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OPTIMAL CONTROL FOR A STATIONARY MHD SYSTEM IN VELOCITY-CURRENT FORMULATION*

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Abstract. An optimal control problem for the equations governing the stationary problem of magnetohydrodynamics (MHD) is considered. Control mechanisms by external and injected currents and magnetic fields are treated. An optimal control problem is formulated. First order necessary and second order sufficient conditions are developed. An operator splitting scheme for the numerical solution of the MHD state equations is analyzed.

Key words. optimal control, magnetohydrodynamics, necessary optimality conditions, operator splitting

AMS subject classifications. 49J20, 49K20, 76D55, 35Q30, 35Q60

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1. Introduction.

Magnetohydrodynamics, or MHD, deals with the mutual interaction of electrically conducting fluids and magnetic fields. In particular the magnetic field interacts with the current in the fluid by exerting a Lorentz force. This feature renders it phenomenally attractive for exploitation, especially in metallurgical processes. The Lorentz force offers a unique possibility of generating a volume force in the fluid and hence to control its motion in a contactless fashion and without any mechanical interference. Therefore MHD technology is used routinely today by engineers, for instance, to stir molten metals during solidification, dampen their undesired convection-driven flow during casting, filter out impurities, and melt and even levitate metals.

With the present paper, we wish to contribute to the application of the powerful methods from mathematical optimization to compute tailored magnetic fields for MHD flow control. Although this work intends primarily to lay the mathematical foundations of MHD optimal control, we believe we have chosen a problem setup of practical relevance, allowing our results to be directly exploited in numerical methods and applications.

Before we turn to the problem description, let us put our work into perspective. Throughout we always refer to stationary incompressible MHD involving viscous fluids. Instationary problems will require an investigation in their own right, and compressible MHD mostly occurs in the realm of plasma physics, whereas we focus on the engineering aspects of MHD. While a remarkable amount of attention in the past decade was devoted to the analysis of optimal control of the Navier–Stokes equations (see, e.g., [9, 10, 11, 14] and the references therein), we are aware of only a few contributions so far concerning the optimal control of the MHD system [13, 16, 17, 18, 19, 24]. The majority of these papers treat the case of low magnetic Reynolds numbers or use either the velocity-potential or the velocity-magnetic field

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formulation, which both result in the necessity of using artificial boundary conditions. The MHD state equations alone have been investigated in a number of papers, including [22, 21, 12].

We organized the material as follows: In the remainder of this section, we briefly recall the stationary MHD state equations and their velocity-current formulation. In section 2 the variational form of the state equations following [22] are introduced. Our main results are given in section 3, where we consider an optimal control problem for the MHD system. We derive the first order necessary optimality system and establish second order sufficient conditions. In section 4, we analyze an operator splitting scheme for the solution of the MHD state equation which makes use of existing solvers for the Navier–Stokes equations and div-curl systems. We conclude with an outlook on follow-up work in section 5.

Essentially, the MHD system consists of the Navier–Stokes equation with Lorentz force, yielding the fluid velocity $\boldsymbol{u}$ and its pressure p, plus Maxwell's equations describing the interaction of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$.[1] In the stationary case, the complete MHD system is given by

$$\nabla \cdot \boldsymbol{J} = 0 \quad \text{(charge conservation)} \qquad \nabla \times \boldsymbol{E} = 0 \quad \text{(Faraday's law)}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{(No magnetic monopoles)} \qquad \nabla \times (\mu^{-1}\boldsymbol{B}) = \boldsymbol{J} \quad \text{(Ampère's Law)}, \tag{1.2}$$

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \quad \text{(Ohm's law)}, \tag{1.3}$$

together with the Navier–Stokes system with Lorentz force

$$\varrho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \eta\Delta\boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{1.4}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1.5}$$

We refer to [25, 4] for more details. Here μ denotes the magnetic permeability of the matter occupying a certain point in space, and ϱ, η, and σ denote the fluid's density, viscosity, and conductivity. All of these numbers are positive. We emphasize that we consider μ to be constant throughout space; hence we assume a nonmagnetic fluid and no magnetic material present in its relevant vicinity.

It is an outstanding feature in magnetohydrodynamics that, from the set of state variables $(\boldsymbol{u}, p, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{J})$, the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ extend to all of $\mathbb{R}^3$, whereas the velocity $\boldsymbol{u}$ and pressure p are confined to the bounded region $\Omega \subset \mathbb{R}^3$ occupied by the fluid. The current density $\boldsymbol{J}$ is defined within the fluid region and possibly also in external conductors.

Rather than treating the full set of variables $(\boldsymbol{u}, p, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{J})$, researchers often describe MHD systems by a properly chosen subset, which is frequently taken as the pair of primal variables $(\boldsymbol{u}, \boldsymbol{B})$. This entails that either $\boldsymbol{B}$ has to be considered on all of $\mathbb{R}^3$, or that artificial shielding boundary conditions have to be assigned on $\partial\Omega$ so that the coupled system can be considered on the fluid region Ω alone. Physically, shielding boundary conditions represent a fluid being surrounded on all sides by a perfectly conducting vessel. Such boundary conditions exclude the control action

[1] Strictly speaking, $\boldsymbol{B}$ should be called the magnetic induction, while $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$ is the magnetic field. It is, however, common usage in MHD literature to call $\boldsymbol{B}$ the magnetic field.

by means of distant magnetic fields. This is an especially attractive feature of MHD control. In what appear to be the practically more relevant cases, where the outside of the fluid region Ω is finitely conducting or nonconducting, hence permitting control by distant magnetic fields, the proper boundary condition for $\boldsymbol{B}$ is an interface condition requiring $\boldsymbol{B}$ to be continuous across $\partial\Omega$ in both its normal and tangential components, i.e.,

$$[\boldsymbol{B}]_{\partial\Omega} = 0, \tag{1.6}$$

where $[\cdot]_{\partial\Omega}$ denotes the jump of any quantity when going from the interior of Ω to its exterior. As a consequence, $\boldsymbol{B}$ has to be considered on all of $\mathbb{R}^3$.

These shortcomings of the $(\boldsymbol{u}, \boldsymbol{B})$ formulation are not present in the velocity-current formulation in the variables $(\boldsymbol{u}, \boldsymbol{J})$ of the state equation system (1.1)–(1.5) as introduced in [22]. In this formulation, the magnetic field $\boldsymbol{B}$ is eliminated by means of a solution operator $\widetilde{\mathcal{B}}(\boldsymbol{J})$ which solves the div-curl system (1.2) for divergence-free currents $\boldsymbol{J}$ and respects the interface condition (1.6). With the condition that it vanishes at infinity, the solution is unique. Moreover, the irrotational electric field $\boldsymbol{E}$ is replaced with its potential ϕ (unique only up to a constant). In our case of constant permeability μ, the operator $\widetilde{\mathcal{B}}(\boldsymbol{J})$ is given by the Biot–Savart law,

$$\widetilde{\mathcal{B}}(\boldsymbol{J})(\boldsymbol{x}) = -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} \times \boldsymbol{J}(\boldsymbol{y})\, d\boldsymbol{y}. \tag{1.7}$$

Inserting $\boldsymbol{B} = \widetilde{\mathcal{B}}(\boldsymbol{J})$ into (1.1)–(1.5) results in the velocity-current formulation of the stationary MHD system,

$$\varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \eta\Delta\boldsymbol{u} + \nabla p - \boldsymbol{J} \times \widetilde{\mathcal{B}}(\boldsymbol{J}) = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1.8}$$

$$\sigma^{-1}\boldsymbol{J} + \nabla\phi - \boldsymbol{u} \times \widetilde{\mathcal{B}}(\boldsymbol{J}) = 0, \qquad \nabla\cdot\boldsymbol{J} = 0 \tag{1.9}$$

for the unknowns $(\boldsymbol{u}, p, \boldsymbol{J}, \phi)$. Here $\boldsymbol{u}$ and p and the electric potential ϕ are confined to the region Ω occupied by our conducting fluid, while $\boldsymbol{J}$ may additionally extend to external conductors.

In general, the total magnetic field $\boldsymbol{B}$ is a superposition of the induced magnetic field $\widetilde{\mathcal{B}}(\boldsymbol{J})$ and other magnetic fields, for instance, those belonging to permanent magnets or fields generated by *given* electric currents; see (3.7). We note in passing that in the case of weakly conducting fluids, such as, for example, salt water, or more generally, in the case of low magnetic Reynolds numbers $R_m = \mu\sigma u l$ (where u and l are typical velocity and length scales), the magnetic field associated with the induced current is negligible in comparison with an imposed field [4]. Hence, $\widetilde{\mathcal{B}}(\boldsymbol{J})$ can be replaced with a *given* field $\boldsymbol{B}_0$. We refer to [19, 24, 17] concerning the optimal control of weakly conducting fluids and to [15] for control approaches concerning the instationary von Kármán flow for a weakly conducting fluid with given Lorentz force.

2. Function spaces and operators. In this section, we present the proper functional analytic setting for the stationary MHD problem following [22]. Throughout, let Ω denote a bounded domain in $\mathbb{R}^3$ with Lipschitzian boundary, and let $L^2(\Omega)$, $H^1(\Omega)$, and $H^1_0(\Omega)$ and, in general, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Sobolev spaces [1] for $1 < p < \infty$. In addition, for $l = 1, 2$, let $V^l(\mathbb{R}^3)$ stand for the completion of $H^l(\mathbb{R}^3)$ with respect to the seminorm which measures only the lth order derivatives [21]. Furthermore, $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, endowed with the norm

$$\|\phi\|_{H^{1/2}(\partial\Omega)} = \inf \|\Phi\|_{H^1(\Omega)},$$

where the infimum extends over all Φ whose trace coincides with ϕ. The norm duals of $H_0^1(\Omega)$ and $H^{1/2}(\partial\Omega)$ are $H^{-1}(\Omega)$ and $H^{-1/2}(\partial\Omega)$, respectively. The norm dual of $W_0^{1,p}(\Omega)$ is $W^{-1,p'}(\Omega)$, where p' is the dual of p, i.e., $p' = p/(p-1)$. Boldface notation indicates the triple Cartesian product of a space with itself, e.g., $\boldsymbol{L}^2(\Omega) = [L^2(\Omega)]^3$, and the symbol $\boldsymbol{L}^2_{\text{div}}(\Omega)$ denotes the subspace of divergence-free (solenoidal) functions in $\boldsymbol{L}^2(\Omega)$. Finally, we denote by $A^\star$ the adjoint of a bounded linear operator A and by X' the dual of a normed linear space X.

We will consider solutions

$$\begin{aligned} &\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^2_{\text{div}}(\Omega), && \boldsymbol{J} \in \boldsymbol{L}^2_{\text{div}}(\Omega), \\ &p \in L^2(\Omega)/\mathbb{R}, && \phi \in H^1(\Omega)/\mathbb{R}, \end{aligned}$$

which satisfy (1.8)–(1.9) in variational form. In order to obtain the variational formulation, we multiply (1.8)–(1.9) by smooth test functions $(\boldsymbol{v}, q, \boldsymbol{K}, \psi)$, where $\boldsymbol{v}$ has zero Dirichlet boundary values. Integration by parts yields

(2.1)
$$\begin{aligned} \varrho \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} + \eta\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} - \int_\Omega (\nabla\cdot\boldsymbol{v})\,\mathcal{P}(p) - \int_\Omega (\boldsymbol{J}\times\widetilde{\mathcal{B}}(\boldsymbol{J}))\cdot\boldsymbol{v} - \int_\Omega (\nabla\cdot\boldsymbol{u})\,q \\ + \sigma^{-1}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{K} + \int_\Omega \boldsymbol{K}\cdot(\nabla\phi) - \int_\Omega (\boldsymbol{u}\times\widetilde{\mathcal{B}}(\boldsymbol{J}))\cdot\boldsymbol{K} + \int_\Omega \boldsymbol{J}\cdot\nabla\psi = \int_{\partial\Omega} j\,\psi, \end{aligned}$$

where $j = \boldsymbol{J}\cdot\boldsymbol{n}$ denotes the given boundary values in the normal direction for the current $\boldsymbol{J}$. In (2.1),

$$\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} = \sum_{i=1}^3 \int_\Omega \nabla\boldsymbol{u}_i\cdot\nabla\boldsymbol{v}_i,$$

and $\mathcal{P}(p)$ denotes the projection of p on the functions with zero mean,

$$\mathcal{P}(p) = p - \frac{1}{|\Omega|}\int_\Omega p.$$

Based on (2.1), we introduce the bilinear forms

$$\begin{aligned} a_1(\boldsymbol{u},\boldsymbol{v}) &= \eta\int_\Omega (\nabla\boldsymbol{u} : \nabla\boldsymbol{v}), & a_2(\boldsymbol{J},\boldsymbol{K}) &= \sigma^{-1}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{K}, \\ d_1(\boldsymbol{u},p) &= -\int_\Omega (\nabla\cdot\boldsymbol{u})\,\mathcal{P}(p), & d_2(\boldsymbol{J},\phi) &= \int_\Omega \boldsymbol{J}\cdot(\nabla\phi) \end{aligned}$$

and trilinear forms

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \varrho\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}, \qquad c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w},$$

where $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{J},\boldsymbol{K} \in \boldsymbol{L}^2(\Omega)$, $p \in L^2(\Omega)/\mathbb{R}$, and $\phi \in H^1(\Omega)/\mathbb{R}$. Note that c is defined on any $\boldsymbol{L}^{p_1}(\Omega)\times\boldsymbol{L}^{p_2}(\Omega)\times\boldsymbol{L}^{p_3}(\Omega)$, where $1/p_1 + 1/p_2 + 1/p_3 = 1$.

Let us now turn to the forms introduced above. Besides the obvious continuity properties, they satisfy the following.

