Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Pures Appl. 85 (2006) 415–449
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Parabolic variational inequalities: The Lagrange multiplier approach

Kazufumi Ito [a,*], Karl Kunisch [b,1]

[a] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA*
[b] *Institute of Mathematics, Karl-Franzens University Graz, Austria*

Received 5 April 2005
Available online 19 October 2005

Abstract

Parabolic variational inequalities are discussed and existence and uniqueness of strong as well as weak solutions are established. Our approach is based on a Lagrange multiplier treatment. Existence is obtained as the unique asymptotic limit of solutions to a family of appropriately regularized nonlinear parabolic equations. Two regularization techniques are presented resulting in feasible and unfeasible approximations respectively. Monotonicity results of the regularized solutions and convergence rate estimate are established. The results are applied to the Black–Scholes model for American options. The case of the bilateral constraints is also treated. Numerical results for the Black–Scholes model are presented and prove the practical efficiency of our results.

Résumé

Des inégalités variationnelles paraboliques sont discutées et l'existence et l'unicité des solutions fortes faibles sont établies. Notre approche des solutions utilise une méthode de multiplicateur de Lagrange. L'existence est obtenue comme limite asymptotique unique des solutions à une famille d'équations paraboliques non linéaires convenablement régularisées. Deux techniques de régularisation sont présentées ayant pour résultat des approximations acceptées ou rejetées. Des résultats de monotonie des solutions régularisées et d'évaluation de taux de convergence sont établis. Les résultats sont appliqués au modèle Black–Scholes pour des options américaines. Le cas des contraintes

* Corresponding author.
E-mail address: kito@math.ncsu.edu (K. Ito).
1 Research in part supported by the Fonds zur Förderung der wissenschaftlichen Forschung under SFB 03 "Optimierung und Kontrolle".

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2005.08.005

bilatérales est également traité. Des résultats numériques pour le modèle Black–Scholes sont présentés et prouvent l'efficacité pratique de nos techniques.

Keywords: Parabolic variational inequalities; Lagrange multipliers; Feasible and unfeasible regularization; Black–Scholes

1. Introduction

In this paper we discuss parabolic variational inequalities in the Hilbert space $H = L^2(\Omega)$ of the type,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}y^*(t) - Ay^*(t) - f(t), y - y^*(t)\right)_H \geqslant 0, \quad y^*(t) \in \mathcal{C}, \tag{1.1}$$

for all $y \in \mathcal{C}$, where the closed convex subset $\mathcal{C}$ of H is defined by:

$$\mathcal{C} = \{y \in H\colon\ y \leqslant \psi\},$$

A is a closed elliptic operator in H, Ω denotes a bounded domain in R^n, and $y \leqslant \psi$ must be interpreted in the pointwise a.e. sense. In [4,7] existence of strong and weak solutions is established using elliptic regularization techniques with respect to the operator $\frac{\mathrm{d}}{\mathrm{d}t} + A$. If the solution satisfies $y^* \in L^2(0, T; \mathrm{dom}(A)) \cap H^1(0, T; H)$, then (1.1) can equivalently be expressed as variational inequality of the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}y^*(t) - Ay^*(t) - f(t) = -\lambda^*(t) \leqslant 0, \\ y^*(t) \leqslant \psi, \quad (y^*(t) - \psi, \lambda^*(t))_H = 0, \quad \text{a.e. in } t > 0. \end{cases} \tag{1.2}$$

The Black–Scholes model for America options, see [14,16] for example, can be formulated as (1.2) (see Section 3).

Our objective is to construct solutions to (1.1) and (1.2) as the asymptotic limit of solutions to regularized problems based on a Yosida–Moreau approximation of (1.1), see Section 2. Hence it is distinctly different from the techniques used in [4,7,13] and follows the abstract treatment in [9], and the treatise of elliptic variational inequalities in [10] and [11]. For fixed $\bar\lambda \in L^2(0, T; H)$ satisfying $\bar\lambda(t) \geqslant 0$ a.e. and $c > 0$, we consider the family of nonlinear parabolic equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}y_c(t) - Ay_c(t) + \max\bigl(0, \bar\lambda(t) + c\,(y_c(t) - \psi)\bigr) - f(t) = 0, \tag{1.3}$$

where the max operation is defined pointwise a.e. in Ω. The motivation for introducing the term $\bar\lambda(t)$ is twofold. First we show that under appropriate assumptions the choice $\bar\lambda(t) \geqslant \max(0, A\psi + f(t))$ (in the variational sense) guarantees that $\lambda_c(t) = \max(0, \bar\lambda(t) + c\,(y_c(t) - \psi)) \to \lambda^*(t)$ in $L^2(0, T; H)$, $y_c(t) \to y^*(t)$ as $c \to \infty$ and (y^*, λ^*) is the solution to (1.2). In particular, this choice of $\bar\lambda$ guarantees the existence of a Lagrange multiplier. Here $y_c(t)$ denotes the solution to (1.3). Secondly, for the above choice of $\bar{\lambda}$ (appropriately modified if $A\psi$ is a distribution) the approximate solutions $y_c(t)$ are feasible, i.e., $y_c(t) \leqslant \psi$, as well as monotone with respect to c, i.e., we have:

$$y_c(t) \leqslant y_{\hat{c}}(t) \leqslant y^*(t)$$

and the bound

$$0 \leqslant \lambda_c(t) \leqslant \bar{\lambda}(t)$$

holds for all $0 < c < \hat{c}$. An analogous result was established for elliptic variational inequalities in [10].

For the penalty method case where $\bar{\lambda} = 0$, see also [4], we can establish monotonicity of the family of solutions $\tilde{y}_c$:

$$\tilde{y}_c(t) \geqslant \tilde{y}_{\hat{c}}(t) \geqslant y^*(t),$$

but no upper bound on $\lambda_c(t)$ can be obtained. In conclusion:

$$y_c(t) \leqslant y^*(t) \leqslant \tilde{y}_c(t).$$

Moreover, for second order elliptic operators A we establish in Section 4 the convergence rate estimate:

$$\left|y_c(t) - y^*(t)\right|_{L^\infty(\Omega)}, \left|\tilde{y}_c(t) - y^*(t)\right|_{L^\infty(\Omega)} \leqslant \frac{M}{c}.$$

For elliptic regularization method square root convergence with respect to the regularization parameter was proved in [4]. These convergence results are particularly important for the Black–Scholes model for American options since the free surface $S(t) = \{y^*(t) = \psi\}$ defines the optimal stopping time [14,16]. That is, we can approximate $S(t)$ with the rate $1/c$ by letting:

$$S_c(t) = \left\{\lambda_c(t) = 0\right\} \quad \text{or} \quad S_c(t) = \{\tilde{y}_c = \psi\}.$$

The paper also contains a discussion of weak solutions and in particular a new result on the uniqueness of the weak solution is obtained in Section 2.3. While most of the paper concentrates on the case where the obstacle is independent of t, we also treat time-dependent constraints $\psi(t)$ in Section 2.4. Section 3 is devoted to some aspects related to the Black–Scholes equation. Convergence rate estimates with respect to c are the subject of Section 4. In Section 5 we discuss the case of bilateral constraints, i.e., the case when:

$$\mathcal{C} = \{y \in H\colon \phi \leqslant y \leqslant \psi\}.$$

As in the unilateral case again our treatment depends, in an essential manner, on an appropriate choice of $\bar{\lambda}$. Lastly, in Section 6 we report on a numerical result for the solutions to a one-dimensional Black–Scholes model.

2. Strong and weak solutions, and existence of Lagrange multipliers

We discuss parabolic variational inequalities in the Hilbert space $H = L^2(\Omega)$. Let X be a Hilbert space that is densely, compactly embedded into H and let V be a closed linear subspace of X endowed with the norm of X. For $\psi \in H$ let $\mathcal{C}$ be the closed convex set in V given by:

$$\mathcal{C} = \{y \in H\colon\ y \leqslant \psi\} \cap V,$$

where we assume that ψ is such that $\mathcal{C}$ is nonempty. The problem that we shall investigate consists in finding $y^*(t) \in \mathcal{C}$ such that for a.e. $t \in (0, T)$,

$$\begin{cases} \langle \frac{\mathrm{d}}{\mathrm{d}t} y^*(t), y(t) - y^*(t)\rangle + a(y^*(t), y(t) - y^*(t)) - \langle f(t), y(t) - y^*(t)\rangle \geqslant 0 \\ \quad \text{for all } y \in \mathcal{C}, \\ y^*(0) = y_0, \end{cases} \tag{2.1}$$

where $y_0 \in \mathcal{C}$, $f \in L^2(0, T; V^*)$, and $a(\cdot,\cdot)$ is a bounded bilinear form on $X \times X$, i.e.,

$$\big|a(y, \phi)\big| \leqslant M\,|y|_X |\phi|_X, \quad y, \phi \in X,$$

which is coercive on V:

$$a(\phi, \phi) \geqslant \omega|\phi|_V^2 - \rho|\phi|_H^2, \quad \phi \in V,$$

with $\omega > 0$ and $\rho \geqslant 0$. Here $(\cdot,\cdot)$ denotes the inner product on H and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V^*,V}$ stands for the duality pairing between V and V^*. While we frequently set $\rho = 0$ for the sake of simplicity of presentation, but we indicate the dependency on ρ when it is necessary.

Let us define $A \in \mathcal{L}(X, V^*)$ by:

$$-\langle Ay, v\rangle_{V^* \times V} = a(y, v) \quad \text{for } y \in X,\ v \in V.$$

Then the restriction of A to V is a closed linear operator in H with,

$$\mathrm{dom}(A) = \big\{y \in V\colon \text{ there exists } \alpha_y \text{ such that } \big|a(y, \phi)\big| \leqslant \alpha_y |\phi|_H \text{ for all } \phi \in V\big\},$$

and $\mathrm{dom}(A)$ is a Hilbert space equipped with the graph norm of $-A$.

Definition 1 *(Strong solution).* Given $y_0 \in \mathcal{C}$ and $f \in L^2(0, T; H)$, a function $y^* \in H^1(0, T; H) \cap C(0, T; V)$ is called strong solution of (2.1) if $y^*(t) \in \mathcal{C}$ and (2.1) is satisfied for a.e. $t \in (0, T)$.

Note that if the strong solution satisfies $y^* \in L^2(0, T; \mathrm{dom}(A))$, then (2.1) can equivalently be written as a variational inequality of the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y^*(t) - Ay^*(t) - f(t) = -\lambda(t) \leqslant 0, \\ y^*(t) \leqslant \psi, \quad (y^*(t) - \psi, \lambda(t))_H = 0, \quad \text{for a.e. } t > 0. \end{cases} \tag{2.2}$$

Remark 1. (1) Let Φ be the convex functional on H defined by:

$$\Phi(y)=\begin{cases}0 & \text{if } y\leqslant\psi \text{ a.e.},\\ \infty & \text{otherwise.}\end{cases}$$

Then (2.2) can be written as

$$-\frac{\mathrm{d}}{\mathrm{d}t}y^*(t)+Ay^*(t)+f(t)\in\partial\Phi\big(y^*(t)\big),$$

where $\partial\Phi$, the sub-differential of Φ. Equivalently this can be expressed as $\lambda^*(t)\in\partial\Phi(y^*(t))$. In this sense $\lambda^*(t)$ is the Lagrange multiplier associated to the constrained $y\leqslant\psi$.

(2) The family of the regularized problems that we shall utilize in this paper is given by:

$$\begin{cases}\langle\frac{\mathrm{d}}{\mathrm{d}t}y(t),\phi\rangle+a(y(t),\phi)+(\max(0,\bar\lambda(t)+c(y(t)-\psi)),\phi)-\langle f(t),\phi\rangle=0,\\ \quad\text{for all } \phi\in V \text{ and a.e. } t\in(0,T),\\ y(0)=y_0,\end{cases}\tag{2.3}$$

with $c>0$, it is based on the Yosida–Moreau approximation [9] of the complementarity condition $\lambda^*(t)\in\partial\Phi(y^*(y))$. Different choices for $\bar\lambda$ will be used.

(3) If there exits a Lagrange multiplier $\lambda^*(t)\in L^2(0,T;H)$ satisfying (2.2), then $y^*(t)$ is a solution to (2.3) with $\bar\lambda=\lambda^*(t)$. In fact, $\lambda^*(t)$ satisfies the complementarity condition,

$$\lambda^*(t)=\max\big(0,\lambda^*(t)+c\big(y^*(t)-\psi\big)\big),\tag{2.4}$$

for each $c>0$. It is shown in [9] (and can also be checked easily by a direct computation) that $\lambda^*(t)\in\partial\Phi(y^*(t))$ if and only if (2.4) is satisfied for some $c>0$.

2.1. Strong solution

In this section we prove existence of strong solutions to (2.1) by means of a finite difference approximation scheme.

Theorem 1. *We consider the regularized problem* (2.3) *with* $f\in L^2(0,T;V^*)$, $y_0\in H$, *and* $\bar\lambda\in L^2(0,T;H)$. *Then for each* $c>0$ *there exists a unique solution* y_c *in* $W(0,T)=H^1(0,T;V^*)\cap L^2(0,T;V)$ *to* (2.3) *and we have the estimates*:

$$\begin{aligned}&\big|y_c(t)-\hat\psi\big|_H^2+\int_0^t\Big(\omega\big|y_c(s)-\hat\psi\big|_V^2+\frac{1}{c}\big|\lambda_c(s)\big|_H^2\Big)\,\mathrm{d}s\\ &\quad\leqslant\big|y_0-\hat\psi\big|^2+\frac{2tM^2}{\omega}\big|\hat\psi\big|_V^2+\int_0^t\Big(\frac{2}{\omega}\big|f(s)\big|_{V^*}^2+\frac{1}{c}\big|\bar\lambda(s)\big|_H^2\Big)\,\mathrm{d}s,\end{aligned}\tag{2.5}$$

for all $\hat\psi \in \mathcal{C}$, *where* $\lambda_c(t) = \max(0, \bar\lambda(t) + c(y_c(t) - \psi))$, *a.e. on* $(0, T)$.

