &\le \frac{\nu}{2}|x - x^*|_V^2 + \frac{c^2}{\nu}|x^*|_V^2|x - x^*|^2 + \frac{1}{\nu}\|B\|_{\mathcal{L}(U,V^*)}^2|u - u^*|^2.\end{aligned}$$
By Gronwall's inequality
$$|x(t) - x^*(t)|^2 \le \frac{2}{\nu^2}\|B\|_{\mathcal{L}(U,V^*)}^2\, exp\left(\frac{2}{\nu}c^2\int_0^t |x^*(s)|_V^2\,ds\right)\int_0^t |u(s) - u^*(s)|^2\,ds \tag{4.3}$$

and

$$\int_0^T |x(t)-x(t)|_V^2\,dt \le M\int_0^T |u(t)-u^*(t)|^2\,dt \tag{4.4}$$

with

$$M=\frac{2}{\nu^2}\|B\|^2_{\mathcal{L}(U,V^*)}\left(1+\frac{2}{\nu}c^2\left(\int_0^T |x^*(t)|_V^2\,dt\right)e^{\frac{2}{\nu}c^2\int_0^T |x^*(t)|_V^2\,dt}\right).$$

Note that

$$\begin{aligned}J(u)-J(u^*)&=\beta(u-u^*,u^*)_{L^2(0,T;U)}+(Q(x^*-\bar{x}),x-x^*)_{L^2(0,T;X)}\\&\quad+\frac{1}{2}\left(\beta|u-u^*|^2_{L^2(0,T;U)}+(Q(x-x^*),(x-x^*))_{L^2(0,T;X)}\right),\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\tfrac{d}{dt}(x(t)-x^*(t),\chi^*(t))=\langle F(x)-F(x^*)-F'(x^*)(x-x^*),\chi^*\rangle\\&\quad-(Q(x^*-\bar{x}),x-x^*)+\langle B(u-u^*),\chi^*\rangle.\end{aligned}$$

Integrating this expression on $[0,T]$,

$$\int_0^T (Q(x^*-\bar{x}),x-x^*)\,dt=\int_0^T (B^*\chi^*,u-u^*)+(H(x-x^*,x-x^*),\chi^*)\,dt,$$

where

$$H(x-x^*,x-x^*)=F(x)-F(x^*)-F'(x^*)(x-x^*).$$

Then, from (4.5)

(4.6)

$$\begin{aligned}J(u)-J(u^*)&=(\beta u^*+B^*\chi^*,u-u^*)_{L^2(0,T;U)}+\int_0^T (H(x-x^*,x-x^*),\chi^*)\,dt\\&\quad+\frac{1}{2}\left(\beta|u-u^*|^2_{L^2(0,T;U)}+(Q(x-x^*),(x-x^*))_{L^2(0,T;X)}\right),\end{aligned}$$

where

$$\begin{aligned}&\left|\int_0^T (H(x-x^*,x-x^*),\chi^*)\,dt\right|\\&\quad\le c\left(\int_0^T |x-x^*|^2|x-x^*|_V^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |\chi^*|_V^2\,dt\right)^{\frac{1}{2}}.\end{aligned}\tag{4.7}$$

Let $u=u^*+\tau\,v,\ v\in L^2(0,T;U)$. Then

$$0\le J(u^*+\tau v)-J(u^*)=(\beta u^*+B^*\chi^*,\tau v)_{L^2(0,T;U)}+o(\tau).$$

Dividing by τ and letting τ tend to 0^+ implies that $(\beta u^*+B^*\chi^*,v)\le 0$ for all $v\in L^2(0,T;X)$. Hence the necessary optimality condition is given by

$$\beta u^*+B^*\chi^*=0\quad\text{in } L^2(0,T;U),\tag{4.8}$$

and the optimality system consists of the primal equation (2.2), the adjoint equation (4.2), and the optimality condition (4.8).

4.2. Second order sufficient optimality. Suppose that there exists a constant $\sigma > 0$ such that

$$\begin{aligned}&\frac{1}{2}\left(|u-u^*|^2_{L^2(0,T;U)} + (Q(x-x^*),(x-x^*))_{L^2(0,T;X)}\right)\\ &\qquad + \int_0^T (H(x-x^*,x-x^*),\chi^*)\,dt \geq \sigma\,|u-u^*|^2_{L^2(0,T;U)}\end{aligned} \tag{4.9}$$

in a neighborhood of u^*, where $x = x(u)$ denote the solution to (2.2). Then the second order sufficient optimality condition (2.16) follows. This follows from (4.6) and (4.8) above.