LEMMA 2.1 (LBB conditions [7]). *The constraint forms d_1 and d_2 satisfy the following Ladyzhenskaya–Babuska–Brezzi (LBB) conditions on $\boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, and on $\boldsymbol{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, respectively:*

$$\inf_{p \in L^2(\Omega)/\mathbb{R}} \sup_{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)} \frac{d_1(\boldsymbol{u}, p)}{\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|p\|_{L^2(\Omega)/\mathbb{R}}} \geq \beta_1,$$
$$\inf_{\phi \in H^1(\Omega)/\mathbb{R}} \sup_{\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)} \frac{d_2(\boldsymbol{J}, \phi)}{\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \|\phi\|_{H^1(\Omega)/\mathbb{R}}} \geq \beta_2$$

for some $\beta_1, \beta_2 > 0$.

Let us define the following spaces associated with the constraint forms d_1 and d_2:

$$\begin{aligned} V_1 &= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,:\, d_1(\boldsymbol{v}, p) = 0 \text{ for all } p \in L^2(\Omega)/\mathbb{R}\}, \\ V_1^0 &= \{\Phi_1 \in \boldsymbol{H}^{-1}(\Omega) \,:\, \langle \Phi_1, \boldsymbol{v} \rangle = 0 \text{ for all } \boldsymbol{v} \in V_1\}, \\ V_2 &= \{\boldsymbol{K} \in \boldsymbol{L}^2(\Omega) \,:\, d_2(\boldsymbol{K}, \phi) = 0 \text{ for all } \phi \in H^1(\Omega)/\mathbb{R}\}, \\ V_2^0 &= \{\Phi_2 \in \boldsymbol{L}^2(\Omega)' \,:\, \langle \Phi_2, \boldsymbol{K} \rangle = 0 \text{ for all } \boldsymbol{K} \in V_2\}. \end{aligned}$$

Note that [26]

$$\begin{aligned} V_1 &= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,:\, \nabla \cdot \boldsymbol{v} = 0 \text{ on } \Omega\}, \\ V_2 &= \{\boldsymbol{K} \in \boldsymbol{L}^2(\Omega) \,:\, \nabla \cdot \boldsymbol{K} = 0 \text{ on } \Omega \text{ and } \boldsymbol{K} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

LEMMA 2.2 (properties of constraint forms). *If $\Phi_1 \in V_1^0$ and $\Phi_2 \in V_2^0$, then the equations*

$$\begin{aligned} d_1(\boldsymbol{v}, p) &= \langle \Phi_1, \boldsymbol{v} \rangle \ \textit{ for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ d_2(\boldsymbol{K}, \phi) &= \langle \Phi_2, \boldsymbol{K} \rangle \ \textit{ for all } \boldsymbol{K} \in \boldsymbol{L}^2(\Omega) \end{aligned}$$

are uniquely solvable for $p \in L^2(\Omega)/\mathbb{R}$ and $\phi \in H^1(\Omega)/\mathbb{R}$, and $\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c_1 \|\Phi_1\|_{\boldsymbol{H}^{-1}(\Omega)}$ and $\|\phi\|_{H^1(\Omega)/\mathbb{R}} \leq c_2 \|\Phi_2\|_{\boldsymbol{L}^2(\Omega)'}$ hold for some $c_1, c_2 > 0$.

Proof. See [7, Chap. I, Lem. 4.1]. ∎

LEMMA 2.3 (passing to the limit in c).

(1) *Let $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{v}^n \to \boldsymbol{v}$ in $\boldsymbol{L}^3(\Omega)$, and $\boldsymbol{w} \in \boldsymbol{L}^6(\Omega)$. Then $c(\boldsymbol{u}^n, \boldsymbol{v}^n, \boldsymbol{w}) \to c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.*

(2) *Let $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{v}^n \to \boldsymbol{v}$ in $\boldsymbol{L}^3(\Omega)$, and $\boldsymbol{w}^n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{L}^6(\Omega)$. Then $c(\boldsymbol{u}, \boldsymbol{v}^n, \boldsymbol{w}^n) \to c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.*

Proof. For the first claim, we use the estimate

$$|c(\boldsymbol{u}^n, \boldsymbol{v}^n, \boldsymbol{w}) - c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq \left| \int_\Omega (\boldsymbol{u}^n \times (\boldsymbol{v}^n - \boldsymbol{v})) \cdot \boldsymbol{w} \right| + \left| \int_\Omega ((\boldsymbol{u}^n - \boldsymbol{u}) \times \boldsymbol{v}) \cdot \boldsymbol{w} \right|.$$

We apply Hölder's inequality to the first term, using the native norms of all three factors involved. It converges to zero since $\|\boldsymbol{v}^n - \boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}$ converges to zero and $\|\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}$ is bounded by assumption (1). Hölder's inequality again shows that $\int_\Omega (\cdot \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is a continuous linear functional on $\boldsymbol{L}^2(\Omega)$ so that also the second term converges to zero. The second claim follows similarly, using the splitting

$$|c(\boldsymbol{u}, \boldsymbol{v}^n, \boldsymbol{w}^n) - c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq \left| \int_\Omega (\boldsymbol{u} \times (\boldsymbol{v}^n - \boldsymbol{v})) \cdot \boldsymbol{w}^n \right| + \left| \int_\Omega (\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{w}^n - \boldsymbol{w}) \right|. \qquad ∎$$

LEMMA 2.4 (properties of $\widetilde{\mathcal{B}}$). *The following properties hold:*

(1) *The Biot–Savart operator* (1.7) *maps any given* $\boldsymbol{J} \in \boldsymbol{L}^2(\mathbb{R}^3)$ *to* $\widetilde{\mathcal{B}}(\boldsymbol{J}) \in \boldsymbol{V}^1(\mathbb{R}^3)$*. The restriction of* $\widetilde{\mathcal{B}}(\boldsymbol{J})$ *to* Ω *lies in* $\boldsymbol{H}^1(\Omega)$*.*
In this sense, the Biot–Savart operator defines a continuous linear map between $\boldsymbol{L}^2(\mathbb{R}^3)$ *and* $\boldsymbol{H}^1(\Omega)$*, i.e.,*

$$\|\widetilde{\mathcal{B}}(\boldsymbol{J})\|_{\boldsymbol{H}^1(\Omega)} \le c_B \|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)}$$

for all $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ *and some* $c_B > 0$*.*

(2) *If* $\boldsymbol{J} \in \boldsymbol{L}^2_{\mathrm{div}}(\Omega)$*, then* $\widetilde{\mathcal{B}}(\boldsymbol{J})$ *is the unique solution of the* div-curl *system* (1.2) *vanishing at infinity.*
The operator $\widetilde{\mathcal{B}}$ *is self-adjoint in* $\boldsymbol{L}^2(\mathbb{R}^3)$*.*

Remark 2.5. Whenever $\boldsymbol{J}$ has compact support, as will be the case in our applications, $\widetilde{\mathcal{B}}(\boldsymbol{J})$ belongs to $\boldsymbol{H}^1(\mathbb{R}^3)$. However, it is sufficient for our purpose that the restriction of $\widetilde{\mathcal{B}}(\boldsymbol{J})$ to Ω is in $\boldsymbol{H}^1(\Omega)$, as guaranteed by the lemma.

Proof of Lemma 2.4. Consider the Newton potential

$$(\mathcal{L}_0\boldsymbol{J})(\boldsymbol{x}) = -\frac{\mu}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{J}(\boldsymbol{y})\,d\boldsymbol{y},$$

which defines an isomorphism from $\boldsymbol{L}^2(\mathbb{R}^3)$ to $\boldsymbol{V}^2(\mathbb{R}^3)$ [21, 3]. One argues that $\widetilde{\mathcal{B}}(\boldsymbol{J}) = \nabla \times \mathcal{L}_0(\boldsymbol{J})$ holds for all $\boldsymbol{J} \in \boldsymbol{L}^2(\mathbb{R}^3)$. This implies that $\boldsymbol{B}(\boldsymbol{J}) \in \boldsymbol{V}^1(\mathbb{R}^3)$.

Since $\boldsymbol{V}^1(\mathbb{R}^3)$ embeds into $\boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ [3], claims (1) and (2) follow. Theorem 2.7 and Remark 2.8(b) in [21] imply that (3) holds. To prove self-adjointness, we multiply $\widetilde{\mathcal{B}}(\boldsymbol{J})$ by a function $\boldsymbol{C} \in \boldsymbol{L}^2(\Omega)$ and integrate over $\mathbb{R}^3$:

$$\begin{aligned}
\int_{\mathbb{R}^3} \boldsymbol{C}\cdot\widetilde{\mathcal{B}}(\boldsymbol{J}) &= -\frac{\mu}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{C}(\boldsymbol{x})\cdot\left(\int_{\mathbb{R}^3}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\times\boldsymbol{J}(\boldsymbol{y})\,d\boldsymbol{y}\right)d\boldsymbol{x}\\
&= -\frac{\mu}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \boldsymbol{C}(\boldsymbol{x})\cdot\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\times\boldsymbol{J}(\boldsymbol{y})\right)d\boldsymbol{x}\,d\boldsymbol{y}\\
&= -\frac{\mu}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{J}(\boldsymbol{y})\cdot\left(\int_{\mathbb{R}^3}\boldsymbol{C}(\boldsymbol{x})\times\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,d\boldsymbol{x}\right)d\boldsymbol{y}.\\
&= -\frac{\mu}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{J}(\boldsymbol{x})\cdot\left(\int_{\mathbb{R}^3}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\times\boldsymbol{C}(\boldsymbol{y})\,d\boldsymbol{y}\right)d\boldsymbol{x}.
\end{aligned}$$

Since the left-hand side by definition equals $\int_{\mathbb{R}^3}\boldsymbol{J}\cdot\widetilde{\mathcal{B}}^\star(\boldsymbol{C})$, we have found $\widetilde{\mathcal{B}}^\star(\boldsymbol{C}) = \widetilde{\mathcal{B}}(\boldsymbol{C})$. ∎

We conclude this section by a compact embedding result whose proof can be found in [7].

LEMMA 2.6 (compact maps and embeddings). *For a bounded domain* $\Omega \subset \mathbb{R}^3$ *with Lipschitz boundary, the embeddings* $H^1(\Omega) \hookrightarrow L^{6-\varepsilon}(\Omega)$ *and* $L^2(\Omega) \hookrightarrow W^{-1,6-\varepsilon}(\Omega)$ *are compact for all* $\varepsilon > 0$*. In addition, the pointwise product map* $(u, v) \mapsto u\,v$ *is continuous from* $H^1(\Omega) \times H^1(\Omega)$ *to* $W^{1,3/2}(\Omega)$*, and the latter embeds compactly into* $L^{3-\varepsilon}(\Omega)$ *for all* $\varepsilon > 0$*.*

3. The optimal control problem. We analyze an optimal control problem for the stationary MHD system motivated by the applications described in [4]. A typical geometry that we have in mind is depicted in Figure 3.1. We assume that the electrically conducting fluid, e.g., a liquid metal, is contained in a vessel occupying

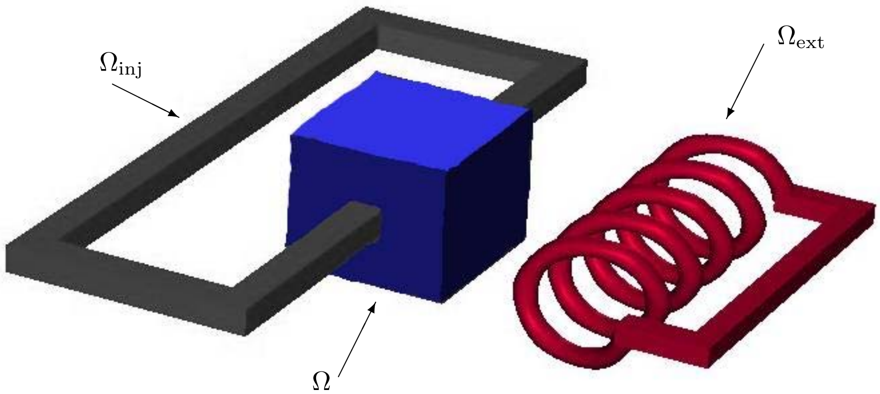

FIG. 3.1. *General setup: Fluid region Ω (cube), external conductor Ω_{inj} attached to the fluid region, and external conductor Ω_{ext} separate from the fluid region.*

the domain Ω. On part of the boundary $\partial\Omega$, an external conductor Ω_{inj} is attached. The current distribution in Ω_{inj} is assumed to be known, but its magnitude can be adjusted. The purpose of such a propulsion device is to drive the fluid in Ω in a desired way, both through the action of the magnetic field induced by the current in Ω_{inj} and through the current, which is "injected" into the fluid region Ω through the electrodes attached to its surface. The same assembly can be found in electromagnetic filtration devices. Since in some cases it may be undesirable to attach the external conductor to the surface of the fluid vessel, we included a second conductor Ω_{ext} separately from the fluid region in which, again, the current distribution is given but its magnitude can be controlled. This external conductor has an impact on the fluid motion in Ω solely through its induced magnetic field. An assembly in which a number of such coils is distributed around the fluid vessel can be found, e.g., in electromagnetic stirring devices [4, 23], albeit their magnetic field is usually amplified by yokes in the coil centers, which are currently not included in our model in view of the assumption that the permeability μ is constant. In addition, we allow for another external magnetic field $\boldsymbol{B}_{\text{ext}}$ which is subject to optimization. In practice such a field cannot be shaped at will. We consider it as originating from a permanent magnet whose field is again known except for its magnitude, which serves as an optimization parameter.