If in addition for some $M_1 \geqslant 0$,

$$\left|\frac{a(y,\phi) - a(\phi, y)}{2}\right| \leqslant M_1 |y|_V |\phi|_H \quad \text{for all } y, \phi \in V, \tag{2.6}$$

$\bar\lambda(t) = \bar\lambda \in H$, $y_0 \in V$ *and* $f \in L^2(0, T; H)$, *then* $y_c \in H^1(0, T; H) \cap C(0, T; V) \cap L^2(0, T; \mathrm{dom}(A))$ *and we have the estimate*:

$$a\big(y_c(t), y_c(t)\big) + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y_c(s)\right|_H^2 \mathrm{d}s$$
$$\leqslant M_2\left(|y_0|_V^2 + c\left|\left(y_0 - \psi + \frac{\bar\lambda}{c}\right)^+\right|_H^2 + \int_0^t \big|f(s)\big|_H^2 \,\mathrm{d}s\right), \tag{2.7}$$

on $[0, T]$, *where* $M_2 > 0$ *is independent of* $c > 0$.

Proof. Consider the finite difference approximation of (2.3):

$$\left(\frac{y^{k+1} - y^k}{\Delta t}, \phi\right) + a\big(y^{k+1}, \phi\big) + \big(\max\big(0, \bar\lambda^k + c\big(y^{k+1} - \psi\big)\big), \phi\big) - \big\langle f^k, \phi\big\rangle = 0,$$
$$\text{for all } \phi \in V, \tag{2.8}$$

with $y^0 = y_0 \in H$, $f^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} f(t)\,\mathrm{d}t$, $\bar\lambda^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} \bar\lambda(t)\,\mathrm{d}t$, $\Delta t = \frac{T}{N}$ and $k = 0, 1, \ldots.$ Note that

$$y \in H \to \max\big(0, \bar\lambda^k + c(y - \psi)\big) \in H$$

is Lipschitz continuous and monotone. Hence, since $B : V \to V^*$ defined by,

$$B(y) = \frac{y}{\Delta t} - Ay + \max\big(0, \bar\lambda^k + c(y - \psi)\big),$$

is coercive, monotone, hemicontinuous [2] for all sufficiently small $\Delta t > 0$ independently of $c > 0$, (2.8) has a unique solution y^{k+1} in V for every k.

We let $\lambda^{k+1} = \max(0, \bar\lambda^k + c(y^{k+1} - \psi))$. Then for all $\hat\psi \in C$:

$$\big(\lambda^{k+1}, y^{k+1} - \hat\psi\big) = \left(\lambda^{k+1}, \frac{\bar\lambda^k}{c} + y^{k+1} - \psi + \psi - \hat\psi - \frac{\bar\lambda^k}{c}\right)$$
$$\geqslant \frac{1}{c}\big|\lambda^{k+1}\big|_H^2 - \frac{1}{c}\big(\lambda^{k+1}, \bar\lambda^k\big) \geqslant \frac{1}{2c}\big|\lambda^{k+1}\big|_H^2 - \frac{1}{2c}\big|\bar\lambda^k\big|_H^2.$$

Setting $\phi = y^{k+1} - \hat\psi$ in (2.8), we obtain for $k = 0, 1, \ldots$

$$\frac{1}{2\Delta t}\big(\big|y^{k+1}-\hat\psi\big|_H^2-\big|y^k-\hat\psi\big|_H^2+\big|y^{k+1}-y^k\big|_H^2\big)+\omega\big|y^{k+1}-\hat\psi\big|_V^2$$
$$-\rho\big|y^{k+1}-\hat\psi\big|_H^2-M\big|y^{k+1}-\hat\psi\big|_V\big|\hat\psi\big|_V+\frac{1}{2c}\big|\lambda^{k+1}\big|_H^2-\frac{1}{2c}\big|\bar\lambda^k\big|_H^2$$
$$\leqslant\big|f^k\big|_{V^*}\big|y^{k+1}-\hat\psi\big|_V$$

and thus

$$\big|y^k-\hat\psi\big|_H^2+\sum_{i=1}^k\Big(\Big(\omega\big|y^i-\hat\psi\big|_V^2+\frac{1}{c}\big|\lambda^i\big|_H^2\Big)\Delta t+\big|y^i-y^{i-1}\big|_H^2\Big)$$
$$\leqslant\big|y_0-\hat\psi\big|_H^2+\sum_{i=1}^k\Big(\frac{2M^2}{\omega}\big|\hat\psi\big|_V^2+\frac{2}{\omega}\big|f^{i-1}\big|_{V^*}^2+\frac{1}{c}\big|\bar\lambda^{i-1}\big|_H^2\Big)\Delta t,\tag{2.9}$$

for $k=1,2,\ldots$. We let:

$$y_{\Delta t}^{(1)}(t)=y^k+\frac{t-k\Delta t}{\Delta t}\big(y^{k+1}-y^k\big)\quad\text{on }\big[k\Delta t,(k+1)\Delta t\big].$$

Then from (2.8), (2.9) the family $y_{\Delta t}^{(1)}$ is bounded in $W(0,T)=H^1(0,T;V^*)\cap L^2(0,T;V)$ and from the Aubin lemma, see, e.g., [6,12] it has a subsequence that converges to some y_c weakly in $W(0,T)$ and strongly in $L^2(0,T;H)$. Moreover for,

$$y_{\Delta t}^{(2)}(t)=y^{k+1}\quad\text{on }\big(k\Delta t,(k+1)\Delta t\big],$$

we have:

$$\int_0^T\big|y_{\Delta t}^{(1)}(t)-y_{\Delta t}^{(2)}(t)\big|_H^2\,\mathrm{d}t=\frac{\Delta t}{3}\sum_{k=1}^N\big|y^k-y^{k-1}\big|_H^2\to 0$$

as $\Delta t\to 0$. Hence without loss of generality the subsequence of $y_{\Delta t}^{(2)}$ converges to the same y_c weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$. Thus the limit y_c satisfies (2.3) and estimate (2.5) holds.

Uniqueness. Note that for $y\in W(0,T)$ we have $\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|_H^2=2\langle\frac{\mathrm{d}}{\mathrm{d}t}y(t),y(t)\rangle$ for a.e. t. Let $y_i\in W(0,T)$ denote solutions to (2.3) with initial condition $y_i(0)\in H$ and $f_i\in L^2(0,T;V^*)$, $i=1,2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|y_1(t)-y_2\big|_H^2+\omega\big|y_1(t)-y_2(t)\big|_V^2-2\rho|y_1-y_2|_H^2\leqslant\frac{1}{\omega}\big|f_1(t)-f_2(t)\big|_{V^*}^2.$$

This implies the existence of $M_1>0$ such that

$$
\begin{aligned}
&\left|y_1(t)-y_2(t)\right|_H^2+\omega\int_0^t\left|y_1(s)-y_2(s)\right|_V^2\,\mathrm{d}s\\
&\quad\leqslant M_1\left(\left|y_1(0)-y_2(0)\right|_H^2+\int_0^t\frac{1}{\omega}\left|f_1(s)-f_2(s)\right|_{V^*}^2\,\mathrm{d}s\right),
\end{aligned}
$$

for all $t\in[0,T]$. Uniqueness of the solution to (2.3) follows.

Strong solution. Define the symmetric part a_s of a by:

$$
a_s(y,\phi)=\frac{a(y,\phi)+a(\phi,y)}{2}+\rho(y,\phi)_H\quad\text{for } y,\phi\in V,
$$

and set

$$
\Psi_c(y)=\int_\Omega\int_{-\bar\lambda/c}^{y}\max\bigl(0,\bar\lambda(x)+cs\bigr)\,\mathrm{d}s\,\mathrm{d}x.
$$

We shall use (2.8) with $\phi=\frac{y^{k+1}-y^k}{\Delta t}$ and observe that

$$
\begin{aligned}
2a_s\bigl(y^{k+1},y^{k+1}-y^k\bigr)=a_s\bigl(y^{k+1},y^{k+1}\bigr)-a_s\bigl(y^k,y^k\bigr)+a_s\bigl(y^k-y^{k+1},y^{k+1}\bigr)\\
-a_s\bigl(y^{k+1},y^k-y^{k+1}\bigr)+a_s\bigl(y^{k+1}-y^k,y^{k+1}-y^k\bigr).
\end{aligned}
$$

Using monotonicity of $y\to\Psi_c(y)$, we obtain:

$$
\begin{aligned}
&\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H^2+\frac{1}{2\Delta t}\bigl(a_s\bigl(y^{k+1},y^{k+1}\bigr)-a_s\bigl(y^k,y^k\bigr)+a_s\bigl(y^{k+1}-y^k,y^{k+1}-y^k\bigr)\bigr)\\
&\quad+\frac{1}{\Delta t}\bigl(\Psi_c(y^{k+1}-\psi)-\Psi_c\bigl(y^k-\psi\bigr)\bigr)\\
&\quad\leqslant\left|\rho\left(y^{k+1},\frac{y^{k+1}-y^k}{\Delta t}\right)\right|_H+M_1\left|y^{k+1}\right|_V\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H+\left|f^k\right|_H\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H\\
&\quad\leqslant\frac12\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H^2+\frac32\bigl(\rho^2\left|y^{k+1}\right|_H^2+M_1^2\left|y^{k+1}\right|_V^2+\left|f^k\right|_H^2\bigr).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{i=1}^k\left(\left|\frac{y^i-y^{i-1}}{\Delta t}\right|_H^2\Delta t+a_s\bigl(y^i-y^{i-1},y^i-y^{i-1}\bigr)\right)+a_s\bigl(y^k,y^k\bigr)+\Psi_c\bigl(y^k-\psi\bigr)\\
&\quad\leqslant a_s\bigl(y^0,y^0\bigr)+\Psi_c\bigl(y^0-\psi\bigr)+3\sum_{i=1}^k\bigl(\rho^2\left|y^i\right|_H^2+M_1^2\left|y^i\right|_V^2+\left|f^i\right|_H^2\bigr)\Delta t.
\end{aligned}
$$

Thus $y^{(1)}_{\Delta t}(\cdot)$ is bounded in $H^1(0,T;H)\cap C(0,T;V)$ and converges weakly in $H^1(0,T;H)$ and weak* in $L^\infty(0,T;V)$ to y_c. Moreover, since y_c satisfies $\frac{\mathrm{d}}{\mathrm{d}t}y_c - Ay_c = \tilde f$ with $\tilde f = \max(0, \bar\lambda + c(y_c - \psi)) - f \in L^2(0,T;H)$ and $y(0) = y_0 \in V$ we have $y_c \in C(0,T;V)\cap L^2(0,T;\mathrm{dom}(A))$. □

Theorem 2. *Assume that $y_0 \in \mathcal{C}$, $f \in L^2(0,T;H)$ and that* (2.6) *holds. Then* (2.1) *has a unique strong solution $y^* \in H^1(0,T;H)$, $t \to y^*(t)$ is right-continuous, and the estimates hold*

$$\big|y^*(t) - \hat\psi\big|_H^2 + \int_0^t \omega\big|y^*(s) - \hat\psi\big|_V^2\,\mathrm{d}s \leqslant \big|y^0 - \hat\psi\big|_H^2 + \frac{2tM^2}{\omega}\big|\hat\psi\big|_V^2 + \int_0^t \frac{2}{\omega}\big|f(s)\big|_{V^*}^2\,\mathrm{d}s$$

for all $\hat\psi \in \mathcal{C}$, and moreover

$$a\big(y^*(t), y^*(t)\big) + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}y^*(s)\right|_H^2 \mathrm{d}s \leqslant M_2\left(|y_0|_V^2 + \int_0^t \big|f(s)\big|_H^2\,\mathrm{d}s\right). \tag{2.10}$$

Proof. Let $\bar\lambda = 0$. Since from (2.7) y_c is bounded in $H^1(0,T,H)\cap C(0,T;V)$, there exists a subsequence that converges to y^* weakly in $H^1(0,T;H)$, weakly star in $L^\infty(0,T;V)$ and strongly in $L^2(0,T;H)$ as $c\to\infty$.

From (2.5) we further deduce that

$$\int_0^T \big|\max\big(0, y_c(t) - \psi\big)\big|_H^2\,\mathrm{d}t \to 0$$

as $c\to\infty$, and consequently $y^*(t)\leqslant\psi$ a.e. Since,

$$\big(\max\big(0, c\big(y_c(t) - \psi\big)\big), y - y_c(t)\big) = \big(\max\big(0, c\big(y_c(t)-\psi\big)\big), y - \psi - \big(y_c(t) - \psi\big)\big) \leqslant 0$$

for all $y \in \mathcal{C}$,

y^* satisfies (2.1).

We turn to the a priori estimates. Since y_c converges to y^* in $L^2(0,T;H)$ as $c\to\infty$ there exists a further subsequence, denoted by $y_{\hat c}$ that converges pointwise a.e. to y^* in H. Due to (2.7) the family $\{|y_{\hat c}(t)|_V\}_{\hat c>0}$ is bounded for every $t\in[0,T]$. Hence, for each $t\in[0,T]$ there exists a subsequence of $\{\hat y_{\hat c}(t)\}$ and $\hat y(t)$ such that $y_{\hat c}(t)$ converges to $\hat y(t)$ weakly in V. We claim that $\hat y(t) = y^*(t)$ for a.e. $t\in(0,T)$ and hence the whole family $\{y_{\hat c}(t)\}_{\hat c>0}$ converges to $y^*(t)$ weakly in V. This follows from the fact that if a sequence $\{z_n\}$ converges strongly in H to z and weakly in V to $\bar z$ then $z = \bar z$. In fact, let $\mathcal{J}: V\to V^*$

denote the Riesz isomorphism and let $\operatorname{dom}\mathcal{J} = \{h \in V\colon\ \mathcal{J}h \in H\}$. For every $h \in \operatorname{dom}\mathcal{J}$ we have:

$$0 = \lim_{n\to\infty}(z_n - \bar{z}, h)_V = \lim_{n\to\infty}\langle z_n - \bar{z}, \mathcal{J}h\rangle_{V,V^*} = \lim_{n\to\infty}(z_n - \bar{z}, h)_H = (z - \bar{z}, h)_H.$$

Since $\operatorname{dom}\mathcal{J}$ is dense in H, and $h \in \operatorname{dom}\mathcal{J}$ is arbitrary, we have $z = \bar{z}$, as desired (see, e.g., [1, pp. 65, 108]). Now, using weak lower semi-continuity of lower semi-continuous convex functionals, we can pass to the limit with respect to $\hat{c}$ in (2.5) to obtain the first a priori estimate in Theorem 2. The second follows from (2.7).