Turning to (4.9), we first note that

$$-\frac{1}{2}\frac{d}{dt}|\chi^*(t)|^2 \leq -\nu|\chi^*(t)|^2_V + c\,|x^*(t)|_V|\chi^*(t)|\,|\chi^*(t)|_V + (Q(x^*(t)-\bar{x}(t)),\chi^*(t)),$$

and hence

$$-\frac{1}{2}\frac{d}{dt}|\chi^*(t)|^2 + \frac{\nu}{2}|\chi^*(t)|^2_V \leq \frac{c^2}{\nu}\,|x^*(t)|^2_V\,|\chi^*(t)|^2 + \frac{1}{\nu}|Q(x^*(t)-\bar{x}(t))|^2_{V^*}.$$

By Gronwall's inequality applied backwards with respect to time and terminal condition $\chi^*(T) = 0$, we find that

$$\int_0^T |\chi^*(t)|^2 \leq \frac{2}{\nu}\int_0^T |Q(x^*-\bar{x})|^2_{V^*}\exp\left(\int_0^T \frac{2c^2}{\nu}|x^*(s)|^2_V\right).$$

Hence there exists a constant $M = M(c, \frac{1}{\nu})$ such that

$$\int_0^T |\chi^*|^2_V \leq M^2\int_0^T |Q(x^*-\bar{x})|^2_{V^*}\,dt. \tag{4.10}$$

By (4.7) and (4.10)

$$\begin{aligned}&\frac{1}{2}\left(|u-u^*|^2_{L^2(U)} + (Q(x-x^*),(x-x^*))_{L^2(X)}\right) + \int_0^T (H(x-x^*,x-x^*),\chi^*)\,dt\\ &\quad \geq \tfrac{1}{2}\left(|u-u^*|^2_{L^2(U)} + (Q(x-x^*),(x-x^*))_{L^2(X)}\right)\\ &\qquad -cM|x-x^*|_{C(X)}\,|x-x^*|_{L^2(V)}\,|Q(\bar{x}-x^*)|_{L^2(V^*)},\end{aligned}$$

where $L^2(U)$ stands for $L^2(0,T;U)$. Let $B(r)$ denote the ball with center 0 and radius r in $L^2(0,T;U)$. Let $r > 0$ and $x_0 \in V$. Then by Lemma 3.2 estimate (4.9) holds for all $u \in B$, provided that

$$|Q(\bar{x}-x^*)|_{L^2(V^*)} \tag{4.11}$$

is sufficiently small.

These results can now be utilized to argue convergence of the reduced optimal controls u to an optimal control u^* of (2.1)–(2.2).

THEOREM 4.1. *Let $x_0 \in V$, J as in (4.1), and let u^* denote an optimal solution to (2.1)–(2.2) and u_N an optimal solution to (2.7)–(2.8), with dynamics given by the Navier–Stokes equations. Then, if $|Q(\bar{x}-x^*)|_{L^2(V^*)}$ is sufficiently small, there exists* *a constant* $k_{11} = k_{11}(C, \sigma, \nu, \bar{c}, M_1, c, k_7, k_8)$, *independent of* N, *such that for all* N *sufficiently large*

$$|u_N - u^*|^2_{L^2(0,T;U)} \le k_{11}\, \frac{L}{(\lambda_{N+1})^{1+\frac{\alpha}{2}}}.$$