From (1.8)–(1.9) we recall the stationary MHD system in velocity-current formulation,

$$\varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \eta\Delta\boldsymbol{u} + \nabla p = \boldsymbol{J}\times\boldsymbol{B}, \qquad \nabla\cdot\boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{3.1}$$

$$\sigma^{-1}\boldsymbol{J} - \boldsymbol{u}\times\boldsymbol{B} + \nabla\phi = 0, \qquad \nabla\cdot\boldsymbol{J} = 0 \qquad \text{on } \Omega. \tag{3.2}$$

We will consider solutions

$$y = (\boldsymbol{u}, p, \boldsymbol{J}, \phi) \tag{3.3}$$

with

$$\begin{aligned} &\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)\cap\boldsymbol{L}^2_{\text{div}}(\Omega), \qquad && \boldsymbol{J}\in\boldsymbol{L}^2_{\text{div}}(\Omega),\\ &p\in L^2(\Omega)/\mathbb{R}, && \phi\in H^1(\Omega)/\mathbb{R}, \end{aligned}$$

which satisfy (3.1)–(3.2) in variational form, together with the following boundary conditions:

$$\boldsymbol{J} \cdot \boldsymbol{n} = \boldsymbol{J}_{\text{inj}} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega_{\text{inj}} \cap \partial\Omega, \tag{3.4}$$

$$\boldsymbol{J} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \setminus \partial\Omega_{\text{inj}}, \tag{3.5}$$

with the injected current $\boldsymbol{J}_{\text{inj}}$ specified below. For the fluid velocity, we impose Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{h} \quad \text{on } \partial\Omega. \tag{3.6}$$

For our setup the total magnetic field $\boldsymbol{B}$ is the superposition of the field $\widetilde{\mathcal{B}}(\boldsymbol{J})$ induced by the current $\boldsymbol{J}$ inside the fluid domain, the fields $\widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}})$ and $\widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}})$ induced by the currents in the external conductors (whether or not attached to the fluid domain), and the magnetic field $\boldsymbol{B}_{\text{ext}}$ associated with the permanent magnet, i.e.,

$$\boldsymbol{B} = \widetilde{\mathcal{B}}(\boldsymbol{J}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}}. \tag{3.7}$$

We repeat that the external magnetic field $\boldsymbol{B}_{\text{ext}}$ and the current fields $\boldsymbol{J}_{\text{ext}}$ and $\boldsymbol{J}_{\text{inj}}$ are assumed known except for their magnitude. For instance, in the case of a smooth wire, the current field simply follows its shape, and thus

$$\boldsymbol{J}_{\text{ext}} = I_{\text{ext}} \cdot \widetilde{\boldsymbol{J}}_{\text{ext}}, \qquad \boldsymbol{J}_{\text{inj}} = I_{\text{inj}} \cdot \widetilde{\boldsymbol{J}}_{\text{inj}}, \qquad \boldsymbol{B}_{\text{ext}} = B_{\text{ext}} \cdot \widetilde{\boldsymbol{B}}_{\text{ext}}, \tag{3.8}$$

where

$$u = (I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in \mathbb{R}^3 \tag{3.9}$$

is the vector of control variables. Herein, I_{ext} and I_{inj} denote the adjustable scalar current strengths and $\widetilde{\boldsymbol{J}}_{\text{ext}}$ and $\widetilde{\boldsymbol{J}}_{\text{inj}}$ are the given solenoidal current field distributions in the external conductors Ω_{ext} and Ω_{inj}, respectively. In practice, these currents must be maintained by an adjustable voltage source. Likewise, B_{ext} relates to the strength of the external magnetic field (associated with a permanent magnet) $\boldsymbol{B}_{\text{ext}}$. The boundary conditions (3.4)–(3.5), together with Assumption 3.1(3) below, close the current loop and ensure that the total current $\boldsymbol{J} + \boldsymbol{J}_{\text{ext}} + \boldsymbol{J}_{\text{inj}}$ is solenoidal on $\mathbb{R}^3$.

The restriction to finite-dimensional controls is motivated by applications. Non-parametrized distributed current and magnetic fields $(\boldsymbol{J}_{\text{ext}}, \boldsymbol{J}_{\text{inj}}, \boldsymbol{B}_{\text{ext}}) \in \boldsymbol{L}^2_{\text{div}}(\Omega) \times \big(\boldsymbol{H}^1(\Omega_{\text{inj}}) \cap \boldsymbol{L}^2_{\text{div}}(\Omega_{\text{inj}})\big) \times \boldsymbol{V}^1(\mathbb{R}^3)$ can be used as controls if the norms in the objective below are adjusted accordingly. Besides technical difficulties, the most significant change that will occur is in the necessary optimality conditions (3.28) (see Theorem 3.11 below), which will involve Poisson equations.

We are now prepared to state our optimal control problem as follows:

$$\begin{aligned} &\text{Minimize } \frac{\alpha_u}{2}\|\boldsymbol{u} - \boldsymbol{u}_{\text{d}}\|^2_{L^2(\Omega_{u,\text{obs}})} + \frac{\alpha_B}{2}\|\boldsymbol{B} - \boldsymbol{B}_{\text{d}}\|^2_{L^2(\Omega_{B,\text{obs}})} + \frac{\alpha_J}{2}\|\boldsymbol{J} - \boldsymbol{J}_{\text{d}}\|^2_{L^2(\Omega_{J,\text{obs}})} \\ \text{(P)} \qquad &\qquad + \frac{\gamma_{\text{ext}}}{2}|I_{\text{ext}}|^2 + \frac{\gamma_{\text{inj}}}{2}|I_{\text{inj}}|^2 + \frac{\gamma_B}{2}|B_{\text{ext}}|^2 \\ &\text{subject to (3.1)–(3.7).} \end{aligned}$$

The control objective reflects the goal of steering the fluid velocities and the magnetic and current fields towards the given desired fields $\boldsymbol{u}_{\text{d}}$, $\boldsymbol{B}_{\text{d}}$, and $\boldsymbol{J}_{\text{d}}$, possibly only on subdomains $\Omega_{u,\text{obs}} \subset \Omega$, $\Omega_{B,\text{obs}} \subset \mathbb{R}^3$, and $\Omega_{J,\text{obs}} \subset \Omega$ of interest. The desired fields are $\boldsymbol{L}^2$ functions in their respective domains of definition. One may choose among those goals by setting the respective weights α equal to zero. The control weights γ are assumed to be positive. Note that due to the "state times control" terms $\boldsymbol{u} \times \boldsymbol{B}$ (through Ohm's law (1.3)) and $\boldsymbol{J} \times \boldsymbol{B}$ (the Lorentz force), problem (P) is a particular type of bilinear control problem.

We shall require the following assumption for the analysis of the MHD system (3.1)–(3.7).

Assumption 3.1 (problem data).

1. Ω, Ω_{inj}, and Ω_{ext} are bounded mutually disjoint domains with $C^{0,1}$ boundary, such that Ω_{inj} and Ω have a part of their boundary of positive surface measure in common; see Figure 3.1.
2. The boundary velocity $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ satisfies $\int_{\partial\Omega} \boldsymbol{h} \cdot \boldsymbol{n} = 0$.
3. $\widetilde{\boldsymbol{J}}_{\text{ext}}$ and $\widetilde{\boldsymbol{J}}_{\text{inj}}$ are (current) fields in $\boldsymbol{L}^2_{\text{div}}(\Omega_{\text{ext}})$ and $\boldsymbol{H}^1(\Omega_{\text{inj}}) \cap \boldsymbol{L}^2_{\text{div}}(\Omega_{\text{inj}})$, respectively, satisfying $\int_{\partial\Omega \cap \partial\Omega_{\text{inj}}} \widetilde{\boldsymbol{J}}_{\text{inj}} \cdot \boldsymbol{n} = 0$.
4. $\widetilde{\boldsymbol{B}}_{\text{ext}}$ is a divergence-free (magnetic) field on $\mathbb{R}^3$ such that its restriction to Ω lies in $\boldsymbol{L}^{3+\varepsilon}(\Omega)$ for some $\varepsilon > 0$.
5. The nonzero fields $\widetilde{\boldsymbol{B}}_{\text{ext}}$, $\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\text{ext}})$, and $\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\text{inj}})$ are linearly independent.

The assumption $\widetilde{\boldsymbol{J}}_{\text{inj}} \in \boldsymbol{H}^1(\Omega_{\text{inj}})$ implies that the normal trace $\widetilde{\boldsymbol{J}}_{\text{inj}} \cdot \boldsymbol{n}$, when restricted to the intersection $\partial\Omega \cap \partial\Omega_{\text{inj}}$ and extended by zero, yields a function $j \in L^2(\partial\Omega)$; hence in particular $j \in H^{-1/2}(\partial\Omega)$ holds. The latter is needed to ensure the existence of a lifting $\boldsymbol{J}_0$ whose normal boundary values coincide with j, see Lemma 3.2 below. Note that $j \in H^{-1/2}(\partial\Omega)$ can in general not be achieved if merely $\widetilde{\boldsymbol{J}}_{\text{inj}} \in \boldsymbol{L}^2_{\text{div}}(\Omega_{\text{inj}})$.

In order to eliminate the boundary conditions for the velocity and current and to homogenize the problem, we introduce liftings $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$ of the given boundary data such that

$$\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega), \qquad \boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{h}, \qquad \nabla \cdot \boldsymbol{u}_0 = 0,$$

$$\boldsymbol{J}_0 \in \boldsymbol{L}^2(\Omega), \qquad \boldsymbol{J}_0 \cdot \boldsymbol{n}|_{\partial\Omega} = j, \qquad \nabla \cdot \boldsymbol{J}_0 = 0,$$

with

$$j = \boldsymbol{J}_{\text{inj}} \cdot \boldsymbol{n} \text{ on } \partial\Omega \cap \partial\Omega_{\text{inj}} \quad \text{and} \quad j = 0 \text{ on } \partial\Omega \setminus \partial\Omega_{\text{inj}}. \tag{3.10}$$

Such a lifting exists according to the following lemma. Note that Assumption 3.1(3) implies that $\int_{\partial\Omega} j = 0$ as required in part (b) of Lemma 3.2 below.

LEMMA 3.2 (lifting). *Let β_i be the constants from the LBB condition (Lemma 2.1). Then we have the following:*

(a) *For every $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, there exists $\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{h}$ and $d_1(\boldsymbol{u}_0, q) = 0$ holds for all $q \in \boldsymbol{L}^2(\Omega)/\mathbb{R}$, i.e., $\nabla \cdot \boldsymbol{u}_0 = 0$. Moreover, the map $\boldsymbol{h} \mapsto \boldsymbol{u}_0$ can be chosen linearly and continuously, such that*

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} \le (1 + \beta_1^{-1}) \|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\partial\Omega)}$$

is satisfied.

(b) *For every $j \in H^{-1/2}(\partial\Omega)$ which satisfies $\langle j, 1\rangle_{\partial\Omega} = 0$, there exists $\boldsymbol{J}_0 \in \boldsymbol{L}^2(\Omega)$ such that $d_2(\boldsymbol{J}_0, \psi) = \langle j, \psi\rangle_{\partial\Omega}$ holds for all $\psi \in H^1(\Omega)/\mathbb{R}$, i.e., $\nabla \cdot \boldsymbol{J}_0 = 0$ and $\boldsymbol{J}_0 \cdot \boldsymbol{n} = j$. Moreover, the map $j \mapsto \boldsymbol{J}_0$ can be chosen linearly and continuously, such that*

$$\|\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} \le \beta_2^{-1} \|j\|_{H^{-1/2}(\partial\Omega)}$$

is satisfied.

Proof. The claim is a consequence of the LBB condition; see [22] for details. Note that for functions $\boldsymbol{J}_0 \in \boldsymbol{L}^2(\Omega)$ such that $\nabla \cdot \boldsymbol{J}_0 \in \boldsymbol{L}^2(\Omega)$, the normal trace $\boldsymbol{J}_0 \cdot \boldsymbol{n}$ exists with values in $H^{-1/2}(\partial\Omega)$ [7]. ◻

As a consequence, the fluid velocity and current can be written as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \widehat{\boldsymbol{u}}, \qquad \boldsymbol{J} = \boldsymbol{J}_0 + \widehat{\boldsymbol{J}},$$

where $\widehat{\boldsymbol{u}} \in \boldsymbol{H}_0^1(\Omega)$ and $\widehat{\boldsymbol{J}} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. It is important to note that in view of the current boundary condition (3.4)–(3.5), the lifting $\boldsymbol{J}_0$ depends on the control variable I_{inj}. We emphasize this dependence whenever appropriate by writing

$$\boldsymbol{J}_0 = \Lambda(I_{\text{inj}} \widetilde{\boldsymbol{J}}_{\text{inj}} \cdot \boldsymbol{n}), \tag{3.11}$$

meaning that $\boldsymbol{J}_0$ is the lifting, according to Lemma 3.2(b), of the function j in (3.10).

We now consider the homogenized state equation to find

$$\widehat{y} = (\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi). \tag{3.12}$$

In its variational form, the homogenized system is given by

$$\begin{aligned} a_1(\widehat{\boldsymbol{u}} + \boldsymbol{u}_0, \boldsymbol{v}) - c(\widehat{\boldsymbol{J}} + \boldsymbol{J}_0, \boldsymbol{B}, \boldsymbol{v}) + b(\widehat{\boldsymbol{u}} + \boldsymbol{u}_0, \widehat{\boldsymbol{u}} + \boldsymbol{u}_0, \boldsymbol{v}) + d_1(\boldsymbol{v}, p) &= 0, \\ d_1(\widehat{\boldsymbol{u}}, q) &= 0, \\ a_2(\widehat{\boldsymbol{J}} + \boldsymbol{J}_0, \boldsymbol{K}) + c(\boldsymbol{K}, \boldsymbol{B}, \widehat{\boldsymbol{u}} + \boldsymbol{u}_0) + d_2(\boldsymbol{K}, \phi) &= 0, \\ d_2(\widehat{\boldsymbol{J}}, \psi) &= 0 \end{aligned} \tag{3.13}$$

for all $(\boldsymbol{v}, q, \boldsymbol{K}, \psi) \in \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \boldsymbol{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, where we set

$$\boldsymbol{B} = \widetilde{\mathcal{B}}(\Lambda(I_{\text{inj}} \widetilde{\boldsymbol{J}}_{\text{inj}} \cdot \boldsymbol{n})) + \widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}} \tag{3.14}$$

as an abbreviation. The homogeneous solution $\widehat{y}$ is sought in the space

$$\widehat{Y} = \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \boldsymbol{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}. \tag{3.15}$$

In its strong form, (3.13) corresponds to

$$\begin{aligned} -\eta\Delta\widehat{\boldsymbol{u}} + \rho(\widehat{\boldsymbol{u}} \cdot \nabla)\widehat{\boldsymbol{u}} + \rho(\widehat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_0 + \rho(\boldsymbol{u}_0 \cdot \nabla)\widehat{\boldsymbol{u}} + \nabla p& \\ = \eta\Delta\boldsymbol{u}_0 - \rho(\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{J}_0 \times \boldsymbol{B}&, \\ \sigma^{-1}\widehat{\boldsymbol{J}} - \widehat{\boldsymbol{u}} \times \boldsymbol{B} + \nabla\phi = -\sigma^{-1}\boldsymbol{J}_0 + \boldsymbol{u}_0 \times \boldsymbol{B}& \end{aligned}$$

plus the incompressibility conditions $\nabla \cdot \widehat{\boldsymbol{u}} = 0$ and $\nabla \cdot \widehat{\boldsymbol{J}} = 0$ and boundary conditions $\widehat{\boldsymbol{J}} \cdot \boldsymbol{n} = 0$ and $\widehat{\boldsymbol{u}} = 0$ on $\partial\Omega$. We note that the velocity boundary condition is incorporated into the space $\boldsymbol{H}_0^1(\Omega)$, whereas the boundary condition for the current is of variational type.