Next we show that $t \to y^*(t)$ is right-continuous from $[0, T)$ to V. Since $y^* \in C(0, T; H)$ and $y^*(t) \in V$ for every $t \in [0, T]$ by (2.10), we can consider an initial value problem of the type (2.1) with initial condition $y^*(\tau)$ at $t = \tau$. Proceeding as in the last step of the proof of Theorem 1, we have:

$$\begin{aligned} &a\big(y_c(t), y_c(t)\big) + e\int_\tau^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y_c(s)\right|_H^2 \mathrm{d}s \\ &\quad\leqslant a\big(y^*(\tau), y^*(\tau)\big) + 3\int_\tau^t \big(\rho^2 |y_c(s)|_H^2 + M_1 |y(s)|_V^2 + |f(s)|_H^2\big)\,\mathrm{d}s. \end{aligned}$$

Now we can proceed as in the first part of the proof of Theorem 1 (see (2.5) with $\hat{\psi} = y_o$ and $\bar{\lambda} = 0$) to ascertain the existence of a continuous function $\rho_\tau : [\tau, T] \to \mathbb{R}$ with $\rho_\tau(\tau) = 0$ (depending on $f \in L^2(0, T; H)$, and $y_0 \in V$) such that

$$a\big(y_c(t), y_c(t)\big) + \int_\tau^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y_c(s)\right|_H^2 \mathrm{d}s \leqslant a\big(y^*(\tau), y^*(\tau)\big) + \rho_\tau(t).$$

Passing to the limit w.r.t. c, we have:

$$a\big(y^*(t), y^*(t)\big) + \int_\tau^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y^*(s)\right|_H^2 \mathrm{d}s \leqslant a\big(y^*(\tau), y^*(\tau)\big) + \rho_\tau(t).$$

This implies that

$$\limsup_{t\to\tau^+} a\big(y^*(t), y^*(t)\big) \leqslant a\big(y^*(\tau), y^*(\tau)\big).$$

Since $y^* \in C(0, T; H)$ and $\{y^*(t)\}_{t\in[0,T]}$ is bounded in V, it follows that $y^*(t) \rightharpoonup y^*(\tau)$ weakly as $t \to \tau$ and, hence

$$a\big(y^*(\tau), y^*(\tau)\big) \leqslant \liminf_{t\to\tau} a\big(y^*(t), y^*(t)\big).$$

Consequently $\lim_{t\to\tau^+} a(y^*(t), y^*(t)) = a(y^*(\tau), y^*(\tau))$. Combined with w-$\lim_{t\to\tau} y^*(t) = y^*(\tau)$ this implies $\lim_{t\to\tau^+} y^*(t) = y^*(\tau)$.

Uniqueness. If y_1^* and y_2^* are two solutions, with possibly different initial conditions and inhomogeneities, then from (2.1),

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(y_1^* - y_2^*), y_1^* - y_2^*\right) + a(y_1^* - y_2^*, y_1^* - y_2^*) \leqslant \big(f_1(t) - f_2(t), y_1^* - y_2^*\big)$$

and thus

$$\big|y_1^*(t) - y_2^*(t)\big|_H^2 + \omega \int_0^t \big|y_1^*(s) - y_2^*(s)\big|_V^2 \,\mathrm{d}s \leqslant \left(\big|y_1^*(0) - y_2^*(0)\big|_H^2 + \frac{1}{\omega}\int_0^t \big|f(t)\big|_{V^*}^2 \,\mathrm{d}s\right),$$

on $(0, T]$, which implies that the strong solution is unique. □

The following corollary shows that the strong solution is continuous with respect to the function $\psi \in H$ which defines the convex set $\mathcal{C}$.

Corollary 1. *In addition to the hypotheses of Theorem* 2 *assume that $\psi_1 - \psi_2 \in V$ and let y_i^*, $i = 1, 2$, denote the strong solutions to* (2.1) *corresponding to the closed convex sets $\mathcal{C}_i = \{y \in V\colon\ y \leqslant \psi_i\}$, $i = 1, 2$, respectively. Then,*

$$\big|y_1^*(t) - \psi_1 - \big(y_2^*(t) - \psi_2\big)\big|_H^2 + \omega \int_0^t \big|y_1^*(s) - y_2^*(s)\big|_V^2 \,\mathrm{d}s \leqslant M_5|\psi_1 - \psi_2|_V^2,$$

on $[0, T]$.

Proof. From (2.1) with $\mathcal{C}_1$ we find:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} y_1^*(t), y_2^*(t) + \psi_1 - \psi_2 - y_1^*(t)\right) + a\big(y_1^*(t), y_2^*(t) + \psi_1 - \psi_2 - y_1^*(t)\big)$$
$$- \big(f(t), y_2^*(t) + \psi_1 - \psi_2 - y_1^*(t)\big) \geqslant 0,$$

and similarly,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} y_2^*(t), y_1^*(t) + \psi_2 - \psi_1 - y_2^*(t)\right) + a\big(y_1^*(t), y_1^*(t) + \psi_2 - \psi_1 - y_2^*(t)\big)$$
$$- \big(f(t), y_1^*(t) + \psi_2 - \psi_1 - y_2^*(t)\big) \geqslant 0.$$

Adding these inequalities implies,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(y_1^*(t)-\psi_1-\big(y_2^*(t)-\psi_2\big)\big), y_1^*(t)-\psi_1-\big(y_2^*(t)-\psi_2\big)\right)$$
$$+\,a\big(y_1^*(t)-y_2^*(t), y_1^*(t)-y_2^*(t)-(\psi_1-\psi_2)\big)\leqslant 0,$$

and thus

$$\big|y_1^*(t)-\psi_1-\big(y_2^*(t)-\psi_2\big)\big|_H^2+\omega\int_0^t\big|y_1^*(t)-y_2^*(t)\big|_V^2\,\mathrm{d}t\leqslant\frac{M^2}{\omega}|\psi_1-\psi_2|_V^2. \qquad \square$$

The following two results are corollaries to the first part of Theorem 1.

Corollary 2 (Monotonicity). *Let $\bar\lambda=0$ in* (2.8) *and assume that $a(y,y^+)\geqslant 0$ for all $y\in V$. Then $y_c^k\geqslant y_{\hat c}^k$ and $y_c\geqslant y_{\hat c}$ for $c\leqslant\hat c$ and all $k=1,2,\ldots$.*

Proof. The proof is given by induction. The case $k=1$ will follow from the arguments given below. Suppose that $y_c^k\geqslant y_{\hat c}^k$ for $c\leqslant\hat c$. Then by (2.8),

$$\frac{1}{\Delta t}\big(y_c^{k+1}-y_{\hat c}^{k+1},\big(y_c^{k+1}-y_{\hat c}^{k+1}\big)^-\big)+a\big(y_c^{k+1}-y_{\hat c}^{k+1},\big(y_c^{k+1}-y_{\hat c}^{k+1}\big)^-\big)$$
$$+\big(\max\big(0,c\big(y_c^{k+1}-\psi\big)\big)-\max\big(0,\hat c\big(y_{\hat c}^{k+1}-\psi\big)\big),\big(y_c^{k+1}-y_{\hat c}^{k+1}\big)^-\big)$$
$$=\left(\frac{1}{\Delta t}\big(y_c^k-y_{\hat c}^k,\big(y_c^{k+1}-y_{\hat c}^{k+1}\big)^-\big)\right)\geqslant 0.$$

Since

$$\big(\max\big(0,c\big(y_c^{k+1}-\psi\big)\big)-\max\big(0,\hat c\big(y_{\hat c}^{k+1}-\psi\big)\big),\big(y_c^{k+1}-y_{\hat c}^{k+1}\big)^-\big)\leqslant 0,$$

for $\Delta t>0$ sufficiently small $|(y_c^{k+1}-y_{\hat c}^{k+1})^-|_H^2\leqslant 0$ and thus $y_c^{k+1}\geqslant y_{\hat c}^{k+1}$ for $c\leqslant\hat c$. The last assertion follows from the fact that $y_{\Delta t}^{(2)}$ converges strongly to y_c in $L^2(0,T;H)$, as $\Delta t\to 0^+$. $\square$

Corollary 3 (Perturbation). *Let $\psi,\hat\psi\in H$ and denote by y_c and $\hat y_c$ the corresponding solutions to* (2.6) *with $\bar\lambda=0$ and $c>0$. Assume that $(y-\gamma)^+\in V$ for all $y\in V$ and $\gamma\geqslant 0$, $a(1,\phi)\geqslant 0$ for all $\phi\geqslant 0$, and $a(y,y^+)\geqslant 0$ for all $y\in X$ with $y^+\in V$. Then for $\alpha=\max(0,\sup_{x,t}(\psi-\hat\psi))$ and $\beta=\min(0,\inf_{x,t}(\psi-\hat\psi))$, we have:*

$$\beta\leqslant y_c-\hat y_c\leqslant\alpha.$$

Proof. On $\{y_c>\psi\}\cap\{\hat y_c>\hat\psi\}$ we have:

$$\max\big(0,c(y_c-\psi)\big)-\max\big(0,c\big(\hat y_c-\hat\psi\big)\big)=c(y_c-y_{\hat c})-c\big(\psi-\hat\psi\big)\geqslant c(y-\hat y_c-\alpha),$$

and hence

$$\begin{aligned}&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(\hat y_c-\hat\psi\big)\big)\big)(y_c-\hat y_c-\alpha)^+\geqslant 0,\\&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(y_{\hat c}-\hat\psi\big)\big)\big)(y_c-\hat y_c-\beta)^-\leqslant 0.\end{aligned}$$

On $\{y_c>\psi\}\cap\{\hat y_c\leqslant\hat\psi\}$ we have $\max(0, c(y_c-\psi))-\max(0, c(\hat y_c-\hat\psi))=c(y_c-\psi)$, and hence

$$\begin{aligned}&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(\hat y_c-\hat\psi\big)\big)\big)(y_c-\hat y_c-\alpha)^+\geqslant 0,\\&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(y_{\hat c}-\hat\psi\big)\big)\big)(y_c-\hat y_c-\beta)^-=0.\end{aligned}$$

On $\{y_c\leqslant\psi\}\cap\{\hat y_c>\hat\psi\}$ we have $\max(0, c(y_c-\psi))-\max(0, c(\hat y_c-\hat\psi))=-c(\hat y_c-\hat\psi)$, and hence

$$\begin{aligned}&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(\hat y_c-\hat\psi\big)\big)\big)(y_c-\hat y_c-\alpha)^+=0,\\&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(y_{\hat c}-\hat\psi\big)\big)\big)(y_c-\hat y_c-\beta)^-\leqslant 0.\end{aligned}$$

Therefore, we have on Ω:

$$\begin{aligned}&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(\hat y_c-\hat\psi\big)\big)\big)(y_c-\hat y_c-\alpha)^+\geqslant 0,\\&\big(\max\big(0, c(y_c-\psi)\big)-\max\big(0, c\big(y_{\hat c}-\hat\psi\big)\big)\big)(y_c-\hat y_c-\beta)^-\leqslant 0.\end{aligned}\tag{2.11}$$

We proceed by induction and assume that $y_c^k-\hat y_c^k\leqslant\alpha$. Then, from (2.8),

$$\begin{aligned}&\frac{1}{\Delta t}\big(\big(y_c^{k+1}-\hat y_c^{k+1}-\alpha\big), \big(y^{k+1}-\hat y_c^{k+1}-\alpha\big)^+\big)+a\big(y_c^{k+1}-\hat y_c^{k+1}, \big(y_c^{k+1}-\hat y_c^{k+1}-\alpha\big)^+\big)\\&\quad+\big(\max\big(0, c\big(y_c^{k+1}-\psi\big)\big)-\max\big(0, c\big(\hat y_c^{k+1}-\hat\psi\big)\big), \big(y_c^{k+1}-\hat y_c^{k+1}-\alpha\big)^+\big)\\&=\left(\frac{1}{\Delta t}\big(y_c^k-\hat y_c^k-\alpha\big), \big(y_c^{k+1}-\hat y_c^{k+1}-\alpha\big)^+\right)\leqslant 0.\end{aligned}$$

From the assumptions on the bilinear form a and (2.11) it follows that $|(y_c^{k+1}-\hat y_c^{k+1}-\alpha)^+|_H^2\leqslant 0$ and thus $y_c^{k+1}-\hat y_c^{k+1}\leqslant\alpha$ a.e. Analogously one shows, using $\beta\leqslant 0$ that $y_c^{k+1}-\hat y_c^{k+1}\geqslant\beta$. The claim now follows from the fact that $y_{\Delta t}^{(2)}$ converges strongly to y_c in $L^2(0,T;H)$ as $\Delta t\to 0^+$. $\square$

2.2. Existence of Lagrange multipliers

In this section we prove that for appropriately chosen $\bar\lambda$ the sequence $\lambda_c(t)=\max(0,\bar\lambda(t)+c(y_c(t)-\psi))$ converges to the Lagrange multiplier $\lambda^*(t)$ in $L^2(0,T;H)$ associated to the constraint $y\leqslant\psi$ as $c\to\infty$.