Proof. Following the notation already used at the end of section 2, we denote by

$$x(u) \text{ the solution to (2.2) with } u,$$

$$z(u) \text{ the solution to (2.8) with } u.$$

Note that $J(z(u_N) + \Phi(z(u_N)), u_N) \le J(z(0) + \Phi(z(0)), 0)$. By Lemma 3.2 the right-hand side is bounded and hence there exists $r \in \mathbb{R}$ such that $u_N \in B(r)$ for all N. Thus, if (4.11) is chosen such that the second order sufficient optimality condition (4.9) is satisfied, then from (2.16), (2.14), (2.13), and Lemma 3.1 we have with $\hat{C} = \frac{C}{\sigma}(1 + k_8)$ that

$$\begin{aligned} &|u_N - u^*|^2_{L^2(0,T;U)} \le \hat{C}\big(|p(u_N) - z(u_N)|_{L^2(X)} \\ &+ |q(u_N) - \Phi(z(u_N))|_{L^2(X)} + |p(u^*) - z(u^*)|_{L^2(X)} + |q(u^*) - \Phi(z(u^*))|_{L^2(X)}\big), \end{aligned}$$

where again $L^2(X)$ is short for $L^2(0, T; X)$. The proof can now be completed by Theorem 3.3, (3.5), and Theorem 3.2 in the same manner as that of Corollary 3.1. ☐

5. Convergence rate analysis.

In the previous section convergence and a first rate of convergence estimate were obtained on the basis of the second order sufficient optimality condition. In this section a rate estimate for the solutions of the reduced problem to the original problem is derived by means of the optimality systems. For the sake of simplicity we assume throughout this section that J is of the form given in (4.1) with $Q = I$ and that

$$\text{range}\,(B) \subset X_1. \tag{5.1}$$

Further let (x^*, u^*) denote a solution to (2.1)–(2.2). Then, (x^*, u^*) with $x^*(t) = p(t) + q(t)$ satisfies

$$\frac{d}{dt} p(t) = A_0 p(t) + P_1\, F\big(p(t) + \Phi(p(t)) + \Delta(t)\big) + Bu^*(t), \tag{5.2}$$

where

$$\Delta(t) = q(t) - \Phi(p(t)) \in X_2.$$

Let

$$\tilde{F}(z) = P_1 F(z + \Phi(z)) \text{ for } z \in X_1$$

and recall that $\Phi : X_1 \to X_2$ is C_1 regular.

The first order necessary optimality condition for (2.7), (2.8) is given by

$$\left\{ \begin{aligned} &\frac{d}{dt} z(t) = A_0 z(t) + \tilde{F}(z(t)) + Bu(t), \\ &-\frac{d}{dt} \xi(t) = A_0 \xi(t) + \tilde{F}'(z(t))^* \xi(t) + (z(t) - P_1 \bar{x}) + \Phi'(z(t))^* (\Phi(z(t)) - P_2 \bar{x}), \\ &\beta\, u(t) + B^* \xi(t) = 0, \end{aligned} \right. \tag{5.3}$$

where $z(0) = P_1 x_0$, $\xi(T) = 0$, and $\hat{x}(t) = z(t) + \Phi(z(t))$. Observe that $\beta\, u(t) + B^* P_1 \xi(t) = 0$, and

$$\begin{aligned} &\tilde{F}'(z)h = P_1 F'(\hat{x})(h + \Phi'(z)h) = F'_{11}(\hat{x})h + F'_{12}(\hat{x})\Phi'(z)h, \\ &\tilde{F}'(z)^* h = F'_{11}(\hat{x})^* h + \Phi'(z)^* F'_{12}(\hat{x})^* h, \end{aligned} \tag{5.4}$$

for $h \in X_1$. Here $F_{ij} = P_i F P_j$. The adjoint equation split into the X_1 and X_2 components and the necessary optimality condition for (2.1)–(2.2) are given by

$$\begin{aligned} &-\tfrac{d}{dt}\eta = A_0 \eta + F'_{11}(x^*)^* \eta + F'_{21}(x^*)^* \zeta + p(t) - P_1 \bar{x}, \\ &-\tfrac{d}{dt}\zeta = A_0 \zeta + F'_{12}(x^*)^* \eta + F'_{22}(x^*)^* \zeta + q(t) - P_2 \bar{x}, \\ &\beta u^*(t) + B^* \eta(t) = 0, \end{aligned} \tag{5.5}$$

where $\eta(T) = 0$, $\zeta(T) = 0$. Note that for $\hat{x} = z + \Phi(z) \in X_1 + X_2$ we have

$$\begin{aligned} &\Phi'(z)h_1 = -A_0^{-1} F'_{21}(z) h_1, \quad h_1 \in X_1, \\ &\Phi'(z)^* h_2 = -F'_{21}(\hat{x})^* A_0^{-1} h_2, \quad h_2 \in X_2. \end{aligned} \tag{5.6}$$