We now comment on the solvability of (3.13)–(3.14) and thus of the original system (3.1)–(3.7). As was observed in [22] for the MHD system without $\boldsymbol{J}_{\text{ext}}$ and $\boldsymbol{B}_{\text{ext}}$, the existence of a solution seems to be contingent upon the smallness of the liftings $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$, i.e., smallness of the boundary data $\boldsymbol{h}$ and $\boldsymbol{J}_{\text{inj}} \cdot \boldsymbol{n}$. Note that, while the Navier–Stokes nonlinearity $\rho(\widehat{\boldsymbol{u}} \cdot \nabla)\widehat{\boldsymbol{u}}$ is conservative in the sense that $b(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{u}}) = 0$, this is not the case for the bilinear terms $\boldsymbol{J} \times \boldsymbol{B}$ and $\widehat{\boldsymbol{u}} \times \boldsymbol{B}$. The existence proof given in [22] uses the LBB theory [7, Chap. IV, Thm. 1.2], and it is based on a limiting process of Galerkin approximations. Applying this technique to the present situation likewise yields solvability, provided that the data $\boldsymbol{h}$ and $\boldsymbol{J}_{\text{inj}} \cdot \boldsymbol{n}$ are sufficiently small. Under stronger assumptions involving also the remaining controls $\boldsymbol{J}_{\text{ext}}$ and $\boldsymbol{B}_{\text{ext}}$, uniqueness of the solution can be proved using [7, Chap. IV, Thm. 1.3]. In any case, the required bounds on the data seem not exactly tangible since they involve the embedding constants of $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and the constant in the Poincaré inequality, as well as the norms of the lifting operator Λ and the Biot–Savart operator $\widetilde{\mathcal{B}}$. This is the reason that we refrain from stating the exact conditions here.

We provide here an alternative existence proof based on the Leray–Schauder fixed point theorem; see, for instance, [6, p. 222]. Let us define the operator $A : \widehat{Y} \to \widehat{Y}'$ by its components:

$$\begin{aligned}
A^1(\delta\boldsymbol{u}, \delta p, \delta\boldsymbol{J}, \delta\phi)(\boldsymbol{v}) &= a_1(\delta\boldsymbol{u}, \boldsymbol{v}) + d_1(\boldsymbol{v}, \delta p), \\
A^2(\delta\boldsymbol{u}, \delta p, \delta\boldsymbol{J}, \delta\phi)(q) &= d_1(\delta\boldsymbol{u}, q), \\
A^3(\delta\boldsymbol{u}, \delta p, \delta\boldsymbol{J}, \delta\phi)(\boldsymbol{K}) &= a_2(\delta\boldsymbol{J}, \boldsymbol{K}) + d_2(\boldsymbol{K}, \delta\phi), \\
A^4(\delta\boldsymbol{u}, \delta p, \delta\boldsymbol{J}, \delta\phi)(\psi) &= d_2(\delta\boldsymbol{J}, \psi).
\end{aligned}$$

We observe that A is an isomorphism since $(A_1, A_2)(\cdot, \cdot, 0, 0)$ defined between $\boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \{0\} \times \{0\}$ and its dual is an isomorphism, and so is $(A_3, A_4)(0, 0, \cdot, \cdot)$, defined between $\{0\} \times \{0\} \times \boldsymbol{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ and its dual; see [22].

Proposition 3.3 (state equation). *Suppose that Assumption* 3.1 *holds and that* $\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}$ *and* $\|\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)}$ *are sufficiently small. Then the homogenized state equations* (3.13)–(3.14), *and hence the original system* (3.1)–(3.7), *possess at least one variational solution. Every such solution satisfies the a priori bound*

$$\begin{aligned}
\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \le{}& c_1 \|\boldsymbol{J}_0\|^2_{\boldsymbol{L}^2(\Omega)} \left(1 + \|\boldsymbol{J}_0\|^2_{\boldsymbol{L}^2(\Omega)} + |u|^2\right) \\
&+ c_2 \|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} \left(1 + \|\boldsymbol{J}_0\|^2_{\boldsymbol{L}^2(\Omega)} + |u|^2\right).
\end{aligned} \tag{3.16}$$

Moreover, we have the bound

$$\|p\|_{L^2(\Omega)/\mathbb{R}} + \|\phi\|_{H^1(\Omega)/\mathbb{R}} \le c_3 \left(\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}\right). \tag{3.17}$$

Proof. The proof can be found in the appendix. ∎

Remark 3.4. The constants c_1 and c_2 are of the form $\overline{c}_i(1 - \|\boldsymbol{J}_0\| - \|\boldsymbol{u}_0\|)^{-1}$. From the proof we infer that the larger the viscosity η of the fluid is and the smaller the conductivity σ is, the larger the liftings $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$ in Proposition 3.3 are allowed to become, and thus the larger the boundary data $\boldsymbol{h}$ and the control I_{inj} are allowed to become.

The variational form of the state equation (3.13) gives rise to the definition of the PDE constraint operator

$$e : \widehat{Y} \times \mathbb{R}^3 \to \widehat{Y}'. \tag{3.18}$$

The components $e^1(\widehat{y},u)(\boldsymbol{v}),\dots,e^4(\widehat{y},u)(\psi)$ are defined through the left-hand sides of (3.13). This concise form of the MHD system

$$e(\widehat{y},u)=0 \qquad \text{in } \widehat{Y}' \tag{3.19}$$

will be used below to argue existence of the Lagrange multipliers in the optimality system, based on the following results on the linearization of e whose proofs are only given as necessary.

LEMMA 3.5 (linearized PDE constraint). *The operator e is infinitely Fréchet differentiable. Its first order partial derivative with respect to the state variables in the direction of $\delta y=(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)$ is given by*

$$\begin{aligned}
e_y^1(\widehat{y},u)(\delta y)(\boldsymbol{v}) &= a_1(\delta\boldsymbol{u},\boldsymbol{v})-c(\delta\boldsymbol{J},\boldsymbol{B},\boldsymbol{v})-c(\widehat{\boldsymbol{J}}+\Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}),\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\boldsymbol{v})\\
&\qquad +b(\delta\boldsymbol{u},\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\boldsymbol{v})+b(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\delta\boldsymbol{u},\boldsymbol{v})+d_1(\boldsymbol{v},\delta p),\\
e_y^2(\widehat{y},u)(\delta y)(q) &= d_1(\delta\boldsymbol{u},q),\\
e_y^3(\widehat{y},u)(\delta y)(\boldsymbol{K}) &= a_2(\delta\boldsymbol{J},\boldsymbol{K})+c(\boldsymbol{K},\boldsymbol{B},\delta\boldsymbol{u})+c(\boldsymbol{K},\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)+d_2(\boldsymbol{K},\delta\phi),\\
e_y^4(\widehat{y},u)(\delta y)(\psi) &= d_2(\delta\boldsymbol{J},\psi),
\end{aligned} \tag{3.20}$$

where we have set again $\boldsymbol{B}=\widetilde{\mathcal{B}}(\Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))+\widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}})+\widetilde{\mathcal{B}}(\boldsymbol{J}_{\mathrm{ext}})+\widetilde{\mathcal{B}}(\boldsymbol{J}_{\mathrm{inj}})+\boldsymbol{B}_{\mathrm{ext}}$. As for the control variables, the first order derivative in the direction of $\delta u=(\delta I_{\mathrm{ext}},\delta I_{\mathrm{inj}},\delta B_{\mathrm{ext}})$ is

(3.21)

$$\begin{aligned}
e_u^1(\widehat{y},u)(\delta u)(\boldsymbol{v}) &= -c(\delta I_{\mathrm{inj}}\cdot\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}),\boldsymbol{B},\boldsymbol{v})-c(\widehat{\boldsymbol{J}}+\Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}),\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))\\
&\qquad +\delta I_{\mathrm{ext}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}})+\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}})+\delta B_{\mathrm{ext}}\cdot\widetilde{\boldsymbol{B}}_{\mathrm{ext}},\boldsymbol{v}),\\
e_u^2(\widehat{y},u)(\delta u)(q) &= 0,\\
e_u^3(\widehat{y},u)(\delta u)(\boldsymbol{K}) &= a_2(\delta I_{\mathrm{inj}}\cdot\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}),\boldsymbol{K}),\\
&\qquad +c(\boldsymbol{K},\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))+\delta I_{\mathrm{ext}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}})+\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}})+\delta B_{\mathrm{ext}}\cdot\widetilde{\boldsymbol{B}}_{\mathrm{ext}},\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)\\
e_u^4(\widehat{y},u)(\delta u)(\psi) &= 0.
\end{aligned}$$

We note that in its strong form, the system $e_y(\widehat{y},u)(\delta y)=(\bar{e},\bar{f},\bar{g},\bar{h})\in\widehat{Y}'$ corresponds to

$$\begin{aligned}
-\eta\Delta\delta\boldsymbol{u}+\rho(\delta\boldsymbol{u}\cdot\nabla)(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)+\rho((\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)\cdot\nabla)\delta\boldsymbol{u}+\nabla\delta p&\\
=\delta\boldsymbol{J}\times\boldsymbol{B}+(\widehat{\boldsymbol{J}}+\boldsymbol{J}_0)\times\widetilde{\mathcal{B}}(\delta\boldsymbol{J})+\bar{e}&,\\
\nabla\cdot\delta\boldsymbol{u}=\bar{f}&,\\
\sigma^{-1}\delta\boldsymbol{J}-\delta\boldsymbol{u}\times\boldsymbol{B}-(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)\times\widetilde{\mathcal{B}}(\delta\boldsymbol{J})+\nabla\delta\phi=\bar{g}&,\\
\nabla\cdot\boldsymbol{J}=\bar{h}&.
\end{aligned}$$

The first and third components of $e_u(\widehat{y},u)\delta u$ must be read as

$$\begin{aligned}
&-\delta I_{\mathrm{inj}}\cdot\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n})\times\boldsymbol{B}-(\widehat{\boldsymbol{J}}+\Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))\\
&\qquad\times\Big(\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))+\delta I_{\mathrm{ext}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}})+\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}})+\delta B_{\mathrm{ext}}\cdot\widetilde{\boldsymbol{B}}_{\mathrm{ext}}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{-1}\delta I_{\mathrm{inj}}\cdot\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n})-(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)\\
&\qquad\times\Big(\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\Lambda(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n}))+\delta I_{\mathrm{ext}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}})+\delta I_{\mathrm{inj}}\cdot\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}})+\delta B_{\mathrm{ext}}\cdot\widetilde{\boldsymbol{B}}_{\mathrm{ext}}\Big),
\end{aligned}$$

respectively.

In preparation for the following proposition, let us define the operator $C : \widehat{Y} \to \widehat{Y}'$ by its components:

$$C^1(\widehat{y},u)(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(\boldsymbol{v}) = -c(\delta\boldsymbol{J},\boldsymbol{B},\boldsymbol{v}) - c(\widehat{\boldsymbol{J}}+\boldsymbol{J}_0,\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\boldsymbol{v}) + b(\delta\boldsymbol{u},\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\boldsymbol{v}) + b(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\delta\boldsymbol{u},\boldsymbol{v}),$$

$$C^2(\widehat{y},u)(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(q) = 0,$$

$$C^3(\widehat{y},u)(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(\boldsymbol{K}) = c(\boldsymbol{K},\boldsymbol{B},\delta\boldsymbol{u}) + c(\boldsymbol{K},\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\widehat{\boldsymbol{u}}+\boldsymbol{u}_0),$$

$$C^4(\widehat{y},u)(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(\psi) = 0.$$

The quantities $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$ are the liftings according to Lemma 3.2 for given arbitrary but fixed controls u and boundary data $\boldsymbol{h}$.

PROPOSITION 3.6 (linearized state equation). *For any given $\widehat{y} \in \widehat{Y}$, $u \in \mathbb{R}^3$, and $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, the linearization with respect to the state variables of the operator e can be decomposed as*

$$e_y(\widehat{y},u) = A + C(\widehat{y},u), \tag{3.22}$$

where $A : \widehat{Y} \to \widehat{Y}'$ is an isomorphism, independent of $(\widehat{y},u)$, and $C(\widehat{y},u) : \widehat{Y} \to \widehat{Y}'$ is a compact linear operator.