Throughout this subsection we assume that

$$\begin{aligned}
&\psi \in X \text{ and } (y-\psi)^+ \in V \quad \text{for all } y \in V,\\
&\bar\lambda \in L^2(0,T;H),\ \bar\lambda \geqslant 0,\ \text{and } \bar\lambda(t) \geqslant A\psi + f(t) \quad \text{for a.e. } t,\\
&a(y,y^+) \geqslant 0 \quad \text{for all } y \in X \text{ with } y^+ \in V.
\end{aligned} \tag{2.12}$$

In (2.12) the condition $\bar\lambda(t) \geqslant A\psi + f(t)$ must be interpreted in the sense that for a.e. t,

$$\langle \bar\lambda(t) - (A\psi + f(t)), \phi\rangle \geqslant 0, \quad \text{for all } \phi \in V,\ \phi \geqslant 0.$$

Theorem 3. *If* (2.12) *holds and $y_0 \in \mathcal{C}$, then the solution to y_c^k to* (2.8) *satisfies $y_c^k \in \mathcal{C}$ for each $c > 0$ and $y_c^k \leqslant y_{\hat c}^k$ for $c \leqslant \hat c$ for all $k \geqslant 0$.*

Proof. For $k > 0$ define $\lambda_c^k \geqslant 0$ by:

$$\lambda_c^{k+1} = \max\big(0, \bar\lambda^k + c\big(y_c^{k+1} - \psi\big)\big),$$

where y_c^k is the solution to (2.8). We first show that $y_c^k \in \mathcal{C}$ for all k. The proof is given by induction. For $y_c^k \in \mathcal{C}$, we have from (2.8):

$$\begin{aligned}
&\frac{1}{\Delta t}\big(y_c^{k+1} - \psi, \big(y_c^{k+1} - \psi\big)^+\big) + a\big(y_c^{k+1} - \psi, \big(y_c^{k+1} - \psi\big)^+\big)\\
&\quad + \big\langle -\big(A\psi + f^k\big) + \lambda_c^{k+1}, \big(y_c^{k+1} - \psi\big)^+\big\rangle = \frac{1}{\Delta t}\big(y_c^k - \psi, \big(y_c^{k+1} - \psi\big)^+\big) \leqslant 0,
\end{aligned}$$

where

$$\big\langle\big(-\big(A\psi + f^k\big) + \lambda_c^{k+1}, \big(y_c^{k+1} - \psi\big)^+\big)\big\rangle \geqslant c\big|\big(y_c^{k+1} - \psi\big)^+\big|_H^2,$$

since $\bar\lambda^k - (A\psi + f^k) \geqslant 0$. Hence for $\Delta t > 0$ sufficiently small $|(y^{k+1} - \psi)^+|_H^2 \leqslant 0$ and thus $y_c^{k+1} \in \mathcal{C}$. Similarly, for $y_c^k \leqslant y_{\hat c}^k$ and $c \leqslant \hat c$,

$$\begin{aligned}
&\frac{1}{\Delta t}\big(y_c^{k+1} - y_{\hat c}^{k+1}, \big(y_c^{k+1} - y_{\hat c}^{k+1}\big)^+\big) + a\big(y_c^{k+1} - y_{\hat c}^{k+1}, \big(y_c^{k+1} - y_{\hat c}^{k+1}\big)^+\big)\\
&\quad + \big(\lambda_c^{k+1} - \lambda_{\hat c}^{k+1}, \big(y_c^{k+1} - y_{\hat c}^{k+1}\big)^+\big) = \left(\frac{1}{\Delta t}\big(y_c^k - y_{\hat c}^k, \big(y_c^{k+1} - y_{\hat c}^{k+1}\big)^+\big)\right) \leqslant 0,
\end{aligned}$$

where

$$\big(\lambda_c^{k+1} - \lambda_{\hat c}^{k+1}, \big(y_c^{k+1} - y_{\hat c}^{k+1}\big)^+\big) \geqslant 0.$$

Hence for $\Delta t > 0$ sufficiently small $|(y_c^{k+1} - y_{\hat c}^{k+1})^+|_H^2 \leqslant 0$ and thus $y_c^{k+1} \leqslant y_{\hat c}^{k+1}$ for $c \leqslant \hat c$. □

Corollary 4. *If in addition to the assumptions of Theorem* 3, (2.6) *holds and* $f \in L^2(0,T;H)$ *then* $y_c(t) = \lim y^{(1)}_{\Delta t} \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A)) \cap C(0,T;V)$ *as* $\Delta t \to 0^+$, (2.5) *is satisfied and*

$$\frac{\mathrm{d}}{\mathrm{d}t} y_c(t) - A y_c(t) + \max\big(0, \bar{\lambda}(t) + c\big(y_c(t) - \psi\big)\big) = f(t). \tag{2.13}$$

Moreover $y_c(t) \in \mathcal{C}$ *for each for* $c > 0$, $y_c(t) \leqslant y_{\hat{c}}(t)$ *for* $c \leqslant \hat{c}$, *and*

$$0 \leqslant \lambda_c(t) = \max\big(0, \bar{\lambda}(t) + c\big(y_c(t) - \psi\big)\big) \leqslant \bar{\lambda}(t) \quad \text{a.e. in } (0,T) \times \Omega. \tag{2.14}$$

Proof. From Theorem 3 we deduce that

$$0 \leqslant \lambda_c^{k+1} = \max\big(0, \bar{\lambda}^k + c\big(y_c^{k+1} - \psi\big)\big) \leqslant \bar{\lambda}^k \quad \text{a.e.}$$

and λ_c^{k+1} monotonically nondecreasing as c increases to ∞. From the proof of Theorem 1 it follows that $y_c(t) = \lim y^{(1)}_{\Delta t}$ strongly $L^2(0,T;H)$ and weakly in $W(0,T)$, and y_c satisfies (2.13) and (2.14). The regularity property $y_c(t) \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A)) \cap C(0,T;V)$ follows from the estimates developed in the part on strong solutions in Theorem 1 with f replaced by $\tilde{f} = f - \lambda_c \in L^2(0,T;H)$ and $\Psi_c = 0$. □

Theorem 4. *If in addition to the assumptions of Theorem* 3, (2.6) *is satisfied and* $f \in L^2(0,T;H)$ *then* (2.1) *has a unique strong solution* $y^* \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A)) \cap C(0,T;V)$ *and there exists a Lagrange multiplier* $\lambda^* \in L^2(0,T;H)$ *such that*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} y^*(t) - A y^*(t) - f(t) + \lambda^*(t) = 0, \\ &\lambda^*(t) = \max\big(0, \lambda^*(t) + \big(y^*(t) - \psi\big)\big). \end{aligned} \tag{2.15}$$

Moreover, $y_c(t) \uparrow y^*(t)$ *a.e. pointwise as* $c \to \infty$.

Proof. From Corollary 4 it follows that $\{y_c\}_{c \geqslant 1}$ is bounded in $W(0,T)$. Hence there exists a subsequence and $y^* \in L^2(0,T;H)$ with $y^*(0) = y_0$ such that $y_c \to y^*$ weakly in W and strongly in $L^2(0,T;H)$. Since $y_c \leqslant \psi$ for all $c > 0$, we have $y^* \leqslant \psi$. Moreover $\lambda_c(t)$ is bounded in $L^2(0,T;H)$ and consequently there exists $\lambda^*(t) \in L^2(0,T;H)$ such that $\lambda^*(t) \geqslant 0$ a.e. and a subsequence of $\lambda_c(t)$ converges weakly to λ^* in $L^2(0,T;H)$. Since

$$0 \leqslant \int_0^T \left(\lambda_c(t), \big(y_c(t) - \psi\big) + \frac{1}{c}\bar{\lambda}(t)\right) \mathrm{d}t \to \int_0^T \big(\lambda^*(t), y^*(t) - \psi\big)\,\mathrm{d}t,$$

$$\int_0^T \big(\lambda^*(t), y^*(t) - \psi\big)\,\mathrm{d}t = 0,$$

and thus $(y^*(t), \lambda^*(t))$ satisfies the complementarity condition. Taking the limit in

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_c(t), \phi \right\rangle + a\big(y_c(t), \phi\big) + \big(\lambda_c(t), \phi\big) - \big\langle f(t), \phi \big\rangle = 0,$$

for all $\phi \in V$ and a.e. $t \in (0, T]$, we have:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y^*(t), \phi \right\rangle + a\big(y^*(t), \phi\big) + \big(\lambda^*(t), \phi\big) - \big\langle f(t), \phi \big\rangle = 0,$$

and hence

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y^*(t), y - y^*(t) \right\rangle + a\big(y^*(t), y - y^*(t)\big) - \big\langle f(t), y - y^*(t) \big\rangle = 0,$$

for all $\phi \in V$ and a.e. $t \in (0, T]$. Moreover $\frac{\mathrm{d}}{\mathrm{d}t} y^*(t) - Ay^*(t) - \tilde{f}(t) = 0$ in V^*, where $\tilde{f} = \lambda^* - f$. Since $\tilde{f} \in L^2(0, T; H)$ and $y_0 \in V$, we have $y^* \in H^1(0, T; H) \cap L^2(0, T; \mathrm{dom}(A)) \cap C(0, T; V)$. □

Corollary 5. *In addition to the assumptions in Theorem* 4 *assume* $\bar{\lambda}(t) \in L^p((0, T) \times \Omega)$, $2 \leqslant p \leqslant \infty$. *Then* $\lambda^* \in L^p((0, T) \times \Omega)$.

2.3. Weak solutions

In this section we consider weak solutions to (2.1).

Definition 2 *(Weak solution).* Assume that $y_0 \in H$ and $f \in L^2(0, T, V^*)$. Then a function $y^* \in L^2(0, T; V)$ satisfying $y^*(t, x) \leqslant \psi(x)$ a.e. in $(0, T) \times \Omega$ is called weak solution to (2.1) if,

$$\int_0^T \left[\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y(t), y(t) - y^*(t) \right\rangle + a\big(y^*(t), y(t) - y^*(t)\big) - \big\langle f(t), y(t) - y^*(t) \big\rangle \right] \mathrm{d}t + \frac{1}{2} \big| y(0) - y_0 \big|_H^2 \geqslant 0 \tag{2.16}$$

is satisfied for all $y \in \mathcal{K}$, where

$$\mathcal{K} = \big\{ y \in W(0, T) \colon\ y(t, x) \leqslant \psi(x) \text{ a.e. in } (0, T) \times \Omega \big\}.$$

If $\bar{y}$ is a strong solution, then

$$\int_0^T \left[\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \bar{y}(t), y(t) - \bar{y}(t) \right\rangle + a\big(\bar{y}(t), y(t) - \bar{y}(t)\big) - \big\langle f(t), y(t) - \bar{y}(t) \big\rangle \right] \mathrm{d}t \geqslant 0,$$

for all $y \in \mathcal{K}$. Setting $y = \bar{y}$ in (2.16) and $y = y^*$ in the above inequality we have $a(\bar{y} - y^*, \bar{y} - y^*) = 0$ for a.e. t. Consequently, if (2.1) admits a strong solution then it is a weak solution and the weak solution is unique. We have also the following stronger result.

Theorem 5. *Assume that $y_0 \in H$ and $f \in L^2(0, T, V^*)$. Then there exists a unique weak solution to y^* to* (2.1).

Proof. For each $c > 0$, let $y_c^k = y^k$ be the unique solution to (2.8) with $\bar{\lambda} = 0$. From (2.9) it follows that for each $k \geqslant 1$ the families $|y_c^k|_V$ and $c|(y_c^k - \psi)^+|_H^2$ are bounded in $c > 0$. Thus there exists a subsequence of $\{y_c^k\}$ that converges to some $y^k \in V$, weakly in V and strongly in H as $c \to \infty$. Moreover $|(y^k - \psi)^+|_H = 0$ and hence $y^k \in \mathcal{C}$. Since

$$\big(\max\big(0, c\big(y_c^k - \psi\big)\big), y - y_c^k\big) = \big(\max\big(0, c\big(y_c^k - \psi\big)\big), y - \psi - \big(y_c^k - \psi\big)\big) \leqslant 0 \quad \text{for all } y \in \mathcal{C},$$

we obtain from (2.8) that y^k, $k \geqslant 0$, satisfies,

$$\left(\frac{y^{k+1} - y^k}{\Delta t}, y - y^{k+1}\right) + a\big(y^{k+1}, y - y^{k+1}\big) - \big\langle f^k, y - y^{k+1}\big\rangle \geqslant 0, \tag{2.17}$$

for all $y \in \mathcal{C}$. Moreover it follows from (2.9) that

$$\sum_{k=1}^{N} \big(\big|y^k - y^{k-1}\big|_H^2 + \big|y^k\big|_V^2 \Delta t\big) \text{ is bounded,} \tag{2.18}$$

with respect to N, where $N\Delta t = T$. Thus it follows from the proof of Theorem 1 that there exist subsequences of $y_{\Delta t}^{(1)}, y_{\Delta t}^{(2)}$ (denoted by the same symbols) and $y^*(t) \in L^2(0, T; V)$ such that

$$y_{\Delta t}^{(1)}(t), y_{\Delta t}^{(2)}(t) \to y^*(t) \quad \text{weakly in } L^2(0, T; V) \text{ and strongly in } L^2(0, T; H),$$

as $\Delta t \to 0$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} = \frac{y^{k+1} - y^k}{\Delta t} \quad \text{on } \big(k\Delta t, (k+1)\Delta t\big].$$

Thus we have from (2.17) for every $y \in \mathcal{K}$,

$$\begin{aligned} &\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y, y - y_{\Delta t}^{(2)} \right\rangle + a\big(y_{\Delta t}^{(2)}, y - y_{\Delta t}^{(2)}\big) - \big\langle f^k, y - y_{\Delta t}^{(2)}\big\rangle \\ &\quad + \left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} y, y - y_{\Delta t}^{(2)} \right\rangle \geqslant 0, \end{aligned} \tag{2.19}$$

a.e. in $(0, T)$. We have:

$$\begin{aligned}&\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} y, y - y_{\Delta t}^{(2)} \right\rangle \\ &\quad = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} y, y - y_{\Delta t}^{(1)} \right\rangle + \left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} y, y_{\Delta t}^{(1)} - y_{\Delta t}^{(2)} \right\rangle, \end{aligned} \tag{2.20}$$

where

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} y, y - y_{\Delta t}^{(1)} \right\rangle \mathrm{d}t \leqslant \frac{1}{2} \left| y(0) - y_0 \right|_H^2 \tag{2.21}$$

and

$$\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)}, y_{\Delta t}^{(1)} - y_{\Delta t}^{(2)} \right) \mathrm{d}t = -\frac{1}{2} \sum_{k=1}^N \left| y^k - y^{k-1} \right|^2. \tag{2.22}$$

Since,

$$\int_0^T a\big(y^*(t), y^*(t)\big)\,\mathrm{d}t \leqslant \liminf_{\Delta t \to 0} \int_0^T a\big(y_{\Delta t}^{(2)}(t), y_{\Delta t}^{(2)}(t)\big)\,\mathrm{d}t,$$

it follows from (2.19)–(2.22) that every weak cluster point y^* of $y_{\Delta t}^{(2)}$ in $L^2(0, T; V)$ is a weak solution.