From the second equation of (5.5)

$$\zeta(t) = -A_0^{-1}(F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x}) - E(t)$$

with

$$E(t) = A_0^{-1} \left(\frac{d}{dt}\zeta + F'_{22}(x^*)^* \zeta \right). \tag{5.7}$$

This quantity E will appear in the final estimate. So let us discuss the effect of the terms which enter into the second equation in (5.5), which is the equation for the fast modes in the adjoint system. The input to the system is $F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x}$ which is small if the effect of the X_2-component of the primal equation and the target state $\bar{x}$, as well as the coupling of the nonlinearity from X_1 to X_2 are small. This results in smallness of the term $\int_0^T |\frac{d}{dt}\zeta + F'_{22}(x^*)^* \zeta|_X^2 \, dt$ is small. Further A_0 acts on the fast components ζ of the adjoint state and thus

$$|E(t)|_X \le \frac{1}{\lambda_{N+1}} \left| \frac{d}{dt}\zeta + F'_{22}(x^*)^* \zeta \right|_X .$$

Inserting E into the first equation of (5.5) we find that

$$-\frac{d}{dt}\eta = A_0 \eta + F'_{11}(x^*)^* \eta - F'_{21}(x^*)^* \big(A_0^{-1}(F'_{12}(x^*)^* \eta + q(t) - P_2 \bar{x}) + E \big) + p(t) - P_1 \bar{x},$$

and thus by the second equations of (5.4) and (5.6)

$$\begin{aligned}
-\frac{d}{dt}(\xi-\eta) &= A_0(\xi-\eta) + \tilde{F}'(z)^*(\xi) - F'_{11}(x^*)^*\eta + z - p + \Phi'(z)^*(\Phi(z) - P_2\bar{x})\\
&\quad +(F'_{21}(x^*)^*A_0^{-1})(F'_{12}(x^*)^*\eta + q - P_2\bar{x}) + F'_{21}(x^*)^*E\\
&= A_0(\xi-\eta) + \tilde{F}'(z)^*(\xi-\eta) + (\tilde{F}'(z)^* - \tilde{F}'(p)^*)\eta + \tilde{F}'(x^*)^*\eta - F'_{11}(x^*)^*\eta + z - p\\
&\quad +\Phi'(z)^*\Phi(z) + F'_{21}(\hat{x})^*A_0^{-1}(q - P_2\bar{x}) + (F'_{21}(x^*)^* - F'_{21}(\hat{x})^*)A_0^{-1}(q - P_2\bar{x})\\
&\quad +F'_{21}(x^*)^*A_0^{-1}F_{12}(x^*)^*\eta - \Phi'(z)^*P_2\bar{x} + F'_{21}(x^*)^*E\\
&= A_0(\xi-\eta) + \tilde{F}'(z)^*(\xi-\eta) + (\tilde{F}'(z)^* - \tilde{F}'(p)^*)\eta + z - p\\
&\quad +\Phi'(z)^*(\Phi(z) - \Phi(p) - \Delta) + (F'_{21}(x^*)^* - F'_{21}(\hat{x})^*)A_0^{-1}(q - P_2\bar{x}) + F'_{21}(x^*)^*E.
\end{aligned} \tag{5.8}$$

Similarly we find

$$\begin{aligned}
\frac{d}{dt}(z-p) &= A_0(z-p) + \tilde{F}'(z)(z-p) + B(u-u^*)\\
&\quad +P_1F(\hat{x}) - P_1F(x^*) - P_1F'(\hat{x})(\hat{x}-x^*)\\
&\quad +F'_{12}(\hat{x})\big(\Phi(z) - \Phi(p) - \Phi'(\hat{x})(z-p) - \Delta\big).
\end{aligned} \tag{5.9}$$

Here we used that

$$\begin{aligned}
P_1F'(\hat{x})(\hat{x}-x^*) &= F'_{11}(\hat{x})(z-p) + F'_{12}(\hat{x})(\Phi(z) - q)\\
&= F'_{11}(\hat{x})(z-p) + F'_{12}(\hat{x})(\Phi(z) - \Phi(p) - \Delta),
\end{aligned}$$

$$\tilde{F}'(z)(z-p) = F'_{11}(\hat{x})(z-p) + F'_{12}(\hat{x})(\Phi'(\hat{x})(z-p)).$$