Proof. The isomorphism property of A has been noted previously; see the definition of A immediately preceding Proposition 3.3. As for compactness of C, we recall that $\boldsymbol{B} \in \boldsymbol{L}^{3+\varepsilon}(\Omega)$ (see Lemma 2.4 and Assumption 3.1) and infer from Lemma 2.6 that $\delta\boldsymbol{J} \mapsto \delta\boldsymbol{J} \times \boldsymbol{B}$ is compact from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}^{-1}(\Omega)$. Hence $\delta\boldsymbol{J} \mapsto c(\delta\boldsymbol{J},\boldsymbol{B},\cdot)$ is compact from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}^{-1}(\Omega)$. Similarly, by Lemmas 2.4 and 2.6, $\delta\boldsymbol{J} \mapsto \boldsymbol{J} \times \widetilde{\mathcal{B}}(\delta\boldsymbol{J})$ is compact from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}^{-1}(\Omega)$, and hence $\delta\boldsymbol{J} \mapsto c(\boldsymbol{J},\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\cdot)$ is compact from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}^{-1}(\Omega)$. In addition, $\delta\boldsymbol{u} \mapsto b(\delta\boldsymbol{u},\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\cdot)$ and $\delta\boldsymbol{u} \mapsto b(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\delta\boldsymbol{u},\cdot)$ are continuous from $\boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{L}^{3/2}(\Omega)$, which embeds compactly into $\boldsymbol{W}^{-1,3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ and thus into $\boldsymbol{H}^{-1}(\Omega)$. This completes the proof of compactness for C^1. As for C^3, we let $p = 2(3+\varepsilon)/(1+\varepsilon) < 6$ and observe that $\delta\boldsymbol{u} \mapsto c(\cdot,\boldsymbol{B},\delta\boldsymbol{u})$ is continuous from $\boldsymbol{L}^p(\Omega)$ to $\boldsymbol{L}^2(\Omega)$. In view of the compact embedding $\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, this map is compact from $\boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{L}^2(\Omega)$. Finally, since $\delta\boldsymbol{J} \mapsto \widetilde{\mathcal{B}}(\delta\boldsymbol{J})$ is continuous from $\boldsymbol{L}^2(\Omega)$ to $V^1(\Omega)$, which embeds compactly into $\boldsymbol{L}^3(\Omega)$, the map $\delta\boldsymbol{J} \mapsto c(\cdot,\widetilde{\mathcal{B}}(\delta\boldsymbol{J}),\widehat{\boldsymbol{u}}+\boldsymbol{u}_0)$ is compact from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{L}^2(\Omega)$, which completes the proof. ∎

The previous proposition allows us to draw the following conclusions about the properties of the linearized state operator $e_y(\widehat{y},u)$.

PROPOSITION 3.7 (bounded invertibility of $e_y(\widehat{y},u)$). *Except for a countable set of (η,σ) values, the operator $e_y(\widehat{y},u)$ is an isomorphism. Moreover, $e_y(\widehat{y},u)$ is an isomorphism whenever η is sufficiently large and σ is sufficiently small.*

Proof. For $(\bar{e},\bar{f},\bar{g},\bar{h}) \in \widehat{Y}'$, consider the equation

$$\left(A + C(\widehat{y},u)\right)(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi) = (\bar{e},\bar{f},\bar{g},\bar{h}) \tag{3.23}$$

and define $\mathcal{A} : \widehat{Y} \to \widehat{Y}'$ through its coordinates

$$\begin{aligned}
\mathcal{A}^1(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(\boldsymbol{v}) &= \int_\Omega \nabla\delta\boldsymbol{u} : \nabla\boldsymbol{v} - \int_\Omega (\nabla\cdot\boldsymbol{v})\,\mathcal{P}(\delta p),\\
\mathcal{A}^2 &= A^2,\\
\mathcal{A}^3(\delta\boldsymbol{u},\delta p,\delta\boldsymbol{J},\delta\phi)(\boldsymbol{K}) &= \int_\Omega \delta\boldsymbol{J}\cdot\boldsymbol{K} + \int_\Omega \boldsymbol{K}\cdot\nabla(\delta\phi),\\
\mathcal{A}^4 &= A^4,
\end{aligned}$$

where $(\boldsymbol{v}, \boldsymbol{K}) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$, i.e., $\mathcal{A}$ arises from A by setting $\eta = \sigma = 1$. Multiplying the first two equations in (3.23) by η^{-1} and the last two by σ, we find that the following equation is equivalent to (3.23):

$$(\mathcal{A} + \eta^{-1}\mathcal{C}_1 + \sigma\mathcal{C}_2)(\delta\boldsymbol{u}, \widehat{\delta p}, \delta\boldsymbol{J}, \widehat{\delta\phi}) = (\eta^{-1}\bar{e}, \bar{f}, \sigma\bar{g}, \bar{h}), \tag{3.24}$$

where $\widehat{\delta p} = \eta^{-1}\delta p$ and $\widehat{\delta\phi} = \sigma\delta\phi$ and $\mathcal{C}_1 = (C_1, C_2, 0, 0)^\top$, $\mathcal{C}_2 = (0, 0, C_3, C_4)^\top$. From the proof of Proposition 3.6 we have that $\mathcal{A} : \widehat{Y} \to \widehat{Y}'$ is an isomorphism and that $\mathcal{C}_1$ and $\mathcal{C}_2$ are compact operators from $\widehat{Y}$ to $\widehat{Y}'$. Moreover, (3.24) is equivalent to

$$(I + \eta^{-1}\mathcal{K}_1 + \sigma\mathcal{K}_2)(\delta\boldsymbol{u}, \widehat{\delta p}, \delta\boldsymbol{J}, \widehat{\delta\phi}) = (e, f, g, h), \tag{3.25}$$

where $(e, f, g, h) = \mathcal{A}^{-1}(\eta^{-1}\bar{e}, \bar{f}, \sigma\bar{g}, \bar{h})$, and

$$\mathcal{K}_1 = \mathcal{A}^{-1}\mathcal{C}_1, \qquad \mathcal{K}_2 = \mathcal{A}^{-1}\mathcal{C}_2.$$

Hence, $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact operators in $\widehat{Y}$. Therefore the spectrum of $\mathcal{K}_1$, denoted by $\sum(\mathcal{K}_1)$, consists of 0 and at most countably many eigenvalues, with 0 being the only possible accumulation point. For $-\eta \notin \sum(\mathcal{K}_1)$ we have

$$(I + \sigma(I + \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)(\delta\boldsymbol{u}, \widehat{\delta p}, \delta\boldsymbol{J}, \widehat{\delta\phi}) = (I + \eta^{-1}\mathcal{K}_1)^{-1}(e, f, g, h). \tag{3.26}$$

Since $(I + \eta^{-1}\mathcal{K}_1)$ has a continuous inverse, we find that (3.26), and hence (3.23) are solvable if $-\sigma^{-1} \notin \sum((I + \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)$. Since the set of points $\mathcal{S} := \{(\eta, \sigma^{-1}) : -\eta \in \Sigma(\mathcal{K}_1), -\sigma^{-1} \in \Sigma((I - \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)\}$ is countable in $\mathbb{R}^2$, the first claim follows. The second claim is a consequence of a Neumann series argument applied to (3.25). ∎

From the proof of Proposition 3.7 we conclude that $e_y(\widehat{y}, u)$, and therefore $e_x(\widehat{y}, u)$, is generically surjective. Lack of surjectivity of $e_y(\widehat{y}, u)$ occurs if and only if $(\eta, \sigma^{-1}) \in \mathcal{S}$. In that case, Fredholm's alternative provides another way to prove surjectivity of $e_x(\widehat{y}, u)$. Note that $e_u(\widehat{y}, u)\,\delta u$ can be written in the form

$$e_u(\widehat{y}, u)\,\delta u = \delta I_{\text{inj}}\,\psi_1 + \delta I_{\text{ext}}\,\psi_2 + \delta B_{\text{ext}}\,\psi_3$$

with $\psi_i \in \widehat{Y}'$. In the following proposition, $(A + C)^\star : \widehat{Y}' \to \widehat{Y}$ refers to the Hilbert space adjoint.

PROPOSITION 3.8 (surjectivity of $e_x(\widehat{y}, u)$). *If span* $\{\psi_1, \psi_2, \psi_3\} \supseteq \ker(A + C)^\star$, *then* $e_x(\widehat{y}, u)$ *is surjective.*

Proof. Since $A : \widehat{Y} \to \widehat{Y}'$ is an isomorphism and C is compact, the Fredholm alternative implies that the range of $A + C$ is closed and $R(A + C)^\perp = \ker(A + C)^\star$, with $\dim \ker(A + C)^\star =: L < \infty$. Let $\{\omega_i\}_{i=1}^L \subset \widehat{Y}'$ be a basis for $\ker(A + C)^\star$, orthonormalized such that $\langle \omega_i, \omega_j \rangle_{\widehat{Y}'} = \delta_{ij}$. For arbitrary $\widehat{f} \in \widehat{Y}'$, define $\widetilde{f} = \widehat{f} - \sum_{i=1}^L \langle \widehat{f}, \omega_i \rangle_{\widehat{Y}'} \cdot \omega_i$. Then $\widetilde{f} \in \ker(A + C)^{\star,\perp}$ and there exists $x_{\widetilde{f}}$ such that $(A + C)x_{\widetilde{f}} = \widetilde{f}$. By assumption there exist $(\delta I_{\text{inj}}, \delta I_{\text{ext}}, \delta\boldsymbol{B}_{\text{ext}})$ such that $e_u(\widehat{y}, u)\delta u = \widehat{f} - \widetilde{f}$ and the claim follows. ∎

Having established these properties for the stationary MHD system and its linearization, we now return to the optimal control problem. We recall that we aim to

$$
\text{(P)} \qquad
\begin{aligned}
\text{minimize } f(\widehat{y}, u) = {} & \frac{\alpha_u}{2}\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \frac{\alpha_B}{2}\|\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} \\
& + \frac{\alpha_J}{2}\|\boldsymbol{J} - \boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} + \frac{\gamma_{\mathrm{ext}}}{2}|I_{\mathrm{ext}}|^2 + \frac{\gamma_{\mathrm{inj}}}{2}|I_{\mathrm{inj}}|^2 + \frac{\gamma_B}{2}|B_{\mathrm{ext}}|^2 \\
& \text{over } \widehat{y} \in \widehat{Y} \text{ and } u \in U_{\mathrm{ad}} \\
& \text{subject to } e(\widehat{y}, u) = 0,
\end{aligned}
$$

where $\boldsymbol{u} = \widehat{\boldsymbol{u}} + \boldsymbol{u}_0$ and $\boldsymbol{J} = \widehat{\boldsymbol{J}} + \boldsymbol{J}_0$. $\boldsymbol{B}$ is defined in (3.14), and U_{ad} in Assumption 3.9 below. Recall also that the lifting $\boldsymbol{J}_0$, and hence $\boldsymbol{B}$, depends on the control I_{inj}. In order to ensure well-posedness of problem (P), we need the following assumption on the problem data.

Assumption 3.9 (control problem data). Assume that the boundary data

$$\|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\partial\Omega)}$$

is sufficiently small, and that for some fixed $r > 0$ and all controls in the set

$$U_{\mathrm{ad}} = \{(I_{\mathrm{ext}}, I_{\mathrm{inj}}, B_{\mathrm{ext}}) \in \mathbb{R}^3 \,:\, |I_{\mathrm{inj}}| \le r\},$$

the liftings $\boldsymbol{u}_0$ and $\boldsymbol{J}_0 = \Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}} \cdot \boldsymbol{n})$ allow the existence of a solution to the stationary MHD system.

In fact, smallness of $\boldsymbol{h}$ and I_{inj} implies by Lemma 3.2 smallness of $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$, which in turn implies existence of solutions according to Proposition 3.3.

PROPOSITION 3.10 (existence of a global minimum). *Under Assumptions* 3.1 *and* 3.9, *problem* (P) *possesses at least one global optimal solution in* $\widehat{Y} \times U_{\mathrm{ad}}$.

Proof. The proof follows along the usual lines. We set $m = \inf f(\widehat{y}, u)$, where the infimum extends over all state/control pairs $(\widehat{y}, u) \in \widehat{Y} \times U_{\mathrm{ad}}$ which satisfy the state equation (admissible pairs). Note that m is nonnegative and finite since f is nonnegative and the set of admissible pairs is nonempty (Assumption 3.9 and Proposition 3.3). Now if $\{(\widehat{y}^n, u^n)\}$ is a minimizing sequence, we can infer from the cost functional that the controls u^n are bounded in $\mathbb{R}^3$. By the a priori estimate (3.16), $\widehat{y}^n$ is bounded in $\widehat{Y}$. We extract weakly convergent subsequences, still denoted by index n, such that

$$
\begin{aligned}
&\widehat{\boldsymbol{u}}^n \rightharpoonup \widehat{\boldsymbol{u}} \quad \text{in } \boldsymbol{H}^1_0(\Omega), && p^n \rightharpoonup p \quad \text{in } L^2(\Omega)/\mathbb{R}, \\
&\widehat{\boldsymbol{J}}^n \rightharpoonup \widehat{\boldsymbol{J}} \quad \text{in } \boldsymbol{L}^2(\Omega), && \phi^n \rightharpoonup \phi \quad \text{in } H^1(\Omega)/\mathbb{R}, \\
&u^n \to u \quad \text{in } \mathbb{R}^3.
\end{aligned}
$$

Note that the lifting $\boldsymbol{u}_0$ is independent of n and that $\boldsymbol{J}_0^n = \Lambda(I^n_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}} \cdot \boldsymbol{n})$ converges strongly to some $\boldsymbol{J}_0$ in $\boldsymbol{L}^2(\Omega)$. In order to pass to the limit in $e(\widehat{y}^n, u^n)$, we consider the individual terms in (3.13). For the terms involving the bilinear forms a_i and d_i, the convergence is evident. In addition, $b(\widehat{\boldsymbol{u}}^n, \widehat{\boldsymbol{u}}^n, \boldsymbol{v}) \to b(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{u}}, \boldsymbol{v})$ is known from the theory of Navier–Stokes problems, see [7, Chap. IV, Thm. 2.1]. The convergence of all terms involving the trilinear form c follows from Lemmas 2.3 and 2.6. Consequently, the weak limit $(\widehat{y}, u)$ satisfies the state equation $e(\widehat{y}, u) = 0$, and hence the weak limit $(\widehat{\boldsymbol{u}} + \boldsymbol{u}_0, p, \widehat{\boldsymbol{J}} + \boldsymbol{J}_0, \phi)$ satisfies our inhomogeneous MHD system. The claim now follows from the weak lower semicontinuity of the objective, by which

$$m \le f(\widehat{y}, u) \le \liminf_{n\to\infty} f(\widehat{y}^n, u^n) = m. \qquad \square$$