Uniqueness. Let y^* be a weak solution. Setting $y = y_{\Delta t}^{(1)} \in \mathcal{K}$ in (2.16) and $y = y^*(t) \in \mathcal{C}$ in (2.17), we have:

$$\begin{aligned}&\int_0^T \left[\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)}, y_{\Delta t}^{(1)} - y^* \right\rangle + a\big(y^*, y_{\Delta t}^{(1)} - y^*\big) - \big\langle f, y_{\Delta t}^{(1)} - y^* \big\rangle \right] \mathrm{d}t \geqslant 0, \\ &\int_0^T \left[\left\langle \frac{\mathrm{d}}{\mathrm{d}t} y_{\Delta t}^{(1)}, y^* - y_{\Delta t}^{(2)} \right\rangle + a\big(y_{\Delta t}^{(2)}, y^* - y_{\Delta t}^{(2)}\big) - \big\langle f, y^* - y_{\Delta t}^{(2)} \big\rangle \right] \mathrm{d}t \geqslant 0.\end{aligned}$$

Summing up these inequalities, and using (2.22) implies that

$$\begin{aligned}&\int_0^T \Big(a\big(y^*, y_{\Delta t}^{(1)} - y_{\Delta t}^{(2)}\big) - \big\langle f, y_{\Delta t}^{(1)} - y_{\Delta t}^{(2)} \big\rangle \Big)\,\mathrm{d}t \\ &\quad \geqslant \frac{1}{2} \sum_{k=1}^N \left| y^k - y^{k-1} \right|_H^2 + \int_0^T a\big(y^* - y_{\Delta t}^{(2)}, y^* - y_{\Delta t}^{(2)}\big)\,\mathrm{d}t.\end{aligned}$$

Letting $\Delta t \to 0^+$ we obtain $0 \geqslant a(y^*(t) - \hat{y}(t), y^*(t) - \hat{y}(t))$ a.e. on $(0, T)$, for every weak cluster point $\hat{y}$ of $y^{(2)}_{\Delta t}$ in $L^2(0, T; V)$. This implies that the weak solution is unique. □

2.4. Time dependent obstacles

In this subsection we discuss the extension of the previous sections to the case that the obstacle depends on t.

(1) If $\psi \in L^2(0, T; H)$ and

$$\mathcal{K} = \big\{ y \in W(0, T)\colon\ y(t, x) \leqslant \psi(t, x) \text{ a.e. in } (0, T) \times \Omega \big\}$$

is nonempty, then (2.3) has a unique solution $y_c(t)$ in $W(0, T)$ for each $c > 0$ and there exits a weak solution $y^* \in L^2(0, T; V)$ to (2.1) satisfying $y^* \leqslant \psi(t)$. Here $\mathcal{C}$ in (2.1) has to be replaced by $\mathcal{C}(t) = \{h \in H\colon\ y \leqslant \psi(t)\} \cap V$ for a.e. t. For the proof we consider the modified finite difference approximation from the proof of Theorem 1:

$$\left(\frac{y^{k+1} - y^k}{\Delta t}, \phi\right) + a\big(y^{k+1}, \phi\big) + \big(\max\big(0, \bar{\lambda}^k + c\big(y^{k+1} - \psi^{k+1}\big)\big), \phi\big) - \big\langle f^k, \phi\big\rangle = 0, \quad \text{for all } \phi \in V, \tag{2.23}$$

with $y^0 = y_0 \in H$ and $\psi^{k+1} = \frac{1}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \psi(s)\,\mathrm{d}s$. If we replace $\hat{\psi}$ by $\hat{\psi}^{k+1} = \frac{1}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \hat{\psi}(s)\,\mathrm{d}s$ for $\hat{\psi} \in \mathcal{K}$ and let,

$$\lambda^{k+1} = \max\big(0, \bar{\lambda}^k + c\big(y^{k+1} - \psi^{k+1}\big)\big),$$

in the proof of Theorem 1, we obtain for $k = 0, 1, \ldots$

$$\begin{aligned}
&\frac{1}{2\Delta t}\big(\big|y^{k+1} - \hat{\psi}^{k+1}\big|_H^2 - \big|y^k - \hat{\psi}^k\big|_H^2 + \big|\big(y^{k+1} - \hat{\psi}^{k+1}\big) - \big(y^k - \hat{\psi}^k\big)\big|_H^2\big) \\
&\quad + \omega\big|y^{k+1} - \hat{\psi}^{k+1}\big|_V^2 - \rho\big|y^{k+1} - \hat{\psi}^{k+1}\big|_H^2 + \frac{1}{2c}\big|\lambda^{k+1}\big|_H^2 \\
&\leqslant \frac{1}{2c}\big|\bar{\lambda}^k\big|_H^2 + \big|y^{k+1} - \hat{\psi}^{k+1}\big|\big|f^k\big|_{V^*} \\
&\quad + \big|y^{k+1} - \hat{\psi}^{k+1}\big|_V\left(M\big|\hat{\psi}^{k+1}\big|_V + \left|\frac{\hat{\psi}^{k+1} - \hat{\psi}^k}{\Delta t}\right|_{V^*}\right),
\end{aligned}$$

where $\hat{\psi}^0 := \hat{\psi}^1$. Here we used the fact that

$$(a - b)(a - c) = \frac{1}{2}|a - c|^2 - \frac{1}{2}|b - d|^2 + \frac{1}{2}\big|(a - c) - (b - d)\big|^2 + (a - c)(c - d).$$

The previous estimate implies the analog of (2.9) for t-dependent obstacles:

$$
\begin{aligned}
&\big|y^k-\hat\psi^k\big|_H^2+\sum_{i=1}^k\bigg(\bigg(\omega\big|y^i-\hat\psi^i\big|_V^2+\frac1c\big|\lambda^i\big|_H^2\bigg)\Delta t+\big|\big(y^i-\psi^i\big)-\big(y^{i-1}-\psi^{i-1}\big)\big|_H^2\bigg)\\
&\quad\leqslant\big|y_0-\hat\psi(0)\big|_H^2+\sum_{i=1}^k\bigg(\frac{2M^2}{\omega}\big|\hat\psi^{i-1}\big|_V^2+\frac1c\big|\bar\lambda^{i-1}\big|_H^2+\frac2\omega\big|f^{i-1}\big|_{V^*}^2\\
&\qquad+\frac4\omega\bigg|\frac{\hat\psi^i-\hat\psi^{i-1}}{\Delta t}\bigg|_{V^*}\bigg)\Delta t.
\end{aligned}
\tag{2.24}
$$

Proceeding as in the proof of Theorem 1 we obtain the existence of a unique $y_c\in W(0,T)$ satisfying (2.3) with ψ replaced by $\psi(t)$, and

$$
\begin{aligned}
&\big|y_c(t)-\hat\psi(t)\big|_H^2+\int_0^t\bigg(\omega\big|y_c(s)-\hat\psi(s)\big|_V^2+\frac1c\big|\lambda_c(s)\big|_H^2\bigg)\,\mathrm ds\\
&\quad\leqslant\big|y_0-\hat\psi(0)\big|^2+\int_0^t\bigg(\frac{2M^2}{\omega}\big|\hat\psi(s)\big|_V^2+\frac1c\big|\hat\lambda(s)\big|_H^2+\frac4\omega\big|f(s)\big|_{V^*}^2\\
&\qquad+\frac4\omega\bigg|\frac{\mathrm d}{\mathrm ds}\hat\psi(s)\bigg|_{V^*}^2\bigg)\,\mathrm ds,
\end{aligned}
\tag{2.25}
$$

where $\lambda_c(t)=\max(0,\bar\lambda(t)+c(y_c(t)-\psi(t)))$ for every $\hat\psi\in\mathcal K$. The existence of a weak solution to (2.1) is verified as in the proof of Theorem 5, replacing $\mathcal C$ by $\mathcal C^{k+1}=\{y\in V\colon\ y\leqslant\psi^{k+1}\}$ and (2.9) by (2.24).

Note that by means of the transformations $\hat y=\mathrm e^{-\rho t}y^*$, $\hat f=\mathrm e^{-\rho t}f$, and $\hat\psi=\mathrm e^{-\rho t}\psi$ the variational inequality is transformed into:

$$
\begin{cases}
\langle\frac{\mathrm d}{\mathrm dt}\hat y(t),y-\hat y(t)\rangle+a(\hat y(t),y-\hat y(t))+\rho(\hat y(t),y-\hat y(t))_H-\langle f(t),y-\hat y(t)\rangle\geqslant0,\\
y(0)=y_0,
\end{cases}
$$

for all $y\in V$ with $y\leqslant\hat\psi(t)$. Here the bilinear form $\hat a(\cdot,\cdot)=a(\cdot,\cdot)+\rho(\cdot,\cdot)_H$ satisfies $\hat a(\phi,\phi)\geqslant|\phi|_V^2$ for all $\phi\in V$.

(2) If $\frac{\mathrm d}{\mathrm dt}\psi\in L^2(0,T;V)$, then Theorems 1 and 2 remain valid with appropriately modified a priori estimates. In this case $\psi\in C(0,T;H)$ and hence in the estimates in (1) above the values for ψ^k can be defined by $\psi(k\Delta t)$ and analogously $\hat\psi^k=\hat\psi(k\Delta t)$, for $k=0,1,\ldots$. Setting $\phi=\frac{y^{k+1}-\psi^{k+1}-(y^k-\psi^k)}{\Delta t}\in V$ in (2.8) we find,

$$
\begin{aligned}
&\frac12\bigg|\frac{y^{k+1}-y^k}{\Delta t}\bigg|_H^2+\frac{1}{2\Delta t}\big(a_s\big(y^{k+1},y^{k+1}\big)-a_s\big(y^k,y^k\big)+a_s\big(y^{k+1}-y^k,y^{k+1}-y^k\big)\big)\\
&\quad+\Psi_c\big(y^{k+1}-\psi^{k+1}\big)-\Psi_c\big(y^k-\psi^k\big)
\end{aligned}
$$

$$\leqslant \left|\rho\left(y^{k+1}, \frac{(y^{k+1}-\psi^{k+1})-(y^k-\psi^k)}{\Delta t}\right)\right| + M_1\left|y^{k+1}\right|_V\left|\frac{(y^{k+1}-\psi^{k+1})-(y^k-\psi^k)}{\Delta t}\right|_H + \left|f^k\right|_H\left(\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H + \left|\frac{\psi^{k+1}-\psi^k}{\Delta t}\right|_H\right)\mathrm{d}s + \frac{1}{2}\left|\frac{\psi^{k+1}-\psi^k}{\Delta t}\right|_H + \frac{M}{2}\left(\left|y^{k+1}\right|_V^2 + \left|\frac{\psi^{k+1}-\psi^k}{\Delta t}\right|_V^2\right)$$

$$\leqslant \frac{1}{4}\left|\frac{y^{k+1}-y^k}{\Delta t}\right|_H^2 + C\left(\rho^2\left|y^{k+1}\right|_H^2 + M_1^2\left|y^{k+1}\right|_V^2 + \left|f^k\right|_H^2 + \left|\frac{\psi^{k+1}-\psi^k}{\Delta t}\right|_V^2\right),$$

for some constant C independent of c and k. Hence $y_{\Delta t}^{(1)}$ is bounded in $H^1(0,T;H) \cap C(0,T;V)$, and the conclusion of the second part of Theorem 1 remains valid with (2.7) replaced by:

$$a\big(y_c(t), y_c(t)\big) + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y_c(s)\right|_H^2 \mathrm{d}s \leqslant M_2\left(|y_0|_V^2 + c\left|\left(y_0 - \psi^0 + \frac{\bar{\lambda}}{c}\right)^+\right|_H^2 + \int_0^t \left(\left|f(s)\right|_H^2 + \left|\frac{\mathrm{d}}{\mathrm{d}s}\psi(s)\right|_{V^*}^2\right)\mathrm{d}s\right), \tag{2.26}$$

on $[0,T]$, with $M_2 > 0$ independent of $c > 0$.

As in the proof of Theorem 2 we now obtain the existence of a strong solution $y^* \in H^1(0,T;H) \cap C(0,T;V)$ satisfying:

$$\left|y^*(t) - \hat{\psi}(t)\right|_H^2 + \int_0^t \omega\left|y_c(s) - \hat{\psi}(s)\right|_V^2 \mathrm{d}s \leqslant \left|y_0 - \hat{\psi}(0)\right|^2 + \int_0^t \left(\frac{2M^2}{\omega}\left|\hat{\psi}(s)\right|_V^2 + \frac{4}{\omega}\left|f(s)\right|_{V^*}^2 + \frac{4}{\omega}\left|\frac{\mathrm{d}}{\mathrm{d}s}\hat{\psi}(s)\right|_{V^*}^2\right)\mathrm{d}s, \tag{2.27}$$

and

$$a\big(y^*(t), y^*(t)\big) + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s} y^*(s)\right|_H^2 \mathrm{d}s \leqslant M_2\left(|y_0|_V^2 + \int_0^t \left(\left|f(s)\right|_H^2 + \left|\frac{\mathrm{d}}{\mathrm{d}s}\psi(s)\right|_{V^*}^2\right)\mathrm{d}s\right). \tag{2.28}$$

(3) If $\psi \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A))$, $y_0 \in V$, $y_0 \leqslant \psi(0)$, and

$$\begin{aligned}
&\big(y-\psi(t)\big)^+ \in V \quad \text{for all } y \in V, \text{ and } t \in [0,T],\\
&\bar\lambda \in L^2(0,T;H),\ \bar\lambda \geqslant 0,\ \text{and } \bar\lambda(t) \geqslant A\psi(t) + f(t) - \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) \quad \text{for a.e. } t,\\
&a(y,y^+) \geqslant 0 \quad \text{for all } y \in X \text{ satisfying } y^+ \in V,
\end{aligned} \tag{2.29}$$

then Theorems 3 and 4, and Corollary 4 hold. In fact, for $k = 0,1,\dots$ define $f^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} f(s)\,\mathrm{d}s$, $\psi^k = \psi(k\Delta t)$, and

$$\bar\lambda^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} \bar\lambda(s)\,\mathrm{d}s - \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} A\psi(s)\,\mathrm{d}s + A\psi^{k+1}.$$

Then by (2.29):

$$\bar\lambda^k \geqslant -\frac{\psi^{k+1}-\psi^k}{\Delta t} + A\psi^{k+1} + f^k.$$

Let y_c^k denote the solution to (2.23) and define $\lambda_c^k \geqslant 0$ by:

$$\lambda_c^{k+1} = \max\big(0, \bar\lambda^k + c\big(y^{k+1}-\psi^{k+1}\big)\big).$$

As in the proof of Theorem 3, we can show by induction that $y_c^{k+1} \leqslant \psi^{k+1}$, since

$$\begin{aligned}
&\frac{1}{\Delta t}\big(y^{k+1}-\psi^{k+1}, \big(y^{k+1}-\psi^{k+1}\big)^+\big) + a\big(y^{k+1}-\psi^{k+1}, \big(y^{k+1}-\psi^{k+1}\big)^+\big)\\
&\qquad + \left\langle -\left(A\psi^{k+1} + f^k - \frac{\psi^{k+1}-\psi^k}{\Delta t}\right) + \lambda_c^{k+1}, \big(y^{k+1}-\psi^{k+1}\big)^+\right\rangle\\
&\quad = \frac{1}{\Delta t}\big(y^k-\psi^k, \big(y^{k+1}-\psi^{k+1}\big)^+\big) \leqslant 0.
\end{aligned}$$

Similarly, it follows that $y_c^{k+1} \leqslant y_{\hat c}^{k+1}$ for $0 < c \leqslant \hat c$. Now the same arguments as in the proofs of Corollaries 4 and 5 can be used to extend these results to case of t-dependent ψ.