For the following discussion let $f : L^2(0,T;U) \times L^2(0,T;X_2) \times L^2(0,T;X_2) \to L^2(0,T;U)$ be defined by the forward integration of the control dynamics for $p(t)$, then the backward integration of the adjoint equation for $\xi(t)$, followed by the application of B^*, where $(u,(\Delta,E)) \in L^2(0,T;U) \times L^2(0,T;X_2) \times L^2(0,T;X_2)$ are given. Thus f is the composite $u \to p(u) \to \xi(p) \to B^*\xi = B^*P_1\xi$. Then the optimality system for (2.1)–(2.2) can be expressed as

$$\beta\, u^* + f(u^*,\Delta,E) = 0,$$

and, using (5.8), (5.9) the optimality system for (2.7), (2.8) is

$$\beta\, u + f(u,0,0) = 0.$$

Now, if f is C^1 and

$$\beta\, I + H, \quad H = f_u(u,0,0)$$

is an isomorphism on $L^2(0,T;U)$, then an argument based on the implicit function theorem implies that

$$|u^* - u|_{L^2(0,T;U)} \le k_{12}\,(|\Delta|_{L^2(0,T;X_2)} + |E|_{L^2(0,T;X_2)}),$$

provided that $u = u_N$ is sufficiently close to u^*. Sufficient conditions for convergence of u_N to u^* were already given in Theorem 4.1. We prove such an estimate directly. First, observe that

$$\beta\,(u - u^*, u - u^*) + (B^*(\xi - \eta), u - u^*) = 0. \tag{5.10}$$

From (5.8), (5.9), and (5.4) we find that

$$\begin{aligned}\frac{d}{dt}(z - p, \xi - \eta) &= \langle F(\hat{x}) - F(x^*) - F'(\hat{x})(\hat{x} - x^*), \xi(t) - \eta(t)\rangle \\ &\quad - \langle \tilde{F}'(z) - \tilde{F}'(p)(z - p), \eta\rangle - |z - p|^2 - (\Phi'(z)^*(\Phi(z) - \Phi(p) - \Delta), z - p) \\ &\quad + ((\Phi'(z)^* - \Phi'(p)^*)(q - P_2\bar{x}), z - p) - \langle F_{21}'(x^*)^*E, z - p\rangle + (B^*(\xi - \eta), u - u^*) \\ &\quad + \langle F_{12}'(x)(\Phi(z) - \Phi(p) - \Phi'(\hat{x})(z - p) - \Delta), \xi - \eta\rangle.\end{aligned}$$

Integrating this expression onto $[0, T]$,

$$\begin{aligned}&\int_0^T (B^*(\xi - \eta), u - u^*)\, dt \\ &= \int_0^T \big[|z - p|^2 + \Phi'(z)^*(\Phi(z) - \Phi(p) - \Delta, z - p) \\ &\qquad + (\Phi'(z)^* - (\Phi'(p)^*)(q - P_2\bar{x}), z - p)\big]\, dt \\ &\quad - \int_0^T [\langle F(\hat{x}) - F(x^*) - F'(\hat{x})(\hat{x} - x^*), \xi - \eta\rangle + \langle(\tilde{F}'(z) - \tilde{F}'(p)(z - p), \eta\rangle \\ &\quad + \langle F_{21}'(x^*)^*E, z - p\rangle]\, dt - \textstyle\int_0^T \langle F_{12}'(x)(\Phi(z) - \Phi(p) - \Phi'(z)(z - p) - \Delta), \xi - \eta\rangle\, dt.\end{aligned} \tag{5.11}$$