THEOREM 3.11 (optimality system). *Let Assumptions* 3.1 *and* 3.9 *hold, and let the state* $\widehat{y} = (\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi) \in \widehat{Y}$ *and control* $u = (I_{\mathrm{ext}}, I_{\mathrm{inj}}, B_{\mathrm{ext}}) \in U_{\mathrm{ad}}$ *constitute a local optimal pair for problem* (P). *In addition, let* $e_x(\widehat{y}, u)$ *be surjective. Then there exists a unique Lagrange multiplier*

$$\lambda = (\boldsymbol{v}, q, \boldsymbol{K}, \psi) \in \widehat{Y}$$

which satisfies the adjoint equations

$$\begin{aligned} &a_1(\delta\boldsymbol{u}, \boldsymbol{v}) + b(\delta\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \delta\boldsymbol{u}, \boldsymbol{v}) + d_1(\delta\boldsymbol{u}, q) + c(\boldsymbol{K}, \boldsymbol{B}, \delta\boldsymbol{u}) \\ &\qquad + \alpha_u \int_{\Omega_{u,\mathrm{obs}}} (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}) \cdot \delta\boldsymbol{u} = 0, \end{aligned} \tag{3.27a}$$

$$d_1(\boldsymbol{v}, \delta p) = 0, \tag{3.27b}$$

$$\begin{aligned} &-c(\delta\boldsymbol{J}, \boldsymbol{B}, \boldsymbol{v}) - c(\boldsymbol{J}, \widetilde{\mathcal{B}}(\delta\boldsymbol{J}), \boldsymbol{v}) + a_2(\delta\boldsymbol{J}, \boldsymbol{K}) + c(\boldsymbol{K}, \widetilde{\mathcal{B}}(\delta\boldsymbol{J}), \boldsymbol{u}) \\ &\quad + d_2(\delta\boldsymbol{J}, \psi) + \alpha_B \int_{\Omega_{B,\mathrm{obs}}} (\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}) \cdot \widetilde{\mathcal{B}}(\delta\boldsymbol{J}) + \alpha_J \int_{\Omega_{J,\mathrm{obs}}} (\boldsymbol{J} - \boldsymbol{J}_{\mathrm{d}}) \cdot \delta\boldsymbol{J} = 0, \end{aligned} \tag{3.27c}$$

$$d_2(\boldsymbol{K}, \delta\phi) = 0 \tag{3.27d}$$

for all $(\delta\boldsymbol{u}, \delta p, \delta\boldsymbol{J}, \delta\phi) \in \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \boldsymbol{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, *and which satisfy the three scalar optimality conditions*

(3.28a)
$$- c(\boldsymbol{J}, \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}}), \boldsymbol{v}) + c(\boldsymbol{K}, \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}}), \boldsymbol{u}) + \gamma_{\mathrm{ext}} I_{\mathrm{ext}} + \alpha_B \int_{\Omega_{B,\mathrm{obs}}} (\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}) \cdot \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}}) = 0,$$

(3.28b)
$$\begin{aligned} &\Big[- c(\boldsymbol{J}, \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}), \boldsymbol{v}) + c(\boldsymbol{K}, \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}), \boldsymbol{u}) + \gamma_{\mathrm{inj}} I_{\mathrm{inj}} + \alpha_B \int_{\Omega_{B,\mathrm{obs}}} (\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}) \cdot \widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}) \\ &- \big\langle \widetilde{\boldsymbol{J}}_{\mathrm{inj}} \cdot \boldsymbol{n}, \psi \big\rangle_{\partial\Omega \cap \partial\Omega_{\mathrm{inj}}} \Big] (\overline{I}_{\mathrm{inj}} - I_{\mathrm{inj}}) \ge 0 \quad \textit{for all } |\overline{I}_{\mathrm{inj}}| \le r, \end{aligned}$$

(3.28c)
$$- c(\boldsymbol{J}, \widetilde{\boldsymbol{B}}_{\mathrm{ext}}, \boldsymbol{v}) + c(\boldsymbol{K}, \widetilde{\boldsymbol{B}}_{\mathrm{ext}}, \boldsymbol{u}) + \gamma_{\mathrm{B}} B_{\mathrm{ext}} + \alpha_B \int_{\Omega_{B,\mathrm{obs}}} (\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}) \cdot \widetilde{\boldsymbol{B}}_{\mathrm{ext}} = 0.$$

Proof. Our proof relies on a classical abstract multiplier result; see, e.g., Maurer and Zowe [20]. Since f is Fréchet differentiable, e is continuously Fréchet differentiable, and e_x is assumed surjective at $(\widehat{y}, u)$, it follows that there exists a Lagrange multiplier $\lambda \in \widehat{Y}$, which satisfies

$$\begin{aligned} &f_y(\widehat{y}, u)(\delta y) + \big\langle \lambda, e_y(\widehat{y}, u)(\delta y) \big\rangle = 0 && \text{for all } \delta y \in \widehat{Y}, \\ &f_u(\widehat{y}, u)(\overline{\delta u} - \delta u) + \big\langle \lambda, e_u(\widehat{y}, u)(\overline{\delta u} - \delta u) \big\rangle \ge 0 && \text{for all } \overline{\delta u} \in U_{\mathrm{ad}}. \end{aligned} \tag{3.29}$$

Above, the duality holds in $\widehat{Y} \times \widehat{Y}'$. It is now straightforward to verify that (3.29) is nothing else but (3.27)–(3.28). ∎

The elements of the Lagrange multiplier λ are termed the adjoint velocity $\boldsymbol{v}$, the adjoint pressure q, the adjoint current density $\boldsymbol{K}$, and the adjoint potential ψ, respectively, all defined on Ω.

In order to improve our understanding of the adjoint system (3.27), we also paraphrase it in its strong form. Exploiting the self-adjointness of the linear Biot–Savart operator $\widetilde{\mathcal{B}}$, we obtain the dual state equations on Ω,

$$\varrho(\nabla \boldsymbol{u})^\top \boldsymbol{v} - \varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{v} - \eta\Delta\boldsymbol{v} + \nabla q - \boldsymbol{B}\times\boldsymbol{K} = -\alpha_u \chi_{\Omega_{u,\mathrm{obs}}}(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{d}}), \tag{3.30}$$

$$\begin{aligned} &\sigma^{-1}\boldsymbol{K} - \widetilde{\mathcal{B}}(\boldsymbol{K}\times\boldsymbol{u} + \boldsymbol{v}\times\boldsymbol{J}) + \nabla\psi - \boldsymbol{B}\times\boldsymbol{v} \\ &\qquad = -\alpha_J \chi_{\Omega_{J,\mathrm{obs}}}(\boldsymbol{J}-\boldsymbol{J}_{\mathrm{d}}) - \alpha_B \chi_{\Omega_{B,\mathrm{obs}}\cap\Omega}\widetilde{\mathcal{B}}(\boldsymbol{B}-\boldsymbol{B}_{\mathrm{d}}), \end{aligned} \tag{3.31}$$

plus incompressibility conditions

$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{on } \Omega, \qquad\qquad \nabla\cdot\boldsymbol{K} = 0 \quad \text{on } \Omega \tag{3.32}$$

and boundary conditions

$$\boldsymbol{v} = 0 \quad \text{on } \partial\Omega, \qquad\qquad \boldsymbol{K}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{3.33}$$

The symbol χ_A denotes the characteristic function of a set A. Note that the controls I_{ext}, I_{inj}, and B_{ext} appear (hidden within $\boldsymbol{B}$) in the adjoint equations (3.27a) and (3.27c), or in (3.30)–(3.31) for that matter. The reason is that the state equations (3.1)–(3.2) contain the "state times control" terms $\boldsymbol{u}\times\boldsymbol{B}$ (through Ohm's Law (1.3)) and $\boldsymbol{J}\times\boldsymbol{B}$ (the Lorentz force).

Let us now turn to a condition which ensures the strict local optimality of a given point (y,u,λ) satisfying the first order optimality condition set forth in Theorem 3.11. To this end, we define the Lagrangian function for problem (P) as

$$\begin{aligned} \mathcal{L}(\widehat{y},u,\lambda) &= \frac{\alpha_u}{2}\|\boldsymbol{u}-\boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \frac{\alpha_B}{2}\|\boldsymbol{B}-\boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} + \frac{\alpha_J}{2}\|\boldsymbol{J}-\boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} \\ &+ \frac{\gamma_{\mathrm{ext}}}{2}|I_{\mathrm{ext}}|^2 + \frac{\gamma_{\mathrm{inj}}}{2}|I_{\mathrm{inj}}|^2 + \frac{\gamma_B}{2}|B_{\mathrm{ext}}|^2 + a_1(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\boldsymbol{v}) - c(\widehat{\boldsymbol{J}}+\boldsymbol{J}_0,\boldsymbol{B},\boldsymbol{v}) \\ &+ b(\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\widehat{\boldsymbol{u}}+\boldsymbol{u}_0,\boldsymbol{v}) + d_1(\boldsymbol{v},p) - d_1(\widehat{\boldsymbol{u}},q) + a_2(\widehat{\boldsymbol{J}}+\boldsymbol{J}_0,\boldsymbol{K}) \\ &+ c(\boldsymbol{K},\boldsymbol{B},\widehat{\boldsymbol{u}}+\boldsymbol{u}_0) + d_2(\boldsymbol{K},\phi) - d_2(\widehat{\boldsymbol{J}},\psi) + \left\langle I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n},\psi\right\rangle_{\partial\Omega\cap\partial\Omega_{\mathrm{inj}}}, \end{aligned}$$

where again $\boldsymbol{u} = \widehat{\boldsymbol{u}}+\boldsymbol{u}_0$, $\boldsymbol{J} = \widehat{\boldsymbol{J}} + \Lambda(I_{\mathrm{inj}}\widetilde{\boldsymbol{J}}_{\mathrm{inj}}\cdot\boldsymbol{n})$, and $\boldsymbol{B}$ is defined in (3.14). Moreover, $\boldsymbol{u}_0$ is the fixed lifting of the velocity boundary data $\boldsymbol{h}$ from Lemma 3.2, and $\boldsymbol{B}$ is still taken according to (3.14). It is readily checked that the following quadratic form is the Hessian of $\mathcal{L}$ with respect to the state/control pair:

$$\begin{aligned} \mathcal{L}''(\widehat{y},u,\lambda)[(\delta y,\delta u)]^2 &= \alpha_u\|\delta\boldsymbol{u}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \alpha_J\|\delta\boldsymbol{J}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} + \alpha_B\|\delta\widetilde{\mathcal{B}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} \\ &+ \gamma_{\mathrm{ext}}|\delta I_{\mathrm{ext}}|^2 + \gamma_{\mathrm{inj}}|\delta I_{\mathrm{inj}}|^2 + \gamma_B|\delta B_{\mathrm{ext}}|^2 + 2b(\delta\boldsymbol{u},\delta\boldsymbol{u},\boldsymbol{v}) \\ &- 2c(\delta\boldsymbol{J},\delta\widetilde{\mathcal{B}},\boldsymbol{v}) + 2c(\boldsymbol{K},\delta\widetilde{\mathcal{B}},\delta\boldsymbol{u}) \end{aligned}$$

with the abbreviation

$$\delta\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}(\delta\boldsymbol{J}) + \delta I_{\mathrm{inj}}\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{inj}}) + \delta I_{\mathrm{ext}}\widetilde{\mathcal{B}}(\widetilde{\boldsymbol{J}}_{\mathrm{ext}}) + \delta B_{\mathrm{ext}}\widetilde{\boldsymbol{B}}_{\mathrm{ext}}.$$

PROPOSITION 3.12 (second order sufficient conditions). *Suppose that* $(\widehat{y},u,\lambda)$ *satisfies the optimality system consisting of* (3.13)–(3.14), (3.27)–(3.28), *and that* $e_y(\widehat{y},u)$ *is boundedly invertible. If, moreover,*

$$\alpha_u\|\boldsymbol{u}-\boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \alpha_B\|\boldsymbol{B}-\boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} + \alpha_J\|\boldsymbol{J}-\boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})}$$

is sufficiently small, then there exists a neighborhood $\mathcal{U}$ *of* $(\widehat{y}, u)$ *and* $\kappa > 0$ *such that*

$$f(\overline{y}, \overline{u}) \geq f(\widehat{y}, u) + \kappa \left(\|\overline{u} - u\|^2 + \|\overline{y} - \widehat{y}\|^2_{\widehat{Y}} \right)$$

holds for all $(\overline{y}, \overline{u}) \in \mathcal{U}$ *satisfying the state equation. In particular,* $(\widehat{y}, u)$ *is a strict local optimum for* (P).

Proof. We shall argue that there exists $\rho > 0$ such that the coercivity condition

$$\mathcal{L}''(\widehat{y}, u, \lambda)[(\delta y, \delta u)]^2 \geq \rho \left(\|\delta y\|^2_{\widehat{Y}} + |\delta u|^2 \right) \tag{3.34}$$

holds for all $(\delta y, \delta u) \in \widehat{Y} \times \mathbb{R}^3$, which satisfy the linear MHD system (see Lemma 3.5)

$$e_y(\widehat{y}, u)\, \delta y + e_u(\widehat{y}, u)\, \delta u = 0. \tag{3.35}$$

The claim then follows from a Taylor series expansion of $\mathcal{L}$ at $(\widehat{y}, u, \lambda)$; see, e.g., [20, Thm. 5.6]. In fact, since $\|\delta p\|_{L^2(\Omega)/\mathbb{R}} + \|\delta \phi\|_{H^1(\Omega)/\mathbb{R}} \leq c\, (\|\delta \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\delta \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)})$ holds (cf. Lemma 2.2), we need to verify (3.34) only for the components $\delta \boldsymbol{u}$ and $\delta \boldsymbol{J}$ of δy.