(4) If $\psi \in C(0,T;H)$ is nondecreasing and concave, then the weak solution to (2.1) is unique. In fact, we can repeat the argument in the proof of Theorem 5 with (2.8), (2.9) replaced by (2.23), (2.24), $\mathcal{C}$ replaced by $\mathcal{C}^{k+1} = \{y \in V\colon y \leqslant \psi^{k+1}\}$, where $\psi^k = \psi(k\Delta t)$. Then the uniqueness argument remains applicable since $y_{\Delta t}^{(1)} \in \mathcal{K}$ due to concavity and since $y^*(t) \leqslant \psi^{k+1}$ on $(k\Delta t, (k+1)\Delta t)$.

3. Black–Scholes model for American options

We consider the Black–Scholes model for American options, which is a variational inequality of the form:

$$\begin{aligned} &\frac{d}{dt}v(t,S)+\frac{\sigma^2}{2}S^2v_{SS}+(r-\delta)Sv_S-rv\leqslant 0 \quad \perp \quad v(t,S)\geqslant \psi(S),\\ &v(T,S)=\psi(S) \end{aligned} \tag{3.1}$$

for a.e. $(t,S)\in(0,T)\times(0,\infty)$, where $\perp$ indicates that both inequalities are satisfied with at least one of them holding as equality for a.e. (t,S). For the put option $\psi(S)=\max(0,K-S)$ and for the call $\psi(S)=\max(0,S-K)$. Here $S\geqslant 0$ denotes the price, v the value of the share, $r>0$ is the interest rate, δ models the influence of dividends, $\sigma>0$ is the volatility of the market and K is the strike price. Further T is the maturity date and ψ the pay-off function. Note that (3.1) is a backwards equation with respect to the time variable. The complementarity system (3.1) has the following interpretation [14,16] in mathematical finance. The price process S_t is governed by the Ito's stochastic differential equation,

$$dS_t=rS_t\,dt+\sigma S_t\,dB_t,$$

where B_t denotes Brownian motion and the value function v is represented by:

$$v(t,S)=\sup_\tau E^{t,x}\big[e^{-r(\tau-t)}\psi(S_\tau)\big],\quad \text{over all stopping times } \tau\leqslant T. \tag{3.2}$$

To express (3.1) in variational form, we define,

$$a(v,\phi)=\int_{S_{\min}}^{S_{\max}}\left(\left(\frac{\sigma^2}{2}S^2v_S+\big(r-\delta-\sigma^2\big)Sv\right)\phi_S+\big(2r-\delta-\sigma^2\big)v\phi\right)dS, \tag{3.3}$$

for $v,\phi\in V$, where V is the completion of the space:

$$\left\{\phi\in H\colon \phi \text{ is absolutely continuous on } (S_{\min},S_{\max}),\ \int_{S_{\min}}^{S_{\max}}S^2|\phi_S|^2\,dS<\infty \text{ and } \phi(S)\to 0 \text{ as } S\to S_{\max} \text{ and } S\to S_{\min}\right\}$$

under the norm

$$|\phi|_V^2=\int_{S_{\min}}^{S_{\max}}\big(S^2|\phi_S|^2+|\phi|^2\big)\,dS.$$

We have the following estimates:

$$a(v,\phi) \leqslant \frac{\sigma^2}{2}|v|_V|\phi|_V + \left|r-\sigma^2-\delta\right||v|_H|\phi|_V + \left|2r-\sigma^2-\delta\right||v|_H|\phi|_H$$

and

$$\begin{aligned} a(v,v) &\geqslant \frac{\sigma^2}{2}|v|_V^2 + \left(2r-\frac{3}{2}\sigma^2-\delta\right)|v|_H^2 - \left|r-\sigma^2-\delta\right||v|_V|v|_H \\ &\geqslant \frac{\sigma^2}{4}|v|_V^2 + \left(2r-\frac{3}{2}\sigma^2-\delta-\frac{(r-\sigma^2-\delta)^2}{\sigma^2}\right)|v|_H^2, \end{aligned}$$

where $H = L^2(S_{\min}, S_{\max})$. The solution to (3.1) satisfies $v-\psi \in V$. Setting $y(t,S) = v(T-t,S)-\psi$ we arrive at

$$\begin{cases} \langle \frac{\mathrm{d}}{\mathrm{d}t} y^*(t), y(t)-y^*(t)\rangle + a(y^*(t), y(t)-y^*(t)) - a(\psi, y(t)-y^*(t)) \geqslant 0 \\ \quad \text{for all } y \in \mathcal{C}, \\ y^*(0,S) = y_0, \end{cases} \tag{3.4}$$

where $\mathcal{C} = \{y \in H\colon\ y \geqslant 0\}$, or in strong form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y^*(t) - Ay^*(t) - A\psi \geqslant 0, \\ y^*(0,S) = y_0, \end{cases}$$

where $Ay = \frac{\sigma^2}{2}S^2 v_{SS} + (r-\delta)Sv_S - rv$. Note that compared to (2.2) the sign is reversed.

Let us briefly comment on the call and put cases. For the call case with $\delta = 0$ we have $\langle A\psi, \phi\rangle \geqslant 0$ for all $\phi \in \mathcal{C}$ and hence it can argued that European options (i.e., the variational inequality in (3.1) is replaced by a parabolic equation without constraints) coincide with American options. Turning to the case with dividends we note that (3.1) has the equilibrium solution,

$$\bar{v} = \begin{cases} K+S, & S \geqslant \bar{S}, \\ S^\rho, & S \leqslant \bar{S}, \end{cases} \tag{3.5}$$

where

$$\bar{S} = \frac{K\gamma}{\gamma-1}, \quad \gamma = \frac{(\delta+\sigma^2/2-r)+\sqrt{(\delta+\sigma^2/2-r)^2+2\sigma^2 r}}{\sigma^2}.$$

The equilibrium solution ($V_t = 0$) satisfies the Cauchy–Euler equation,

$$\frac{\sigma^2}{2}S^2 v_{SS} + (r-\delta)Sv_S - rv = 0,$$

on $(0, \bar{S})$ and thus has the form:

$$V(S) = CS^{\gamma},$$

where γ must satisfy $\frac{\sigma^2}{2}\gamma(\gamma-1)(r-\delta)\gamma - r = 0$: This equation admits the solution $\gamma > 1$ given above. Since $v \in H^2(0,\infty)$ we must have:

$$v(\bar{S}) = \bar{S} - K = C\bar{S}^{\gamma}, \qquad v_S(\bar{S}) = C\gamma\bar{S}^{\gamma-1} = 1,$$

which yields (3.5). It can be verified that $v(t,S) \leqslant \bar{S}$ for all $t \leqslant T$ and $S \geqslant 0$ and hence in the call case with dividends one can choose $S_{\max} = \bar{S}$, while $S_{\min} = 0$.

For the put case $(0,\infty)$ can be replaced by $(\bar{S},\infty)$. In fact (3.1) with $\delta = 0$ has the equilibrium solution of the form:

$$\bar{v} = \begin{cases} (K-\bar{S})(\frac{S}{\bar{S}})^{-\gamma}, & S \geqslant \bar{S}, \\ (K-S), & S \leqslant \bar{S}, \end{cases} \tag{3.6}$$

where

$$\gamma = \frac{2r}{\sigma^2}, \qquad \bar{S} = \frac{K\gamma}{1+\gamma}.$$

The equilibrium solution satisfies the Cauchy–Euler equation,

$$\frac{\sigma^2}{2}S^2 v_{SS} + rSv_S - rv = 0,$$

on $(\bar{S},\infty)$ and thus can be written as

$$v(S) = C_1 S^{s_1} + C_2 S^{s_2},$$

where s_1, s_2 satisfy

$$\frac{\sigma^2}{2}s(s-1) + rs - r = \left(\frac{\sigma^2}{2}s + r\right)(s-1) = 0.$$

That is, $s = -\gamma$ and $s = 1$. Since $v \to 0$ as $S \to \infty$, we have $v = C_1 S^{-\gamma}$. Since $v \in H^2(0,\infty)$ we must have:

$$v(\bar{S}) = K - \bar{S}, \qquad v_S(\bar{S}) = (K-\bar{S})\frac{-\gamma}{\bar{S}} = -1,$$

which yields (3.6). It can be argued that $v(t,S) \leqslant \bar{v}(S)$ for all $t \leqslant T$ and $v(t,S) \to \bar{v}(S)$ monotonically as $t \to -\infty$, for all $S \geqslant 0$. Hence in the put case we can choose $S_{\min} = \bar{S}$, which allows to avoid the singularity at 0, while $S_{\max} = \infty$.

4. Convergence rate

In this section we study the convergence of the solutions y_c of the regularized problem (2.3) to the solution y^* of (2.1) as $c \to 0$. We assume that the bilinear form a is:

$$a(y,\phi) = \int_\Omega \left[a_{ij}\partial_{x_i} y \partial_{x_j}\phi + (b_i \partial_{x_i} y + \mathrm{d}y)\phi\right] \mathrm{d}x,$$

for $y, \phi \in X = H^1(\Omega)$, where we use the summation convention. The leading differential operator is assumed to be uniformly elliptic, all coefficients are in $L^\infty(\Omega)$ and $d \geqslant 0$. Moreover we assume that

$$\mathrm{dom}(A) \subset C(\bar{\Omega}). \tag{4.1}$$

This is the case, for example, if $V = H_0^1(\Omega)$, where Ω is a polyhedron or it has a $C^{1,1}$ boundary, and $a_{ij} \in W^{1,p}(\Omega)$, $p > n$, $b_i \in L^p$, $p > n$, $d \in L^{\hat{p}}$, $\hat{p} \geqslant \max(p,4)/2$.

Our objective is to prove convergence of y_c to y^* in $L^\infty((0,T) \times \Omega)$ with rate $1/c$, provided certain regularity conditions are satisfied. Some preliminary considerations are required. Let $K = \{v \in V\colon\ v \geqslant 0, \text{a.e. in } \Omega\}$, and let $K^* = \{v^* \in V^* = H^{-1}(\Omega)$: $\langle v^*, v\rangle \geqslant 0 \text{ for all } v \in K\}$ denote the dual cone. Then V^* is a Hilbert lattice with respect to the ordering induced by K^*, and every $v^* \in V^*$ can be uniquely decomposed as $v^* = (v^*)^+ - (v^*)^-$ with $(v^*)^{+/-} \in K^*$ [3,15]. We say that $v^* \in K^*$ is bounded above by the constant $|v^*|_\infty \in [0,\infty)$, if

$$\left\langle |v^*|_\infty - v^*, v\right\rangle_{V^*,V} \geqslant 0 \quad \text{for all } v \in K.$$

We say that $v^* \in V^*$ is bounded by a constant if $(v^*)^+$ and $(v^*)^-$ are bounded above by constants and we set:

$$|v^*|_\infty = \max\left(\left|(v^*)^+\right|, \left|(v^*)^-\right|\right).$$

For example, consider the case $\Omega = (-1,1)$, let $\psi(x) = |x|$ and $\Delta\psi \in V^*$, where $\Delta : V \to V^*$ is the Laplacian with Dirichlet boundary conditions. Then $|(\Delta\psi)^+|_\infty = \infty$ and $|(\Delta\psi)^-|_\infty = 0$. If $v^* \in L^\infty(\Omega) \subset V^*$ then $|(v^*)^+|_\infty = |(v^*)^+|_{L^\infty(\Omega)}$, $|(v^*)^-|_\infty = |(v^*)^-|_{L^\infty(\Omega)}$ and $|v^*|_\infty = |v^*|_{L^\infty(\Omega)}$.

We assume throughout this section that

$$\begin{aligned} &y_0 \in V, \quad y_0 \leqslant \psi, \quad \psi \in X, \quad \left|\left(A\psi + f(\cdot)\right)^+\right|_\infty \in L^\infty(0,T), \\ &f \in L^2(0,T;H), \end{aligned} \tag{4.2}$$

and that

$$\begin{aligned} &y^+ \in V,\ (y-\psi)^+ \in V, \quad \text{for all } y \in V, \text{ and} \\ &a(y,y^+) \geqslant 0, \quad \text{for all } y \in X \text{ satisfying } y^+ \in V. \end{aligned} \tag{4.3}$$

With these conditions satisfied, Theorems 1 and 4, with $\bar{\lambda} = |(A\psi + f(\cdot))^+|_\infty$ imply the existence of $y_c \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A))$, $y^* \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A))$, and $\lambda^* \in L^2(0,T;H)$, which are solutions to (2.3) and (2.15), respectively. Moreover, using (4.1) we have:

$$y_c \in L^2\big(0,T;C(\bar{\Omega})\big) \cap H^1\big((0,T)\times\Omega\big) \quad \text{and}$$
$$y^* \in L^2\big(0,T;C(\bar{\Omega})\big) \cap H^1\big((0,T)\times\Omega\big). \tag{4.4}$$

We require a technical lemma which we describe next. For this purpose let Q denote a non-cylindrical open subset of $(0,T)\times\Omega$ and define $\Omega_t = \{x\colon (t,x)\in Q\}$, for $t\in(0,T)$, and $\Omega_0 = \{x\colon (0,x)\in\bar{Q}\}$, $\Omega_T = \{x\colon (T,x)\in\bar{Q}\}$. Let $(\cdot,\cdot)_{\Omega_t}$ denote the standard inner product on Ω_t. The restriction of a to $H^1(\Omega_t)\times H^1(\Omega_t)$ will again be denoted by a.