Let us now assume that the assumptions of Theorem 4.1 are satisfied, and let U denote a neighborhood of u^* in $L^2(0, T; U)$. Let M_1 be a bound for $|\hat{x}|_{C(0,T;V)}$ with $u \in U$, and let N be sufficiently large such that $u = u_N \in U$. Using the properties of F specified at the beginning of section 4, we verify from (5.9) with techniques as in the proof of Theorem 3.3 that for some constant k_1,

$$|z(t) - p(t)|^2 + \int_0^T |z(s) - p(s)|_V^2\, ds \le k_1 \int_0^T (|u(s) - u^*(s)|^2 + L\,|\Delta(s)|^2)\, ds \tag{5.12}$$

for all $t \in [0, T]$ and N sufficiently large. Similarly by (5.8), if $x^* \in L^2(0, T; dom(-A))$, then there exists a constant k_2 such that for all N sufficiently large so that $u \in U$,

$$\begin{aligned}&|\xi(t) - \eta(t)|^2 + \int_0^T |\xi(s) - \eta(s)|_V^2\, ds \\ &\qquad \le k_2 \textstyle\int_0^T \big(|u(s) - u^*(s)|_U^2 + L\,|\Delta(s)|_V^2 + |E(s)|^2\big)\, ds.\end{aligned} \tag{5.13}$$

Applying the last two estimates to (5.11) we find with Lemma 3.1 and (3.2) the existence of a constant k_3 such that

$$\begin{aligned}\int_0^T (B^*(\xi - \eta), u - u^*)\, dt \le k_3 \int_0^T \big(&|u(s) - u^*(s)|_U^2 \\ &+ L\,|\Delta(s)|^2 + |E(s)|^2\big)\, ds,\end{aligned} \tag{5.14}$$

for u sufficiently close to u^*. The detailed estimates are quite similar to those necessary for proving (5.12) and (5.13). Together with (5.10) we obtain the following result.

Theorem 5.1. *Under the assumptions of Theorem* 4.1, *if β is sufficiently large, then there exists a constant k_4 such that*

$$|u_N - u^*|_{L^2(0,T;U)} \le \frac{k_4}{\beta}\left(L^{\frac{1}{2}}\, |\Delta|_{L^2(0,T;X_2)} + |E|_{L^2(0,T;X_2)}\right),$$

where Δ defines the gap and E is defined by (5.7)*.*

Referring to (5.11) let us discuss qualitatively which terms enter into the assumption that β is sufficiently large. These are clearly the first and second terms, without Δ, on the right-hand side of (5.11). The third term is quadratic in $|z-p|_{L^2(0,T;V)}$ multiplied by $|q - P_2\bar{x}|_{L^2(0,T;V)}$, which is small for small residue problems. Similarly, the fifth term is a quadratic multiplied by $|\eta|_{L^2(0,T;V)}$, which in view of (5.5) is again small for small residue problems when p is close to $P_1\bar{x}$. The fourth and seventh terms are of cubic nature. The remaining terms involving Δ and E imply no assumption on size of β.

6. Appendix.

Here we sketch the proof for Lemma 3.2. The estimate for x is well known [23]. We turn to the estimate for $\hat{x}$. Let $f = Bu$ and $r = \Phi(z)$. Then $\hat{x} = z + r$ satisfies

$$\begin{aligned}&\left(\frac{d}{dt}z, \psi_1\right) - \nu(Az,\psi_1) + (b(z+r,z+r),\psi_1) = (f,\psi_1),\\ &\nu\,(Ar,\psi_2) - (b(z,z),\psi_2) = 0\end{aligned}$$

for all $\psi_1 \in X_1$ and $\psi_2 \in X_2$. Letting $\psi_1 = z$ and $\psi_2 = r$, we have

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\,(|z|_V^2 + |r|_V^2) + b(r,r,z) = (f,z), \tag{6.1}$$

where we used

$$(b(z+r,z+r),z) + (b(z,z),r) = (b(r,r),z).$$

Assume that $|z(t)|_V \le M$ on $[0,T]$. The definition of M will be given at the end of the proof. From (3.2)

$$|r|_V \le c_4 L^{\frac{1}{2}} M \frac{1}{\nu\sqrt{\lambda_{N+1}}}|z|_V, \quad |r|_X \le \frac{1}{\sqrt{\lambda_{N+1}}}|r|_V, \quad |r|_X \le c_4 L^{\frac{1}{2}} M \frac{1}{\nu\sqrt{\lambda_{N+1}}}|z|_X. \tag{6.2}$$