Since $e_y(\widehat{y}, u)$ is surjective, $e_y^\star(\widehat{y}, u) : \widehat{Y} \to \widehat{Y}'$ has closed range and is continuously invertible on its range [2]. Hence in view of (3.30)–(3.33), there exists $\kappa_1 > 0$ such that

$$\begin{aligned} &\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{K}\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \leq \kappa_1 \left(\alpha_u \|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \alpha_B \|\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} + \alpha_J \|\boldsymbol{J} - \boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} \right) \end{aligned}$$

holds. From (3.35) and the bounded invertibility of $e_y(\widehat{y}, u)$, we have

$$\|\delta \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq 2 \|\delta y\|_{\widehat{Y}} \leq \kappa_2 |\delta u| \tag{3.36}$$

for a constant $\kappa_2 > 0$ independent of $\delta u \in \mathbb{R}^3$. Hence

(3.37)
$$\begin{aligned} &|b(\delta \boldsymbol{u}, \delta \boldsymbol{u}, \boldsymbol{v})| \leq \kappa_3 |\delta u|^2 \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq \kappa_1 \kappa_3 |\delta u|^2 \left(\alpha_u \|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \alpha_B \|\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} + \alpha_J \|\boldsymbol{J} - \boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} \right). \end{aligned}$$

Further there exists κ_4 independent of δu such that

$$\begin{aligned} &|c(\delta \boldsymbol{J}, \delta \widetilde{\mathcal{B}}, \boldsymbol{v})| + |c(\boldsymbol{K}, \delta \widetilde{\mathcal{B}}, \delta \boldsymbol{u})| \\ &\quad \leq \kappa_4 \left(\|\delta \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} + |\delta u| \right) \left(\|\delta \boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{K}\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \right) \\ &\quad \leq C \left(\alpha_u \|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}\|^2_{L^2(\Omega_{u,\mathrm{obs}})} + \alpha_B \|\boldsymbol{B} - \boldsymbol{B}_{\mathrm{d}}\|^2_{L^2(\Omega_{B,\mathrm{obs}})} + \alpha_J \|\boldsymbol{J} - \boldsymbol{J}_{\mathrm{d}}\|^2_{L^2(\Omega_{J,\mathrm{obs}})} \right), \end{aligned}$$

where $C = \kappa_1 \kappa_2 (\kappa_2 + 1) \kappa_4 |\delta u|^2$. This last estimate, together with (3.36) and (3.37), implies (3.34). ∎

4. An operator splitting scheme. In this section, we address an operator splitting scheme for the numerical solution of the MHD state equations (3.13)–(3.14). Our approach is based on the hypothesis that one wants to decouple the system and put to use existing and validated solvers for the Navier–Stokes equations and for div-curl systems. The same idea can be applied to the adjoint system. Different iterative schemes for MHD have been proposed; see, for instance [22, 5]. In [5] two decoupling algorithms for the velocity-magnetic field formulation are discussed and numerical examples are given. This work contains a stability result for a special geometry but no convergence analysis. The classical proofs (see, e.g., [8]) for general purpose decomposition techniques rely on monotonicity of the differential operators, not only in each variable separately, but also jointly in all variables simultaneously, to obtain unconditional convergence with respect to the problem data. This behavior is not possible for MHD systems, and localization naturally leads to smallness requirements.

With this in mind, we analyze the following iterative scheme to compute a solution of the MHD system for given controls $u = (I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in U_{\text{ad}}$ and given velocity boundary data $\boldsymbol{h}$. As before, $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$ denote the liftings according to Lemma 3.2.

ALGORITHM 4.1 (operator splitting scheme).

(1) *Choose an initial guess* $\widehat{\boldsymbol{J}}^0 \in \boldsymbol{L}^2_{\text{div}}(\Omega)$*; set* $n = 0$.

(2) *Solve the* div-curl *system for* $\boldsymbol{B}^{n+1} \in \boldsymbol{V}^1(\mathbb{R}^3)$,

$$\nabla \cdot \boldsymbol{B}^{n+1} = 0, \qquad \nabla \times (\mu^{-1}\boldsymbol{B}^{n+1}) = \widehat{\boldsymbol{J}}^n,$$

with the interface condition $[\boldsymbol{B}^{n+1}]_{\partial\Omega} = 0$.

(3) *Solve the Navier–Stokes system with Lorentz force for* $\widehat{\boldsymbol{u}}^{n+1} \in \boldsymbol{H}^1_0(\Omega)$ *and* $p^{n+1} \in L^2(\Omega)/\mathbb{R}$,

$$\begin{aligned} -\eta\Delta\widehat{\boldsymbol{u}}^{n+1} + \rho(\widehat{\boldsymbol{u}}^{n+1}\cdot\nabla)\widehat{\boldsymbol{u}}^{n+1} + \rho(\widehat{\boldsymbol{u}}^{n+1}\cdot\nabla)\boldsymbol{u}_0 + \rho(\boldsymbol{u}_0\cdot\nabla)\widehat{\boldsymbol{u}}^{n+1} + \nabla p^{n+1} \\ = \eta\Delta\boldsymbol{u}_0 - \rho(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 + (\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0) \times (\boldsymbol{B}^{n+1} + \boldsymbol{B}_0), \\ \nabla\cdot\widehat{\boldsymbol{u}}^{n+1} = 0 \end{aligned}$$

with homogeneous Dirichlet boundary data on $\partial\Omega$.

(4) *Solve for* $\widehat{\boldsymbol{J}}^{n+1} \in \boldsymbol{L}^2_{\text{div}}(\Omega)$ *and* $\phi^{n+1} \in H^1(\Omega)/\mathbb{R}$

$$\begin{aligned} \sigma^{-1}\widehat{\boldsymbol{J}}^{n+1} + \nabla\phi^{n+1} = (\widehat{\boldsymbol{u}}^{n+1} + \boldsymbol{u}_0) \times (\boldsymbol{B}^{n+1} + \boldsymbol{B}_0) - \sigma^{-1}\boldsymbol{J}_0, \\ \nabla\cdot\widehat{\boldsymbol{J}}^{n+1} = 0 \end{aligned}$$

with boundary condition $\widehat{\boldsymbol{J}}^{n+1}\cdot\boldsymbol{n} = 0$ *on* $\partial\Omega$.

(5) *Unless* $\|\widehat{\boldsymbol{J}}^{n+1} - \widehat{\boldsymbol{J}}^n\|_{\boldsymbol{L}^2(\Omega)}$ *is sufficiently small, increase* n *and go to* (2).

Note that the solution to step (2) is given by the Biot–Savart operator $\widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}}^n)$. In steps (3) and (4), $\boldsymbol{B}_0 = \widetilde{\mathcal{B}}(\boldsymbol{J}_0) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}}$ collects the constant contributions to the total magnetic field. Obviously, instead of computing the liftings $\boldsymbol{u}_0$ and $\boldsymbol{J}_0$ and repeatedly solving homogeneous problems in steps (3) and (4), one may directly address the inhomogeneous ones with unknowns $\widehat{\boldsymbol{u}}^{n+1} + \boldsymbol{u}_0$ and $\widehat{\boldsymbol{J}}^{n+1} + \boldsymbol{J}_0$. The same applies to the div-curl system in step (2), which yields $\boldsymbol{B}^{n+1} + \boldsymbol{B}_0 - \boldsymbol{B}_{\text{ext}}$ if $\widehat{\boldsymbol{J}}^n$ is replaced with $\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0 + \boldsymbol{J}_{\text{ext}} + \boldsymbol{J}_{\text{inj}}$.

Remark 4.2 (alternative form of step (4)). Note that step (4) in Algorithm 4.1 above is equivalent to the solution of the div-curl system on Ω,

$$\begin{aligned} \nabla\cdot\widehat{\boldsymbol{J}}^{n+1} = 0, \\ \nabla\times(\sigma^{-1}\widehat{\boldsymbol{J}}^{n+1}) = \nabla\times\left[(\widehat{\boldsymbol{u}}^{n+1} + \boldsymbol{u}_0) \times (\boldsymbol{B}^{n+1} + \boldsymbol{B}_0)\right] - \nabla\times(\sigma^{-1}\boldsymbol{J}_0) \end{aligned}$$

with boundary condition $\widehat{\boldsymbol{J}}^{n+1} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, provided that the right-hand side is in $\boldsymbol{L}^2(\Omega)$. This can be guaranteed if $\boldsymbol{J}_0$, $\boldsymbol{B}_{\text{ext}}$, and $\partial\Omega$ are smooth enough.

For the proposed scheme, we have the following conditional convergence result.

PROPOSITION 4.3 (convergence of the operator splitting scheme). *Let $u \in \mathbb{R}^3$ and $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ be a given control vector and boundary data and suppose that η is sufficiently large and σ is sufficiently small. Then there exists $\rho_J > 0$ such that whenever the initial iterate $\widehat{\boldsymbol{J}}^0 \in \boldsymbol{L}^2_{\text{div}}(\Omega)$ satisfies $\|\widehat{\boldsymbol{J}}^0 + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} < \rho_J$, then the iterates $(\widehat{\boldsymbol{J}}^n, \widehat{\boldsymbol{u}}^n)$ of Algorithm* 4.1 *converge in $\boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^1(\Omega)$ to the necessarily unique solution of* (3.13)–(3.14) *which satisfies $\|\widehat{\boldsymbol{J}} + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} \le \rho_J$.*

Proof. The proof uses the Banach fixed point theorem. This choice is due to the fact that the nonlinearities in (3.13)–(3.14) are not of strictly monotone or energy preservation type, so that techniques analogous to those developed for decomposition methods, e.g., in [8], cannot be used. Let $T : \boldsymbol{L}^2_{\text{div}}(\Omega) \to \boldsymbol{L}^2_{\text{div}}(\Omega)$ denote the operator which assigns to $\widehat{\boldsymbol{J}}^n$ the value $\widehat{\boldsymbol{J}}^{n+1}$ defined by steps (2)–(4) of Algorithm 4.1. Let us denote by ρ_I a common bound for the inhomogeneities $\widetilde{u} = (I_{\text{ext}}, I_{\text{inj}})$ and $\boldsymbol{h}$, i.e., $|\widetilde{u}| \le \rho_I$ and $\|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\partial\Omega)} \le \rho_I$. Given the solenoidal current field $\widehat{\boldsymbol{J}}^n$, we infer from Lemma 2.4 the existence of $\boldsymbol{B}^{n+1}$ satisfying the equations in step (2) and the a priori estimate

$$\|\boldsymbol{B}^{n+1} + \boldsymbol{B}_0\|_{\boldsymbol{L}^3(\Omega)} \le c_1\, \mu \left(\|\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} + |\widetilde{u}| \right) + c_1 |\boldsymbol{B}_{\text{ext}}|.$$

Here and below, the constants c_i are independent of μ, η, σ, iteration index n, and controls u. Let us further assume that $\|\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} \le \rho_J$. Then we have

$$\|\boldsymbol{B}^{n+1} + \boldsymbol{B}_0\|_{\boldsymbol{L}^3(\Omega)} \le c_1 \big(\mu(\rho_J + \rho_I) + |B_{\text{ext}}| \big). \tag{4.1}$$

Standard estimates for the Navier–Stokes equations in step (3) imply that

$$\begin{aligned} \|\widehat{\boldsymbol{u}}^{n+1} + \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} &\le \mu\eta^{-1} c_2 \left(\|\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0\|^2_{\boldsymbol{L}^2(\Omega)} + |\widetilde{u}|^2 \right) \\ &\quad + c_2 \|\widehat{\boldsymbol{J}}^n + \boldsymbol{J}_0\| |B_{\text{ext}}| + c_2 \left(\|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\partial\Omega)} + \|\boldsymbol{h}\|^2_{\boldsymbol{H}^{1/2}(\partial\Omega)} \right) \\ &\le c_2 \left(\mu\eta^{-1}(\rho_J^2 + \rho_I^2) + \rho_J |B_{\text{ext}}| + \rho_I + \rho_I^2 \right). \end{aligned} \tag{4.2}$$

By Lemma 2.1 and a direct computation (or by [7, Chap. I, Cor. 4.1]), the system

$$\sigma^{-1} \widehat{\boldsymbol{J}} + \nabla\phi = f \text{ on } \Omega, \qquad \nabla \cdot \widehat{\boldsymbol{J}} = 0 \text{ on } \Omega$$

with given $f \in \boldsymbol{L}^2(\Omega)$ has a unique solution $\boldsymbol{J} \in \boldsymbol{L}^2_{\text{div}}(\Omega)$ and $\phi \in H^1(\Omega)/\mathbb{R}$ which satisfies $\|\boldsymbol{J}\|_{\boldsymbol{L}^2(\Omega)} \le \sigma \|f\|_{\boldsymbol{L}^2(\Omega)}$. Hence we conclude from step (4) and (4.1) and (4.2) that

$$\begin{aligned} \|\widehat{\boldsymbol{J}}^{n+1} + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} &\le \sigma\, c_3' \,\|\widehat{\boldsymbol{u}}^{n+1} + \boldsymbol{u}_0\|_{\boldsymbol{L}^6(\Omega)} \|\boldsymbol{B}^{n+1} + \boldsymbol{B}_0\|_{\boldsymbol{L}^3(\Omega)} \\ &\le \sigma\, c_3' \left(\mu\eta^{-1}(\rho_J^2 + \rho_I^2) + \rho_J |B_{\text{ext}}| + \rho_I + \rho_I^2 \right) \left(\mu(\rho_J + \rho_I) + |B_{\text{ext}}| \right) + c_3' |I_{\text{inj}}| \\ &\le \sigma\, c_3 \Big(\mu^2 \eta^{-1} (\rho_J^3 + \rho_I^3) + |B_{\text{ext}}| \left(\mu\rho_J(\rho_J + \rho_I) + \mu\eta^{-1}(\rho_J^2 + \rho_I^2) + \rho_J |B_{\text{ext}}| \right) \\ &\qquad + (\rho_I + \rho_I^2)\big(\mu(\rho_J + \rho_I) + |B_{\text{ext}}|\big) \Big) + c_3 |I_{\text{inj}}|. \end{aligned} \tag{4.3}$$