Lemma 4.1. *Assume that $Q = \{(t,x)\colon t\in(0,T), x\in\Omega_t\}$ is a sub-domain of $(0,T)\times\Omega$ with Lipschitzian boundary, with $g\in L^\infty(0,T;H^{-1}(\Omega_t))$ with $\operatorname{ess\,sup}_{t\in(0,T)}|g(t)|_{\infty,\Omega_t}<0$, and that $a(1,\phi^+)\geqslant 0$, $a(\phi,\phi^+)\geqslant 0$, for all $\phi\in H^1(\Omega_t)$, $t\in(0,T)$. Let $c>0$, and assume that $y\in Y=\{y\in H^1(Q)\colon y(t,x)=0$ for $t\in(0,T), x\in\partial\Omega_t\}$ satisfies $y(0,\cdot)=0$ a.e. in Ω_0, and*

$$\int_0^t \bigg[\Big(\frac{\mathrm{d}}{\mathrm{d}s}y(s),\phi(s)\Big)_{\Omega_s} + a\big(y(s),\phi(s)\big) + c\big(y(s),\phi(s)\big)_{\Omega_s} - \big\langle g(s),\phi(s)\big\rangle_{H^{-1}(\Omega_s),H_0^1(\Omega_s)}\bigg]\,\mathrm{d}s = 0, \tag{4.5}$$

for all $t\in[0,T]$ and $\phi\in Y$. Then $y\in L^\infty(Q)$ and for all $t\in[0,T]$:

$$-\frac{1}{c}\operatorname*{ess\,sup}_{t\in(0,T)}\big|g(t)^-\big|_\infty \leqslant y(t,x) \leqslant \frac{1}{c}\operatorname*{ess\,sup}_{t\in(0,T)}\big|g(t)^+\big|_\infty \quad \text{for a.e. } x\in\Omega_t.$$

Proof. Let $\bar{g} = \operatorname{ess\,sup}_{t\in(0,T)}|g(t)^+|_{\infty,\Omega_t}$, set $\phi = (y-\bar{g}/c)^+$ and observe that $\phi\in Y$. Below we shall use repeatedly that for $y\in Y$, the traces $y(t)=y(t,\cdot)\in L^2(\Omega_t)$ for each $t\in[0,T]$, and $y(t,\cdot)\in H^1(\Omega_t)$ for a.e. $t\in(0,T)$. Since $a(1,\phi(t))\geqslant 0$, it follows from (4.5) that

$$\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big(y(s)-\frac{\bar{g}}{c}\Big),\phi(s)\Big)_{\Omega_t} + \int_0^t a\Big(y(s)-\frac{\bar{g}}{c},\phi(s)\Big) + c\int_0^t \Big(y(s)-\frac{\bar{g}}{c},\phi(s)\Big)_{\Omega_s}$$
$$\leqslant \int_0^t \big\langle g(s)-\bar{g},\phi\big\rangle_{H^{-1}(\Omega_s),H_0^1(\Omega_s)} \leqslant 0,$$

and thus by Green's formula [8]:

$$\frac{1}{2}\left|\left(y(t)-\frac{\bar{g}}{c}\right)^+\right|^2_{\Omega_t}+\int_0^t a\left(y(s)-\frac{\bar{g}}{c},\left(y(s)-\frac{\bar{g}}{c}\right)^+\right)\mathrm{d}s\leqslant 0.$$

Since by assumption $a(v,v^+)\geqslant 0$ for $v\in H^1(\Omega_t)$, this implies that for each $t\in[0,T]$, we have:

$$y(t,x)\leqslant \operatorname*{ess\,sup}_{t\in(0,T)}\left|g^+(t)\right|_\infty,$$

for a.e. $x\in\Omega_t$. The estimate from below can be verified analogously. □

Let us introduce the active and inactive sets associated to the solution y^* of (2.1):

$$\mathcal{A}^*=\left\{(t,x)\in(0,T)\times\Omega\colon\ y^*(t,x)=\psi(x)\right\},$$
$$\mathcal{I}^*=\left\{(t,x)\in(0,T)\times\Omega\colon\ y^*(t,x)<\psi(x)\right\},$$

with boundaries $\partial\mathcal{A}^*$ and $\partial\mathcal{I}^*$, respectively.

4.1. Case I

Here we consider the case when $\bar{\lambda}=0$. Recall that by the monotonicity result Corollary 2 we have:

$$y^*\leqslant y_{\hat{c}}\leqslant y_c,$$

for $0<c\leqslant\hat{c}<\infty$. Define

$$\mathcal{A}_c=\left\{(t,x)\in(0,T)\times\Omega\colon\ y_c(t,x)>\psi(x)\right\}.$$

Then for $0<c\leqslant\hat{c}<\infty$

$$\mathcal{A}^*\subset\mathcal{A}_{\hat{c}}\subset\mathcal{A}_c.$$

This inclusion holds in the a.e. sense. If $\psi\in C(\bar{\Omega})$, then due to (4.4) we have that for a.e. t the inclusion $\mathcal{A}^*(t)=\{x\in\Omega\colon\ y^*(t,x)>\psi(x)\}\subset\mathcal{A}_c(t)=\{x\in\Omega\colon\ y_c(t,x)>\psi(x)\}$ holds for all $x\in\Omega$.

Theorem 4.1. *Assume that* (4.1)–(4.3) *hold, that* $\psi\in C(\bar{\Omega})$ *and that* $\mathcal{A}^*$ *and* $\mathcal{A}_c$, $c>0$, *are domains in* R^{n+1} *with Lipschitz continuous boundaries. Then, for every* $c>0$ *and* $t\in[0,T]$,

$$\left|y_c(t)-y^*(t)\right|_{L^\infty(\Omega)}\leqslant\frac{1}{c}\operatorname*{ess\,sup}_{t\in(0,T)}\left|\left(A\psi+f(\cdot)\right)^+\right|_\infty.$$

Proof. We recall the regularity properties (4.4) as well as that $\mathcal{A}^* \subset \mathcal{A}_c$ for every $c > 0$. From the definition of $\mathcal{A}_c$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_c = A(y_c - \psi) - c(y_c - \psi) + A\psi + f(t) \quad \text{in } \mathcal{A}_c,$$
$$y_c - \psi = 0 \quad \text{on } \partial\mathcal{A}_c \setminus \{(T, x) \in \bar{\mathcal{A}}_c\}.$$

From the proof of Lemma 4.1 with $Q = \mathcal{A}_c$ and $g = A\psi + f$, we find:

$$\sup_{t\in[0,T]} \left|y_c(t) - \psi\right|_{L^\infty(\Omega_t)} \leqslant \frac{1}{c} \operatorname*{ess\,sup}_{t\in(0,T)} \left|\left(A\psi + f(\cdot)\right)^+\right|_{\infty,\Omega_t}$$
$$\leqslant \frac{1}{c} \operatorname*{ess\,sup}_{t\in(0,T)} \left|\left(A\psi + f(\cdot)\right)^+\right|_{\infty}, \tag{4.6}$$

where $\Omega_t = \{x\colon (t, x) \in \mathcal{A}_c\}$.

We turn to the estimate on $\mathcal{I}^*$. Let $\Sigma = \{(t, x) \in \partial\mathcal{I}^*\colon t \in (0, T)\}$ denote the lateral boundary of $\mathcal{I}^*$ and set $\Sigma_t = \{x\colon (t, x) \in \Sigma\}$. Note that Σ_t is defined in the pointwise everywhere sense for a.e. $t \in (0, T)$, since $y^*(t) - \psi \in C(\bar{\Omega})$ for a.e. $t \in (0, T)$. For a.e. t we have $y_c(t, \cdot) - \psi \geqslant 0$ on Σ_t. Therefore $\alpha = \operatorname{ess\,sup}_{t\in(0,T)} |y_c(t, \cdot) - \psi|_{L^\infty(\Sigma_t)}$ and $y_c - y^* = y_c - \psi \geqslant 0$ a.e. on $\partial\mathcal{I}^*$ are well defined. Note that

$$\alpha \leqslant \operatorname*{ess\,sup}_{t\in(0,T)} \left|y_c(t, \cdot) - \psi\right|_{L^\infty(\mathcal{A}_{c,t})} \leqslant \frac{1}{c} \left|\max(0, A\psi + f)\right|_{L^\infty(Q)},$$

where $\mathcal{A}_{c,t} = \{x\colon (t, x) \in \mathcal{A}_c\}$. On $\mathcal{I}^*$ we have:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(y_c - y^*) - A(y_c - y^*)+ = \lambda^* - \lambda_c \leqslant 0, & \text{on } \mathcal{I}^*, \\ y_c - y^* = y_c - \psi \geqslant 0 \text{ on } \partial\mathcal{I}^*, & y_c - y^* = 0 \text{ on } \{(0, x) \in \mathcal{I}^*\}, \end{cases}$$

and therefore $y = y_c - y^*$ satisfies (4.5) with $c = 0$, $g = \lambda^* - \lambda_c \leqslant 0$ and $Q = \mathcal{I}^*$. Setting $\phi = (y(t) - \alpha)^+ \in Y$ in (4.5), it follows with the arguments as in the proof of Lemma 4.1 that for all $t \in [0, T]$:

$$\left|y_c(t) - y^*(t)\right|_{L^\infty(\mathcal{I}_t^*)} \leqslant \alpha \leqslant \frac{1}{c} \operatorname*{ess\,sup}_{t\in(0,T)} \left|\left(A\psi + f(\cdot)\right)^+\right|_{\infty}. \tag{4.7}$$

where $\mathcal{I}_t^* = \{x\colon (t, x) \in \mathcal{I}^*\}$. Combining (4.6) and (4.7) implies the desired estimate. □

4.2. Case II

We choose $\bar{\lambda}(t) = |(A\psi + f(t))^+|_\infty$ and note that by (4.2) we have $\bar{\lambda} \in L^\infty(0, T)$. The monotonicity result Corollary 4 implies that $y_c \leqslant y_{\hat{c}} \leqslant y^*$, for $0 < c \leqslant \hat{c} < \infty$. Define

$$\mathcal{A}_c = \left\{(t, x) \in (0, T) \times \Omega\colon \lambda_c(t, x) > 0\right\},$$

where $\lambda_c(t) = \max(0, \bar{\lambda}(t) + c(y_c(t) - \psi)) \leqslant \max(0, \bar{\lambda}(t))$. Then $\mathcal{A}_c \subset \mathcal{A}^*$.

Theorem 4.2. *Assume that* (4.1)–(4.3) *hold, and that $\mathcal{A}^*$ and $\mathcal{A}_c$ are domains with Lipschitz continuous boundaries. Then*

$$|y_c - y^*|_{L^\infty((0,T)\times\Omega)} \leqslant \frac{1}{c} \operatorname*{ess\,sup}_{t\in(0,T)} \left|\left(A\psi + f(\cdot)\right)^+\right|_\infty.$$

Proof. On $\mathcal{A}_c$ we have $\bar{\lambda} + c(y_c - \psi) \geqslant 0$, $y^* = \psi$ and $y_c \leqslant \psi$ a.e. Hence

$$|y^* - y_c|_{L^\infty(\mathcal{A}_c)} \leqslant \frac{1}{c}|\bar{\lambda}|_{L^\infty(0,T)}.$$

Note that $\alpha := \operatorname{ess\,sup}_{(t,x)\in L^\infty(\mathcal{A}_c)} |y^*(t,x) - y_c(t,x)| = \operatorname{ess\,sup}_{(0,T)} |y^*(t) - y_c(t)|_{L^\infty(\Omega_t)}$, where $\Omega_t = \{x\colon (t,x) \in \mathcal{A}_c\}$, and that $y^*(t) - y_c(t) \in C(\bar{\Omega})$, for a.e. $t \in (0,T)$. Consequently, for a.e. $t \in (0,T)$, we have:

$$|y^*(t) - y_c(t)|_{L^\infty(\Sigma_t)} \leqslant |y^*(t) - y_c(t)|_{C(\bar{\Omega}_t)} \leqslant \alpha,$$

where $\Sigma_t = \{x\colon (t,x) \in \Sigma\}$ and $\Sigma = \{(t,x) \in \partial\mathcal{I}^*\colon t \in (0,T)\}$. On $\mathcal{I}^*$ we have:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(y^* - y_c) - A(y^* - y_c) = \lambda_c - \lambda^* \leqslant 0 \quad \text{in } \mathcal{I}_c, \\ y^* - y_c \geqslant 0 \text{ on } \partial\mathcal{I}_c \quad \text{and} \quad y^* - y_c = 0 \text{ on } \{(0,x) : \bar{\mathcal{I}}_c\}. \end{cases}$$

Taking the inner product with $\phi = (y^* - y_c - \alpha)^+$ implies that

$$|y^* - y_c|_{L^\infty(\mathcal{I}_c)} \leqslant \alpha \leqslant \frac{1}{c}|\bar{\lambda}|_{L^\infty(0,T)}. \qquad \square$$

Remark. If $A\psi \in L^\infty(\Omega)$, then Theorem 4.2 holds when $\bar{\lambda}(t,x) = \max(0, A\psi(x) + f(t,x))$, when max defined pointwise a.e. in Ω.