Since $|(b(r,r),z)| \le c_2 |r||r|_V|z|_V$,

$$|b(r,r,z)| \le c_2 c_4^2 L M^3 \frac{1}{\nu^2(\lambda_{N+1})^{3/2}}|z|_V^2.$$

We select N so that $c_2 c_4^2 L M^3 \frac{1}{\nu^2(\lambda_{N+1})^{3/2}} \le \frac{\nu}{2}$. Then from (6.1)

$$|z(t)|^2 + \nu \int_0^t |z|_V^2\, ds \le e^t |z(0)|^2 + \int_0^t e^{t-s}|f(s)|^2\, dx \le C, \tag{6.3}$$

for $t \in [0,T]$, where

$$C = e^T |z(0)|^2 + \int_0^T e^{T-s} |f(s)|^2 \, dx. \tag{6.4}$$

Next, letting $\psi_1 = -Az$ and $\psi_2 = -Ar$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|z|_V^2 + \nu\,|Az|^2 - (b(z+r,z+r),Az) = (f,Az), \\ &\nu\,|Ar|^2 - (b(z,z),Ar) = 0. \end{aligned} \tag{6.5}$$

From the second equation

$$\nu\,|Ar| \le c_4 L^{\frac{1}{2}} |z|_V^2.$$

Thus,

$$|(b(z+r,r),Az)| \le |z+r|_{L^\infty} |r|_V |Az| \le \bar{c} c_4 L M^3 \frac{1}{\nu\sqrt{\lambda_{N+1}}} |z|_V |Az|,$$

where, as shown in section 3, we can assume that $\bar{c} = \max c_3, \frac{c_4}{\nu}$ is chosen such that

$$|z(t)+r(t)|_{L^\infty} \le \bar{c} M^2 L^{\frac{1}{2}}.$$

From (3.2) and (6.2) we deduce that

$$|(b(z+r,z),Az)| \le 2c_1 |z|^{\frac{1}{2}} |z|_V |Az|^{\frac{3}{2}},$$

where we assumed

$$c_4 L^{\frac{1}{2}} M \frac{1}{\nu\sqrt{\lambda_{N+1}}} \le 1.$$

From the first equation in (6.5) we obtain

$$\frac{1}{2}\frac{d}{dt}|z|_V^2 + \frac{\nu}{2}\,|Az|^2 \le \left(\frac{2}{\nu} d^2 + \frac{108}{\nu^3} c_1^4 |z|^2 |z|_V^2\right) |z|_V^2 + \frac{2}{\nu}\,|f|^2,$$

where

$$d = c_4^2 L M^3 \frac{1}{\nu^2\sqrt{\lambda_{N+1}}}.$$

We select N such that $d^2 = d(N)^2 \le \frac{\nu}{4}$. Then

$$\frac{1}{2}\frac{d}{dt}|z|_V^2 + \frac{\nu}{2}\,|Az|^2 \le \left(1 + \frac{216}{\nu^3} c_1^4 C |z|_V^2\right) |z|_V^2 + \frac{4}{\nu}\,|f|^2.$$

By Gronwall's lemma we obtain

$$|z(t)|_V^2 + \nu \int_0^t |Az|^2 \, ds \le exp\left(\int_0^T \left(1 + \frac{216}{\nu^3} c_1^4 C |z|_V^2\right) ds\right) \left(|z(0)|_V^2 + \frac{4}{\nu}\int_0^T |f|^2 \, ds\right),$$

and hence

$$|z(t)|_V^2 + \nu \int_0^t |Az|^2\, ds \le \exp\left(T + \frac{216}{\nu^4} c_1^4 C^2\right) \left(|z(0)|_V^2 + \frac{4}{\nu}\int_0^T |f|^2\, ds\right) =: M. \tag{6.6}$$

Note that the estimate for $|z(t)|_X$ by C in (6.3) involves $|z(0)|$ and $|f|_{L^2(0,T;X)} = |B(u)|_{L^2(0,T;X)}$ but is independent of M for all N sufficiently large. Similarly, the estimate for $|z(t)|_V$ in (6.6) involves C, $|z(0)|$, and $|B(u)|_{L^2(0,T;X)}$, but is independent of M for all N sufficiently large. Hence, the theory of ordinary differential equations on the ball with radius M in $V \cap X_1$ allows one to construct a solution to (2.5) with a priori bound for the $z(t)$ component given by M. The uniform bound for the $\Phi(z(t))$ component follows immediately from (6.2).

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