Concerning the initialization, note that $\widehat{\boldsymbol{J}}^0$ can be taken as zero. Then $\|\widehat{\boldsymbol{J}}^0+\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)}$ is bounded by $c_3|I_{\text{inj}}|$. Choosing $\rho_J := 2c_3|I_{\text{inj}}|$ and assuming that $\|\widehat{\boldsymbol{J}}^n+\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} \le \rho_J$ by induction, we obtain from (4.3) that $\|\widehat{\boldsymbol{J}}^{n+1}+\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} \le \rho_J$, provided that, for instance, σ is sufficiently small, or μ and $|B_{\text{ext}}|$ are sufficiently small. From (4.1) and (4.2) follows the existence of constants ρ_B and ρ_u independent of n such that

$$\|\boldsymbol{B}^n+\boldsymbol{B}_0\|_{\boldsymbol{L}^3(\Omega)} \le \rho_B \quad \text{and} \quad \|\widehat{\boldsymbol{u}}^n+\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} \le \rho_u \tag{4.4}$$

for all n. To prove that T is a contraction, let $\widehat{\boldsymbol{J}}_i \in \boldsymbol{L}^2_{\text{div}}(\Omega)$, $i = 1, 2$, and let $\boldsymbol{K}_i$ be their images under T. Further let $\boldsymbol{B}_i$ and $\widehat{\boldsymbol{u}}_i$ denote the associated magnetic and velocity fields according to Algorithm 4.1. Then

$$\|\boldsymbol{B}_1 - \boldsymbol{B}_2\|_{\boldsymbol{H}^1(\Omega)} \le \mu\, c_4 \|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.5}$$

Here and below, the constants are also independent of $\widehat{\boldsymbol{J}}_1$ and $\widehat{\boldsymbol{J}}_2$. Moreover, $U = \widehat{\boldsymbol{u}}_1 - \widehat{\boldsymbol{u}}_2 \in \boldsymbol{H}^1_0(\Omega)$ satisfies

$$\begin{aligned} &- \eta\Delta U + \rho(U\cdot\nabla)(\widehat{\boldsymbol{u}}_1 + \boldsymbol{u}_0) + \rho((\widehat{\boldsymbol{u}}_2 + \boldsymbol{u}_0)\cdot\nabla)U + \nabla P \\ &\qquad = (\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2) \times (\boldsymbol{B}_1 + \boldsymbol{B}_0) + (\widehat{\boldsymbol{J}}_2 + \boldsymbol{J}_0) \times (\boldsymbol{B}_1 - \boldsymbol{B}_2) \end{aligned}$$

for some $P \in L^2(\Omega)/\mathbb{R}$. This implies that

$$\begin{aligned} \eta\|\nabla U\|^2_{\boldsymbol{L}^2(\Omega)} &\le c_5\Big(\rho_u\|\nabla U\|^2_{\boldsymbol{L}^2(\Omega)} + \rho_B\|\nabla U\|_{\boldsymbol{L}^2(\Omega)}\|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|_{\boldsymbol{L}^2(\Omega)} \\ &\qquad\qquad + \rho_J\|\nabla U\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{B}_1 - \boldsymbol{B}_2\|_{\boldsymbol{L}^3(\Omega)}\Big) \\ &\le c_5\Big(\rho_u\|\nabla U\|^2_{\boldsymbol{L}^2(\Omega)} + \big(\rho_B + \mu\, c_4\,\rho_J\big)\|\nabla U\|_{\boldsymbol{L}^2(\Omega)}\|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|_{\boldsymbol{L}^2(\Omega)}\Big) \\ &\le c_5\Big(\rho_u\|\nabla U\|^2_{\boldsymbol{L}^2(\Omega)} + \eta/2\|\nabla U\|^2_{\boldsymbol{L}^2(\Omega)} + \big(\rho_B + \mu\, c_4\,\rho_J\big)^2/(2\eta)\|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|^2_{\boldsymbol{L}^2(\Omega)}\Big). \end{aligned}$$

Hence if η is sufficiently large, or ρ_u is sufficiently small (which can be achieved by μ, $|B_{\text{ext}}|$, and ρ_I sufficiently small), we have

$$\|\nabla U\|_{\boldsymbol{L}^2(\Omega)} \le c_6\,\eta^{-1}(\rho_B + \mu\, c_4\,\rho_J)\|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.6}$$

Finally, we obtain from step (4)

$$\begin{aligned} \|\boldsymbol{K}_1 - \boldsymbol{K}_2\| &\le \sigma\, c_7'\Big(\|U\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{B}_1 + \boldsymbol{B}_0\|_{\boldsymbol{L}^3(\Omega)} + \|\widehat{\boldsymbol{u}}_2 + \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{B}_1 - \boldsymbol{B}_2\|_{\boldsymbol{L}^3(\Omega)}\Big) \\ &\le \sigma\, c_7\Big(\eta^{-1}\rho_B(\rho_B + \mu\,\rho_J) + \mu\, c_4\,\rho_u\Big)\|\widehat{\boldsymbol{J}}_1 - \widehat{\boldsymbol{J}}_2\|_{\boldsymbol{L}^2(\Omega)}. \qquad (4.7) \end{aligned}$$

Hence we conclude that if σ is sufficiently small, or if μ is sufficiently small and η sufficiently large, then T is a contraction on the ball $\{\boldsymbol{J} : \|\widehat{\boldsymbol{J}} + \boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)} < \rho_J\}$. ☐

5. Conclusion and outlook. In this paper, we have presented and analyzed an optimal control problem for the stationary MHD system. We derived necessary and sufficient conditions for local optimal solutions. In addition, we analyzed an iterative scheme for the numerical solution of the MHD state equations which is tailored to make use of existing Navier–Stokes and div-curl solvers. We believe that in the face of industrial MHD applications, there is ample room to extend our results in several directions. Of particular interest are the cases of instationary MHD flows, unknown current densities in external conductors, flows with thermal coupling and Ohmic heating, and the case of material-dependent magnetic permeability. All of the above present additional technical difficulties, which are the subject of future investigations. Finally, devising an efficient numerical algorithm to solve optimal control problems involving MHD flows presents another challenging task.

Appendix. Proof of Proposition 3.3. Let us define $T : \widehat{Y} \to \widehat{Y}$ according to

$$(\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi) = T(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$$

if and only if

$$A(\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi) = R(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi) \tag{A.1}$$

holds in $\widehat{Y}'$. That is, T is the solution operator of a linear PDE, which depends nonlinearly on the data $(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$. Defining the components of R as

$$\begin{aligned}
R^1(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)(\boldsymbol{v}) &= c(\widehat{\boldsymbol{J}} + \boldsymbol{J}_0, \widetilde{\mathcal{B}}(\boldsymbol{J}_0) + \widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}}, \boldsymbol{v}) \\
&\quad - a_1(\boldsymbol{u}_0, \boldsymbol{v}) - b(\widehat{\boldsymbol{u}} + \boldsymbol{u}_0, \widehat{\boldsymbol{u}} + \boldsymbol{u}_0, \boldsymbol{v}), \\
R^2(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)(q) &= 0, \\
R^3(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)(\boldsymbol{K}) &= -c(\boldsymbol{K}, \widetilde{\mathcal{B}}(\boldsymbol{J}_0) + \widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{ext}}) + \widetilde{\mathcal{B}}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}}, \widehat{\boldsymbol{u}} + \boldsymbol{u}_0) \\
&\quad - a_2(\boldsymbol{J}_0, \boldsymbol{K}), \\
R^4(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)(\psi) &= 0,
\end{aligned}$$

we easily verify that the solutions to the homogenized problem (3.13)–(3.14) are exactly the fixed points of T. In view of Proposition 3.6 below, $A : \widehat{Y} \to \widehat{Y}'$ is an isomorphism, and hence T is well defined from $\widehat{Y}$ to itself. We now confirm that T is compact. To this end, we consider a bounded and weakly convergent sequence $(\widehat{\boldsymbol{u}}^n, p^n, \widehat{\boldsymbol{J}}^n, \phi^n) \rightharpoonup (\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$ in $\widehat{Y}$. Since the norm in $\widehat{Y}'$ of the right-hand side in (A.1) is a quadratic polynomial in the norms of $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{J}}$, the sequence $(\delta \boldsymbol{u}^n, \delta p^n, \delta \boldsymbol{J}^n, \delta\phi^n) := T(\widehat{\boldsymbol{u}}^n, p^n, \widehat{\boldsymbol{J}}^n, \phi^n)$ is bounded in $\widehat{Y}$ and thus possesses a weakly convergent subsequence in $\widehat{Y}$, i.e., $(\delta \boldsymbol{u}^n, \delta p^n, \delta \boldsymbol{J}^n, \delta\phi^n) \rightharpoonup (\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi)$. Using Lemma 2.3, one confirms that the weak limit $(\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi)$ satisfies $(\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi) = T(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$; i.e., T is weakly continuous. The difference $(\delta \boldsymbol{u}^n, \delta p^n, \delta \boldsymbol{J}^n, \delta\phi^n) - (\delta \boldsymbol{u}, \delta p, \delta \boldsymbol{J}, \delta\phi)$ satisfies (A.1) with right-hand side

$$R(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi) - R(\widehat{\boldsymbol{u}}^n, p^n, \widehat{\boldsymbol{J}}^n, \phi^n),$$

which converges to zero strongly in $\widehat{Y}'$, as a straightforward application of Lemmas 2.3, 2.4, and 2.6 shows. Hence T is indeed compact.

Now let $(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$ be a fixed point of $s \cdot T$ for any $s \in [0, 1]$; i.e., $(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$ satisfies (A.1) with the right-hand side multiplied by s. Testing this system with $(\widehat{\boldsymbol{u}}, p, \widehat{\boldsymbol{J}}, \phi)$, we obtain

$$\begin{aligned}
\eta \, \|\nabla \widehat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + \sigma^{-1} \|\widehat{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)} = s\big(c(\boldsymbol{J}_0, \boldsymbol{B}, \widehat{\boldsymbol{u}}) - c(\widehat{\boldsymbol{J}}, \boldsymbol{B}, \boldsymbol{u}_0) - b(\widehat{\boldsymbol{u}}, \boldsymbol{u}_0, \widehat{\boldsymbol{u}}) \\
- a_1(\boldsymbol{u}_0, \widehat{\boldsymbol{u}}) - a_2(\boldsymbol{J}_0, \widehat{\boldsymbol{J}})\big)
\end{aligned}$$

with $\boldsymbol{B}$ according to (3.14). By Poincaré's inequality $\|\widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \le c_P \|\nabla \widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}$, one obtains

$$\text{(A.2)} \quad \frac{\eta}{1+c_P^2}\|\widehat{\boldsymbol{u}}\|^2_{\boldsymbol{H}^1(\Omega)} + \sigma^{-1}\|\widehat{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)} \le s\big(c(\boldsymbol{J}_0, \boldsymbol{B}, \widehat{\boldsymbol{u}}) - c(\widehat{\boldsymbol{J}}, \boldsymbol{B}, \boldsymbol{u}_0) - b(\widehat{\boldsymbol{u}}, \boldsymbol{u}_0, \widehat{\boldsymbol{u}}) \\ - a_1(\boldsymbol{u}_0, \widehat{\boldsymbol{u}}) - a_2(\boldsymbol{J}_0, \widehat{\boldsymbol{J}})\big).$$

The application of the Leray–Schauder fixed point theorem requires that the left-hand side be a priori bounded uniformly in $s \in [0,1]$. The bound may depend on the controls $(I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}})$ and the boundary data $\boldsymbol{h}$. We observe that the right-hand side in (A.2) is bounded above by

$$\text{(A.3)} \quad \|\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)}\|\widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}})\|_{\boldsymbol{L}^3(\Omega)}\|\widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^6(\Omega)} + \|\widehat{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}\|\widetilde{\mathcal{B}}(\widehat{\boldsymbol{J}})\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^6(\Omega)} \\ + \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}\|\widehat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^4(\Omega)}$$

plus a number of terms which are at most linear in $\|\widehat{\boldsymbol{u}}\|$ and $\|\boldsymbol{J}\|$. The latter can be treated using Young's inequality according to the pattern $c\,\|\widehat{\boldsymbol{u}}\| \le \varepsilon\,\|\widehat{\boldsymbol{u}}\|^2 + c/(4\varepsilon)$, and $\varepsilon\,\|\widehat{\boldsymbol{u}}\|^2$ can then be absorbed into the left-hand side of (A.2) for sufficiently small $\varepsilon > 0$. However, in order for the terms in (A.3) to be likewise absorbed in the left-hand side of (A.2), the coefficients $\|\boldsymbol{J}_0\|_{\boldsymbol{L}^2(\Omega)}$, $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^6(\Omega)}$, and $\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}$ must be sufficiently small. In this case, $\|\widehat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\widehat{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}$ are indeed a priori bounded by the right-hand side in (3.16). In view of Lemma 3.2, the same bound holds for the inhomogeneous solution $\boldsymbol{u}$ and $\boldsymbol{J}$. Finally, the bounds for the pressure p and potential ϕ follow from Lemma 2.2. Hence we conclude the applicability of the Leray–Schauder theorem which yields the existence of a fixed point of T.

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