5. Bilateral constraints

In this section we consider (2.1) with bilateral constraints, i.e., the closed convex set $\mathcal{C}$ is given by:

$$\mathcal{C} = \{y \in H\colon \varphi \leqslant y \leqslant \psi\} \cap V,$$

and it is assumed to be nonempty. We assume that $f \in C([0,T]; H)$, and that $\varphi, \psi \in X$ satisfy $A\varphi \in H$, $A\psi \in H$,

$$S_1(t) = \left\{x \in \Omega\colon A\psi + f(t) > 0\right\} \cap S_2(t) = \left\{x \in \Omega\colon A\varphi + f(t) < 0\right\} \text{ is empty,} \tag{5.1}$$

for all $t \in [0, T]$ and that there exists a $c_0 > 0$ such that

$$-A(\psi - \varphi) + c_0(\psi - \varphi) \geqslant 0 \quad \text{a.e. in } \Omega. \tag{5.2}$$

In (5.1) the inequalities must be interpreted in the a.e. sense with respect to $x \in \Omega$. Let $\bar{\lambda}(t) \in H$ be defined by:

$$\bar{\lambda}(t) = \begin{cases} A\psi + f(t), & x \in S_1(t), \\ A\varphi + f(t), & x \in S_2(t), \\ 0, & \text{otherwise}. \end{cases} \tag{5.3}$$

We consider the regularized finite difference equations:

$$\left(\frac{y^{k+1} - y^k}{\Delta t}, \phi\right) + a\big(y^{k+1}, \phi\big) + \big(\lambda_c^{k+1}, \phi\big) - \big(f^k, \phi\big) = 0, \quad \text{for all } \phi \in V, \tag{5.4}$$

where $y^0 = y_0$, $f^k = f((k+1)\Delta t)$, and

$$\lambda_c^{k+1} = \max\big(0, \bar{\lambda}^k + c\big(y^{k+1} - \psi\big)\big) + \min\big(0, \bar{\lambda}^k + c\big(y^{k+1} - \varphi\big)\big), \tag{5.5}$$

with $\bar{\lambda}^k$ to be defined below. Then we have:

Theorem 5.1. *Assume that $\varphi, \psi \in X$ satisfy* (5.1)–(5.3)*, $(y - \psi)^+, (y - \varphi)^- \in V$ for all $y \in V$ and*

$$a(y, y^+) \geqslant 0 \quad \textit{for all } y \in X \textit{ with } y^+ \in V.$$

If $y_0 \in \mathcal{C}$, then the solution y_c^k to (5.4) *with*

$$\bar{\lambda}^k = \begin{cases} A\psi + f^k, & \textit{if } A\psi + f^k > 0, \\ A\varphi + f^k, & \textit{if } A\varphi + f^k < 0, \\ 0, & \textit{otherwise}, \end{cases}$$

defined a.e. with respect to $x \in \Omega$, satisfies $y_c^k \in \mathcal{C}$ for each $c > 0$ and all $k \geqslant 0$.

Proof. Since $y \to \max(0, \bar{\lambda}^k + c(y - \psi)) + \min(0, \bar{\lambda}^k + c(y - \varphi)) \in H$ is Lipschitz continuous and monotone, existence of a solution to (5.4) follows with the same arguments as in the proof of Theorem 1, provided that Δt is sufficiently small. We now show by induction that $y^k \in \mathcal{C}$ for all k. For $y^k \in \mathcal{C}$, we have:

$$\frac{1}{\Delta t}\big(y^{k+1} - \psi, \big(y^{k+1} - \psi\big)^+\big) + a\big(y^{k+1} - \psi, \big(y^{k+1} - \psi\big)^+\big)$$
$$+ \big(-\big(A\psi + f^k\big) + \lambda_c^{k+1}, \big(y^{k+1} - \psi\big)^+\big) = \frac{1}{\Delta t}\big(y^k - \psi, \big(y^{k+1} - \psi\big)^+\big) \leqslant 0.$$

On the set $\{x\colon\ y^{k+1} \geqslant \psi\}$ we have pointwise a.e.:
If $\bar\lambda^k(x) > 0$, then

$$-\big(A\psi + f^k\big) + \lambda_c^{k+1} = c\big(y^{k+1} - \psi\big) \geqslant 0,$$

if $\bar\lambda^k(x) = 0$, then

$$-\big(A\psi + f^k\big) + \lambda_c^{k+1} \geqslant c\big(y^{k+1} - \psi\big) \geqslant 0,$$

if $\bar\lambda^k(x) < 0$, then $(A\psi + f^k)(x) \leqslant 0$ and

$$-\big(A\psi + f^k\big) + \lambda_c^{k+1} \geqslant \min\big(0, -A(\psi - \varphi) + c(\psi - \varphi)\big) \geqslant 0,$$

for $c \geqslant c_0$. Thus $(-(A\psi + f^k) + \lambda_c^{k+1}, (y^{k+1} - \psi)^+) \geqslant 0$ and for $\Delta t > 0$ sufficiently small, $|(y^{k+1} - \psi)^+|_H^2 \leqslant 0$ and thus $y^{k+1} \leqslant \psi$. Similarly, one can prove that $y^{k+1} \geqslant \varphi$ a.e. in Ω by choosing the test function as $(y^{k+1} - \varphi)^- \in V$, and thus $y^{k+1} \in \mathcal{C}$. □

Theorem 5.2. *If the assumptions of Theorem* 5.1 *and* (2.6) *hold, then* $y_c = \lim_{\Delta t \to 0} y_{\Delta t}^{(1)}$ *weakly in* $W(0,T)$ *as* $\Delta t \to 0^+$, $y_c \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A)) \cap C(0,T;V)$, *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} y_c(t) - Ay_c(t) + \lambda_c(t) = f(t), \quad \textit{with } y_c(0) = y_0, \tag{5.6}$$

where

$$\lambda_c(t) = \max\big(0, \bar\lambda(t) + c\big(y_c(t) - \psi\big)\big) + \min\big(0, \bar\lambda(t) + c\big(y_c(t) - \varphi\big)\big),$$

$y_c(t) \in \mathcal{C}$ *for all* $c > 0$ *and* $t \in [0,T]$. *Moreover,* $y^* = \lim_{c \to 0} y_c$ *weakly in* $W(0,T)$, *and* $\lambda^* = \lim_{c\to 0} \lambda_c$ *weakly in* $L^2(0,T;H)$, *satisfy* $y^* \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A)) \cap C(0,T;V)$ *satisfy*:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y^*(t) - Ay^*(t) + \lambda^*(t) = f(t), \quad y^*(0) = y_0, \\ \lambda^*(t) = \max(0, \lambda^*(t) + c(y^*(t) - \psi)) + \min(0, \lambda^*(t) + c(y^*(t) - \varphi)). \end{cases} \tag{5.7}$$

Proof. From (5.5) and Theorem 5.1 it follows that $|\lambda_c^{k+1}| \leqslant |\bar\lambda^k|$ a.e. in Ω for all k. Thus we can proceed as in the proof of Corollary 4 and obtain the existence of a unique $y_c(t) \in H^1(0,T;H) \cap L^2(0,T;\mathrm{dom}(A))$ satisfying $y_c(t) \in \mathcal{C}$ for each for $c > 0$ and $t \in [0,T]$. Moreover,

$$\big|\lambda_c(t)\big| \leqslant \big|\bar\lambda(t)\big| \quad \text{for a.e. } t \in (0,T).$$

Proceeding as in the proof of Theorem 4 we obtain y^* and λ^*, with the specified regularity properties and such that the first equation in (5.7) is satisfied. Moreover $y_c \to y^*$ strongly in $L^2(0,T;H)$ and $\lambda_c \to \lambda^*$ weakly in $L^2(0,T;H)$ as $c \to \infty$. It remains to verify the complementarity conditions.

Without loss of the generality y_c converges to $y^*(t)$ a.e. in $(0, T) \times \Omega$ and hence $y^*(t) \in \mathcal{C}$, since $y_c(t) \in \mathcal{C}$, for all $t \in [0, T]$. Moreover $\lambda_c(t) \geqslant 0$ a.e. on $S_1(t)$ for a.e. t implies that $\lambda^*(t) \geqslant 0$ a.e. on $S_1(t)$ for a.e. t. Thus $\int_0^T (\lambda^*(t), y^*(t) - \psi)_{L^2(S_1(t))}\, \mathrm{d}t \leqslant 0$. Since

$$0 \leqslant \int_0^T \left(\lambda_c(t), \left(y_c(t) - \psi\right) + \frac{1}{c}\bar{\lambda}(t)\right)_{L^2(S_1(t))} \mathrm{d}t \to \int_0^T \left(\lambda^*(t), y^*(t) - \psi\right)_{L^2(S_1(t))} \mathrm{d}t,$$

it follows that

$$\int_0^T \left(\lambda^*(t), y^*(t) - \psi\right)_{L^2(S_1(t))} \mathrm{d}t = 0.$$

Similarly, we have:

$$\int_0^T \left(\lambda^*(t), y^*(t) - \varphi\right)_{L^2(S_2(t))} \mathrm{d}t = 0.$$

Hence $(y^*(t), \lambda^*(t))$ satisfies the complementarity condition. □

6. Numerical result for Black–Scholes model

In this section we present a numerical result for the Black–Scholes model for the American put option. We let $\sigma = 0.3$, $r = 0.06$, $\delta = 0$ and $K = 10$. For these parameter choices we have $\bar{S} \sim 5.7$ according to Section 3 and thus we take $[5, \infty)$ as our computational domain. In order deal with the semi-infinite domain we use a decomposition technique. That is, on $[5, 15]$ we use the original coordinate and on $[15, \infty)$ we employ the coordinate transform $S = \mathrm{e}^x$. The resulting transformed equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t} v + \frac{\sigma^2}{2} v_{xx} + r v_x - r v = 0, \tag{6.1}$$

on $x \in (\log(15), \infty)$. An advantage of the equation in transformed coordinates is that it allows to effectively treat the far-field condition. As boundary condition we use:

$$\frac{\sigma^2}{2} v_x + r v = 0, \quad x = \bar{X},$$

for sufficiently large $\bar{X}$. This boundary condition is satisfied asymptotically by the asymptotic solution $\bar{v}$ in Section 3. We use the central difference schemes space-wise with uniform grids on $[5, 15]$, and with non-uniform grids (successively doubling the step lengths towards infinity) for (6.1). For time discretisation the Crank–Nicolson scheme is

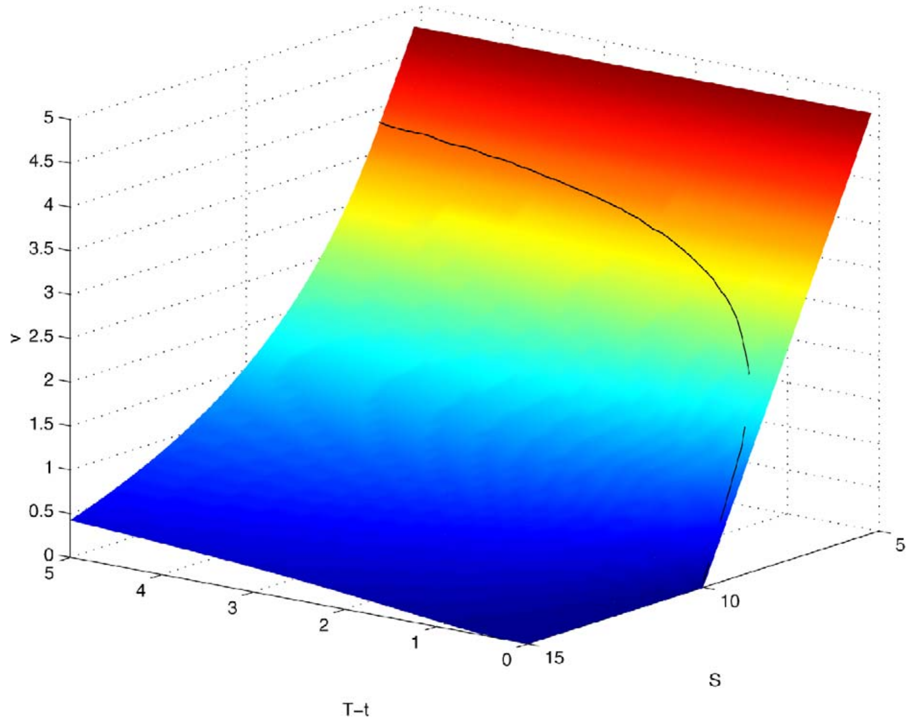

Fig. 1.

used and thus the method is second-order in time and space. We implement the feasible approximation method and which leads to solving nonlinear equations of the form:

$$V - AV + \min\bigl(0, \bar{\lambda} + c(V - \psi)\bigr) = F, \quad \text{with } \bar{\lambda} = -rK, \tag{6.2}$$

on $[5, \bar{X})$. The semi-smooth Newton method [5,11] is used to solve (6.2). As expected it converges in finite step. In Fig. 1 the value function v and the free curve $S(t)$ based on

$$\lambda_c(t) = \min\bigl(0, \bar{\lambda} + c\bigl(V\bigl(t, S(t)\bigr) - \psi\bigr)\bigr)$$

are shown.

References

[1] J.-P. Aubin, Applied Functional Analysis, John Wiley and Sons, New York, 1979.

[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff Int. Publ., Leyden, 1976.

[3] C. Baiocchi, A. Capelo, Variational an Quasivariational Inequalities, John Wiley and Sons, New York, 1984.

[4] A. Bensoussan, J.L. Lions, Applications of Variational Inequalities in Stochastic Control, North-Holland, Amsterdam, 1982.

[5] M. Hintermüller, K. Ito, K. Kunisch, The primal-dual active set strategy as a semi-smooth Newton method, SIAM J. Optim. 13 (2003) 865–888.
[6] R. Dautray, J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vol. 5, Springer-Verlag, Berlin, 1992.
[7] R. Glowinski, J.L. Lions, R. Tremolierès, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.
[8] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, London, 1985.
[9] K. Ito, K. Kunisch, Augmented Lagrangian methods for nonsmooth convex optimization in Hilbert spaces, Nonlinear Anal. 41 (2000) 573–589.
[10] K. Ito, K. Kunisch, An augmented Lagrangian technique for variational inequalities, Appl. Math. Optim. 21 (1990) 223–241.
[11] K. Ito, K. Kunisch, Semi-smooth Newton methods for variational inequalities of the first kind optimal control of elliptic variational inequalities, RAIRO Math. Model. Numer. Anal., in press.
[12] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.
[13] M. Mignot, J.-P. Puel, Solution maximum de certaines inéquations d'évolution paraboliques, et inéquations quasi variationelles paraboliques, C. R. Acad. Sci. Paris 280 (1975) 259–262.
[14] B. Oksendal, Stochastic Differential Equations, Springer-Verlag, Berlin, 1998.
[15] H.H. Schäfer, Banach Lattices and Positive Operators, Springer-Verlag, Berlin, 1974.
[16] R. Seydel, Tools for Computational Finance, Springer-Verlag, Berlin, 2002.